The behavior of quadratic and differential forms under function field extensions in characteristic two

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Contents

Abstract

Let F be a field of characteristic 2. Let Ω_F^n be the F-space of absolute differential forms over F. There is a homomorphism $\wp : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$ given by

$$\wp(x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) = (x^2 - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \quad \mod d\Omega_F^{n-1}$$

Let $H^{n+1}(F) = \operatorname{Coker}(\wp)$. We study the behavior of $H^{n+1}(F)$ under the function field $F(\phi)/F$, where $\phi = \ll b_1, \ldots, b_n \gg$ is a n-fold Pfister form and $F(\phi)$ is the function field of the quadric $\phi = 0$ over F. We show that

$$\ker(H^{n+1}(F) \to H^{n+1}(F(\phi)) = \overline{F \cdot \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}}$$

Using Kato's isomorphism of $H^{n+1}(F)$ with the quotient $I^n W_q(F)/I^{n+1}W_q(F)$, where $W_q(F)$ is the Witt group of quadratic forms over F and $I \subset W(F)$ the maximal ideal of even dimensional bilinear forms over F, we deduce from the above result the analogue in characteristic 2 of Knebusch's degree conjecture, i.e. $I^n W_q(F)$ is the set of all classes $\overline{q}$ with $\deg(q) \geq n$.

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Introduction

Since Knebusch's seminal papers on generic splitting of quadratic forms appeared (see [Kn 1], [Kn 2]), few work until recently has been done on his degree conjecture, which asserts that the n-th power I^n of the ideal I of even dimensional forms in the Witt-ring $W(F)$ of symmetric non singular bilinear forms of a field F with $2 \neq 0$, coincides with the ideal of forms of degree $\geq n$. Compare [Kn 1], [Ar-Kn], the 1998-preprint [OVV] and [A-Ba 2]. A similar theory of generic splitting of quadratic forms over a field of characteristic 2 can be developed (see section 6 of this work) and the corresponding degree conjecture can be stated. The aim of this work is to prove this analogue of Knebusch's conjecture for fields with 2=0.

The advantage of working with fields of characteristic 2 is the fact discovered by K. Kato (see [Ka 1]), that there is a strong relationship between quadratic forms and differential forms defined over such fields (see section 5 of this work). Thus many problems concerning quadratic forms in characteristic 2 can be translated into the language of differential forms, which are sometimes easier to handle, in particular choosing a suitable 2-basis of the ground field. Let us briefly recall Kato's above mentioned correspondence. Let F be a field of characteristic 2 and let Ω_F^n be the space of n-differential forms over F (see [Ca], [Groth]). Let $d : \Omega_F^{n-1} \to \Omega_F^n$ be the usual differential operator which extends $d : F \to \Omega_F^1$, $a \mapsto da$. Then there is a well defined homomorphism $\wp : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$ given by

$$\wp(x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) = (x^2 - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \mod d\Omega_F^{n-1}$$

(see [Ka 1], [Mi] or section 1 of this paper).

Choosing a 2-basis of F one can easily lift $\wp$ to a homomorphism $\wp : \Omega_F^n \to \Omega_F^n$ which of course depends on the 2-basis (see section 1). Let $\nu_F(n) = \ker(\wp)$ and $H^{n+1}(F) = \mathrm{Coker}(\wp)$, so that there is a exact sequence

$$0 \to \nu_F(n) \to \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1} \to H^{n+1}(F) \to 0.$$

In [Ka 1] it is shown that there exists a natural isomorphism of groups

$$H^{n+1}(F) \xrightarrow{\sim} I^n W_q(F) / I^{n+1} W_q(F)$$

given by

$$\overline{b\frac{ba_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}} \mapsto \overline{\ll a_1, \ldots, a_n; b]]}.$$

Here $W_q(F)$ denotes the $W(F)$-module of non singular quadratic forms over F and $\ll a_1, \dots , a_n; b]]$ denotes the quadratic n-fold Pfister form defined by $a_1, \dots , a_n \in F^*$, $b \in F$ (see [Ka 1], [A-Ba 1] or section 5 of this work).

The generic splitting theory developed in section 6 enables us to define the degree $\deg(q)$ of a non singular quadratic form over a field F with 2=0 along the same lines as Knebusch does for fields with $2 \neq 0$. We show also that $I^nW_q(F) \subseteq \{\overline{q} \in W_q(F) | \deg(q) \geq n\}$. In [A-Ba 2] we have shown that equality holds (i.e. the degree conjecture holds) if one has the following equality:

$$\ker \left[\overline{I^nW_q(F)} \to \overline{I^nW_q(F(\phi))}\right] = \overline{\phi \cdot W_q(F)}$$

for any anisotropic n-fold bilinear Pfister form ϕ over F. Here $\overline{I^nW_q(F)}$ means $I^nW_q(F)/I^{n+1}W_q(F)$ and $F(\phi)$ is the function field of ϕ over F.

This last equality is according to Kato's isomorphism equivalent with

$$(*) \qquad \ker \left[H^{n+1}(F) \to H^{n+1}(F(\phi)\right] = \overline{F \cdot \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}}$$

in $H^{n+1}(F)$, where $\phi = \ll b_1, \dots , b_n \gg$. The proof of (*) will be finished in section 4 of this work. The corresponding result for fields with $2 \neq 0$ has been announced in [OVV].

In section 1 we review some well known definitions and results concerning differential forms over a field with $2 = 0$. We introduce the Cartier operator C and the $\wp$-homomorphism and we prove some technical results about divisibility of forms by forms in the F-algebra $\Omega_F = \bigoplus_{n=0}^{\infty} \Omega_F^n$ (see (1.16)). In section 2 we start to study the behavior of Ω^n and H^{n+1} under field extensions. The computation of $\ker(\Omega_F^n \to \Omega_E^n)$ for some field extension E/F is not so difficult if one can choose a suitable 2-basis of E. We prove for example $\ker(\Omega_F^m \to \Omega_{F(\phi)}^m) = \Omega_F^{m-n} \wedge db_1 \wedge \cdots \wedge db_n$ if $m \geq n$ (and = 0 otherwise). This is the first evidence for the equality (*). The computation of $\ker(H^{n+1}(F) \to H^{n+1}(E))$ is much harder, even for very simple extensions E/F. We use frequently Kato's fundamental lemma (see [Ka 2]), which is stated here without proof as lemma (2.15). As an important consequence of these computations we obtain that a form $w \in \Omega_F^n$ belongs to $\ker(H^{n+1}(F) \to H^{n+1}(F(\phi)))$ if and only if w satisfies a certain "differential equation" in the space Ω_L^n where L/F is a purely transcendental extension.

In fact, let S_n be the set of all maps $\mu : \{1, \dots , n\} \to \{0, 1\}$ with $\mu(i) = 1$ for at least one i. Let $L = F(X_\mu, \mu \in S_n)$ where X_μ are independent variables. Let $M = F(X_\mu^2, \mu \in S_n) \subset L$ and set $\Omega_F[M]$ for the sub-space of Ω_L (over M) generated by the forms $db, b \in F$, over M. For example if $T = \sum_{\mu\in S_n} b^\mu X_\mu^2$ with $b^\mu = \prod_{i=1}^n b_i^{\mu(i)}$, then $T \in M$ and $dT \in \Omega_F[M]$. Then we show that $w \in \Omega_F^n$ is contained in $\ker(H^{n+1}(F) \to H^{n+1}(F(\phi)))$ if and only if w satisfies an equation in Ω_L of the form

$$w = \wp(u) + dv + \lambda \wedge dT$$

with u, v, $\lambda \in \Omega_F[M]$ (see (2.25)). Section 3 is of technical nature and prepares the way for the proof of our main result in section 4. Section 4 is the heart of this work. We start with a relation $w = \wp(u)+dv+\lambda\wedge dT$ where u, v, $\lambda \in \Omega_F[M]$ and we develop a descent procedure to finally end with a relation $w = \wp(u_0) + dv_0 + \lambda_0 \wedge db_1 \wedge \cdots \wedge db_n$ where u_0, v_0, $\lambda_0 \in \Omega_F$. This is exactly the content of (*). In section 5 we explain briefly the basic relations between quadratic and differential forms. It is of expository character and details can be found in [Ka 1], [A-Ba 2]. Finally in section 6, as mentioned above, we extend Knebusch's generic splitting theory to fields with $2 = 0$ and prove the analogue of his degree conjecture.

1 The algebra of differential forms

We will consider in this paper only fields of characteristic 2. Let F be such a field. Let Ω^1_F be the F-vector space of absolute differential 1-forms, i.e. Ω^1_F is the F-vector space generated by the symbols da, $a \in F$, with the relations $d(a+b) = da + db$, $d(ab) = adb + bda$ for $a, b \in F$. In particular $d(F^2) = 0$, where $F^2 = \{a^2 | a \in F\}$ and $d : F \to \Omega^1_F$ is a F^2-derivation.

Let us denote by Ω^n_F the n-exterior power $\bigwedge^n(\Omega^1_F)$. Thus Ω^n_F is a F-vector space generated by the forms $da_1 \wedge \cdots \wedge da_n$. The operator d can be extended to a F^2-linear map $d : \Omega^n_F \to \Omega^{n+1}_F$ by $d(ada_1 \wedge \cdots \wedge da_n) = da \wedge da_1 \wedge \cdots \wedge da_n$. We will write Ω_F or Ω^*_F for the F-algebra $\oplus_{n=0}^{\infty} \Omega^n_F$. We denote by Z_F the F^2-sub algebra $\{w \in \Omega^*_F | dw = 0\}$. Since $d^2 = 0$, we obtain the ideal $B_F = d\Omega^*_F$ in Z_F of exact forms. Now let us fix a 2-basis $\mathcal{B} = \{b_i, i \in I\}$ of F over F^2, i.e. if we take an ordering of I, then the monomials $b_{i_1} \cdots b_{i_r}, i_1 < \cdots < i_r$, form a F^2 -basis of F (see [Ca] or [Groth] for details about p-basis). Then it is well known that the (logarithmic) differential forms $\left\{\frac{db_{i_1}}{b_{i_n}} \wedge \cdots \wedge \frac{db_{i_n}}{b_{i_n}}, i_1 < \cdots < i_n\right\}$ form a $F-$basis of Ω^n_F. The following fact is obvious.

(1.1) **Lemma.** *Let $\eta \in \Omega_F$ be a form which does not contain db for some $b \in \mathcal{B}$, in its expansion with respect to the above basis. Then $\eta \wedge db = 0$ implies $\eta = 0$.*

Let us denote by $\Omega^{[2]}_F$ the F^2-sub algebra of Ω^*_F generated by the logarithmic differential $db/b, b \in \mathcal{B}$. $\Omega^{[2]}_F$ depends on the choice of the 2-basis $\mathcal{B}$. Then a well known result of Cartier (see [Ca]) asserts that as an F^2- algebra we have

(1.2) $$Z_F = B_F \oplus \Omega^{[2]}_F$$

Moreover this decomposition is compatible with the graduation of Ω^*_F. Any $w \in Z_F$ can be written uniquely as

$$w = d\eta + \sum_{i_1 < \cdots < i_n} a^2_{i_1 \cdots i_n} \frac{db_{i_1}}{b_{i_1}} \wedge \cdots \wedge \frac{db_{i_n}}{b_{i_n}}$$

so that we can define a homomorphism

(1.3) $$C : Z_F \to \Omega^*_F$$

by

$$C(w) = \sum_{i_1 < \cdots < i_n} a_{i_1 \cdots i_n} \frac{db_{i_1}}{b_{i_1}} \wedge \cdots \wedge \frac{db_{i_n}}{b_{i_n}}$$

C is the well known Cartier-operator (see [Ca]) and it is uniquely determined by the following properties

$$C(a^2 \frac{db}{b}) = a \frac{db}{b} \tag{1.4}$$

$$C(d\eta) = 0 \quad \text{for} \quad \eta \in \Omega^{\cdot}_F \tag{1.5}$$

$$C(a^2 w) = aC(w) \ , \ a \in F \ , \ w \in Z_F \tag{1.6}$$

$$C(w \wedge \lambda) = C(w) \wedge C(\lambda) \tag{1.7}$$

In particular C does not depend on the choice of the $2-$basis and $\ker(C) = B_F$, $\mathrm{Im}(C) = \Omega^*_F$. Thus we get a ring isomorphism (compatible with the graduation)

$$C : Z_F / B_F \xrightarrow{\sim} \Omega^*_F \tag{1.8}$$

For a fixed 2-basis $\mathcal{B} = \{b_i, i \in I\}$ of F we define the square operator

$$s : \Omega^*_F \to \Omega^*_F \tag{1.9}$$

by

$$s\left(\sum_\sigma a_\sigma \frac{db_\sigma}{b_\sigma}\right) = \sum_\sigma a_\sigma^2 \frac{db_\sigma}{b_\sigma}$$

where σ runs over tuples of indices $i_1 < \cdots < i_p$ and $b_\sigma = b_{i_1} \cdots b_{i_p}$, $db_\sigma = db_{i_1} \wedge \cdots \wedge db_{i_p}$. Of course s depends on the choice of $\mathcal{B}$. We will write $w^{[2]}$ instead of $s(w)$ when we have a fixed 2-basis of F. Using (1.9) we also define the following operator

$$\begin{aligned} \wp : \ & \Omega^*_F \to \Omega^*_F \\ & \wp(w) = w^{[2]} + w \end{aligned} \tag{1.10}$$

Since s is additive, $\wp$ is additive too and depends on the choice of the 2-basis $\mathcal{B}$. But any other choice of a 2-basis changes $\wp(w)$ by an exact form, i.e. we get a well defined group-homomorphism

$$\wp : \Omega_F^* \to \Omega_F^*/d\Omega_F^*$$

which for every n defines a homomorphism

$$\wp : \Omega_F^n \to \Omega_F^n/d\Omega_F^{n-1} \tag{1.11}$$

Using (1.8) one easily checks that

$$\wp = C^{-1} - id$$

We now derive some useful properties of the Cartier operators, which will be used frequently in the next sections.

(1.12) **Proposition.** *Let $\{b_1, \dots, b_n\}$ be elements of a 2-basis $\mathcal{B}$ of F. Let $\lambda \in \Omega_F^*$ be such that $d(\lambda \wedge db_1 \wedge \cdots \wedge db_n) = 0$. Then there is $\delta \in \Omega_F^*$ with*

$$C(\lambda \wedge db_1 \wedge \cdots \wedge db_n) = \delta \wedge db_1 \wedge \cdots \wedge db_n.$$

Proof. We apply induction with respect to n. Assume first $n = 1$, i.e. let $u = \lambda \wedge db$ be a closed form and $b \in \mathcal{B}$. From $du = 0$ we conclude from (1.2) that $u = dn + m^{[2]}$ with some forms m, n. Write $m = m_0 + m_1 \wedge \frac{db}{b}$ where m_0, m_1 are forms which do not contain db in the basis representation with respect to $\mathcal{B}$. We can also write $n = n_0 + bn_1 + n_2 \wedge db + bn_3 \wedge db$ where n_0, n_1, n_2, n_3 do not contain odd powers of b in their coefficients nor db in their 2-basis expansion. Thus

$$\begin{aligned} u &= m_0^{[2]} + m_1^{[2]} \wedge \tfrac{db}{b} + dn_0 + bdn_1 + n_1 \wedge db \\ &+ dn_2 \wedge db + bdn_3 \wedge db \\ &= \lambda \wedge db \end{aligned}$$

Comparing terms with db we obtain

$$m_0^{[2]} + dn_0 + bdn_1 = 0$$

$$u = \lambda \wedge db = m_1^{[2]} \wedge \tfrac{db}{b} + n_1 \wedge db + dn_2 \wedge db + bdn_3 \wedge db$$

The second equation and the assumption $du = 0$ imply $d(n_1 \wedge db) = dn_1 \wedge db = 0$.

But since n_1 does not contain db as well b as odd power in its coefficients, we see that dn_1 does not contain db in its basis expansion. Thus (1.1) implies $dn_1 = 0$. Thus $n_1 \wedge db = d(bn_1)$, and since $dn_2 \wedge db = d(n_2 \wedge db)$, $bdn_3 \wedge db = d(bn_3 \wedge db)$ are exact, we conclude

$$u = m_1^{[2]} \wedge \frac{db}{b} + dv$$

with some form v. Applying the Cartier-operator to u we obtain

$$C(u) = m_1 \wedge \frac{db}{b}$$

and $\delta = b^{-1}m_1$ does the job. Let us now assume the proposition for all integers less than n. Let $u = \lambda \wedge db_1 \wedge \cdots \wedge db_n$ with $du = 0$ and $b_1, \ldots, b_n \in \mathcal{B}$. By induction

$$C(u) = \delta \wedge db_2 \wedge \cdots \wedge db_n = \mu \wedge db_1 \wedge db_3 \wedge \cdots \wedge db_n$$

with some forms δ, μ. We can assume that δ is free from terms containing $db_2, \ldots, db_n$, respectively μ is free from terms containing $db_1, db_3, \ldots, db_n$, in their basis expansion. Set $\delta = \delta_0 + \delta_1 \wedge db_1$ where δ_0, δ_1 are free from terms containing db_1. Then

$$\begin{aligned} \delta_0 \wedge db_2 \wedge \cdots \wedge db_n \quad &+ \quad \delta_1 \wedge db_1 \wedge db_2 \wedge \cdots \wedge db_n \\ &= \quad \mu \wedge db_1 \wedge db_3 \wedge \cdots \wedge db_n. \end{aligned}$$

Since $\delta_0 \wedge db_2 \wedge \cdots \wedge db_n$ is free from terms containing db_1 we conclude $\delta_0 \wedge db_2 \wedge \cdots \wedge db_n = 0$ and $C(u) = \delta_1 \wedge db_1 \wedge db_2 \wedge \cdots \wedge db_n$. This proves the claim. □

(1.13) **Proposition.** *Let $\{b_1, \ldots, b_n\}$ be elements belonging to a 2-basis $\mathcal{B}$ of F. Let $v \in \Omega_F$ be such that*

$$dv \in \Omega_F \wedge db_1 \wedge \cdots \wedge db_n.$$

Then there exist forms $z_\mu \in Z_F$ and $u \in \Omega_F$ with

$$dv = \left(\sum_{\mu \neq \mathbf{1}} b^\mu z_\mu + b_1 \cdots b_n du \right) \wedge db_1 \wedge \cdots \wedge db_n$$

where μ runs over all functions $\{1, \dots, n\} \to \{0,1\}$ distinct from $\mathbf{1}$ given by $\mathbf{1}(i) = i$ for all $1 \le i \le n$. Here we have set $b^\mu = \prod_{i=1}^n b_{\mu(i)}$. Moreover the forms z_μ and u can be chosen free from terms containing $db_1, \dots, db_n$ as well as coefficients containing odd powers of $b_1, \dots, b_n$ in their 2-basis expansion.

Proof. We show the claim by induction on n. Let us first assume $n = 1$, i.e. $dv \in \Omega_F \wedge db$ with $b = b_1 \in \mathcal{B}$. We write

$$v = v_0 + bv_1 + v_2 \wedge db + bv_3 \wedge db$$

where v_0, v_1, v_2, v_3 are free from terms containing db and also do not contain odd powers of b in the expansion of their coefficients with respect to the 2-basis $\mathcal{B}$. Then we have

$$\begin{aligned} dv &= dv_0 + bdv_1 + v_1 \wedge db + dv_2 \wedge db + bdv_3 \wedge db \\ &= dv_0 + bdv_1 + (v_1 + dv_2 + bdv_3) \wedge db \end{aligned}$$

By the choice of v_0, v_1 we see that dv_0, dv_1 do not contain terms involving db in their basis expansion with respect to $\mathcal{B}$. But by hypothesis $dv \in \Omega_F \wedge db$, so that we get

$$dv_0 = 0, \;\; dv_1 = 0$$

since dv_0 and dv_1 do not have coefficients containing odd powers of b in their 2-basis expansion. Hence

$$dv = (v_1 + dv_2 + bdv_3) \wedge db$$

is of the desired form.

Let us assume the assertion for any integer less than n. Then from $dv \in \Omega_F \wedge db_1 \wedge \cdots \wedge db_n$ we conclude

$$dv = \left(\sum_{v \in S_{n-1}, \nu \neq \mathbf{1}} b^\nu z_\nu + b_1 \cdots b_{n-1} du \right) \wedge db_1 \wedge \cdots \wedge db_{n-1}$$

where $S_r = \{\nu : \{1, \dots, r\} \to \{0,1\}\}$ is the set of all functions $\{1, \dots, r\} \to \{0,1\}$. Here all z_ν are closed and the z_ν, u are free from $b_1, \dots, b_{n-1}$. We write

$$\begin{aligned} z_\nu &= c_{\nu,0} + b_n c_{\nu,1} + c_{\nu,2} \wedge db_n + b_n c_{\nu,3} \wedge db_n \\ u &= u_0 + b_n u_1 + u_2 \wedge db_n + b_n u_3 \wedge db_n \end{aligned}$$

with $c_{\nu,i}$, u_i free from b_n (i.e. do not contain db_n as well as odd powers of b_n in the 2-basis expansion of their coefficients). Since $dz_\nu = 0$, we have

$$0 = dc_{\nu,0} + b_n dc_{\nu,1} + c_{\nu,1} \wedge db_n + dc_{\nu,2} \wedge db_n + b_n dc_{\nu,2} \wedge db_n$$

and the choice of the $c_{\nu,i}$ is imply

$$dc_{\nu,0} = 0, \quad dc_{\nu,1} = 0$$

$$c_{\nu,1} + dc_{\nu,2} + b_n dc_{\nu,3} = 0.$$

This last equation implies

$$c_{\nu,1} = dc_{\nu,2}, \quad dc_{\nu,3} = 0.$$

Hence

$$dv = \left(\sum_{\nu \in S_{n-1}, \neq \mathbf{1}} b^\nu (c_{\nu,0} + b_n dc_{\nu,2}) + b^{\mathbf{1}} (du_0 + b_n du_1) \right) \wedge db_1 \wedge \cdots \wedge db_{n-1} +$$
$$+ \left(\sum_{\nu \in S_{n-1}, \neq \mathbf{1}} b^\nu (c_{\nu,2} + b_n c_{\nu,3}) + b^{\mathbf{1}} (u_1 + du_2 + b_n du_3) \right) \wedge db_1 \wedge \cdots \wedge db_n$$

The expression in the first parenthesis does not contain $db_1, \ldots, db_n$ and since $dv \in \langle db_1 \wedge \cdots \wedge db_n \rangle$, we conclude (see (1.1))

$$\sum_{v \in S_{n-1}, \neq \mathbf{1}} b^\nu (c_{\nu,0} + b_n dc_{\nu,2}) + b_1 \cdots b_{n-1} du_0 + b_1 \cdots b_n du_1 = 0$$

Since by hypothesis $c_{\nu,0}$, $c_{\nu,2}$, u_0, u_1 do not contain coefficients with odd powers of b_n in their 2-basis expansion, we conclude

$$\sum_{\nu \in S_{n-1}, \neq \mathbf{1}} b^\nu dc_{\nu,2} + b_1 \cdots b_{n-1} du_1 = 0$$

By induction all $c_{\nu,2}, u_1$ are free from $b_1, \ldots, b_{n-1}$. Then we obtain $dc_{\nu,2} = 0$, $du_1 = 0$. Since

$$\begin{aligned} dv \;&=\; \Big(\textstyle\sum_{\nu \in S_{n-1}, \neq \mathbf{1}} b^\nu c_{\nu,2} + \sum_{\nu \in S_{n-1}, \neq \mathbf{1}} b_n b^\nu c_{\nu,3} \\ &\quad + b^{\mathbf{1}} (u_1 + du_2) + b^{\mathbf{1}} b_n du_3 \Big) \wedge db_1 \wedge \cdots \wedge db_n \end{aligned}$$

is of desired form, this concludes the proof of the lemma. □

(1.14) **Remark.** Under the hypothesis of the lemma above, one can give a more precise description of the form v. By the lemma we have

$$dv = \left(\sum_{\mu \in S_n, \neq \mathbf{1}} b^\mu z_\mu + b^{\mathbf{1}} du \right) \wedge db_1 \wedge \cdots \wedge db_n$$

where $b^{\mathbf{1}} = b_1 \cdots b_n$, $dz_\mu = 0$ for all $\mu \neq \mathbf{1}$. Let μ be such a function and let $1 \leq i \leq n$ with $\mu(i) = 0$. Then

$$d(b_i b^\mu z_\mu \wedge db_1 \wedge \cdots \wedge db_{i-1} \wedge db_n)$$

$$= b^\mu z_\mu \wedge db_1 \wedge \cdots \wedge db_n + b_i db^\mu \wedge z_\mu \wedge db_1 \wedge \cdots \wedge db_{i-1} \wedge db_{i+1} \wedge \cdots \wedge db_n$$

Since b^μ contains only b_j with $j \neq i$, we see that the last term is 0 and hence

$$d(b_i b^\mu z_\mu \wedge db_1 \wedge \cdots \wedge db_i \wedge \cdots \wedge db_n) = b^\mu z_\mu \wedge db_1 \wedge \cdots \wedge db_n.$$

Also $b^{\mathbf{1}} du \wedge db_1 \wedge \cdots \wedge db_n = d(b^{\mathbf{1}} u \wedge db_1 \wedge \cdots \wedge db_n)$ and we conclude

$$dv = d\left(\sum\nolimits_{\substack{\mu \in S_n, \neq \mathbf{1} \\ \mu(i)=0}} b_i b^\mu z_\mu \wedge db_1 \wedge \cdots \wedge \hat{db_i} \wedge \cdots \wedge db_n + b^{\mathbf{1}} u \wedge db_1 \wedge \cdots \wedge db_n \right)$$

i.e.

(1.15)

$$v = \sum\nolimits_{\substack{\mu \in S_n, \neq \mathbf{1} \\ \mu(i)=0}} b_i b^\mu z_\mu \wedge db_1 \wedge \cdots \wedge \hat{db_i} \wedge \cdots \wedge db_n + b^{\mathbf{1}} u \wedge db_1 \wedge \cdots \wedge db_n + z$$

with $z \in Z_F$ a closed form.

The next result characterizes divisibility by pure forms and will be useful in the next sections. We say that a form $\omega \in \Omega_F$ divides the form λ if $\lambda = \eta \wedge \omega$ with some form $\eta \in \Omega_F$. Then we have

(1.16) **Proposition.** *Let $b_1, \ldots, b_n$ be elements of F contained in a 2-basis $\mathcal{B}$ of F. Let $w \in \Omega_F^m$ be a m-form. If $db_1, \ldots, db_n$ divide w, then $db_1 \wedge \cdots \wedge db_n$*

divides w.

Proof. Let $\mathcal{B} = \{b_1, \dots, b_n, \dots\}$ be the given 2-basis. We show the claim by induction on n. For $n = 1$ the claim is obvious. Assume the proposition for $n-1$. Then we have $w = \eta \wedge db_1 \wedge \cdots \wedge db_{n-1}$. Write $\eta = \eta_0 + \eta_1 \wedge db_n$ with forms η_0, η_1 which do not contain db_n in their basis expansion. Then $w = \eta_0 \wedge db_1 \wedge \cdots \wedge db_{n-1} + \eta_1 \wedge db_1 \wedge \cdots \wedge db_n$ is divisible by db_n by hypothesis. Since $\eta_0 \wedge db_1 \wedge \cdots \wedge db_{n-1}$ does not contains db_n, we conclude $\eta_0 \wedge db_1 \wedge \cdots \wedge db_{n-1} = 0$ and $w = \eta_1 \wedge db_1 \wedge \cdots \wedge db_n$. $\square$

(1.17) **Remark.** Let us consider a rational function field $L = F(X_\mu; \mu \in A))$ with A finite. Let $\mathcal{B} = \{b_i | i \in I\}$ be a 2-basis of F. Then $\mathcal{B} \cup \{X_\mu, \mu \in A\}$ is a 2-basis of L. Let $B \subseteq A$ be a subset and $N = F(X_\mu, \mu \in B)$, respectively $M = F(X_\mu^2, \mu \in B) \subset N$. Let X be any variable X_μ with $\mu \notin B$. We will be later interested in forms contained in Ω_L which are generated over M by the differentials db, $b \in F$. Thus we will define $\Omega_F M = \Omega_F \otimes M \subset \Omega_L$, respectively $\Omega_F M[X^2] = \Omega_F \otimes M[X^2] \subset \Omega_L$. This last set is the $M[X^2]$-submodule

$$\bigoplus_{i_1 < \cdots < i_m} M[X^2] db_{i_1} \wedge \cdots \wedge db_{i_m}$$

(for some ordering $<$ in I) of Ω_L. For every $p(X) \in N[X]$ irreducible and monic we set for every $n \geq 0$

$$(1.18) \qquad p^{-\infty}\Omega_F^n M[X^2] = \left\{ \frac{w}{p^s} | w \in \Omega_F^n M[X^2], s \geq 1, \deg_X w \leq s \deg_X p \right\}$$

if $p \in M[X^2]$, and

$$(1.19) \qquad p^{-\infty}\Omega_F^n M[X^2] = \left\{ \frac{w}{p^{2s}} | w \in \Omega_F^n M[X^2], s \geq 1, \deg_X w \leq 2s \deg_X p \right\}$$

if $p \notin M[X^2]$.

Here we have set $\deg_X w = 2t$ whenever we have $w = w_0 + w_1 X^2 + \cdots + w_t X^{2t}$ with $w_0, \dots, w_t \in \Omega_F^n$, $w_t \neq 0$ for some $w \in \Omega_F^n M[X^2]$.

Now we have

(1.20) **Lemma.** *The sum*

$$\Omega_F^n M[X^2] + \sum_p p^{-\infty}\Omega_F^n M[X^2] \subseteq \Omega_L^n$$

is direct. Here p runs over all irreducible polynomials contained in $N[X]$.

Proof. Let us assume

$$u_0 + \sum_p \frac{u_p}{p^{s_p}} = 0$$

in Ω_L, with u_0, $u_p \in \Omega_F M[X^2]$, $\deg_X u_p < s_p \deg_X p$ for all p. Thus

$$\prod_p p^{s_p} u_o + \sum_p \left(\prod_{q \neq p} q^{s_q} \right) u_p = 0$$

holds in $\Omega_F M[X^2]$. Recall that s_p is even if $p \notin M[X^2]$, and hence p^{s_p} is contained always in $M[X^2]$.

We fix now some p_0 and we get in $\Omega_F M[X^2]$

$$\left(\prod_p p^{s_p} \right) u_0 = \left(\prod_{q \neq p_0} q^{s_q} \right) u_{p_0} + \sum_{p \neq p_0} \left(\prod_{q \neq p} q^{s_q} \right) u_p.$$

This implies that $p_0^{s_{p_0}}$ divides the term $\left(\prod_{q \neq p_0} q^{s_q} \right) u_{p_0}$, and since $\Omega_F M[X^2]$ is a free module, $p_0^{s_{p_0}}$ divides u_{p_0} in $\Omega_F M[X^2]$. Since $\deg_X u_{p_0} < s_{p_0} \deg_X p_0$, it follows $u_{p_0} = 0$. This proves the lemma. $\square$

The relevant point in the above decomposition is that the operators $\wp$ and d respect this decompositions, whenever $\wp$ is defined with respect to a 2-basis containing a 2-basis of F. Thus we have:

$$\begin{aligned} d\Omega_F M[X^2] &\subseteq \Omega_F M[X^2] \\ \wp\Omega_F M[X^2] &\subseteq \Omega_F M[X^2] \\ d(p^{-\infty}\Omega_F M[X^2]) &\subseteq p^{-\infty}\Omega_F M[X^2] \\ \wp(p^{-\infty}\Omega_F M[X^2]) &\subseteq p^{-\infty}\Omega_F M[X^2] \end{aligned} \tag{1.21}$$

If we do not specify a particular 2-basis, the relations above must be understood as follows. For any field F the maps $\Omega_F \to \Omega_F$ given by $w \mapsto w^{[2]}$

depends on the choice of a 2-basis. If we choose another 2-basis of F and we denote by $w^{(2)}$ the same operation with respect to this new 2-basis, we have

$$w^{[2]} = w^{(2)} + dv$$

with some $v \in \Omega_F$. Therefore expressions of the form $w^{[2]} + dv$, which will often occur in the sequel, make sense if the particular choice of the form v does not matter. Thus, we will not make sometimes an explicit choice of a 2-basis when dealing with such expressions. Of course the same remark applies to expressions of the form $\wp w + dv$. For example if $u \in \Omega_F$ and $f \in F^*$, then for any choice of a 2-basis, the forms $\wp(u \wedge \frac{df}{f})$ and $\wp(u) \wedge \frac{df}{f}$ differ by an exact form.

In the concrete situation above, if we choose $u \in \Omega_F M[X^2]$ and $f \in M[X^2]$, then there is some $v \in \Omega_F M[X^2]$ with

$$\wp\left(u \wedge \frac{df}{f}\right) = \wp(u) \wedge \frac{df}{f} + \frac{dv}{f^2}$$

2 The behavior of differential forms under some field extensions

We continue in this section the algebraic study of differential forms by considering their behavior under field extension. Any field extension $F \hookrightarrow L$ induces a natural homomorphism $\Omega_F^n \to \Omega_L^n$ for all $n \geq 0$. We will denote by $\Omega^n(L/F)$ the kernel of this homomorphism. It is clear that $d(\Omega^n(L/F))$ is contained in $\Omega^{n+1}(L/F)$. Fixing any 2-basis of F we see that the same holds true for the operator $\wp$ and s(= square). We are particularly interested in the following field extension of F. Let $\phi = \ll b_1, \dots, b_n \gg = \langle 1, b_1 \rangle \otimes \cdots \otimes \langle 1, b_n \rangle$ be an anisotropic bilinear Pfister form (see [Ba 1], [A-Ba 1]). The fact that ϕ is anisotropic means that $\{b_1, \dots, b_n\}$ are algebraically independent over F^2 and hence can be chosen as part of a 2-basis of F. This will be always assumed in the sequel. The function field $F(\phi)$ of the quadric $\phi = 0$ is constructed as follows. Let S_n be the set of all maps $\mu : \{1, \dots, n\} \to \{0, 1\}$ with $\mu(i) = 1$ for at least one index i and choose a variable X_μ for each $\mu \in S_n$. Let $L = F(X_\mu, \mu \in S_n)$ the rational function field over F in the variables X_μ and set $T = \sum_{\mu \in S_n} b^\mu X_\mu^2$, where $b^\mu = \prod_{i=1}^n b_i^{\mu(i)}$ (T is the so called pure part of ϕ). Then the field

$$F(\phi) = L(\sqrt{T}) \tag{2.1}$$

is the function field of ϕ. Obviously $\phi \otimes F(\phi)$ is isotropic, although it is not necessarily hyperbolic, but metabolic. We are interested in the behavior of Ω^m under the field extension $F \hookrightarrow F(\phi)$. To this end we will compute $\Omega^m(L/F)$, $\Omega^m(F(\phi)/L)$ and finally $\Omega^m(F(\phi)/F)$. We will denote by K the field $F(\phi)$ in what follows.

(2.2) **Lemma.** *Let $F(X)/F$ be a pure transcendental extension of F. Then $\Omega^m(F(X)/F) = 0$.*

Proof. We may assume that $F(X)/F$ has transcendence degree one. Let $\mathcal{B} = \{b_i, i \in I\}$ be a 2-basis of F. Then $\mathcal{B} \cup \{X\}$ is a 2-basis of $F(X)$. If

$$w = \sum c_{i_1, \dots, i_m} \frac{db_{i_1}}{b_{i_m}} \wedge \cdots \wedge \frac{db_{i_m}}{b_{i_m}}$$

is in $\Omega^m(F(X)/F)$, we have $w = 0$ in $\Omega^m_{F(X)}$. But since

$$\left\{ \frac{db_{i_1}}{b_{i_1}} \wedge \cdots \wedge \frac{db_{i_m}}{b_{i_m}}, i_1 < \cdots < i_m \right\}.$$

is part of a $F(X)$-basis of $\Omega^m_{F(X)}$, we conclude $c_{i_1\cdots i_m} = 0$ for all $i_1 < \cdots < i_m$, i.e. $w = 0$ in Ω^m_L. $\square$

(2.3) **Remark.** If $\mathcal{B}$ is a 2-basis of F, then $\mathcal{B} \cup \{X_\mu, \mu \in S_n\}$ is a 2-basis of L.

(2.4) **Lemma.** *Let k be any field of characteristic 2 and $t \in k \setminus k^2$. Then for $m \geq 1$*

$$\Omega^m(k(\sqrt{t})/k) = \Omega^{m-1}_k \wedge dt.$$

Proof. Since $t = (\sqrt{t})^2$ we have $dt = 0$ in $\Omega_{k(\sqrt{t})}$, and therefore $\Omega^{m-1}_k \wedge dt \subseteq \Omega^m(k(\sqrt{t})/k)$. Since $t \notin k^2$, we can choose a 2-basis of k, say $\mathcal{B}$, such that $t \in \mathcal{B}$, i.e. $\mathcal{B} = \{t, c_j, j \in J\}$. Then $\{\sqrt{t}, c_j, j \in J\}$ is a 2-basis of $k(\sqrt{t})$. Let $w \in \Omega^m(k(\sqrt{t})/k)$. Then we have

$$w = \sum_\sigma a_\sigma dc_\sigma + (\sum b_\tau dc_\tau) \wedge dt$$

where σ runs over all maps $\{1, \ldots, m\} \to J$ and τ over all maps $\{1, \ldots, m-1\} \to J$ which are monotone (i.e. $i < j$ implies $\sigma(i) < \sigma(j)$ in some ordering $<$ of J). Moreover dc_σ means $dc_{\sigma(1)} \wedge \cdots \wedge dc_{\sigma(m)}$, etc. and $a_\sigma, b_\tau \in k$. Then in $\Omega^m_{k(\sqrt{t})}$ we get $\sum_\sigma a_\sigma dc_\sigma = 0$. Since all c_i, $i \in J$ are part of a 2-basis of $k(\sqrt{t})$ we conclude $a_\sigma = 0$ for all σ, i.e. $w = (\sum b_\tau dc_\tau) \wedge dt$. This proves the claim. $\square$

In particular we obtain

(2.5) **Corollary.** $\Omega^m(K/L) = \Omega^{m-1}_L \wedge dT$.

(2.6) **Remark.** In the case of a quadratic separable extension of k, say $E = k + kz$ with $z^2 + z = a$, we may assume that a is a square in k and hence $dz = 0$. In this case z is a square in E and a 2-basis of k remains a 2-basis of E. Therefore the argument in the proof of (2.4) shows $\Omega^m(E/k) = \{0\}$.

(2.7) **Lemma.** *For all* $m \geq n$

$$\Omega^m(F(\phi)/F) = \Omega_F^{m-n} \wedge db_1 \wedge \cdots \wedge db_n$$

Otherwise $\Omega^m(F(\phi)/F) = 0.$

Proof. We choose a 2-basis of F containing $\{b_1, \dots, b_n\}$. Let us take w in $\Omega^m(F(\phi)/F)$, i.e. $w \in \Omega_F^m$ with $w = 0$ in $\Omega_{F(\phi)}^m$. From (2.4) we infer that $w = u \wedge dT$ with some $u \in \Omega_L^{m-1}$. Now $dT = k_1 db_1 + \cdots + k_n db_m$ with some polynomials $k_1, \dots, k_n$ (see (2.9) below). Thus replacing db_1 in u we see that one can assume that u does not contain db_1 in its basis expansion with respect to the 2-basis $\mathcal{B} \cup \{X_\mu, \mu \in S_n\}$. Let us write in Ω_F^m

$$w = w_0 + w_1 \wedge db_1$$

with forms w_0, w_1 not containing db_1. Then in Ω_L^m

$$\begin{aligned} w_0 + w_1 \wedge db_1 &= u \wedge k_1 db_1 + u \wedge (k_2 db_2 + \cdots + k_n db_n) \\ (w_1 + k_1 u) \wedge db_1 &= w_0 + u \wedge (k_2 db_2 + \cdots + k_n db_n) \end{aligned}$$

Since b_1 is part of the 2-basis of L and the right hand side of this equation does not contain db_1, we obtain $(w_1 + k_1 u) \wedge db_1 = 0$ in Ω_L^m. But $w_1 + k_1 u$ also does not contain db_1, so that $w_1 + k_1 u = 0$. Thus $u = k_1^{-1} w_1$, and $w = k_1^{-1} w_1 \wedge dT$. We get

$$k_1 w = w_1 \wedge dT \tag{2.8}$$

which is a relation between forms all of whose coefficients are polynomials. Now on such differential forms we can specialize the values of the variables X_μ, obtaining forms defined over F. Letting $X_\mu \to 0$ for all $\mu \neq 1$ and $X_1 \to 1$ and using $k_1 \to 1$, $k_i \to 0$ for $i \geq 2$, we get from (2.8) in Ω_F^m

$$w = \bar{w}_1 \wedge db_1$$

with some form $\bar{w}_1$. Thus $db_1, \dots, db_n$ divide the form w . The lemma follows from (1.16). $\square$

(2.9) **Remark.** The differential of the polynomial T is

$$dT = k_1 db_1 + \cdots + k_n db_n$$

where $k_1, \dots, k_n$ are polynomials in $F[X^2]$ given by

$$k_i = \sum_{\mu \in S_n, \mu(i)=1} b_i^{-1} b^{\mu} X_{\mu}^2, \qquad 1 \le i \le n.$$

Let us recall that the Cartier-operator defines an isomorphism $C : Z_F^n / B_F^n \to \Omega_F^n$ where Z_F^n are the closed forms in Ω_F^n and $B_F^n = d\Omega_F^{n-1}$ are the exact forms. Following Kato (see [Ka 1] and [Mi]) we introduce the operator

$$\wp : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1} \tag{2.10}$$

given by

$$\wp(w) = C^{-1}(w) - \overline{w}$$

i.e. $\wp = C^{-1} - \overline{id}$.

For example

$$\wp\left(a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\right) = \overline{(a^2 - a)\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}} \tag{2.11}$$

In order to complete (2.10) to an exact sequence we introduce the groups (see [Ka 1])

$$\nu_F(n) = \ker(\wp) \tag{2.12}$$

$$H^{n+1}(F) = \text{ Coker } (\wp) \tag{2.13}$$

and we have the exact sequence

$$0 \to \nu_F(n) \to \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1} \to H^{n+1}(F) \to 0. \tag{2.14}$$

Taking a 2-basis $\mathcal{B} = \{b_i, i \in I\}$ of F we get the square-operator $s : \Omega_F^n \to \Omega_F^n$ given by $s : \left(\sum_\sigma a_\sigma \frac{db_\sigma}{b_\sigma}\right) = \sum_\sigma a_\sigma^2 \frac{db_\sigma}{b_\sigma}$ (see section 1). We define the basis-dependent operator $\wp : \Omega_F^n \to \Omega_F^n$ by $\wp(w) = s(w) - w$ and easily check that

$$H^{n+1}(F) = \Omega_F^n / (d\Omega_F^{n-1} + \wp\Omega_F^n).$$

Another choice of a 2-basis would change $\wp(w)$ by an exact form. Thus $d\Omega_F^{n-1} + \wp\Omega_F^n$ is basis independent.

The next two propositions are due to Kato (see [Ka 1]) and will be used continuously in what follows.

Let $\{b_i\}_{i\in I}$ be a 2-basis of F and endow I with the structure of a totally ordered set. For $j \in I$ let F_j (resp. $F_{<j}$) be the subfield of F generated over F^2 by the elements b_i with $i \leq j$ (resp. $i < j$). For fixed n let $\sum_n$ be the set of all functions $\alpha : \{1, \dots, n\} \to I$ with $\alpha(i) < \alpha(j)$ whenever $1 \leq i < j \leq n$. We endow $\sum_n$ with the lexicographic ordering, namely $\alpha < \beta$ ($\alpha,\ \beta \in \sum_n$) if and only if there exists some i such that $\alpha(i) < \beta(i)$ and $\alpha(j) \leq \beta(j)$ for all $j \leq i$. The F-vector space Ω_F^n has the basis $\{db_{\alpha(1)} \wedge \cdots \wedge db_{\alpha(n)}, \alpha \in \sum_n\}$ and we can introduce in Ω_F^n the following filtration: for $\alpha \in \sum_n$ let $\Omega^n_{F,\alpha}$, (resp.$\Omega^n_{F,<\alpha}$), be the subspace of Ω_F^n generated by the elements $db_{\beta(1)} \wedge \cdots \wedge db_{\beta(n)}$ with $\beta \leq \alpha$, (resp. $\beta < \alpha$). Using this notation we formulate the following basic result due to Kato (see also [Ka 2]).

(2.15) **Lemma.** *Let $y \in F, \alpha \in \sum_n$ and*

$$w_\alpha = \frac{db_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{db_{\alpha(n)}}{b_{\alpha(n)}} \in \Omega_F^n$$

be such that

$$(y^2 - y)w_\alpha \in \Omega^n_{F,<\alpha} + d\Omega_F^{n-1}$$

Then

$$yw_\alpha = v + \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$$

for some $v \in \Omega^n_{F,<\alpha}$ and some $a_i \in F^*_{\alpha(i)}$, $1 \leq i \leq n$.

We will refer to this result in the sequel as Kato's lemma. An immediate consequence of (2.15) is

(2.16) **Corollary.**

$$\nu_F(n) = \left\{ \sum_i \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}} | a_{i_k} \in F^* \right\}.$$

We will now study the behavior of H^{n+1} under field extensions. If $F \hookrightarrow L$ is a field extension we get the obvious maps $\nu_F(n) \to \nu_L(n)$ and $H^{n+1}(F) \to H^{n+1}(L)$. We write $H^{n+1}(L/F)$ for the kernel of $H^{n+1}(F) \to H^{n+1}(L)$. The main goal of this paper is the computation of $H^{n+1}(F(\phi)/F)$ for an anisotropic bilinear n-fold Pfister form ϕ defined over F. This will be done in section 4. We will now consider only some easier field extensions.

(2.17) **Lemma.** *Let $F(X)/F$ be a pure transcendental extension. Then*

$$H^{n+1}(F(X)/F) = 0$$

Proof. Let $\mathcal{B} = \{b_i,\ i \in I\}$ be a fixed 2-basis of F. We may assume that $F(X)/F$ has transcendence degree one. Then $\mathcal{B} \cup \{X\}$ is a 2-basis of $F(X)$. We fix an ordering in I (and hence in $\mathcal{B}$) and we choose the ordering of $\mathcal{B} \cup \{X\}$ with $X > b_i$ for all $i \in I$. Let $\bar{w} \in H^{n+1}(F)$ with $\bar{w} = 0$ in $H^{n+1}(F(X))$. Thus in $\Omega^n_{F(X)}$ we have

$$w = \wp u + dv$$

for some $u \in \Omega^n_{F(X)}$ and $v \in \Omega^{n-1}_{F(X)}$. Here $\wp$ is defined with respect to the 2-basis $\mathcal{B} \cup \{X\}$. Hence $\wp u = w + dv$ with $w \in \Omega^n_F$. Let $\alpha \in \Sigma_{n,F} \subset \Sigma_{n,F(X)}$ be the leading index of w, that is $w = \sum_{\gamma \in \Sigma_{n,F}} w_\gamma \frac{db_\gamma}{b_\gamma}$ with $w_\alpha \neq 0$ and $w_\gamma = 0$ for all $\gamma > \alpha$, and $\beta \in \Sigma_{n,F(X)}$ the leading index of u. If $\beta > \alpha$, since $\wp(u_\beta)\frac{db_\gamma}{b_\gamma} \in d\Omega^{n-1}_{F(X)} + \Omega^n_{F(X),<\beta}$, we apply Kato's lemma and conclude that $u_\beta \frac{db_\gamma}{b_\gamma} = u' + \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$, with $a_i \in F_{\beta(i)}$ and $u' \in \Omega^n_{F(X),<\beta}$. By this way we can replace u by a differential form with lower leading index. This means that we may assume $\beta \leq \alpha$. Then we have

$$(\wp(u_\alpha) + w_\alpha)\,\frac{db_\alpha}{b_\alpha} = d(v) \mod \Omega^n_{F(X),<\alpha}$$

with $v \in \Omega^{n-1}_{F(X)}$. Since $b_{\alpha(i)} < X$ for all $i \in I$, we conclude that we may assume that the coefficients of $d(v)$ are in $F(X^2)$. Because if we write $v = v_0 + Xv_1 + v_2 \wedge dX + Xv_3 \wedge dX$ where v_0, v_1, v_2, v_3 are generated over $F(X^2)$ by the differentials db_i, $i \in I$, we get

$$dv = dv_0 + Xdv_1 + (v_1 + dv_2 + Xdv_3) \wedge dX$$

and hence $v_1 + dv_2 + Xdv_3 = 0$. By the choice of the forms v_1, v_2, v_3 we infer from the last relation that $v_1 + dv_2 = 0$ and $dv_3 = 0$. Therefore inserting

$v_1 = dv_2$ into v we obtain

$$v = v_0 + d(Xv_2) + Xd(Xv_3)$$

and since $d(Xd(Xv_3)) = dX \wedge d(Xv_3) = 0$, it follows $dv = dv_0$. Thus we may replace v by v_0 and we can assume that v is generated over $F(X^2)$ by the differentials db_i, $i \in I$. Then u is also generated over $F(X^2)$ by the differentials db_i, $i \in I$.

Therefore the relation $\wp(u) + w = d(v)$ holds in the subspace

$$\Omega_F^n[X^2] \oplus \bigoplus_p p^{-\infty}\Omega_F^n[X^2] \text{ of } \Omega^n_{F(X)},$$

where p runs over all irreducible polynomials in $F[X^2]$. Let us write

$$u_\alpha = u_{\alpha,0} + \sum_p u_{\alpha,p} \quad , \quad v = v_0 + \sum_p v_p \quad ,$$

with $u_{\alpha,0} \in F[X^2]$, $v_0 \in \Omega_F^{n-1}[X^2]$, resp. $u_p \in p^{-\infty}F[X^2]$, $v_p \in p^{\infty}\Omega_F^{n-1}[X^2]$. Then

$$w_\alpha \frac{db_\alpha}{b_\alpha} = \wp u_{\alpha,0}\frac{db_\alpha}{b_\alpha} + dv_0 + \sum_p \left(\wp u_{\alpha,p}\frac{db_\alpha}{b_\alpha} + dv_p\right) \mod \Omega^n_{F(X),<\alpha}$$

and since $\wp$, d are compatible with the above direct sum, we see that

$$w_\alpha \frac{db_\alpha}{b_\alpha} = \wp u_{\alpha,0}\frac{db_\alpha}{b_\alpha} + dv_0 \mod \Omega^n_{F,<\alpha}[X^2]$$

holds in $\Omega_F^n[X^2]$. We write now $u_{\alpha,0} = \overline{u_{\alpha,0}} + X^{2s}u_s$, $v_0 = \overline{v_0} + X^{2r}v_r$ with $u_s \in F$, $v_r \in \Omega_F^{n-1}$ and $2s = \deg(u_{\alpha,0})$, $2r = \deg(v_0)$, $\deg(\overline{u_{\alpha,0}}) < 2s$, $\deg(\overline{v_0}) < 2r$. Thus

$$w_\alpha \frac{db_\alpha}{b_\alpha} = \wp\overline{u_{\alpha,0}}\frac{db_\alpha}{b_\alpha} + d\overline{v_0} + [X^{4s}u_s^{[2]} + X^{2s}u_s]\frac{db_\alpha}{b_\alpha} + X^{2r}dv_r \mod \Omega^n_{F,<\alpha}[X^2].$$

Let us first assume $4s > 2r$. Then comparing coefficients we get $u_s^{[2]}\frac{db_\alpha}{b_\alpha} \in \Omega^n_{F,<\alpha}$ and applying the Cartier-operator it follows $u_s = 0$, which is a contradiction. Similarly if $2r > 4s$, then $dv_r \in \Omega^n_{F,<\alpha}$, including $X^{2r}dv_r$ in $\Omega^n_{F,<\alpha}$, we conclude that we can lower the degree of v_0. Thus we are lead to the case $4s = 2r$. If $4s = 2r > 0$, then we get $u_s^{[2]} \frac{db_\alpha}{b_\alpha} = dv_r \mod \Omega^n_{F,<\alpha}$ and applying the Cartier operator we obtain $u_s \frac{db_\alpha}{b_\alpha} \in \Omega^n_{F,<\alpha}$, that is $u_s \frac{db_\alpha}{b_\alpha} = 0$, which is a contradiction. Thus we have $s = r = 0$, i.e. $w_\alpha \frac{db_\alpha}{b_\alpha} = \wp u_0 \frac{db_\alpha}{b_\alpha} + dv_0 \mod \Omega^n_{F,<\alpha}$ with $u_0 \in F$ and $v_0 \in \Omega_F$. Replacing in w, this see $\overline{w}$ can be represented by a differential form with lower leading index. This shows $\overline{w} = 0$ in $H^{n+1}(F)$, and concludes the proof of the lemma. $\square$

In particular we have $H^{m+1}(L/F) = (0)$ for all m, where L is the field extension $F(X_\mu, \mu \in S_n)$ introduced at the beginning of this section.

We want now to compute $H^{m+1}(F(\phi)/L)$, where $F(\phi) = L(\sqrt{T})$ (see (2.1)). To this end we prove the following general fact.

(2.18) **Lemma.** *Let k be field of characteristic 2, and $b \in k \setminus k^2$. Then*

$$H^{m+1}(k(\sqrt{b})/k) = \overline{\Omega^{m-1}_k \wedge db}$$

Proof. Since $b \in k \setminus k^2$ we can take b part of a 2-basis of k. Let $\mathcal{B} = \{b_1 = b, b_2, \dots\}$ be a 2-basis of k. Then $\mathcal{B}' = \{\sqrt{b}, b_2, \dots\}$ is a 2-basis of $k(\sqrt{b})$. Take now $\bar{w} \in H^{m+1}(k(\sqrt{b})/k)$, i.e. $\bar{w} \in H^{m+1}(k)$ with $\bar{w} = 0$ in $H^{m+1}(k(\sqrt{b}))$. This means

$$w = \wp u + dv$$

with $u \in \Omega^m_{k(\sqrt{b})}$, $v \in \Omega^{m-1}_{k(\sqrt{b})}$. We order the 2-basis $\mathcal{B}'$ with $\sqrt{b} > b_i$ for all i $(\neq 1)$. Let $\alpha \in \Sigma_{m,k}$, $\alpha(i) > 1$ for all $i = 1, \dots, m$, be the leading index of w and $\beta \in \Sigma_{m,k(\sqrt{b})}$ the leading index of u. Thus we can assume, as in the proof of (2.17), $\beta \leq \alpha$. In this case we have

$$(\wp(u_\alpha) + w_\alpha) \frac{db_\alpha}{b_\alpha} = d(v) \mod \Omega^n_{k(\sqrt{b}),<\alpha}$$

with $v \in \Omega^{n-1}_{k(\sqrt{b})}$. Since $b_{\alpha(i)} < \sqrt{b}$ for all i, we conclude that the leading coefficient of $d(v)$ is in k, then u_α is also in k and we may assume that $v \in \Omega^{m-1}_k$.

Since $\ker[\Omega^m_k \to \Omega^m_{k(\sqrt{b})}] = \Omega^{m-1}_k \wedge d(b)$, we conclude that

$$w_\alpha \frac{db_\alpha}{b_\alpha} = \wp(u_\alpha) \frac{db_\alpha}{b_\alpha} + d(v) \mod \Omega^n_{k,<\alpha} + \Omega^{m-1}_k \wedge d(b)$$

in Ω_k^m. Replacing in w, this shows $\overline{w}$ can be represented by a differential form with lower leading index. This shows $\overline{w} \in \overline{\Omega_k^{m-1} \wedge db}$ in $H^{n+1}(k)$, and concludes the proof of the lemma. $\square$

(2.19) **Corollary.**

$$H^{m+1}(F(\phi)/L) = \overline{\Omega_L^{m-1} \wedge dT}$$

Let us close this section with some remarks concerning the computation of $H^{n+1}(F(\phi)/F)$. We write $y = \sqrt{T}$, so that $K = F(\phi) = L[y]$. Fix a 2-basis $\mathcal{B} = \{b_i, i \in I\}$ of F, so that $\mathcal{B} \cup \{X_\mu, \mu \in S_n\}$ is a 2-basis of L. We order the elements of $\mathcal{B}$ according to an order of I and the elements of $\{X_\mu, \mu \in S_n\}$ for example using the lexicographic ordering and we set $\mathcal{B} < \{X_\mu, \mu \in S_n\}$ for an ordering in $\mathcal{B} \cup \{X_\mu, \mu \in S_n\}$. Since $y^2 = T = \sum_\mu b^\mu X_\mu^2$ in K, we see that the elements of $\mathcal{B}$ are not 2-independent over K^2. Let us fix some $b = b_1 \in \mathcal{B}$. Then $\mathcal{B} \setminus \{b_1\} \cup \{X_\mu, \mu \in S_n\} \cup \{y\}$ is a 2-basis of K. We order this basis such that y is the maximal element. In particular we have the operator $\wp$ on Ω_K defined with respect to this 2-basis. Take now $\bar{w} \in H^{n+1}(K/F)$. Then

$$w = \wp u + dv \tag{2.20}$$

with $u \in \Omega_K^n$ and $v \in \Omega_K^{n-1}$. From $\wp u = w + dv$, and using Kato's lemma with a filtration defined by the above ordering, we see that one can assume in (2.20) that u and v are differential forms generated over K by the differential db_i, $i \in I$ $(i \neq 1)$, i.e. they do not contain differentials of the type dX_μ or dy. Looking at the 2-basis expansion of both sides of (2.20) we easily conclude that u and v do not contain y in their coefficients, i.e. they are contained in L. Therefore the forms u, v (and w) are defined over L and are generated over L by the differentials db_i, $i \in I$. From (2.4) we conclude that $w + \wp u + dv \in \Omega^n(K/L) \subset \Omega_L^n$ as a form in Ω_L^n, and using (2.4) we obtain in Ω_L^n

$$w = \wp u + dv + \lambda \wedge dT \tag{2.21}$$

with $\lambda \in \Omega_L^{n-1}$. We will show below that the coefficients of u, v, λ in the 2-basis expansion of L do not contain odd powers of the variables X_μ and that λ also is generated by the differentials db_i, $i \in I$. Therefore if we define the subfield of L

$$M = F(X_\mu^2, \mu \in S_n) \subset L$$

we see that all forms u, v, λ, dT and w are generated over M by the differentials db_i, $i \in I$, i.e. they are contained in the subspace $\Omega_F \otimes M$ of Ω_L. We show now this assertion. The fact that λ does not contain differentials dX_μ, $\mu \in S_n$ follows from (2.21) and the fact that all other forms, including dT, do not contain such forms. Let us write $u = u_0 + Xu_1$, $v = v_0 + Xv_1$, $\lambda = \lambda_0 + X\lambda_1$, for some fixed $X = X_\mu$, and where u_i, v_i, λ_i $(i = 0, 1)$ do not contain odd powers of X in their coefficients, with respect to their 2-basis representation. Then in Ω_L^n

$$w = \wp u_0 + X^2 u_1^{[2]} + Xu_1 + dv_0 + Xdv_1 + v_1 \wedge dX + \lambda_0 \wedge dT + X\lambda_1 \wedge dT$$

and this implies

$$w = \wp u_0 + X^2 u_1^{[2]} + dv_0 + \lambda_0 \wedge dT \tag{2.22}$$

$$u_1 + dv_1 + \lambda_1 \wedge dT = 0 \tag{2.23}$$

$$v_1 \wedge dX = 0. \tag{2.24}$$

Since v_1 does not contain dX, we obtain from (2.24) $v_1 = 0$ and therefore $u_1 = \lambda_1 \wedge dT$. Inserting this in (2.22) it follows

$$w = \wp u_0 + dv_0 + (\lambda_0 + X^2 T \lambda_1^{[2]}) \wedge dT,$$

since from $u_1 = T\lambda_1 \wedge \frac{dT}{T}$ we get $u_1^{[2]} = T^2 \lambda_1^2 \wedge \frac{dT}{T} = T\lambda_1^{[2]} \wedge dT$. The form $\lambda_0 + X^2 T \lambda_1^{[2]}$ does not contain odd powers of X , and the above equation therefore shows that we can eliminate all odd powers of X from u, v and λ. Since $X = X_\mu$ was arbitrary, this proves the claim. Therefore we have shown

(2.25) **Proposition.** *Let $\bar{w} \in H^{n+1}(K/F)$ and fix a 2-basis $\mathcal{B} = \{b_i, i \in I\}$ of F. Then there exist forms u, v, λ in $\Omega_F \otimes M \subset \Omega_L$, i.e. generated over $M = F(X_\mu^2, \mu \in S_n)$ by the differentials db_i, $i \in I$ with*

$$w = \wp u + dv + \lambda \wedge dT$$

where $\wp$ is defined with respect to any 2-basis of L containing the basis $\mathcal{B}$

The next two sections will be devoted to the study of this equation.

(2.26) **Remark.** The behavior of $\nu_F(m)$ under field extensions is easier to handle than that of $H^{m+1}(F)$. For example (2.7) implies $\Omega^m(F(\phi)/F) = \Omega_F^{m-n} \wedge db_1 \wedge \cdots \wedge db_n$ if $\phi = \ll b_1, \ldots , b_n \gg$. Hence $\ker(\nu_F(m) \to \nu_{F(\phi)}(m)) = \Omega_F^{m-n} \wedge db_1 \wedge \cdots \wedge db_n \cap \nu_F(m)$. Thus this kernel is determined by the following

(2.27) **Lemma.** *For any $a \in \Omega_F^{m-n}$ the following statements are equivalent:*

(i) $$a \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} \in \nu_F(m)$$

(ii) $$\wp(a) \in \sum_{\epsilon \in S_n, \epsilon \neq 0} b^\epsilon [\Omega_F^{m-n}]^{[2]} + d\Omega_F^{m-n-1} + \sum_{i=1}^{n} \Omega_F^{m-n-1} \wedge db_i$$

where $\epsilon = (\epsilon_1, \ldots , \epsilon_n)$ runs over all sequences with $\epsilon_i = 0$ or 1.

Proof. Choose a 2-basis $\mathcal{B}$ of F with $b_1, \ldots , b_n \in \mathcal{B}$. Without restriction we can assume $\mathcal{B}$ finite, i.e. $\mathcal{B} = \{b_1, \ldots , b_n, \ldots , b_N\}$ and let us denote by η the differential form $\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$.

1. First we prove that (ii) implies (i). Let a be an element in Ω_F^{m-n} such that $\wp(a) \in \sum_{\epsilon \in S_n, \epsilon \neq 0} b^\epsilon [\Omega_F^{m-n}]^{[2]} + d\Omega_F^{m-n-1} + \sum_{i=1}^{n} \Omega_F^{m-n-1} \wedge db_i$, i.e. a can be written as

$$a = a^{[2]} + \sum_{\epsilon \neq 0} b^\epsilon A_\epsilon^{[2]} + dB + \sum_{i=1}^{n} E_i \wedge db_i$$

with $A_\epsilon \in \Omega_F^{m-n}$, $B \in \Omega_F^{m-n-1}$, $E_i \in \Omega_F^{m-n-1}$. Since $(\sum_{\epsilon \neq 0} b^\epsilon A_\epsilon^{[2]}) \wedge \eta$ and $dB \wedge \eta$ are in $d(\Omega_F^{m-1})$, and $E_i \wedge db_i \wedge \eta = 0$, we obtain that

$$d(a \wedge \eta) = 0$$

and

$$\begin{aligned} C(a \wedge \eta) &= C((a^{[2]} + \sum_{\epsilon \neq 0} b^\epsilon A_\epsilon^{[2]} + dB + \sum_{i=1}^{n} E_i \wedge db_i) \wedge \eta) \\ &= C(a^{[2]} \wedge \eta) + \sum_{\epsilon \neq 0} C(b^\epsilon A_\epsilon^{[2]} \wedge \eta) + C(dB \wedge \eta) = a \wedge \eta \end{aligned}$$

This implies that $a \wedge \eta \in \nu(m)_F$ and proves (i).

2. Now let a be in Ω_F^{m-n} such that $a \wedge \eta \in \nu_F(m)$. We write a as

$$a = \sum_\mu c_\mu \frac{db_\mu}{b_\mu}$$

where $\frac{db_\mu}{b_\mu} = \frac{db_{\mu(1)}}{b_{\mu(1)}} \wedge \cdots \wedge \frac{db_{\mu(m-n)}}{b_{\mu(m-n)}}$, and $c_\mu \in F$. Note that $a \wedge \eta \in \nu_F(m)$ implies that $d(a \wedge \eta) = 0$ and $C(a \wedge \eta) = a \wedge \eta$.

Be k the maximal index $k > n$ such that

$$a = R_0 + b_k R_1$$

where R_0, R_1 are differential forms generated by $\frac{db_1}{b_1}$, ..., $\frac{db_N}{b_N}$, with coefficients in $F^2(b_1, \ldots, b_{k-1})$. Also we decompose R_0 and R_1 as $R_0 = M_0 + M_1 \wedge \frac{db_k}{b_k}$ and $R_1 = M_2 + M_3 \wedge \frac{db_k}{b_k}$. Then $d(a \wedge \eta) = 0$ implies that

$$\left[d(M_0) + b_k d(M_1) + b_k M_1 \wedge \frac{db_k}{b_k} + (d(M_2) + b_k d(M_3)) \wedge \frac{db_k}{b_k} \right] \wedge \eta = 0$$

which means that

$$\begin{aligned} d(M_0) \wedge \eta &= 0 \\ d(M_1) \wedge \eta &= 0 \\ d(M_2) \wedge \eta &= 0 \\ M_1 \wedge \eta &= dM_3 \wedge \eta. \end{aligned}$$

From the last relation we obtain $M_1 = d(M_3) + E$ where E is in $\Omega_F^{m-n-1} \wedge \frac{db_1}{b_1} + \cdots + \Omega_F^{m-n-1} \wedge \frac{db_n}{b_n}$. Replacing in the above decomposition of $a \wedge \eta$ we get

$$a = M_0 + M_2 \wedge \frac{db_k}{b_k} + d(b_k M_3) + E.$$

Now we work with $a' = M_0 + M_2 \wedge \frac{db_k}{b_k}$ which is also generated by $\frac{db_1}{b_1}$, ..., $\frac{db_N}{b_N}$, with coefficients in $F^2(b_1, \ldots, b_{k-1})$. Since $d(M_0) \wedge \eta = 0$ and $d(M_2) \wedge \eta = 0$ we have that $a' \wedge \eta = 0$. By repeating of the above procedure on a' we conclude that a can be written as

$$a = M + d(G) + H$$

where M is generated by $\frac{db_1}{b_1}, \dots, \frac{db_N}{b_N}$ with coefficients in $F^2(b_1, \dots, b_n)$, $G \in \Omega_F^{m-n-1}$ and H in $\Omega_F^{m-n-1} \wedge \frac{db_1}{b_1} + \cdots + \Omega_F^{m-n-1} \wedge \frac{db_n}{b_n}$. This means that a can written as

$$a = \sum_{\mu \in \Sigma_{m-n}} c_\mu \frac{db_\mu}{b_\mu} + d(G) + H$$

where each c_μ is in $F^2(b_1, \dots, b_n)$, i.e. $c_\mu = \sum_\epsilon c_{\mu,\epsilon}^2 b^\epsilon$ with $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ running over all sequences with $\epsilon_i = 0$ or 1. Reordering the above relation we obtain

$$a = \sum_\epsilon b^\epsilon \left(\sum_\mu c_{\mu,\epsilon}^2 \frac{db_\mu}{b_\mu} \right) + d(G) + H.$$

Note that each $\left(\sum_\mu c_{\mu,\epsilon}^2 \frac{db_\mu}{b_\mu} \right)$ is in $\Omega_F^{[2]}$ so we will denote it by $A_\epsilon^{[2]}$. By this way we write a as

$$a = \sum_\epsilon b^\epsilon A_\epsilon^{[2]} + d(G) + H = A_0^{[2]} + \sum_{\epsilon \neq 0} b^\epsilon A_\epsilon^{[2]} + d(G) + H.$$

Finally we compute $C(a \wedge \eta)$. Since $b^\epsilon A_\epsilon^{[2]} \wedge \eta$, $d(G) \wedge \eta \in d(\Omega_F^{m-1})$ and $H \wedge \eta = 0$, we have

$$C(a \wedge \eta) = C(A_0^{[2]} \wedge \eta) = A_0 \wedge \eta.$$

Using that $C(a \wedge \eta) = a \wedge \eta$, we obtain $a = A_0 + H'$ with H' in $\Omega_F^{m-n-1} \wedge \frac{db_1}{b_1} + \cdots + \Omega_F^{m-n-1} \wedge \frac{db_n}{b_n}$. This means that

$$a = a^{[2]} + \sum_{\epsilon \neq 0} b^\epsilon A_\epsilon^{[2]} + d(G) + H''$$

where $H'' = H + H'$, completing the proof. $\square$

Considering the case $m = n$ we obtain

(2.28) **Corollary.**

$$\ker(\nu_F(n) \to \nu_{F(\phi)}(n)) = \left\{ a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} | \wp(a) \in D_F(\phi') \right\}$$

3 Some technical results

Let us first recall some notation. Let S_n be the set of maps $\mu : \{1, \ldots , n\} \to \{0, 1\}$ with $\mu(i) = 1$ for at least some index i. If μ is defined by $\mu(i) = 1$, $\mu(j) = 0$ for $j \neq i$ we write i instead of μ. Let $L = F(X_\mu, \mu \in S_n)$ and for a fixed $0 \leq s \leq n$ we set $N = F(X_\mu|\mu \neq 0, \ldots , s)$, $M = F(X_\mu^2|\mu \neq 0, \ldots , s)$, (if $s = 0$ we get $N = L$ and M is the field introduced in the last section).

(3.1) **Lemma.** *Let $X = X_s$. If $f \in M[X^2]$ is decomposed in monic irreducible polynomials in $M[X]$, say*

$$f = a \prod_p p^{n_p} \qquad\qquad (a \in M)$$

then for each p holds $p \in M[X^2]$ or $n_p \equiv 0$ (mod 2).

Proof. Let $p \in M[X]$ be an irreducible factor of f and assume $p \notin M[X^2]$. This means $D_X(p) \neq 0$, where $D_X(p)$ is the derivative of p with respect to X. Let us assume $n_p = 2t + 1$ for some $t \geq 0$. Since $f \in M[X^2]$ implies $D_X(f) = 0$, we obtain

$$D_X \left(ap^{2t+1} \prod_{q \neq p} q^{n_q} \right) = 0$$

$$ap^{2t+1} D_X \left(\prod_{q \neq p} q^{n_q} \right) + a \left(\prod_{q \neq p} q^{n_q} \right) p^{2t} D_X(p) = 0$$

$$\left(\prod_{q \neq p} q^{n_q} \right) D_X(p) = pD_X \left(\prod_{q \neq p} q^{n_q} \right)$$

But $\deg D_X(p) < \deg_X(p)$ implies that p divides $\prod_{q \neq p} q^{n_q}$ in $M[X]$, which is impossible. This proves the claim. □

For any irreducible monic polynomial $p \in N[X] (X = X_s)$ we will write $N(p)$ for the quotient field $N[X]/(p)$. Let us fix a 2-basis $\mathcal{B} = \{b_i, i \in I\}$ of F, so that $\mathcal{B} \cup \{X_\mu \setminus \mu \neq 1, \ldots , s\}$ is a 2-basis of N. If $p \in M[X^2]$, there is some $b_i \in \mathcal{B}$ such that $\mathcal{B} \setminus \{b_i\} \cup \{\bar{X}_\mu, \mu \neq 1, \dots, s-1\}$ is a 2-basis of $N(p)$, where $\bar{X}_\mu$ denotes the image of X_μ in $N(p)$. If $p \notin M[X^2]$ then $\mathcal{B}$ is part of a 2-basis of $N(p)$ and in fact there is some index $i_0 \neq 1, \dots, s-1$ such that $\mathcal{B} \cup \{\bar{X}_\mu, \mu \neq i_0, 1, \dots, s-1\}$ is a 2-basis of $N(p)$. The natural map $N[X] \to N(p)$ induces a homomorphism $\Omega_F^m M[X^2] \to \Omega_{N(p)}^m$.

(3.2) **Lemma.** *a) If* $p \in M[X^2]$

$$\ker(\Omega_F^m M[X^2] \to \Omega_{N(p)}^m) = p\Omega_F^m M[X^2] + \Omega_F^{m-1} M[X^2] \wedge dp$$

b) If $p \notin M[X^2]$

$$\ker(\Omega_F^m M[X^2] \to \Omega_{N(p)}^m) = p^2 \Omega_F^m M[X^2]$$

Proof. a) Since $p \in M[X^2]$, there is some $b_k \in \mathcal{B}$ such that $\overline{D_{b_k}(p)} \neq 0$ in $N(p)$, because otherwise one would infer that p is a square in $M[X]$. But in $N(p)$ we have $0 = dp = \sum_{b_j} \overline{D_{b_j}(p)} db_j$, where $\overline{D_{b_k}(p)} \neq 0$ is the coefficient of db_k. Choose $\Delta \in M[X^2]$ with $\Delta \cdot D_{b_k}(p) = 1 + p \cdot r$ in $M[X^2]$ and let $\bar{\Delta}$ be the image in $N(p)$. Then

$$db_k = \bar{\Delta} \sum_{i \neq k} \overline{D_{b_i}(p)} db_i \text{ in } \Omega_{N(p)}$$

Let $w \in \Omega_F^m M[X^2]$ be in the kernel of $\Omega_F^m M[X^2] \to \Omega_{N(p)}^m$, i.e. $w = 0$ in $\Omega_{N(p)}$. We have by definition of $\Omega_F^m M[X^2]$

$$w = \sum_\gamma a_\gamma db_\gamma + (\sum_\delta c_\delta db_\delta) \wedge db_k$$

where γ runs over all $\sum_m(I)$ with $k \notin \mathrm{Im}(\gamma)$ and δ runs over $\sum_{m-1}(I)$ with $k \notin \mathrm{Im}(\delta)$ and $a_\gamma,\ c_\gamma \in M[X^2]$. Recall that $\sum_m(I)$ are all maps $\gamma : \{1, \dots, m\} \to I$ with $\gamma(1) < \dots < \gamma(m)$ in a fixed ordering of I. Then in $\Omega_{N(p)}^m$

$$\sum a_\gamma db_\gamma + (\sum c_\delta db_\delta) \wedge \Delta \sum_{i \neq k} D_{bi}(p) db_i = 0$$

(we omit the bars for simplicity).

Thus

$$\sum_{k\notin\gamma}(a_\gamma+\Delta\sum_{\delta\cup j=\gamma}c_\delta D_{b_j}(p))db_\gamma=0$$

Here $k\notin\gamma$ means $\gamma\in\sum_m(I)$ with $k\notin \mathrm{Im}(\gamma)$ and $\delta\cup j=\gamma$ means that $\delta\in\sum_{m-1}(I)$ can be extended to γ with $j\in\mathrm{Im}(\gamma)$.

Since $\mathcal{B}\setminus\{b_k\}$ is part of a 2-basis of $N(p)$ it follows that the db_γ, $k\notin\gamma$ are linear independent over $N(p)$ and hence in $N(p)$

$$a_\gamma=\Delta\sum_{\delta\cup j=\gamma}c_\delta D_{b_j}(p).$$

Then in $N[X]$ we have

$$a_\gamma=\Delta\sum_{\delta\cup j=\gamma}c_\gamma D_{b_j}(p)+t_\gamma\cdot p$$

for each $\gamma\in\sum_m$, $k\notin\gamma$, with $t_\gamma\in N[X]$. It follows easily that $t_\gamma\in M[X^2]$ for all γ, i.e. the above relation holds in $M[X^2]$. Inserting in w we obtain

$$\begin{aligned}
w &= \textstyle\sum_{k\notin\gamma}(\sum_{\delta\cup j=\gamma}\Delta c_\delta D_{b_j}(p)+pt_\gamma)db_\gamma+(\sum_{k\notin\delta}c_\delta db_\delta)\wedge db_k\\
&= \textstyle\sum_{k\notin\delta}(c_\delta db_k+\Delta\sum_{j\neq k}c_\delta D_{b_j}(p)db_j)\wedge db_\delta+p\sum_{k\notin\gamma}t_\gamma db_\gamma\\
&= \textstyle\sum_{k\notin\delta}c_\delta(db_k+\Delta\sum_{j\neq k}D_{b_j}(p)db_j)\wedge db_\delta+p\sum_{k\notin\gamma}t_\gamma db_\gamma
\end{aligned}$$

Replacing the coefficient 1 of db_k by $1=\Delta D_{b_k}(p)+pr$, we get

$$w=\sum_{k\notin\delta}c_\delta\Delta dp\wedge db+p(\sum_{k\notin\gamma}t_\gamma db_\gamma+r\sum_{k\notin\delta}c_\delta db_k\wedge db_\delta)=w_1\wedge dp+pw_2$$

with w_1,w_2 in $\Omega_F M[X^2]$.

b) Let us assume $p\notin M[X^2]$. Then $\mathcal{B}$ is part of a 2-basis of $N(p)$. Let $w\in\ker(\Omega^m_F M[X^2]\to\Omega^m_{N(p)})$ and set $w=\sum_\gamma a_\gamma db_\gamma\in\Omega^m_F M[X^2]$, $\gamma\in\sum_m$, $a_\gamma\in M[X^2]$. Thus $\sum\bar{a_\gamma}db_\gamma=0$ in $\Omega^m_N(p)$, and by the remark above, we have $\bar{a_\gamma}=0$ in $N(p)$ for all γ. Thus $a_\gamma=p\cdot t_\gamma$, $t_\gamma\in N[X]$.

But $a_\gamma\in M[X^2]$ implies $D_{X_\mu}(a_\gamma)=0$ for all $\mu\neq 1,\dots,s-1$. Thus

$$D_{X_\mu}(p)\cdot t_\gamma+pD_{X_\mu}(t_\gamma)=0.$$

We have $p \notin M[X^2]$, so that there is some $\mu \neq 1, \dots, s-1$ with $D_{X_\mu}(p) \neq 0$. Choose this μ in the equation above. Then it follows $p|D_{X_\mu}(p)t_\gamma$ in $N[X]$ for all γ. But p does not divide $D_{X_\mu}(p)$, and hence $t_\gamma = ps_\gamma$, $s_\gamma \in N[X]$. Thus $a_\gamma = p^2 s_\gamma$ for all γ, and since $a_\gamma \in M[X^2]$, it follows $s_\gamma \in M[X^2]$ for all γ. Therefore $w = p^2 s$ with some $s \in \Omega_F^m M[X^2]$. This proves the lemma. $\square$

Let $u \in \Omega_F M[X^2]$ be a form generated over $M[X^2]$ by forms defined over F. The choice of any 2-basis $\mathcal{B}$ of F enable us to define $u^{[2]}$ (resp. $\wp(u)$) and this form is uniquely determined module $d\Omega_F M[X^2]$ (see remark (1.17)). We are interested in the behavior of $u^{[2]}$ under the reduction homomorphism $\Omega_F M[X^2] \to \Omega_{N(p)}$, where p is any irreducible polynomial in $N[X]$. In particular we want to compare $\overline{u^{[2]}}$ with $\overline{u}^{[2]}$, where this last square is taken with respect to the 2-basis of $N(p)$ as defined at the beginning of this section. In this case we have

(3.3) **Lemma.**

$$\overline{u^{[2]}} - \overline{u}^{[2]} \in \overline{d\Omega_F M[X^2]}$$

(3.4) **Lemma.** *Let $u, dv, \lambda \in \Omega_F M[X^2]$ and $T \in M[X^2]$, defined by $T = \sum_{\mu \in S_n, \mu \neq 1, \dots, s-1} b^\mu X_\mu^2$.*

Assume

$$u^{[2]} + dv = \lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$$

in $\Omega_{N(p)}$, where $p \in N[X]$ is irreducible and monic and $b_1, \dots, b_r \in F$. Then

a) *If $p \in M[X^2]$, there exists $\delta, u_1, u_2 \in \Omega_F M[X^2]$ such that*

$$u = \delta \wedge db_1 \wedge \cdots \wedge db_r \wedge dT + pu_1 + u_2 \wedge dp.$$

b) *If $p \notin M[X^2]$, there exists $\delta, u_1 \in \Omega_F M[X^2]$ such that*

$$u = \delta \wedge db_1 \wedge \cdots \wedge db_r \wedge dT + p^2 u_1.$$

Proof. Since $u^{[2]} + dv$ is closed, we can apply the Cartier-operator to this form and we get $C(u^{[2]} + dv) = u$. Thus in $\Omega_{N(p)}$

$$u = C(\lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}).$$

If $db_1 \wedge \cdots \wedge db_r \wedge d\bar{T} = 0$, then $u = 0$ and the lemma follows from (3.2). Thus we can assume $db_1 \wedge \cdots \wedge db_r \wedge d\bar{T} \neq 0$, in $\Omega_{N(p)}$, and therefore we can take $\{b_1, \ldots, b_r, \bar{T}\}$ as part of a 2-basis of $N(p)$. Then (1.12) implies

$$u = \bar{\delta} \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$$

with some $\bar{\delta} \in \Omega_{N(p)}$. We will show that $\bar{\delta}$ is contained in the image $\bar{\Omega}_F M[X^2]$ of $\Omega_F M[X^2]$ in $\Omega_{N(p)}$. It is clear that once we have this, the lemma follows from (3.2).

a) Assume $p \in M[X^2]$. Then there is some $i_0 \in I$ such that $\mathcal{B} \setminus \{b_{i_0}\} \cup \{\bar{X}_\mu,\ \mu \neq 1, \ldots, s-1\}$ is a 2-basis of $N(p)$ (we have chosen a 2-basis of F including $b_1, \ldots, b_r$). We write in $\Omega_{N(p)}$

$$\bar{\delta} = \bar{\delta}_0 + \bar{\delta}_1 \wedge d\bar{X}$$

where $\bar{\delta}_0$, $\bar{\delta}_1$ are forms not containing $d\bar{X} = d\bar{X}_s$. Then

$$u = \delta_0 \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T} + \delta_1 \wedge d\bar{X} \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$$

Since $u \in \Omega_F M[X^2]$ can not contain $d\bar{X}$ in its expansion in this 2-basis of $N(p)$, we conclude that

$$u = \delta_0 \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$$

(Notice that the expansion of db_{i_0} coming from $dp = 0$ in $\Omega_{N(p)}$ does not contain $d\bar{X}$, because $p \in M[X^2]$).

We can proceed in the same way with the other variables $X_\mu, \mu \neq 1, \ldots, s-1$, and we finally obtain that δ_0 is free from all differentials dX_μ, $\mu \neq 1, \ldots, s-1$. Thus δ_0 is generated over $N(p)$ by the differentials db_i, $i \in I \setminus \{i_0\}$. We write now the coefficients of δ_0 in the 2-basis expansion. First we set $\delta_0 = \delta_0' + \bar{X}\delta_0''$ where $\bar{X}$ appears in the coefficients of δ_0', δ_0'' only in even powers. Then

$$u = \delta_0' \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T} + \bar{X}\delta_0'' db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$$

The fact that u is in $\bar{\Omega}_F M[X^2]$ implies that the coefficients of u (in the 2-basis expansion) do not contain odd power of $\bar{X}$. Comparing coefficients we obtain

$$u = \delta_0' \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$$

with δ_0' free from odd powers of $\bar{X}$. Doing the same with the other variables we finally conclude $u = \bar{\delta} \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T}$ with $\bar{\delta} \in \bar{\Omega}_F M[X^2]$. From (3.2) follows the claim. The case (b) i.e. $p \notin M[X^2]$ can be treated in a similar way and we omit the proof. □

(3.5) **Lemma.** *Let $p \in M[X^2]$ be irreducible and monic (in $N[X]$). If*

$$pu = v \wedge dp$$

in $\Omega_F M[X^2]$ with $u, v \in \Omega_F M[X^2]$, then there exist $v_1, v_2 \in \Omega_F M[X^2]$ with

$$v = pv_1 + v_2 \wedge dp.$$

Proof. Since $p \in M[X^2]$, we have $D_{X_i}(p) = 0$ for all $i \neq 1, \ldots, s-1$. Thus

$$dp = \sum_{i \in I} D_{b_i}(p) db_i.$$

The fact that p is irreducible implies $dp \neq 0$ and hence there is some $i_0 \in I$ with $D_{b_{i_0}}(p) \neq 0$. Let us write $p(X) = X^{2n} + \cdots = p_0 + b_{i_0} p_1$, where p_0, p_1 are polynomials whose coefficients do not contain odd powers of b_{i_0} in the 2-basis expansion. Hence $D_{b_{i_0}}(p) = p_1$ and $\deg_X p_1 < \deg p$. In particular $p \not| p_1$. Therefore one can find polynomials $\Delta, t \in M[X^2]$ with $D_{b_{i_0}}(p)\Delta = 1 + p \cdot t$. Let us set $u = u_0 + u_1 \wedge db_{i_0}, v = v_0 + v_1 \wedge db_{i_0}$ with u_0, u_1, v_0, v_1 free from db_{i_0}. Thus

$$p(u_0 + u_1 \wedge db_{i_0}) = (v_0 + v_1 \wedge db_{i_0}) \wedge (D_{b_{i_0}}(p) db_{i_0} + \sum_{i \neq i_0} D_{b_i}(p) db_i)$$

implies

$$\begin{aligned} pu_0 &= v_0 \wedge \sum_{i \neq i_0} D_{b_i}(p) db_i \\ pu_1 &= D_{b_{i_0}}(p) v_0 + v_1 \wedge \sum_{i \neq i_0} D_{b_i}(p) db_i. \end{aligned}$$

Taking modulo p this equations, we obtain in $\Omega_{N(p)}$

$$(D_{b_{i_0}}(p) v_0 = v_1 \wedge \sum_{i \neq i_0} D_{b_i}(p) db_i$$

and since $\bar{\Delta} D_{b_{i_0}}(p) = 1$ in $N(p)$, it follows

$$v_0 = \bar{\Delta} v_1 \wedge \sum_{i \neq i_0} D_{b_i}(p) db_i.$$

But all these forms are contained in $\Omega_F M[X^2]$ so that (3.2) (a) implies

$$v_0 = \Delta v_1 \wedge \sum_{i \neq i_0} D_{b_i}(p) db_i) + p v_3 + v_4 \wedge dp$$

with v_3, $v_4 \in \Omega_F M[X^2]$. Inserting v_0 in $v = v_0 + v_1 \wedge db_{i_0}$ we get

$$\begin{aligned} v &= \Delta v_1 \wedge \sum_{i \neq i_0} D_{b_i}(p) db_i + p v_3 + v_4 \wedge dp + v_1 \wedge db_{i_0} \\ &= v_1 \wedge (\Delta \sum_{i \neq i_0} D_{b_i}(p) db_i + db_{i_0}) + p v_3 + v_4 \wedge dp. \end{aligned}$$

Since $1 = \Delta D b_{i_0}(p) + p \cdot t$ in $M[X^2]$, we get

$$\begin{aligned} v &= v_1 \wedge (\Delta dp + pt db_{i_0}) + p v_3 + v_4 \wedge dp \\ &= p(v_1 \wedge t db_{i_0} + v_3) + (\Delta v_1 + v_4) \wedge dp \end{aligned}$$

which shows that v has the desired form. $\square$

(3.6) **Lemma.** *Let dv, λ be forms in $\Omega_F M[X^2]$, $b_1, \ldots, b_r \in F$ be 2-independent and T as before. Assume*

$$dv = \lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge dT$$

in $\Omega_{N(p)}$, where p is monic and irreducible. Then

a) *If $p \in M[X^2]$, there exist δ, v_1, $v_2 \in \Omega_F M[X^2]$ such that*

$$dv = \delta \wedge db_1 \wedge \cdots \wedge db_r \wedge dT + d(p v_1 + v_2 \wedge dp)$$

b) *If $p \notin M[X^2]$, there exist δ, $v_1 \in \Omega_F M[X^2]$ such that*

$$dv = \delta \wedge db_1 \wedge \cdots \wedge db_r \wedge dT + p^2 dv_1$$

Proof. If $db_1 \wedge \cdots \wedge db_r \wedge dT = 0$ in $\Omega_{N(p)}$, and therefore $dv = 0$, we may use (3.2) to prove the lemma. Hence we will assume $db_1 \wedge \cdots \wedge db_r \wedge dT \neq 0$ in $\Omega_{N(p)}$, and for the time being we set $T = b_{r+1}$. Thus $b_1, \ldots, b_{r+1}$ can be chosen as part of a 2-basis of $N(p)$. From (1.13) and (1.14) we infer in $N(p)$

$$v = \sum_{\mu} b_i b^{\mu} z_{\mu} \wedge db_1 \wedge \cdots \wedge \hat{db_i} \wedge \cdots db_{r+1} + b^{\mathbf{1}} u \wedge db_1 \wedge \cdots \wedge db_{r+1} + z$$

with z_{μ}, z closed forms in $\Omega_{N(p)}$, μ running over the index set indicated by (1.13). We shall next prove that the form z_{μ}, u, z can be chosen in $\bar{\Omega}_F M[X^2]$. Let us first assume $p \in M[X^2]$. We take a 2-basis of $N(p)$ of the form $\mathcal{B} \setminus \{b_{i_0}\} \cup \{X_{\mu}\}$ where X_{μ} are all variable involved ($X = X_s$). Let us fix some variable X_{μ} which we denote by Y. Thus in $N(p)$ we have

$$z_{\mu} = e_{\mu,1} + \bar{Y} e_{\mu,2} + (e_{\mu,3} + \bar{Y} e_{\mu,4}) \wedge d\bar{Y}$$

with forms $e_{\mu,i}$ free from $d\bar{Y}$ and whose coefficients are free from odd powers of Y. There are similar decompositions for u, z. Since z_{μ} is closed we get

$$0 = de_{\mu,1} + \bar{Y} de_{\mu,2} + e_{\mu,2} \wedge d\bar{Y} + de_{\mu,3} \wedge d\bar{Y} + \bar{Y} de_{\mu,4} \wedge d\bar{Y}.$$

Thus we obtain

$$de_{\mu,1} = de_{\mu,4} = 0, \qquad e_{\mu,2} = de_{\mu,3}.$$

Inserting these expressions for z_{μ}, u and z in the equation for v, we get

$$\begin{aligned} v &= \sum_{\mu} b_i b^{\mu} z_{\mu,1} \wedge db_1 \wedge \cdots \wedge \hat{db_i} \wedge \cdots \wedge db_{r+1} \\ &\quad + b^{\mathbf{1}} u_1 \wedge db_1 \wedge \cdots \wedge db_{r+1} + z_1 \end{aligned}$$

when all $z_{\mu,1}$ and z_1 are closed and moreover $z_{\mu,1}, z_1$ and u_1 are free from $d\bar{Y}$ and $\bar{Y}$. Doing the same with the other variables, we finally conclude that the forms z_{μ}, u and z can be taken in $\overline{\Omega_F} M[X^2]$.

Therefore we have $v - \sum_{\mu} b_i b^{\mu} z_{\mu} \wedge db_1 \wedge \cdots \wedge \hat{db_i} \wedge \cdots \wedge db_{r+1} - b^{\mathbf{1}} u \wedge db_1 \wedge \cdots \wedge db_{r+1} - z \in \ker(\Omega^m_F M[X^2] \to \Omega^m_{N(p)})$ and from (3.2) it follows

$$\begin{aligned} v &- \sum_{\mu} b_i b^{\mu} z_{\mu} \wedge db_1 \wedge \cdots \wedge \hat{db_i} \wedge \cdots \wedge db_{r+1} - b^{\mathbf{1}} u \wedge db_1 \wedge \cdots \wedge db_{r+1} - z \\ &= pv_1 + v_2 \wedge dp \end{aligned}$$

with some $v_1, v_2 \in \Omega_F M[X^2]$. We apply now d to this relation and obtain (a).

Let us now assume $p \notin M[X^2]$. Then these is some index μ_0 such that $D_{X_{\mu_0}}(p) \neq 0$ and $\mathcal{B} \cup \{X_\mu, \mu \neq \mu_0\} \cup \{y\}$, where y is the image of X in $N(p)$, is a 2-basis of $N(p)$. All variables $X_\mu, \mu \neq \mu_0$ as well as y can be handled in the same way as in the first case, so that we are led to consider only the variable X_{μ_0}. Via the relation $dp = 0$ we express dX_{μ_0} in terms of the other differentials, so that we may assume that all z_μ, u, z do not contain dX_{μ_0} too. Thus $\bar{X}_{\mu_0}$ may appear in odd powers in the coefficients of z_μ, u, z. Let us write $p = p_0 + X_{\mu_0} p_1$with p_0, p_1 not containing X_{μ_0} in odd powers. Thus in $N(p)$ we have $\bar{X}_{\mu_0} = \bar{p}_0/\bar{p}_1$, so that replacing $\bar{X}_{\mu_0}$ by $\bar{p}_0/\bar{p}_1$ in these coefficients, we get rid of the odd powers of $\bar{X}_{\mu_0}$, but there appear again the variables $X_\mu, \mu \neq \mu_0$, in these coefficients. We apply again the above procedure to get rid of the odd powers of these variable in the forms z_μ, u, z. Therefore we may assume $z_\mu, u, z \in \Omega_F M[X^2]$. The assertion (b) follows again from (3.2). $\square$

(3.7) **Lemma**. *Let $b_1, \dots, b_r \in F$ be 2-independent in F and let T be as before. Assume for a form $\lambda \in \Omega_F M[X^2]$*

$$\lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge dT = 0$$

in $\Omega_{N(p)}$. Then

a) *If $p \in M[X^2]$ and $db_1 \wedge \cdots \wedge db_r \wedge dT \neq 0$ in $\Omega_{N(p)}$, there exist $\lambda_1, \lambda_2 \in \Omega_F M[X^2]$ with*

$$\lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge dT = (p\lambda_1 + \lambda_2 \wedge dp) \wedge db_1 \wedge \cdots \wedge db_r \wedge dT$$

in $\Omega_F M[X^2]$.

b) *If $p \notin M[X^2]$ and $db_1 \wedge \cdots \wedge db_r \wedge dT \neq 0$ in $\Omega_{N(p)}$, there exist $\lambda_1 \in \Omega_F M[X^2]$ with*

$$\lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge dT = p^2 \lambda_1 \wedge db_1 \wedge \cdots \wedge db_r \wedge dT$$

in $\Omega_F M[X^2]$.

c) *If $db_1 \wedge \cdots \wedge db_r \wedge dT = 0$ in $N(p)$, there exists $t \in M[X^2]$ with $\deg_X t < \deg_X p$ and $db_1 \wedge \cdots \wedge db_r \wedge dT = db_1 \wedge \cdots \wedge db_r \wedge d(pt)$ or $= db_1 \wedge \cdots \wedge d(pt) \wedge \cdots \wedge db_r \wedge dT$ for some $1 \leq i \leq r$, this case only occurs for $p \in M[X^2]$.*

Proof. Let us first assume $db_1 \wedge \cdots \wedge db_r \wedge dT \neq 0$ in $\Omega_{N(p)}$. Then $b_1, \ldots, b_r, \bar{T}$ can be chosen as part of a 2-basis of $N(p)$ and the assumption $\lambda \wedge db_1 \wedge \cdots \wedge db_r \wedge d\bar{T} = 0$ implies (see (1.1))

$$\lambda = \sum_1^r \delta_i \wedge db_i + \delta \wedge d\bar{T}$$

with some forms $\delta_i, \delta \in \Omega_{N(p)}$. Since $\lambda \in \bar{\Omega} M[X^2]$, one easily shows that the forms δ_i, δ can be chosen in $\bar{\Omega}_F M[X^2]$ too. Then (3.2) implies

$$\lambda = \sum_{i=1}^r \delta_i \wedge db_i + \delta \wedge dT + p\lambda_1 + \lambda_2 \wedge dp$$

if $p \in M[X^2]$ and $\lambda = \sum \delta_i \wedge db_i + \delta \wedge dT + p^2\lambda_2$ if $p \notin M[X^2]$, where λ_1, $\lambda_2 \in \Omega_F M[X^2]$. Taking the product with $db_1 \wedge \cdots \wedge db_r \wedge dT$ we obtain a) and b). Let us assume now $db_1 \wedge \cdots \wedge db_r \wedge d\bar{T} = 0$ in $\Omega_{N(p)}$. We write $b_{r+1} = \overline{T}$ for the time being, and we choose (after reordering) a maximal 2-independent subset $\{b_1, \ldots, b_{j_0}\}$ of $\{b_1, \ldots, b_{r+1}\}$. For example one could have a relation $\overline{T} = b_{r+1} = \sum_\mu p_\mu^2 b_\mu$, where μ runs over the set of map $\mu : \{1, \ldots, j_0\} \to \{0, 1\}$ and $p_\mu \in N(p)$, $b_\mu = \prod_{i=1}^{j_0} b_i^{\mu(i)}$. Then we can write $T = \sum p_\mu^2 b_\mu + p \cdot t$ in $N[X]$, with $p_\mu, p, t \in N[X]$. Since $T, p_\mu^2 \in M[X^2]$, it follows $p \cdot t \in M[X^2]$, and (3.1) implies $p, t \in M[X^2]$, or $t = p \cdot \ell$ with $\ell \in M[X^2]$. Thus we have

$$T = \sum p_\mu^2 b_\mu + p \cdot t, \;\; p, t \in M[X^2]$$

or

$$T = \sum p_\mu^2 b_\mu + p^2 \cdot \ell, \;\; \ell \in M[X^2]$$

and $p \notin M[X^2]$. But in the last case we see that $\deg_X(p^2 l) > \deg_X T$ and $\deg_X p_\mu^2 b_\mu$ (since p_μ can be chosen with $\deg_X p_\mu < \deg_X p$). This means that $p \notin M[X^2]$ never happen. Therefore

$$dT = \sum p_\mu^2 db_\mu + d(pt)$$

with $t \in M[X^2]$ and $\deg_X t < \deg_X p$. It follows

$$db_1 \wedge \cdots \wedge db_r \wedge dT = db_1 \wedge \cdots \wedge db_r \wedge d(pt).$$

The same argument applies for the case that some b_k $(k \neq r+1)$ is 2-dependent of $\{b_1, \dots, b_{j_0}\}$. We omit the details. $\square$

(3.8) **Lemma.** *Let $p \in M[X^2]$ be a monic irreducible polynomial in $N[X]$. Let $u, v, \lambda \in \Omega_F M[X^2]$ be such that*

$$\wp u + dv = \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \qquad \text{in } \Omega_{N(p)}$$

where $b_1, \dots, b_{s-1} \in F^*$ and $T = b_s X^2 + T'$ as before $(\deg_X T' = 0)$. *Then there exist forms $u_1, u_2, \delta \in \Omega_F M[X^2]$, f_{ij}, $g_{ij} \in M[X^2]$ with* $\deg(f_{ij}) < \deg(p)$ *and $f_{ij} g_{ij} \equiv 1 \pmod p$, such that*

$$\begin{aligned} u = \sum_i (g_{i1} df_{i1}) \wedge \cdots \wedge (g_{im} df_{im}) \quad &+ \quad pu_1 + u_2 \wedge dp \\ &+ \quad \delta \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \end{aligned}$$

holds in $\Omega_F M[X^2]$.

Proof. If $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$ in $\Omega^s_{N(p)}$, then $\wp u = dv$, and we conclude from (2.15)

$$u = \sum_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}}$$

with certain $f_{ij} \in N(p)^*$.
Taking $g_{ij} \in N[X]$ with $g_{ij} f_{ij} \equiv 1 \pmod p$, we obtain $u = \sum_i g_{i1} df_{i1} \wedge \cdots \wedge g_{is} df_{is}$. Since $u \in \Omega_F M[X^2]$ one can show that the f_{ij} and g_{ij} can be chosen in $M[X^2]$, and we can now apply (3.2). Thus we can assume $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \neq 0$ in $\Omega_{N(p)}$, and hence $db_1, \dots, db_{s-1}, T$ are 2-independent in $N(p)$. We choose a 2-basis of F which contains $\{b_1, \dots, b_{s-1}\}$ and we take the constructed 2-basis of $N(p)$ for the case $p \in M[X^2]$ as indicated after the proof of lemma (3.1). The excluded index i_0 can be chosen $\neq 1, \dots, s-1$. Moreover we replace the next element b_s by the image $\bar{T}$ of T in $N(p)$ and we write b_s for $\bar{T}$. Thus the new 2-basis of $N(p)$ is now $\{b_1, \dots, b_{s-1}, b_s = \bar{T}, \dots, \hat{b}_{i_0}, \dots, b_N, X_\mu, \mu \neq 1, \dots, s-1\}$. We order this basis such that all $b_i < X_\mu$ for all μ and all i. In $\Sigma_{m,N(p)}$ we choose the lexicographic ordering and with respect to this ordering choose α minimal with $\alpha > \gamma$ for all γ with $\gamma(1) = 1, \dots, \gamma(s) = s$. Hence if $u \in \Omega^m_{N(p),<\alpha}$ then there exists some δ with $u = \delta \wedge db_1 \wedge \cdots \wedge db_s$ and δ in the image of $\Omega_F M[X^2]$. We choose now

$\beta \in \Sigma_{m,N(p)}$ minimal with the property $u \in \Omega_{N(p),\beta}$. This means that in the representation of u with respect to the above 2-basis of $N(p)$, β corresponds to the leading index of this representation. Since u comes from $\Omega_F M[X^2]$ we see that $b_{\beta(i)} \in M[X^2]$ for all i. Now we conclude that $N(p)_{\beta(i)} \subset \bar{M}[X^2]$.

If $\beta \leq \alpha$ we have $u = \delta \wedge db_1 \wedge \cdots \wedge db_s$ with δ in the image of $\Omega_F M[X^2]$, and we are done by (3.2). Assume now $\alpha < \beta$. Then we have $\lambda \wedge db_1 \wedge \cdots \wedge db_s \in \Omega_{N(p),<\beta}$ and hence

$$\wp(u) \in \Omega_{N(p),<\beta} + d\Omega_{N(p)}.$$

Applying Kato's lemma one gets

$$u = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m} + u'$$

in $\Omega_{N(p)}$ with $u' \in \Omega_{N(p),<\beta}$, $a_i \in N(p)_{\beta(i)} \subset M[X^2]$. The form $u^{'}$ is contained in $\overline{\Omega_F} M[X^2]$ because u and also $\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$ belong to $\overline{\Omega_F} M[X^2]$. We apply now the same procedure to u' until we get

$$u = \sum \frac{df_{i,1}}{f_{i,1}} \wedge \cdots \wedge \frac{df_{i,m}}{f_{i,m}} + v$$

with all $f_{i,j} \in \bar{M}[X^2]$ and $v \in \Omega_{N(p),<\alpha}$. By the remark above we conclude $v = \delta \wedge db_1 \wedge \cdots \wedge db_s$ with δ in the image of $\Omega_F M[X^2]$. All $f_{i,j} \in M[X^2]$ can be chosen with $\deg_X f_{i,j} < \deg_X p$. Let $g_{i,j} \in M[X^2]$ with $f_{i,j} g_{i,j} \equiv 1($ mod $p)$. Inserting in the above equation and applying again lemma (3.2) we finally get the desired result. $\square$

4 The kernel $H^{n+1}(F(\phi)/F)$

In this section we will prove the main result of this paper, namely

(4.1) **Theorem.** *Let $\phi = \ll b_1, \dots, b_n \gg$ be an anisotropic bilinear Pfister form over F. Then for $m \geq n$*

$$H^{m+1}(F(\phi)/F) = \overline{\Omega_F^{m-n} \wedge db_1 \wedge \cdots \wedge db_n}.$$

If $m < n$, then $H^{m+1}(F(\phi)/F) = 0$.

Let us write during this section K for the function field $F(\phi)$ of the conic $\phi = 0$. In section 2 we have shown that $w \in H^{m+1}(F(\phi)/F)$ holds if and only if w satisfies the following equation in Ω_L^m

$$w = \wp u + dv + \lambda \wedge dT \tag{4.2}$$

with $u, v, \lambda \in \Omega_F M' \subset \Omega_L$, where M' is the subfield $F(X_\mu^2 | \mu \in S_n)$ of $L = F(X_\mu | \mu \in S_n)$, and $\Omega_F M'$ denotes the subspace of Ω_L generated by the forms $db, b \in F$ over the field M'. Recall that $T = \sum_{\mu \in S_n} b^\mu X_\mu^2$ and S_n is the set of all maps $u : \{1, \dots, n\} \to \{0, 1\}$ with at least one value $\mu(i) = 1$. We will develop a descent procedure, which starting from (4.2) will led us to an equation $w = \wp u + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_n$ with $u, v, \lambda \in \Omega_F$ in Ω_F.

Let us fix some integer s with $1 \leq s \leq n$. Set $M = F(X_\mu^2 | \mu \in S_n, \mu \neq 1, \dots, s), X = X_s$ and let us consider the equations in Ω_L

$$w = \wp u + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \tag{4.3}$$

with $w \in \Omega_F$, u, v, $\lambda \in \Omega_F M(X^2)$ and $T = b_s X^2 + T'$, $\deg_X T' = 0$ and T' is a polynomial in X_μ^2, $\mu \neq 1, \dots, s$ over F. The equation (4.3) for $s = 1$ is just (4.2). Our strategy is to start with (4.2) and to push up the index s until we get the factor $db_1 \wedge \cdots \wedge db_n$ and then to eliminate the rest of the variables until we get the desired equation $w = \wp u_0 + dv_0 + \lambda_0 \wedge db_1 \wedge \cdots \wedge db_n$ in Ω_F, which is obviously the assertion of (4.1).

Any $u \in \Omega_F M(X^2)$ can be written in the form

$$u = u_0 + \sum_p u_p \tag{4.4}$$

with $u_0 \in \Omega_F M[X^2]$, $u_p \in p^{-\infty} \Omega_F M[X^2]$, p running over all irreducible monic polynomials in $N[X]$, where $N = F(X_\mu, \mu \neq 1, \dots, s)$. Recall that

$p^{-\infty}\Omega_F M[X^2]$ denotes the space of forms u/p^r with $u \in \Omega_F M[X^2]$ and $\deg_X u < \deg_X(p^r)$ (see section 2). Fixing a 2-basis of F and N we have seen that the operators $\wp$ and d leave invariant the spaces $\Omega_F M[X^2]$, $p^{-\infty}\Omega_F M[X^2]$ (see (1.21)). Let us now insert in (4.3) the decompositions $u = u_0 + \sum u_p$, $v = v_0 + \sum v_p$ and $\lambda = \lambda_0 + \sum \lambda_p$. The terms $\lambda_p \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$ can contribute eventually with an integral form (i.e. from $\Omega_F M[X^2]$) , which we will denote by E_p. Thus $\lambda_p \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + E_p \in p^{-\infty}\Omega_F M[X^2]$. Thus we conclude from (4.3), (1.20) and (1.21)

$$(4.5) \qquad w = u_0^{[2]} + u_0 + dv_0 + \lambda_0 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + \sum_p E_p$$

$$(4.6) \qquad 0 = u_p^{[2]} + u_p + dv_p + \lambda_p \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + E_p.$$

(4.7) **Remark.** As noticed in lemma (3.3), under the natural homomorphism

$$\Omega_F M[X^2] \longrightarrow \Omega_{N(p)}$$

($p \in N[X]$ an irreducible polynomial), the operation $u^{[2]}$ behaves well, i.e., $\overline{u^{[2]}} - \overline{u}^{[2]}$ is contained in $\overline{d\Omega_F M[X^2]}$ provided the computations are done with respect to the 2-basis indicated there. In what follows we will frequently reduce modulo p expressions of the form $u^{[2]} + dv$ and then we will lift back to $\Omega_F M[X^2]$. By the above remark, the differential form dv will eventually change, but this will be of no importance for further computations. Because of this reason we will not explicitely mention these changes.

Our next goal is to study the forms $E_p \in \Omega_F M[X^2]$. To this end we will distinguish three types of monic irreducible polynomials in $N[X]$, namely

a) $p \notin M[X^2]$ and $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \neq 0$ in $\Omega_{N(p)}$

b) $p \in M[X^2]$ and $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \neq 0$ in $\Omega_{N(p)}$

c) $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$ in $\Omega_{N(p)}$.

Case (a): We can write (see (4.6))

$$E_p = \frac{\bar{u}^{[2]}}{p^{4r}} + \frac{\bar{u}}{p^{2r}} + \frac{d\bar{v}}{p^{4r}} + \frac{\bar{\lambda}}{p^{4r}} \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with integral forms $\bar{u}, \bar{v}, \bar{\lambda} \in \Omega_F M[X^2]$, and $r \geq 0$. For the time being, let us write u, v, λ instead of $\bar{u}, \bar{v}, \bar{\lambda}$. Then in $\Omega_F M[X^2]$ we get

$$p^{4r} E_p = u^{[2]} + p^{2r} u + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT. \tag{4.8}$$

Next we show that the form E_p can be absorbed by the first sum $u_0^{[2]} + u_0 + dv_0 + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$ in (4.5). Of course we can assume $r \geq 1$ in (4.8). Taking (4.8) modulo p we obtain in $\Omega_{N(p)}$

$$u^{[2]} + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0. \tag{4.9}$$

Since $p \notin M[X^2]$, lemma (3.4) (b) implies

$$u = \delta \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + p^2 u_1$$

in $\Omega_F M[X^2]$. Inserting this expression in (4.8) we get

$$p^{4r} E_p = p^4 u_1^{[2]} + p^{2r+2} u_1 + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \tag{4.10}$$

with some new forms v and $\lambda \in \Omega_F M[X^2]$. It follows $dv = \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$ in $\Omega_{N(p)}$ and lemma (3.5) implies

$$dv = \delta_1 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + p^2 dv_1$$

in $\Omega_F M[X^2]$. Thus (4.10) reads now

$$p^{4r} E_p = p^4 u_1^{[2]} + p^{2r+2} u_1 + p^2 dv_1 + \delta' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

and hence in $\Omega_{N(p)}$ we have

$$\delta' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0.$$

From (3.6) (b) we conclude

$$\delta' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = p^2 \delta'' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with $\delta'' \in \Omega_F M[X^2]$, since $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \neq 0$ in $\Omega_{N(p)}$. In this case we get

$$\begin{aligned} p^{4r} E_p &= p^4 u_1^{[2]} + p^{2r+2} u_1 + p^2 dv_1 + p^2 \delta'' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \\ p^{4r-2} E_p &= p^2 u_1^{[2]} + p^{2r} u_1 + dv_1 + \delta'' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT. \end{aligned}$$

Again one obtains in $\Omega_{N(p)}$

$$dv_1 = \delta'' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

and it follows $dv_1 = \delta''' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + p^2 dv_2$ with δ''', $v_2 \in \Omega_F M[X^2]$. Repeating the last argument we finally obtain a relation

$$p^{4r-4} E_p = u_1^{[2]} + p^{2r-2} u_1 + dv_1 + \lambda_1 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

in $\Omega_F M[X^2]$, i.e.

$$E_p = \wp(\frac{u_1}{p^{2r-2}}) + d(\frac{v_1}{p^{4r-4}}) + \frac{\lambda_1}{p^{4r-4}} \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT.$$

Thus we have reduced the number r by one in (4.8). Iterating this process we finally arrive at a relation with $r = 0$, i.e.

$$E_p = \wp(u_0') + dv_0' + \lambda_0' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with $u_0', v_0', \lambda_0' \in \Omega_F M[X^2]$. This expression can be absorbed by the first integral part of (4.5) and hence we have eliminated E_p from this sum.

Case (b): Thus we assume $p \in M[X^2]$ and $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \neq 0$ in $\Omega_{N(p)}$. We can write

$$E_p = \frac{u^{[2]}}{p^{2r}} + \frac{u}{p^r} + \frac{dv}{p^{2r}} + \frac{\lambda}{p^{2r}} \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with some integer $r \geq 1$. Thus

$$(4.11) \qquad p^{2r} E_p = u^{[2]} + p^r u + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with some forms $u, v, \lambda \in \Omega_F M[X^2]$. In $\Omega_{N(p)}$ we get

$$0 = u^{[2]} + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT.$$

From lemma (3.4) (a) we conclude

$$u = \delta \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + pu_1 + u_2 \wedge dp$$

with $\delta, u_1, u_2 \in \Omega_F M[X^2]$.

We insert now this expression in E_p and obtain

$$\begin{aligned} p^{2r}E_p &= (\delta \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT)^{[2]} + p^r\delta \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \\ &\quad + p^2u_1^{[2]} + pu_2^{[2]} \wedge dp + p^{r+1}u_1 + p^r u_2 \wedge dp \\ &\quad + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \\ &= p^2u_1^{[2]} + pu_2^{[2]} \wedge dp + p^{r+1}u_1 + p^r u_2 \wedge dp \\ &\quad + dv + \lambda' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \end{aligned}$$

with $\lambda' = \lambda + p^r\delta + \delta'$, where δ' is some form in $\Omega_F M[X^2]$ with $(\delta \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT)^{[2]} = \delta' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$. Thus in $\Omega_{N(p)}$ it holds

$$dv = \lambda' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

and lemma (3.5) implies

$$dv = \lambda'' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + d(pv_1 + v_2 \wedge dp)$$

with $\lambda'', v_1, v_2 \in \Omega_F M[X^2]$. Therefore

$$\begin{aligned} p^{2r}E_p &= p^2u_1^{[2]} + pu_2^{[2]} \wedge dp + p^r u_2 \wedge dp + p^{r+1}u_1 \\ &\quad + pdv_3 + v_3 \wedge dp + \lambda''' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \end{aligned}$$

where $v_3 = v_1 + dv_2$ and λ''' is some form in $\Omega_F M[X^2]$. Taking this equation modulo p we obtain $\lambda''' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$ in $\Omega_{N(p)}$. Since $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \neq 0$ in $\Omega_{N(p)}$, we obtain from lemma (3.6) (a)

$$\lambda''' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = (p\lambda_1 + \lambda_2 \wedge dp) \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with $\lambda_1, \lambda_2 \in \Omega_F M[X^2]$. Therefore

$$\begin{aligned} p[p^{2r-1}E_p + pu_1^{[2]} &+ (u_2^{[2]} + p^{r-1}u_2) \wedge dp \\ &\quad + p^r u_1 + dv_3 + \lambda_1 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT)] \\ &= (v_3 + \lambda_2 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT) \wedge dp. \end{aligned}$$

Lemma (3.4) implies

$$v_3 + \lambda_2 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = pv_4 + v_5 \wedge dp$$

with $v_4, v_5 \in \Omega_F M[X^2]$. It follows

$$\begin{aligned} p^{2r-1}E_p &+ pu_1^{[2]} + u_2^{[2]} \wedge dp + p^{r-1}u_2 \wedge dp + p^r u_1 \\ &+ pdv_4 + dv_5 \wedge dp + \lambda_3 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0 \end{aligned}$$

where $\lambda_3 = \lambda_1 + d\lambda_2$. We get again in $\Omega_{N(p)}$

$$\lambda_3 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$$

and (3.6) (a) implies $\lambda_3 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = (p\lambda_4 + \lambda_5 \wedge dp) \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$ in $\Omega_F M[X^2]$. Inserting this expression in the above equation we obtain

(4.12)

$$\begin{aligned} p[p^{2r-2}E_p &+ u_1^{[2]} + p^{r-1}u_1 + \lambda_4 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + dv_4] \\ &= (u_2^{[2]} + p^{r-1}u_2 + dv_5 + \lambda_5 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT) \wedge dp \end{aligned}$$

in $\Omega_F M[X^2]$. Lemma (3.4) implies

$$u_2^{[2]} + p^{r-1}u_2 + dv_5 + \lambda_5 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$$

in $\Omega_{N(p)}$.

We consider now two cases.

Case 1. $r > 1$. Then it holds in $\Omega_{N(p)}$,

$$u_2^{[2]} + dv_5 + \lambda_5 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$$

and we can apply (3.4) (a) to obtain

$$u_2 = \lambda_6 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT + pu_3 + u_4 \wedge dp$$

in $\Omega_F M[X^2]$. Inserting this value of u_2 in the above equation and using a similar argument as in case (a) we easily see that the exponent of p in (4.10) can be lowered. Therefore we are led to consider the next case.

Case 2. $r = 1$. Then we have $\Omega_{N(p)}$,

$$u_2^{[2]} + u_2 + dv_5 + \lambda_5 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0.$$

For the time being we will set $w_s = db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$. From lemma (3.8) we conclude

$$u_2 = \sum_i \bigwedge_j g_{ij} df_{ij} + pu_3 + u_4 \wedge dp + \delta \wedge w_s$$

with $f_{ij}, g_{ij} \in M[X^2]$, $u_3, u_4, \delta \in \Omega_F M[X^2]$ as indicated by the lemma. Therefore the right hand side of the equation (4.12) reads now

$$\begin{aligned} \Big[\sum_i \bigwedge_j (g_{ij} df_{ij})^2 \, \frac{df_{ij}}{f_{ij}} \; &+ \; \sum_i \bigwedge_j g_{ij} f_{ij} \frac{df_{ij}}{f_{ij}} \\ &+ \; p^2 u_3^{[2]} + pu_3 + dv_5 + \lambda_0 \wedge w_s] \wedge dp \end{aligned}$$

with certain form $\lambda_0 \in \Omega_F M[X^2]$.

Since $g_{ij} f_{ij} \equiv 1(\mod p)$, it follows $\prod_j g_{ij} f_{ij} = 1 + ph_i$ with some $h_i \in M[X^2]$ for all i. Thus (4.12) implies

$$\begin{aligned} p[E_p \; &+ \; u_1^{[2]} + u_1 + \lambda_4 \wedge w_s + dv_4] = [(\sum_i ph_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}})^{[2]} \\ &+ \; \sum ph_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}} + \wp(pu_3) + dv_5 + \lambda_0 \wedge w_s] \wedge dp. \end{aligned}$$

But one easily sees that

$$\wp(pu_3) \wedge dp \;\; \equiv \;\; p\wp(u_3 \wedge dp)$$

$$\wp\left[\sum_i ph_i \wedge \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}}\right] \wedge dp \;\; \equiv \;\; p\wp\left[dp \wedge \sum_i h_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}}\right]$$

$(\mod d\Omega_F M[X^2])$.

We bring both terms to the left side of the above equation and we get

$$\begin{aligned} p[E_p + \wp(u_1 + u_3 \wedge dp \; &+ \; \sum_i h_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}} \wedge dp) \\ &+ \; \lambda_4 \wedge w_s + dv_4] = [dv_5 + \lambda_0 \wedge w_s] \wedge dp. \end{aligned}$$

Thus

$$\begin{aligned} E_p \;\; &= \;\; \wp(u_1 + u_3 \wedge dp + \sum_i h_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}} \wedge dp) \\ &+ d(v_4 + v_5 \wedge \frac{dp}{p}) + (\lambda_4 + \lambda_0 \wedge \frac{dp}{p}) \wedge w_s. \end{aligned}$$

Notice that $u_1 + u_3 \wedge dp + \sum_i h_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{im}}{f_{im}} \wedge dp$ contains denominators all of whose prime factors in $M[X^2]$ (and in $N[X]$) are of degree $< \deg(p)$. The other involved forms have at most p in the denominator. Using a similar argument as done at the beginning of this section we get

$$E_p = \wp(u_0') + dv_0' + \lambda_0' \wedge w_s + G_p + \sum_{q \deg(q)<\deg(p)} G_q$$

with $u_0', v_0', \lambda_0' \in \Omega_F M[X^2]$ and integral forms G_p, G_q of the type

$$\begin{aligned} G_p &= dv_p + \lambda_p \wedge w_s \\ G_q &= \wp(u_q) + dv_q + \lambda_q \wedge w_s \end{aligned}$$

where $v_p, \lambda_p \in \frac{1}{p}\Omega_F M[X^2]$ and $u_q, v_q, \lambda_q \in q^{-\infty}\Omega_F M[X^2]$ if $\deg(q) < \deg(p)$.

Since the forms G_q have denominators of degree $< \deg(p)$ and are of the same type as E_q, we can add them to the E_q's, so that we forget then now. Let us consider G_p. Let us write

$$v_p = \frac{v'}{p^2}, \qquad \lambda_p = \frac{\lambda'}{p} \qquad \text{with} \qquad v', \lambda' \in \Omega_F M[X^2].$$

Thus $G_p = \frac{dv'}{p^2} + \frac{\lambda'}{p} \wedge w_s$ and hence

$$p^2 G_p = dv' + p\lambda \wedge w_s$$

holds in $\Omega_F M[X^2]$. Thus $dv' = 0$ in $\Omega_{N(p)}$. Using Cartier's theorem (see [Ca] or section 1) we can write $v' = A^{[2]} + dB$ in $\Omega_{N(p)}$. Since $v' \in \bar{\Omega}_F M[X^2] =$ Image of $\Omega_F M[X^2]$ in $\Omega_{N(p)}$, we easily see that B also can be chosen in $\bar{\Omega}_F M[X^2]$. Therefore in $\Omega_F M[X^2]$ (see (3.2))

$$v' = A^{[2]} + dB + pv_1 + v_2 \wedge dp$$

with $v_1, v_2 \in \Omega_F M[X^2]$. It follows $dv' = d(pv_1 + v_2 \wedge dp) = d(p(v_1 + dv_2)) = d(pv_1')$ where $v_1' = v_1 + dv_2$. Therefore in $\Omega_F M[X^2]$

$$p[pG_p + dv_1' + \lambda \wedge w_s] = v_1' \wedge dp$$

Lemma (3.4) implies $v_1' = pv_2 + v_3 \wedge dp$ in $\Omega_F M[X^2]$ and hence after replacing v_1' in the above equation it follows

$$pG_p = pdv_2 + dv_3 \wedge dp + \lambda \wedge w_s.$$

Thus $\lambda \wedge w_s = 0$ in $\Omega_{N(p)}$. Lemma (3.6) implies $\lambda \wedge w_s = (p\lambda_1 + \lambda_2 \wedge dp) \wedge w_s$ in $\Omega_F M[X^2]$ and therefore

$$pG_p = pdv_2 + dv_3 \wedge dp + p\lambda_1 \wedge w_s + \lambda_2 \wedge dp \wedge w_s$$

$$p[G_p + dv_2 + \lambda_1 \wedge w_s] = (dv_3 + \lambda_2 \wedge w_s) \wedge dp.$$

We apply again (3.4) and obtain $dv_3 + \lambda_2 \wedge w_s = pv_4 + v_5 \wedge dp$ in $\Omega_F M[X^2]$. Hence in $\Omega_{N(p)}$ it holds $dv_3 = \lambda_2 \wedge w_s$. Applying (3.5) we obtain $dv_3 = \delta \wedge w_s + d(pv_6 + v_7 \wedge dp)$. This implies

$$p[G_p + dv_2 + \lambda_1 \wedge w_s] = pdv_6 \wedge dp + \delta' \wedge w_s \wedge dp$$

with $\delta' = \lambda_2 + \delta$. Lemma (3.4) implies $\delta' \wedge w_s = p\lambda_3 + \lambda_4 \wedge dp$ in $\Omega_p M[X^2]$. Hence $\delta' \wedge w_s \wedge dp = p\lambda_3 \wedge dp$. But we are assuming that $w_s \neq 0$ in $\Omega_{N(p)}$. Thus we easily conclude that there exist some $\lambda_3' \in \Omega_F M[X^2]$ with $\delta' \wedge w_s \wedge dp = p\lambda_3' \wedge w_s \wedge dp$. Inserting this in the above relation we get

$$p[G_p + dv_2 + \lambda_1 \wedge w_s] = pdv_6 \wedge dp + p\lambda_3' \wedge w_s \wedge dp$$

$$G_p = d(v_2 + v_6 \wedge dp) + (\lambda_1 + \lambda_3' \wedge dp) \wedge w_s$$

in $\Omega_F M[X^2]$. Thus we also get rid of G_p in the equation for E_p.

Case (c). We assume $db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$ in $\Omega_{N(p)}$. Thus we assume now $p \in M[X^2]$. In $\Omega_F M[X^2]$ we can write

$$p^{2h} E_p = u^{[2]} + p^h u + dv + \lambda \wedge w_s \tag{4.13}$$

with $h \geq 1$, with $u, v, \lambda \in \Omega_F M[X^2]$ and $w_s = db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$. We claim that there is $t \in M[X^2]$, $\deg_X t < \deg_X p$ with

$$t^2 p^{2h-2} E_p = u'^{[2]} + tp^{h-1} u' + dv' + \lambda' \wedge w_s \tag{4.14}$$

with $u', v', \lambda' \in \Omega_F M[X^2]$.

From (4.13) it is clear that we have reduced the exponent of p in the denominators of E_p at the cost that we increase the number of terms of type E_q but with $\deg_X(q) < \deg_X(p)$. Hence, iterating this process and using partial fractions decompositions, we can finally eliminate E_p from the original equation (4.5). Of course we have during this process to modify the other terms E_q for irreducibles q with $\deg_X(q) < \deg_X(p)$.

The assumption $w_s = db_1 \wedge \cdots \wedge db_{s-1} \wedge dT = 0$ in $\Omega_{N(p)}$ implies (see (3.7) (c))

$$w_s = db_1 \wedge \cdots \wedge db_{s-1} \wedge d(pt)$$

or

$$w_s = db_1 \wedge \cdots \wedge db_{i-1} \wedge d(pt) \wedge db_{i+1} \wedge \cdots \wedge db_{s-1} \wedge dT$$

with $t \in M[X^2]$, $\deg_X t < \deg_X p$, and some i, $1 \leq i \leq s-1$. Let us write $w_s = w_{s-1} \wedge d(pt)$ where w_{s-1} denotes $db_1 \wedge \cdots \wedge db_{s-1}$ or $db_1 \wedge \cdots \wedge \hat{i} \wedge \cdots \wedge db_{s-1} \wedge dT$. We will assume $w_{s-1} \neq 0$ in $\Omega_{N(p)}$, and we will omit the proof in the case $w_{s-1} = 0$, which can be treated similarly. Since we can assume $h \geq 1$, we get from (4.13)

$$u^{[2]} + dv = 0$$

in $\Omega_{N(p)}$. It follows $u = 0$ in $\Omega_{N(p)}$ and hence $u = pu_1 + u_2 \wedge dp$ with $u_1, u_2 \in \Omega_F M[X^2]$ (see (3.2)). Then

$$p^{2h} E_p = p^2 u_1^{[2]} + pu_2^{[2]} \wedge dp + p^{h+1} u_1 + p^h u_1 \wedge dp + dv + \lambda \wedge w_s.$$

This implies $dv = 0$ in $\Omega_{N(p)}$ and hence $dv = d(pv_1)$ with some $v_1 \in \Omega_F M[X^2]$ (see proof of case (2) above). Replacing this value of dv in the above equation it follows

$$p[p^{2h-1} E_p + pu_1^{[2]} + u_2^{[2]} \wedge dp + p^h u_1 + p^{h-1} u_2 \wedge dp$$

$$+ dv_1 + \lambda \wedge w_{s-1} \wedge dt] = (v_1 + t\lambda \wedge w_{s-1}) \wedge dp.$$

Lemma (3.4) implies

$$v_1 = t\lambda \wedge w_{s-1} + pv_2 + v_3 \wedge dp$$

with $v_2, v_3 \in \Omega_F M[X^2]$. Thus

$$p[p^{2h-2} E_p + u_1^{[2]} + p^{h-1} u_1 + dv_2] =$$

$$td\lambda \wedge w_{s-1} + [u_2^{[2]} + p^{h-1} u_2 + dv_3] \wedge dp.$$

Hence

$$td\lambda \wedge w_{s-1} = 0$$

in $\Omega_{N(p)}$. Since $\deg_X t < \deg_X p$, it follows $t \neq 0$ in $N(p)$ and we get

$$d\lambda \wedge w_{s-1} = 0.$$

This implies $d\lambda \wedge w_{s-1} = d(p\lambda_1) \wedge w_{s-1}$ in $\Omega_F[X^2]$. Therefore

$$\begin{aligned} p[p^{2h-2}E_p + u_1^{[2]} + p^{h-1}u_1 + dv_2 + td\lambda_1 \wedge w_{s-1}] \\ = [u_2^{[2]} + p^{h-1}u_2 + dv_3 + t\lambda_1 \wedge w_{s-1}] \wedge dp. \end{aligned}$$

Lemma (3.4) implies

$$u_2^{[2]} + p^{h-1}u_2 + dv_3 = t\lambda_1 \wedge w_{s-1} \tag{4.15}$$

in $\Omega_{N(p)}$. Thus we are led to consider two cases

a) $h > 1$. Then

$$u_2^{[2]} + dv_3 = t\lambda_1 \wedge w_{s-1}$$

implies (see (3.4))

$$u_2 = \delta \wedge w_{s-1} + pu_3 + u_4 \wedge dp$$

with $\delta, u_3, u_4 \in \Omega_F M[X^2]$. Inserting this expression for u_2 in the equation for E_p we obtain

$$p[p^{2h-2}E_p + u_1^{[2]} + p^{h-1}u_1 + dv_2 + td\lambda_1 \wedge w_{s-1} + pu_3^{[2]} \wedge dp + p^{h-1}u_3 \wedge dp]$$

$$= [(\delta \wedge w_{s-1})^{[2]} + p^{h-1}\delta \wedge w_{s-1} + dv_3 + t\lambda_1 \wedge w_{s-1}] \wedge dp.$$

It follows

$$(\delta \wedge w_{s-1})^{[2]} + dv_3 + t\lambda_1 \wedge w_{s-1} \in \ker[\Omega_F^m M[X^2] \to \Omega_{N(p)}^m]$$

which means that

$$dv_3 = z \wedge w_{s-1} + pu_4 + u_5 \wedge dp$$

in $\Omega_F^m M[X^2]$. This relation says $dv_3 \in \Omega_{N(p)}^{m-s+1} \wedge w_{s-1}$.

We apply now lemma (1.13) and remark (1.14), and we obtain in $\Omega_F^m M[X^2]$

$$dv_3 = \left(\sum_{\mu \neq 1} b^\mu z_\mu + b^1 dB \right) \wedge w_{s-1} + d(pu_4 + u_5 \wedge dp)$$

with $z_\mu, B \in \Omega_F M[X^2], dz_\mu = 0$ for all μ. Notice that the form $(\sum_{\mu \neq 1} b^\mu z_\mu + b^1 dB) \wedge w_{s-1}$ is exact, i.e. equal to some dH with $H \in \Omega_F M[X^2]$. Inserting this in the above equation it follows

$$p[p^{2h-2}E_p + u_1^{[2]} + p^{h-1}u_1 + dv_2 + td\lambda_1 \wedge w_{s-1} + (pu_3^{[2]} + p^{h-1}u_3 + du_4) \wedge dp]$$

$$= [(\delta \wedge w_{s-1})^{[2]} + p^{h-1}\delta \wedge w_{s-1} + t\lambda_1 \wedge w_{s-1} + dH] \wedge dp.$$

Let us denote by A the expression inside the parenthesis of the right hand side of this equation, which is of the form $A = B \wedge w_{s-1}$, with some $B \in \Omega_F M[X^2]$. The equation $p[\ ] = B \wedge w_{s-1} \wedge dp$ implies by lemma (3.4) $B \wedge w_{s-1} = 0$ in $\Omega_{N(p)}$. Now we apply lemma (3.6) to conclude $B \wedge w_{s-1} = (p\mu_1 + \mu_2 \wedge dp) \wedge w_{s-1}$ in $\Omega_F M[X^2]$. Inserting in the above relation we obtain then that we can assume $B \wedge w_{s-1} = p\mu_1 \wedge w_{s-1}$ with $\mu_1 \in \Omega_F M[X^2]$. Thus let us write $pB \wedge w_{s-1}$ instead of $B \wedge w_{s-1}$. Multiplying the above equation by t and using $tdp = d(pt) + pdt$ as well as $w_{s-1} \wedge d(pt) = w_s$, we get

$$p^{2h-2}E_p + u_1^{[2]} + p^{h-1}u_1 + dv_2 + td\lambda_1 \wedge w_{s-1} + (pu_3^{[2]} + p^{h-1}u_3 + du_4) \wedge dp$$

$$= \frac{pA \wedge dt}{pt} + \frac{pB \wedge w_{s-1}}{pt} = A \wedge \frac{dt}{t} + \frac{B}{t} \wedge w_{s-1}.$$

This equation implies

$$\begin{aligned} E_p &= \wp\left(\frac{u_1}{p^{h-1}}\right) + \wp\left(\frac{u_3 \wedge dp}{p^{h-1}}\right) + d\left(\frac{v_2 + u_2 \wedge dp}{p^{2(h-1)}}\right) \\ &+ \wp\left(\frac{\delta \wedge w_{s-1}}{p^{h-1}} \wedge \frac{dt}{t}\right) + d\left(\frac{H}{p^{2(h-1)}} \wedge \frac{dt}{t}\right) \\ &+ \frac{td\lambda_1 \wedge w_{s-1} + t\lambda_1 \wedge w_{s-1} \wedge \frac{dt}{t}}{p^{2h-2}} + \frac{1}{p^{2h-2}t} B \wedge w_s. \end{aligned}$$

Notice $td\lambda_1 \wedge w_{s-1} + \lambda_1 \wedge w_{s-1} \wedge dt = d(t\lambda_1 \wedge w_{s-1})$, so that

$$\begin{aligned} E_p &= \wp\left(\frac{u_1}{p^{h-1}} + \frac{u_3 \wedge dp}{p^{h-1}} + \frac{\delta \wedge w_{s-1} \wedge dt}{p^{h-1}t}\right) \\ &\quad + d\left(\frac{v_2 + u_4 \wedge dp}{p^{2h-2}} + \frac{(t\lambda_1) \wedge w_{s-1}}{p^{2h-2}} + \frac{H \wedge dt}{p^{2h-2}t}\right) \\ &\quad + \frac{B}{p^{2h-2}t} \wedge w_s. \end{aligned}$$

Thus

$$E_p = \wp\left(\frac{u'}{p^{h-1}t}\right) + d\left(\frac{v'}{p^{2(h-1)}t}\right) + \frac{\lambda'}{tp^{2h-2}} \wedge w_s$$

with forms u', v', $\lambda' \in \Omega_F M[X^2]$. Using partial fraction decomposition of the forms $u'/(p^{h-1}t)$, $v'/(p^{2(h-1)}t)$, $\lambda'/(p^{2h-2}t)$, we see that the exponent h of p in equation (4.13) can be reduced by one, although expressions for polynomials q of lower degree of the same type can appear, which will be absorbed by the corresponding E_q. Thus we are led to consider the case

(b) $h = 1$. Then we have (see (4.15))

$$u_2^{[2]} + u_2 + dv_3 = t\lambda_1 \wedge w_{s-1} \qquad (4.16)$$

in $\Omega_{N(p)}$. We can now apply lemma (3.7) to conclude

$$u_2 = \sum_i \bigwedge_j g_{ij} df_{ij} + pv_1 + v_2 \wedge dp + \delta \wedge w_{s-1}$$

with $v_1, v_2, \delta \in \Omega_F M[X^2]$, $f_{ij}, g_{ij} \in M[X^2]$, $\deg_X f_{ij} < \deg_X p$, $f_{ij}g_{ij} =$

$1+h_{ij}p$ in $M[X^2]$. Next we compute $\wp(\sum_i \bigwedge_j g_{ij}df_{ij})$. We have

$$\begin{aligned}
\wp(\sum_i \bigwedge_j g_{ij}df_{ij}) &\equiv \left[\sum_i \bigwedge_j g_{ij}f_{ij})\frac{df_{ij}}{f_{ij}}\right]^{[2]} + \sum_i \bigwedge_j g_{ij}f_{ij}\frac{df_{ij}}{f_{ij}} \\
&\equiv \sum_i (\prod_j (g_{ij}f_{ij})^2 + \prod_j g_{ij}f_{ij}) \bigwedge_j \frac{df_{ij}}{f_{ij}} \\
&\equiv \sum_i (\prod_j g_{ij}^2 f_{ij} + \prod_j g_{ij}) \bigwedge_j df_{ij} \\
&\equiv \sum_i \prod_j g_{ij}(1+\prod_j (1+ph_{ij})) \bigwedge_j df_{ij} \\
&\equiv \sum_i p \prod_j g_{ij}\cdot h_i \bigwedge_j df_{ij}
\end{aligned}$$

mod $d\Omega_F M[X^2]$, with certain $h_i \in M[X^2]$. Thus we have obtained

$$\wp(\sum_i \bigwedge_j g_{ij}df_{ij}) = p\sum_i g_i \bigwedge_j df_{ij} \qquad \text{mod} \quad d\Omega_F M[X^2]$$

with $g_i \in M[X^2]$. Let us insert u_2 in (4.15). According to the above computation we obtain in $\Omega_F M[X^2]$

$$\wp(\delta \wedge w_{s-1}) + dv_3 + t\lambda_1 \wedge w_{s-1} = pu_4 + u_5 \wedge dp$$

and this implies

$$dv_3 = \lambda \wedge w_{s-1}$$

in $\Omega_{N(p)}$, with some form $\lambda \in \bar{\Omega}_F M[X^2] \subset \Omega_{N(p)}$. From lemma (1.13) and remark (1.14) we obtain in $\Omega_F M[X^2]$

$$dv_3 = \left(\sum_{\mu\neq 1} b^\mu z_\mu + b^1 dB\right) \wedge w_{s-1} + d(pu_4)$$

with $z_\mu, B \in \Omega_F M[X^2], dz_\mu = 0$ for all μ. Notice that the form $(\sum_{\mu\neq 1} b^\mu z_\mu + b^1 dB) \wedge w_{s-1}$ is exact, i.e. equal to some dH with $H \in \Omega_F M[X^2]$. Therefore

$$\begin{aligned}
p[E_p + \wp u_1 + dv_2 \quad + \quad & td\lambda_1 \wedge w_{s-1}] = [\wp(\sum_i \bigwedge_j g_{ij}df_{ij} + pv_1 + \delta \wedge w_{s-1}) + \\
& \left(\sum_{\mu\neq 1} b^\mu z_\mu + b^1 dB\right) \wedge w_{s-1} + d(pu_4) + t\lambda_1 \wedge w_{s-1}] \wedge dp.
\end{aligned}$$

After dividing by p, we are led to consider the following expressions.

a) $\wp(pv_1) \wedge \frac{dp}{p} = \wp(pv_1 \wedge \frac{dp}{p}) = \wp(v_1 \wedge dp) \qquad \mod \quad d\Omega_F M[X^2]$

b)

$$\begin{aligned}
\wp \sum_i \bigwedge_j g_{ij} df_{ij} \wedge \frac{dp}{p} &= \wp\left(\sum_i \bigwedge_j g_{ij} f_{ij} \frac{df_{ij}}{f_{ij}} \wedge \frac{dp}{p}\right) \\
&= \wp\left(\sum_i \bigwedge_j (1+ph_{ij}) \frac{df_{ij}}{f_{ij}} \wedge \frac{dp}{p}\right) \\
&= \wp\left(\sum_i \bigwedge_j \frac{df_{ij}}{f_{ij}} \wedge \frac{dp}{p} + \sum_i H_i \bigwedge_j \frac{df_{ij}}{f_{ij}} \wedge dp\right) \\
&= \wp\left(\sum_i H_i \bigwedge_j \frac{df_{ij}}{f_{ij}} \wedge dp\right) \quad \mod d\Omega_F M[X^2]
\end{aligned}$$

with certain polynomials $H_i \in M[X^2]$.

c) $\wp(\delta \wedge w_{s-1}) \wedge \frac{dp}{p} = \wp(\delta \wedge w_{s-1} \wedge \frac{dt}{t}) + \wp(\delta \wedge w_{s-1} \wedge \frac{d(pt)}{pt}) \qquad \mod \ d\Omega_F M[X^2]$

d) $t\lambda_1 \wedge w_{s-1} \wedge \frac{dp}{p} = \lambda_1 \wedge w_{s-1} \wedge \frac{tdp}{p} = \lambda_1 \wedge w_{s-1} \wedge \frac{d(tp)+pdt}{p} = t\lambda_1 \wedge w_{s-1} \wedge \frac{d(tp)}{tp} + \lambda_1 \wedge w_{s-1} \wedge dt$ (recall $w_s = w_{s-1} \wedge d(pt)$)

e) $\left(\sum_{\mu \neq 1} b^\mu z_\mu + b^1 dB\right) \wedge w_{s-1} \wedge \frac{dp}{p} = \left(\sum_{\mu \neq 1} b^\mu z_\mu + b^1 dB\right) \wedge w_{s-1} \wedge \frac{d(pt)}{pt} + \left(\sum_{\mu \neq 1} b^\mu z_\mu + b^1 dB\right) \wedge w_{s-1} \wedge \frac{d(t)}{t}$

Thus we get

$$\begin{aligned}
E_p &= \wp(u_1 + v_1 \wedge dp) + d(v_2 + u_4 \wedge w_{s-1} \wedge dp) \\
&\quad + \wp(\sum_i H_i \bigwedge_j \frac{df_{ij}}{f_{ij}} \wedge dp) + d(t\lambda_1 \wedge w_{s-1}) \\
&\quad + \wp(\delta \wedge w_{s-1}) \wedge \frac{dt}{t} + \wp(\delta \wedge w_{s-1}) \wedge \frac{d(pt)}{pt} + \frac{\lambda_1}{p} \wedge w_s \\
E_p &= \wp\left(u_1 + v_1 \wedge dp + \sum_i H_i \bigwedge_j \frac{df_{ij}}{f_{ij}} \wedge dp + \delta \wedge w_{s-1} \wedge \frac{dt}{t}\right) \\
&\quad + d(v_2 + u_4 \wedge w_{s-1} \wedge dp + t\lambda_1 \wedge w_{s-1}) + \alpha_3 \wedge w_s
\end{aligned}$$

where $\wp(\delta \wedge w_{s-1}) \wedge \frac{d(pt)}{pt} + \frac{\lambda_1}{p} \wedge w_s = \alpha_3 \wedge w_s$, α_3 being a form of the type μ/pt with $\mu \in \Omega_F M[X^2]$, i.e. α_3 has a denominator containing p at most in the first power, and all other prime factors are of degree (in X) less than $\deg(p)$. Thus we have

$$E_p = \wp(\alpha_1) + d(\alpha_2) + \alpha_3 \wedge w_s$$

where α_1, α_2 have numerator in $\Omega_F M[X^2]$ and denominators with prime factors of degree less than $\deg(p)$. In particular the above equation shows that $\alpha_3 \wedge w_s$ is also a form of the type β/q where q is a product of prime polynomials of degree less than p in $M[X^2]$. We claim

$$\alpha_3 \wedge w_s = \alpha \wedge w_s \tag{4.17}$$

where α is a form of the type γ/q where q is a product of irreducible polynomials of degree $< \deg(p)$ contained in $M[X^2]$. Once we have shown this claim, we see that we can get rid of E_p in the equation (4.5). Thus we must show (after scaling (4.16) with a convenient polynomial q).

(4.18) **Lemma.** *Let λ be a form in $\frac{1}{p}\Omega_F M[X^2]$ such that $\lambda \wedge w_s \in \Omega_F M[X^2]$. Then there is some form $\lambda' \in \Omega_F M[X^2]$ with*

$$\lambda \wedge w_s = \lambda' \wedge w_s.$$

Proof. Set $\lambda = p^{-1}\lambda_0$ with $\lambda_0 \in \Omega_F M[X^2]$. We can obviously assume that $\deg_X \lambda_0 < \deg_X p$. We write $dT = k_s db_s + dT'$, with $\deg_X k_s = 2$, $\deg_X T' = 0$ and dT' not containing db_s. Since $w_s = db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$, we may assume that λ_0 is only generated by forms $db_j, j \geq s$. Set $\lambda_0 = \lambda_s \wedge db_s + \lambda'_s$ with λ_s, λ'_s generated by forms $db_j, j > s$. Then

$$\begin{aligned} \lambda \wedge w_s &= p^{-1}\lambda_0 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge (k_s db_s + dT') \\ &= p^{-1} db_1 \wedge \cdots \wedge db_{s-1} \wedge db_s \wedge (\lambda_s \wedge dT' + k_s \lambda'_s) \\ &\quad + p^{-1} db_1 \wedge \cdots \wedge db_{s-1} \wedge \lambda'_s \wedge dT'. \end{aligned}$$

Because of the above choices, both summands do not interfere with each other, so that both must be integral, since $\lambda \wedge w_s$ is integral. In particular

$$p^{-1} db_1 \wedge \cdots \wedge db_{s-1} \wedge \lambda'_s \wedge dT' \in \Omega_F M[X^2].$$

Since $\deg_X \lambda'_s < \deg_X p$ and $\deg_X T' = 0$, it follows

$$db_1 \wedge \cdots \wedge db_{s-1} \wedge \lambda'_s \wedge dT' = 0.$$

But $\lambda'_s \wedge dT'$ is generated only by differentials $db_j, j \geq s$, so that we obtain $\lambda'_s \wedge dT' = 0$. It follows $\lambda'_s = \delta \wedge dT'$ with some form $\delta \in \Omega_F M[X^2]$, and $\deg_X \delta < \deg_X p$. Therefore

$$\begin{aligned} \lambda \wedge w_s &= p^{-1} db_1 \wedge \cdots \wedge db_s \wedge (\lambda_s \wedge dT' + k_s \delta \wedge dT') \\ &= p^{-1} db_1 \wedge \cdots \wedge db_s \wedge (\lambda_s + k_s \delta) \wedge dT' \end{aligned}$$

Since $db_1 \wedge \cdots \wedge db_s \wedge dT'$ does not contain X we easily see that p divides $\lambda_s + k_s\delta$ i.e. $\lambda_s + k_s\delta = p \cdot \mu$ with $\mu \in \Omega_F M[X^2]$. Hence

$$\begin{aligned} \lambda_0 &= (k_s\delta + p\mu) \wedge db_s + \delta \wedge dT' \\ &= \delta \wedge dT + p\mu \wedge db_s. \end{aligned}$$

This implies

$$\begin{aligned} \lambda \wedge w_s &= p^{-1} \lambda_0 \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \\ &= p^{-1} p\mu \wedge db_s \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT \\ &= \mu \wedge w_s \end{aligned}$$

with $\mu \in \Omega_F M[X^2]$. This proves the claim. $\square$

Therefore, using this descent procedure we have shown the following result.

(4.19) **Proposition.** *If* $w \in \Omega_F^n$ *satisfies an equation*

$$w = \wp u + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with $u, v, \lambda \in \Omega_F M(X^2)$ *and* T *as before, then* w *satisfies an equation*

$$w = \wp u' + dv' + \lambda' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$$

with $u', v', \lambda' \in \Omega_F M[X^2]$.

(4.20) **Lemma.** *Let* $w \in \Omega_F^n$ *be such that there exist* $u, v, \lambda \in \Omega_F M[X^2]$ *with* $w = \wp u + dv + \lambda \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$, $T = b_s X^2 + T'$, $\deg_X T' = 0$. *Then there exist* $u', v', \lambda' \in \Omega_F M$ *with*

$$w = \wp u' + dv' + \lambda' \wedge db_1 \wedge \cdots \wedge db_{s-1} \wedge db_s \wedge dT'.$$

Proof. Set $\deg_X u = 2h$, $\deg_X v = 2k$, $\deg_X \lambda = 2\ell$. We consider the following cases:

a) $4h \geq 2k$. Then the coefficient of X^{4h} in $\wp u + dv + \lambda \wedge w_s$ ($w_s = db_1 \wedge \cdots \wedge db_{s-1} \wedge dT$) is the form $u_{2h}^{[2]} + dv_{4h} + \lambda_{4h-2} \wedge w_{s-1} \wedge db_s$, where $w_{s-1} = db_1 \wedge \cdots \wedge db_{s-1}$. Then we must have

$$u_{2h}^{[2]} + dv_{4h} + \lambda_{4h-2} \wedge w_{s-1} \wedge db_s = 0$$

if $h \geq 1$. Here we have used the following notation: $u = u_0 + u_2 X^2 + \cdots + u_{2h} X^{2h}$, with $u_i \in \Omega_F M$. Then applying the Cartier operator

$$0 = u_{2h} + C(\lambda_{4h-2} \wedge w_{s-1} \wedge db_s).$$

From lemma (1.12) we obtain in $\Omega_{F(X_\mu)}$

$$u_{2h} = \mu \wedge w_{s-1} \wedge db_s.$$

Since $u_{2h} \in \Omega_F M$, it is easy to conclude that we can choose $\mu \in \Omega_F M$. Therefore

$$\begin{aligned} u &= u_0 + u_2 X^2 + \cdots + u_{2h} X^{2h} \\ &= u_0 + \cdots + \mu \wedge w_{s-1} \wedge db_s X^{2h} \\ &= u_0 + \cdots + \mu \wedge w_{s-1} \wedge (dT + dT') X^{2h-2} \\ &= u_0 + \cdots + u'_{2h-2} X^{2h-2} + \mu X^{2h-2} \wedge w_{s-1} \wedge dT \end{aligned}$$

where $u'_{2h-2} = u_{2h-2} + \mu \wedge dT'$. The term $\mu X^{2h-2} \wedge w_{s-1} \wedge dT$ can be added to $\lambda \wedge s_{s-1} \wedge dT$ replacing λ by $\lambda + \mu X^{2h-2}$. Thus we have lowered the degree of u. Iterating this procedure we are led to consider the following case

b) $4h < 2k$. It follows $2\ell \leq 2k-2$, and the coefficient of X^{2k} is $dv_{2k} + \lambda_{2k-2} \wedge w_{s-1} \wedge db_s$. Therefore if $k \geq 1$, we must have

$$dv_{2k} + \lambda_{2k-2} \wedge w_{s-1} \wedge db_s = 0.$$

Then according (1.13) there exist forms $z_\mu \in \Omega_F M$ and $B \in \Omega_F M$ such that

$$dv_{2k} = \sum_{\mu \neq 1_{s-1}} [b^\mu (z_{\mu,2} + b_s z_{\mu,3}) + b^{1_{s-1}}(z_1 + b_s dB)] \wedge w_{s-1} \wedge db_s$$

with all z_μ closed, and $z_{\mu,2}, z_{\mu,3}, z_1, B$ not containing b_s in the 2-expansion of their coefficients. The term $b^\mu z_{\mu,2} \wedge w_{s-1}$ is exact for all $\mu \neq 1_{s-1}$ i.e. $b^\mu z_{\mu,2} \wedge w_{s-1} \in d\Omega_F M$. Also $b_s b^\mu \wedge w_{s-1} \in b_s d\Omega_F M$. The form $b^{1_{s-1}} z_1 \wedge w_{s-1}$ is closed and hence contained in $d\Omega_F M + (\Omega_F M)^{[2]}$ and $b_s b^{1_{s-1}} dB \wedge w_{s-1} \in b_s d\Omega_F M$.

Putting all this together, we obtain

$$dv_{2k} = (A^{[2]} + dC + b_s dE) \wedge db_s$$

with forms $A, C, E \in \Omega_F M$, which moreover are multiples of w_{s-1}. From

$$\lambda_{2k-2} \wedge w_{s-1} \wedge db_s = (A^{[2]} + dC + b_s dE) \wedge db_s$$

we infer

$$\lambda_{2k-2} \wedge w_{s-1} = A^{[2]} + dC + b_s dE + F \wedge db_s$$

with some $F \in \Omega_F M$, which can be chosen as a multiple of w_{s-1}, since F does not have terms containing db_s and $F \wedge db_s \in \langle w_{s-1} \rangle$. Therefore

$$\begin{aligned}
\lambda \wedge w_{s-1} &= (\lambda_0 + \lambda_2 X^2 + \cdots + \lambda_{2k-2} X^{2k-2}) \wedge w_{s-1} \\
\lambda \wedge w_{s-1} \wedge dT &= (\lambda_0 + \lambda_2 X^2 + \cdots + \lambda_{2k-4} X^{2k-4}) \wedge w_{s-1} \wedge dT \\
&\quad + X^{2k-2}(A^{[2]} + dC + b_s dE + F \wedge db_s) \wedge dT.
\end{aligned}$$

The terms $X^{2k-2}A^{[2]} \wedge dT$ and $X^{2k-2}dC \wedge dT$ are exact, and hence can be absorbed by dv without increasing the degree of v. Now

$$\begin{aligned} X^{2k-2}b_s dE \wedge dT &= X^{2k-4}(X^2 b_s)dE \wedge dT \\ &= X^{2k-4}(T+T')dE \wedge dT \\ &= d(X^{2k-4}TE \wedge dT) + X^{2k-4}T'dE \wedge dT. \end{aligned}$$

The first summand can be added to dv without increasing the degree of v. Since dE is a multiple of w_{s-1}, it follows that the second summand in this expression is contained in $X^{2k-4}\Omega_F M \wedge w_{s-1} \wedge dT$ and hence it can be added to the term $\lambda_{2k-4} \cdot X^{2k-4}$ of degree $2k-4$ in $\lambda \wedge w_{s-1} \wedge dT$. Finally the term $X^{2k-2}F \wedge db_s \wedge dT = X^{2k-4}F \wedge dT' \wedge dT$ is contained in $X^{2k-4}\Omega_F M \wedge w_{s-1} \wedge dT = X^{2k-4}\Omega_F M \wedge w_s$, because F is a multiple of w_{s-1}, and therefore this term is absorbed by the term of degree $2k-4$ in $\lambda \wedge w_s$.

This shows that can assume $\deg \lambda \leq 2k-4$. But this implies $dv_{2k} = 0$, i.e. $\deg v < 2k$. Thus we have lowered the degree of v, and iterating this procedure we come again to the case a).

Combining the procedures of cases a) and b) we finally arrive at an equation

$$w = u^{[2]} + u + dv + \lambda \wedge w_{s-1} \wedge dT$$

with $u, v, \lambda \in \Omega_F M$. Since $\lambda \wedge w_{s-1} \wedge dT = X^2 \lambda \wedge w_{s-1} \wedge db_s + \lambda \wedge w_{s-1} \wedge dT'$, we see that $X^2\lambda \wedge w_{s-1} \wedge db_s \in \Omega_F M$, and this is possible only if $\lambda \wedge w_{s-1} \wedge db_s = 0$. We have assumed that λ does not contain terms with $db_1, \ldots, db_{s-1}$, so that it follows$\lambda = \lambda' \wedge db_s$ in $\Omega_F M$. Thus

$$w = \wp u + dv + \lambda' \wedge db_1 \wedge \cdots \wedge db_s \wedge dT$$

and this proves the lemma. $\square$

We are now ready to prove the main result of this paper.

Proof of Theorem (4.1) Let us introduce the following subfields of $L = F(X_\mu | \mu \in S_n)$ and polynomials

$$
\begin{aligned}
L_0 &= L = F(X_\mu | \mu \in S_n) \\
T_0 &= T = \sum_{\mu \in S_n} b^\mu X_\mu^2 \\
M_0 &= M = F(X_\mu^2 | \mu \in S_n) \\
L_1 &= F(X_\mu | \mu \neq e_1), e_1 = (1, 0, \dots, 0) \\
T_1 &= \sum_{\mu \neq e_1} b^\mu X_\mu^2 \\
M_1 &= F(X_\mu^2 | \mu \neq e_1) \\
&\vdots \\
L_j &= F(X_\mu | \mu \neq e_1, \dots, e_j), e_i = (0, \dots 1, \dots, 0) \\
T_j &= \sum_{\mu \neq e_1, \dots, e_j} b^\mu X_\mu^2 \\
M_j &= F(X_\mu^2 | \mu \neq e_1, \dots, e_j)
\end{aligned}
$$

where $j = 1, 2, \dots, n$. We have $L_{j-1} = L_j(X_j)$,$T_{j+1} = T_j + b_{j+1} X_{j+1}^2 \in M_{j+1}$.

Thus equation (4.2) corresponds to (4.3) with $s = 1$, i.e. $w = \wp u + dv + \lambda \wedge dT_0$ with $w \in \Omega_F, u, v, \lambda \in \Omega_F M_0$. The above process implies an equation $w = \wp u' + dv' + \lambda' \wedge dT_0$ with $u', v', \lambda' \in \Omega_F M_1[X_1^2]$. Lemma (4.19) implies now

$$w = \wp u'' + dv'' + \lambda'' \wedge db_1 \wedge dT_1$$

with $u'', v'', \lambda'' \in \Omega_F M_1$. We continue with this process for $s = 2, \dots, n-1$ until we get forms $\bar{u}, \bar{v}, \bar{\lambda} \in \Omega_F M_{n-1}$ such that

$$w = \wp \bar{u} + d\bar{v} + \bar{\lambda} \wedge db_1 \wedge \cdots \wedge db_{n-1} \wedge dT_{n-1}.$$

But we have

$$db_1 \wedge \cdots \wedge db_{n-1} \wedge dT_{n-1} = db_1 \wedge \cdots \wedge db_{n-1} \wedge k_n db_n$$

where $k_n = b_n^{-1} \sum_{\mu(n)=1} b^\mu X_\mu^2$. Therefore we obtain an equation

$$w = \wp\bar{u} + d\bar{v} + \bar{\lambda}' \wedge db_1 \wedge \cdots \wedge db_n \tag{4.21}$$

in $\Omega_F M_{n-1}$. We can now get rid of the remaining variables just by comparing coefficients. Suppose namely that k is a field, $K = k(X)$ a pure transcendental extension in one variable of $k, w \in \Omega_k$ and that there is a relation

$$w = \wp u + dv + \lambda \wedge w_n \tag{4.22}$$

in $\Omega_k(X^2)$, where w_n is now defined over k. We decompose u, v, λ in partial fractions

$$\begin{aligned} u &= u_0 + \sum u_p \\ v &= v_0 + \sum v_p \\ \lambda &= \lambda_0 + \sum \lambda_p \end{aligned}$$

with $u_0, v_0, \lambda_0 \in \Omega_k[X^2], u_p, v_p, \lambda_p \in p^{-\infty}\Omega_k[X^2]$. Since the operators $\wp$ and d respect these decomposition, we conclude from (4.21) that

$$w = \wp u_0 + dv_0 + \lambda_0 \wedge db_1 \wedge \cdots \wedge db_n$$

in $\Omega_k[X^2]$. Letting $X = 0$, we obtain $w = \wp\bar{u}_0 + d\bar{v}_0 + \bar{\lambda}_0 \wedge db_1 \wedge \cdots \wedge db_n$ in Ω_k. Applying this argument to the equation (4.21) we conclude

$$w = \wp\alpha + d\beta + \gamma \wedge db_1 \wedge \cdots \wedge db_n$$

with $\alpha, \beta, \gamma \in \Omega_F$. This finishes the proof of the theorem. □

5 Quadratic forms and differential forms

We review briefly in this section for the sake of completeness some basic notations and results from quadratic form theory and its relations with differential forms over fields of characteristic two. Our basic references will be [A-Ba 1], [Ba 1], [Ka 1] and [Sa]. Let F be a field of characteristic two. We denote by $[a, b]$ the binary nonsingular quadratic form $ax^2+xy+by^2$ $(a, b \in F)$. Any non singular quadratic form over F is of the form $\perp_{i=1}^{n} [a_i, b_i]$, $(a_i, b_i \in F)$, where $\perp$ means orthogonal sum. The form $[0, 0]$ is the hyperbolic plane and any orthogonal sum $\perp [0, 0]$ is called a hyperbolic space. Two quadratic forms q_1, q_2 are called equivalent $(q_1 \sim q_2)$ if $H_1 \perp q_1 \simeq H_2 \perp q_2$, where H_1, H_2 are hyperbolic spaces. A form q is called isotropic if there is a nonzero vector x with $q(x) = 0$, otherwise q is called anisotropic. The set of equivalence classes of anisotropic quadratic forms over F form the Witt-group $W_q(F)$ (with respect to orthogonal sums). Respectively, we denote by $\langle a \rangle$ the one dimensional symmetric bilinear form axy $(a \in F^*)$ and by $\langle a_1, \dots, a_n \rangle$ the orthogonal sum $\langle a_1 \rangle \perp \dots \perp \langle a_n \rangle$. Let $W(F)$ be the Witt ring of F, i.e. the ring of classes of non singular symmetric bilinear forms over F. Then $W_q(F)$ is a $W(F)$-module via the operation $b \otimes q(x \otimes y) = b(x, x) \cdot q(y)$ (see [Ba 1], [Sa]). The maximal ideal $I \subset W(F)$ of even dimensional bilinear forms is additively generated by the 1-Pfister forms $\langle 1, a \rangle$, $a \in F^*$, so that the n-power I^n is additively generated by the n-fold Pfister forms $\ll a_1, \dots, a_n \gg = \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$. We get the submodules $I^n W_q(F)$ of $W_q(F)$, $n \geq 0$, which are additively generated by the quadratic n-fold Pfister forms $\ll a_1, \dots, a_n; b]] = \ll a_1, \dots, a_n \gg \otimes [1, b]$, where $[1, b] = x^2+xy+by^2$ is a 0-fold Pfister form. Thus we have the filtration $W(F) \supset I \supset I^2 \supset \cdots$ and $W_q(F) \supset IW_q(F) \supset I^2W_q(F) \supset \cdots$. In [Ka 1] it is shown that there are a natural isomorphisms (see section 2 for the definition of $\nu_F(n)$ and $H^{n+1}(F)$)

$$\alpha : \nu_F(n) \xrightarrow{\sim} I_F^n / I_F^{n+1} \tag{5.1}$$

$$\beta : H^{n+1}(F) \xrightarrow{\sim} I^n W_q(F) / I^{n+1} W_q(F) \tag{5.2}$$

given on generators by

$$\begin{aligned} \alpha\left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right) &= \ll x_1, \dots, x_n \gg \quad \bmod I^n \\ \beta\left(\overline{b\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}}\right) &= \ll x_1, \dots, x_n; b]] \quad \bmod I^n W_q(F). \end{aligned}$$

If L/F is any field extension, any quadratic form q (or bilinear form b) over F can be viewed as a form over L, which we denote by q_L (or b_L). The natural homomorphisms $W_q(F) \to W_q(L)$, resp. $W(F) \to W(L)$ are compatible with the above isomorphisms, i.e.

$$\begin{array}{ccc} \nu_F(n) & \xrightarrow{\sim} & I_F^n/I_F^{n+1} \\ \downarrow & & \downarrow \\ \nu_L(n) & \xrightarrow{\sim} & I_L^n/I_L^{n+1} \end{array}$$

resp.

$$\begin{array}{ccc} H^{n+1}(F) & \xrightarrow{\sim} & I^nW_q(F)/I^{n+1}W_q(F) \\ \downarrow & & \downarrow \\ H^{n+1}(L) & \xrightarrow{\sim} & I^nW_q(L)/I^{n+1}W_q(L) \end{array}$$

In particular the main isomorphism (4.1) can be restated in terms of quadratic forms as follows.

(5.3) **Theorem.** *Let $\phi = \ll b_1, \dots, b_n \gg$ be an anisotropic bilinear n-fold Pfister form over F. Then*

$$\ker\left[\frac{I^nW_q(F)}{I^{n+1}W_q(F)} \to \frac{I^nW_q(F(\phi))}{I^{n+1}W_q(F(\phi))}\right] = \{\overline{\phi \otimes [1,b]} | b \in F\}.$$

Let $p = \ll b_1, \dots, b_n; b]]$ be an anisotropic quadratic n -fold Pfister form over F. Let $F(p)$ be the function field of the quadric $\{p = 0\}$ over F. In [A-Ba 2] we have shown that (5.3) implies the following result, whose proof will be given here for the sake of completeness.

(5.4) **Theorem.** *Let p be as above. Then*

$$\ker\left[\frac{I^nW_q(F)}{I^{n+1}W_q(F)} \to \frac{I^nW_q(F(p))}{I^{n+1}W_q(F(p))}\right] = \{0, \bar{p}\}$$

Proof. Let $q \in I^n W_q(F)$ be such that $q_{F(p)} \in I^{n+1}W_q(F(p))$. Set $q = \sum_{i=1}^{r} \phi_i[1, a_i]$, ϕ_i an n-fold bilinear Pfister form over F, $1 \le i \le r$. If $r = 1$, i.e. $q = \phi[1, a]$, then the above assumption implies that $\phi[1, a]$ is hyperbolic over $F(p)$ (see [Ba 2]) and then by the norm theorem (see [Ba 3]) we conclude $\phi[1, a] \simeq p$ over F, i.e. $\bar{q} = \bar{p}$. Now assume $r > 1$, and we will prove the assertion by induction on r. Thus we assume the assertion true for any field and any form of length less than r. Thus without restriction $\phi_r[1, a_r]$ is anisotropic. Set $\psi = \phi_r$, and let $F(\psi)$ be its function field. Then $q_{F(\psi)} = \sum_{1}^{r-1} \phi_i[1, a_i] \in I^n W_q(F(\psi))$ and over $F(\psi)(p)$ we get $q_{F(\psi)(p)} \in I^{n+1}W_q(F(\psi)(p))$.

Therefore by induction we obtain

$$q_{F(\psi)} \equiv \epsilon p_{F(\psi)} \bmod I^{n+1}W_q(F(\psi))$$

with $\epsilon = 0$ or 1. Thus

$$(q \perp \epsilon p)_{F(\psi)} \in I^{n+1}W_q(F(\psi)).$$

From (5.1) we conclude $q \perp \epsilon p \equiv \psi[1, c] \bmod I^{n+1}W_q(F)$ with some $c \in F$. Since $q_{F(p)} \in I^{n+1}W_q(F(p))$, it follows $\psi[1, c]_{F(p)} \in I^{n+1}W_q(F(p))$, i.e. $\psi[1, c]$ is hyperbolic over $F(p)$, and hence by the norm theorem (see loc. cit.) $\psi[1, c] \equiv \eta p \bmod I^{n+1}W_q(F)$ with $\eta = 0$ or 1. Therefore $q \equiv (\epsilon + \eta)p \bmod I^{n+1}W_q(F)$. This conclude the proof. □

Using the isomorphism (5.2) we can restate the above result as follows.

(5.5) **Theorem.** *Let $p = \ll b_1, \dots, b_n; b]]$ be an anisotropic quadratic n-fold Pfister form over F. Then*

$$H^{n+1}(F(p)/F) = \left\{0, \overline{b\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}}\right\}.$$

(5.6) **Remark.** The analogue of (5.3) for bilinear forms is the following result

(5.7) **Proposition.** *Let $\phi = \ll b_1, \dots, b_n \gg$ be an anisotropic bilinear n-fold Pfister form. Then for $m \geq n$*

$$\ker\left[\frac{I_F^m}{I_F^{m+1}} \to \frac{I_{F(\phi)}^m}{I_{F(\phi)}^{m+1}}\right] = \left\{\sum I^{m-n}\overline{\ll x_1, \dots, x_n \gg} | x_1, \dots, x_n \in F^2(b_1, \dots, b_n)\right\}.$$

In the case $m = n$, we have

$$\ker\left[\frac{I_F^n}{I_F^{n+1}} \to \frac{I_{F(\phi)}^n}{I_{F(\phi)}^{n+1}}\right] = \left\{\overline{\ll x_1, \dots, x_n \gg} | x_1, \dots, x_n \in F^2(b_1, \dots, b_n)\right\}.$$

We will show the special case $m = n$ for simplicity.

If $x_1, \dots, x_n \in F^2(b_1, \dots, b_n)$, then $dx_i \in Fdb_1 \oplus \cdots \oplus Fdb_n$ and hence

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} = a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

with some $a \in F^2(b_1, \dots, b_n)$. Since $a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} \in \ker(\Omega_F^n \to \Omega_{F(\phi)}^n)$, it follows $\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \ker[\nu(n)_F \to \nu(n)_{F(\phi)}]$. Thus

$$\overline{\ll x_1, \dots, x_n \gg} = \alpha(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) \in \ker\left[\frac{I_F^n}{I_F^{n+1}} \to \frac{I_{F(\phi)}^n}{I_{F(\phi)}^{n+1}}\right].$$

Conversely take any $\overline{\psi} = \alpha(w) \in \ker\left[\frac{I_F^n}{I_F^{n+1}} \to \frac{I_{F(\phi)}^n}{I_{F(\phi)}^{n+1}}\right]$, with $w \in \ker[\nu(n)_F \to \nu(n)_{F(\phi)}]$, i.e. $w = a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$. Using Kato's lemma (see (2.15)) one immediately obtains $w = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ with $x_1, \dots, x_n \in F^2(b_1, \dots, b_n)$ and this implies $\overline{\psi} = \overline{\ll x_1, \dots, x_n \gg}$.

In particular, since this kernel is a group, we obtain

(5.8) **Corollary.** *Given $x_1, \dots, x_n, y_1, \dots, y_n \in F^2(b_1, \dots, b_n)$, then there exist $z_1, \dots, z_n \in F^2(b_1, \dots, b_n)$ with*

$$\ll x_1, \dots, x_n \gg + \ll y_1, \dots, y_n \gg = \ll z_1, \dots, z_n \gg \qquad \bmod I_F^{n+1}$$

This corollary says that the field $F^2(b_1, \dots, b_n)$ is n-linked relative to F.

In particular if $\{b_1, \dots, b_N\}$ is a 2-basis of the field F, i.e. $F = F^2(b_1, \dots, b_N)$, then $I_F^{N+1} = 0$ and it follows

$$I_F^N = \{\ll x_1, \dots, x_N \gg | x_i \in F^*\}\,.$$

6 Generic splitting of quadratic forms and the degree conjecture

Since Knebusch's seminal papers on generic splitting of quadratic forms over fields of characteristic $\neq 2$ (see [Kn 1], [Kn 2]) appeared, few work has been done on the subject (see [Ar-Kn], [F], ...). In particular Knebusch's degree conjecture $I^n = J_n$, where J_n is the ideal of $W(F)$ of forms of degree $\geq n$ remains still open. In what follows we will briefly develop the analog of Knebusch's theory over fields of characteristic 2 and using the results of section 5 we will show that in this case the corresponding degree conjecture is true.

Our main reference will be Knebusch's paper [Kn] on reduction theory of quadratic and bilinear forms, which holds true for fields of any characteristic, as well as his generic splitting papers cited above. Many of the definitions and results of Knebusch's theory can be extended (using [Ba 2], [Ba 3]) mutatis mutandis to the case $2 = 0$, so that we will often refer to the above papers for proofs.

From now on all fields have characteristic 2. The most basic notion in this theory is that of generic zero field of a quadratic form q over F. A field extension L/F is a generic zero field of q if q_L is isotropic and if E/F is any extension with q_E isotropic, then there exists a F -place $\lambda : L \to E \cup \infty$ (see [La]). One easily checks that the function field $F(q)$ of q is a generic field of q. If $q = \langle a_1 \rangle [1, b_1] \perp \cdots \perp \langle a_n \rangle [1, b_n]$, then $F(q) = F(x_1, \ldots, x_n, y_1, \ldots, y_n)$ with the single relation $\sum_1^n a_i(x_i^2 + x_i y_i + b_i y_i^2) = 0$. Also the field $F(q)_0 = F(u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}, z)$ with $a_n(z^2 + z + b_n) + \sum_1^{n-1} a_i(u_i^2 + u_i v_i + b_i v_i^2) = 0$ is a generic zero field of q. $F(q)_0$ is purely transcendental over F if and only if q is isotropic over F. Starting with a non singular quadratic form q over F we can define a field tower $F = F_0 \subset F_1 \subset \cdots \subset F_h$ and forms $q = q_0, q_1, \ldots, , q_h$ defined over $F_0, F_1, \ldots, F_h$ respectively such that q_r is anisotropic over $F_r, 0 \leq r \leq h-1$, q_h is hyperbolic over F_h, and $q_{r-1} \otimes F_r \simeq q_r \perp i_r \times [0, 0]$ with some integer i_r, F_r being a generic zero field of q_{r-1} over F_{r-1} . The sequence $(F_r, q_r, i_r, 0 \leq r \leq h)$ is called a generic splitting tower of q. Recall that two field extensions $L_1/F, L_2/F$ are called F-equivalent if there exist F-places $\lambda_1 : L_1 \to L_2 \cup \infty, \lambda_2 : L_2 \to L_1 \cup \infty$. Then any field L which is F-equivalent to F_h is called a generic splitting field of q and any field F-equivalent to F_{h-1} is called a leading field of q. A generic splitting tower of q is essentially unique in the sense that if $(F_r, q_r, i_r, 0 \leq r \leq h)$,

$(F'_s, q'_s, i'_s, 0 \leq s \leq h')$ are two generic splitting towers of q, then $h = h'$, $i_r = i'_r$, $0 \leq r \leq h$, F_h is equivalent with F'_h. The number $h = h(q)$ is called the height of q. Obviously any form $\langle a \rangle p$, $a \in F^*$ and p a Pfister form, has height 1. Conversely if $h(q) = 1$, then $q_{F(q)}$ is hyperbolic and the norm theorem proved in [Ba 2] implies immediately that $q \simeq \langle a \rangle p$ with $a \in F^*$ and p a Pfister form. In particular for any form q, the form q_{h-1} is similar to a n-fold Pfister form over F_{h-1}. The degree n of this form is uniquely determined and we will call it the degree of q and we denote it by $\deg(q)$. If q is hyperbolic we set $\deg(q) = \infty$. For any extension L/F we have $\deg(q_L) \geq \deg(q)$ and deg is a well defined function on $W_q(F)$, $\deg : W_q(F) \to N \cup \{\infty\}$. We define for any $n \geq 0$

$$J(n) = \{\bar{q} \in W_q(F) | \deg(q) \geq n\}. \tag{6.1}$$

One easily checks that $J(0) = W_q(F)$, $J(1) = IW_q(F)$, $J(2) = I^2W_q(F)$ (see [Ba 1], [A-Ba 1]). First we show that $J(n)$ is a $W(F)$ submodule of $W_q(F)$ and that $I^nW_q(F) \subseteq J(n)$. The key fact is the following result (compare [Kn 1]).

(6.2) **Proposition.** *Let $q = \langle a \rangle p \perp q'$ be a quadratic form over F, where p is an anisotropic Pfister form of degree $n \geq 1$, $a \in F^*$ and $\deg(q') \geq n + 1$. Let L be a leading field of q. Then*

i) $\deg(q) = n$

ii) *p_L is a leading form of q.*

iii) *If $\deg(q') \geq n + 2$, then p_L is anisotropic and q_L is Witt-equivalent to $\langle a \rangle p_L$ with some $a \in L^*$.*

Proof. We may assume that q' is not hyperbolic. We will show that $\deg(q) = n$. Let $(L_i, q'_i, 0 \leq i \leq e)$ be a generic splitting tower of q'. Then p_{L_e} is anisotropic. Otherwise p_{L_e} is hyperbolic and we can choose $0 \leq m \leq e$ maximal with p_{L_m} anisotropic. Then p is hyperbolic over $L_m(q'_m)$ and the norm theorem (see [Ba 2]) together with the sub form theorem (see [Ba 3]) show that $\langle a \rangle q'_m$ is a sub form of p_{L_m} for some $b \in L^*_m$. In particular $\deg(q') = \deg(q'_m) \leq n$ which is a contradiction. Thus p_{L_e} is anisotropic, and therefore $\langle a \rangle p_{L_e}$ is the kernel form of q_{L_e}. This shows $\deg(q) \leq n$. If $\deg(q) = m < n$, let L be a leading field of q with $\ker(q_L) = \langle c \rangle r$,

$c \in L^*$ and r a m-fold Pfister form over L. Thus $q'_L \sim \langle c\rangle r \perp \langle a\rangle p_L$, and since $\dim(\langle c\rangle r \perp \langle a\rangle p_L) = 2^m + 2^n < 2^{n+1}$, it follows $q'_L \sim 0$ because $\deg(q') \geq n+1$. Thus $\langle c\rangle r \sim \langle a\rangle p_L$ over L, and this implies p_L hyperbolic and hence r is hyperbolic, which is a contradiction. Thus $\deg(q) = n$. The rest of the proposition follows easily and we omit the proof. $\square$

(6.3) **Corollary.** *For any two forms q_1, q_2 over F*

$$\deg(q_1 \perp q_2) \geq \min\{\deg(q_1), \deg(q_2)\}$$

and if $\deg(q_1) \neq \deg(q_2)$, the equality holds.

From these results we immediately obtain

(6.4) **Theorem.**

(i) *$J(n)$ is a $W(F)$-sub-module of $W_q(F)$*

(ii) *$I^n W_q(F) \subset J(n)$*

(iii) *$I^m J(n) \subset J(m+n)$.*

Proof. (i) The above corollary shows that $J(n)$ is subgroup of $W_q(F)$. Since for any $a \in F^*$, $\deg(\langle a\rangle q) = \deg(q)$, again the same corollary implies $\deg(\langle a_1, \ldots, a_m\rangle q) \geq \deg(q)$ for any $a_1, \ldots, a_m \in F^*$, i.e. $J(n)$ is a $W(F)$-submodule of $W_q(F)$. This shows (i). Since $I^n W_q(F)$ is additively generated by the n-fold Pfister forms $\ll a_1, \ldots, a_n; a]]$ of degree n, (ii) follows from (i), (iii) is also an immediate corollary of (i). $\square$

If F is a field of characteristic different from 2, then one of the major conjectures of Knebusch's generic splitting theory is the equality $J_n = I^n$ in $W(F)$. In [A-Ba 2] Theorem 1, we have shown that the equality $I^n W_q(F) = J(n)$ for any field F is equivalent with the equality

$$\ker[I^n W_q(F)/I^{n+1} W_q(F) \to I^n W_q(F(p))/I^{n+1} W_q(F(p))] = \{0, \bar{p}\}$$

for any quadratic n-fold Pfister form p over F. Now if F is a field of characteristic 2, theorem (5.4) asserts that this equality is true. Thus combining these results we have the following.

(6.5) **Main theorem.** *Let F be a field of characteristic 2. Then for any $n \geq 0$*

$$I^n W_q(F) = J(n).$$

(6.6) **Remark.** Of course (6.5) implies (6.4), but the above outlined proof of (6.4) is much more elementary.

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