UNRAMIFIED COHOMOLOGY OF CLASSIFYING VARIETIES FOR EXCEPTIONAL SIMPLY CONNECTED GROUPS

R. SKIP GARIBALDI

Abstract. Let BG be a classifying variety for an exceptional simple algebraic group G. We compute the degree 3 unramified Galois cohomology of BG with values in $(\mathbb{Q}/\mathbb{Z})'(2)$ over a nearly arbitrary field F. Combined with a paper by Merkurjev, this completes the computation of these cohomology groups for G semisimple simply connected over (nearly) all fields.

Let G be an algebraic group over a field F with an embedding $\rho\colon G \hookrightarrow SL_n$ over F. The isomorphism class of the variety $X := SL_n/\rho(G)$ depends upon the embedding, but its stable birationality type does not. We call X a *classifying space* of G.

We will compute certain invariants of the stable birationality type of BG, specifically the *unramified cohomology* defined as follows. Let $(\mathbb{Q}/\mathbb{Z})'(d)$ be the module $\varinjlim \boldsymbol{\mu}_n^{\otimes d}$ for n not divisible by $\mathrm{char}(F)$. For each $d > 0$, define $H^d_{\mathrm{nr}}(F(X))$ to be the intersection of the kernels of the residue homorphisms

$$\partial_v\colon H^d(F(X), (\mathbb{Q}/\mathbb{Z})'(d-1)) \to H^{d-1}(F(v), (\mathbb{Q}/\mathbb{Z})'(d-2))$$

as v ranges over the discrete valuations of $F(X)$ over F. If K is a purely transcendental extension of F, then the natural map $H^d_{\mathrm{nr}}(F(X)) \to H^d_{\mathrm{nr}}(K(X))$ is an isomorphism, where the discrete valuation rings for the latter group are those containing K [Mer, 2.3]. That is, the group $H^d_{\mathrm{nr}}(F(X))$ does not depend on ρ, but only upon the stable birationality type BG of X, so we write $H^d_{\mathrm{nr}}(BG)$ for $H^d_{\mathrm{nr}}(F(X))$, or $H^d_{\mathrm{nr}}(B_F G)$ in order to emphasize the base field F.

The natural homomorphism $H^d(F, (\mathbb{Q}/\mathbb{Z})'(d-1)) \to H^d_{\mathrm{nr}}(B_F G)$ is split by the evaluation at the distinguished point of BG, thus,

$$H^d_{\mathrm{nr}}(B_F G) = H^d(F, (\mathbb{Q}/\mathbb{Z})'(d-1)) \oplus H^d_{\mathrm{nr}}(B_F G)_{\mathrm{norm}},$$

where the latter group is the group of *normalized* classes. (For details about all of this, please see [Mer].)

The goal of this paper is to complete the computation of $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ for G simply connected semisimple and F (nearly) arbitrary. The computation of $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ for these groups G is quickly reduced to the case where G is simple simply connected [Mer, §4]. In [Mer], $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ was computed for G simple and classical. We compute it for the remaining cases, where G is *exceptional*, that is, where G is of type G_2, ${}^3D_4$, ${}^6D_4$, F_4, E_6, E_7, or E_8.

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Main Theorem 0.1. *Suppose that G is a simply connected simple* exceptional *algebraic group defined over a field F and that the characteristic of F is not one of the torsion primes for G. Then*

$$H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = \begin{cases} \mathbb{Z}/2 & \text{if } G \text{ is of type } {}^3D_4 \text{ and has a nonsplit Tits algebra} \\ 0 & \text{otherwise.} \end{cases}$$

By *torsion primes*, we mean the primes associated with G in [Ser95, §2.2]:

type of G	torsion primes
G_2	2
${}^3D_4$, ${}^6D_4$, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

(The interested reader may apply Gille's [Gil00, Th. 2] to transfer our Main Theorem to fields with characteristic a torsion prime of G.)

Since H^d_{nr} is homotopy invariant, we have: If BG is stably rational, then $H^d_{\mathrm{nr}}(BG)_{\mathrm{norm}} = 0$. (This is a general motivation for studying $H^d_{\mathrm{nr}}(F(X))$, cf. [CT95, §4.2].) It was an open question whether BG is stably rational. The first counterexamples were provided in [Mer], where a simple group G was given with $H^d_{\mathrm{nr}}(BG)_{\mathrm{norm}} \neq 0$. The results here give another such example.

The computations of $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ in this paper depend heavily on those in Merkurjev's. The crux case for proving our Main Theorem is that of type ${}^2E_6$, the work for which is done in Sections 4 and 5.

Remark 0.2. In contrast the generality of our base field F, computations of $H^d_{\mathrm{nr}}(F(X))$ in the literature for X a smooth variety (e.g., a classifying space) typically assume that F is algebraically closed. The examples of nonrational classifying varieties BG provided here and in [Mer] require that F is not algebraically closed.

Notations and conventions. We say that an algebraic group is *simple* if it is $\neq 1$, is connected, and has no nontrivial connected normal subgroups over an algebraic closure. (These groups are often called "absolutely almost simple".) Simple groups are classified in, e.g., [KMRT98, Ch. VI]. We say that a group is *of type T_n* if it is simple with root system of type T_n and *of type ${}^tT_n$* if additionally the absolute Galois group of F acts as a group of automorphisms of order t on the Dynkin diagram.

We use the standard notation $\boldsymbol{\mu}_n$ for the algebraic groups with F-points the nth roots of unity in F. If G is defined over F and L is an extension of F, we set $G_L := G \times_F L$.

The cohomology here is all Galois cohomology, for which the standard reference is [Ser94, §I.5].

1. Preliminaries

1.1. Tits algebras. Let V be an irreducible representation of a simple group G over F. The F-algebra $\mathrm{End}_G(V)$ is a skew field by Schur's Lemma, and it is finite-dimensional over F; it is called a *Tits algebra* for G. If it is a (commutative) field, we say that it is *split*.

1.2. For G simple simply connected, there is a canonical and nontrivial morphism of functors

$$r_G \colon H^1(*, G) \longrightarrow H^3(*, (\mathbb{Q}/\mathbb{Z})'(2))$$

called the *Rost invariant* [KMRT98, 31.40]. It has finite order n_G, and this order depends only upon the type of G and the (Schur) indices of its Tits algebras. We will repeatedly make us of the fact that the value of n_G is known and can be found by looking in [KMRT98, pp. 437–442] (or see [Mer01] for proofs). The value of n_G for G exceptional is:

type of G	G_2	${}^{3,6}D_4$, all Tits alg's split	${}^{3,6}D_4$, some Tits alg's nonsplit	F_4	${}^1E_6$	${}^2E_6$	E_7	E_8
n_G	2	6	12	6	6	12	12	60

2. Ramification

We say that a scalar multiple mr_G is *ramified* (resp. *unramified*) if there is some (resp. no) field extension E of F such that the image of the composition

$$H^1(E((t)),G) \xrightarrow{mr_G} H^3(E((t)),(\mathbb{Q}/\mathbb{Z})'(2)) \xrightarrow{\partial} H^2(E,(\mathbb{Q}/\mathbb{Z})'(1))$$

is nontrivial for ∂ the residue with respect to the canonical discrete valuation of $E((t))$ over E.

Lemma 2.1. [Mer] *$H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ is isomorphic to the set of unramified multiples of the Rost invariant.*

Since the Rost invariant is torsion, $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ is necessarily finite.

Strongly Inner Lemma 2.2. *Let G be a simply connected simple group over F. Fix $\alpha \in H^1(F,G)$ and $m \in \mathbb{Z}$. Then mr_G is ramified if and only if mr_{G_α} is. Moreover,*

$$H^d_{\mathrm{nr}}(BG)_{\mathrm{norm}} = H^d_{\mathrm{nr}}(BG_\alpha)_{\mathrm{norm}}.$$

The notation G_α means an algebraic group with the same set of points as G over a separable closure F_{sep} of F (i.e., $G_\alpha(F_{\mathrm{sep}}) = G(F_{\mathrm{sep}})$), but with a different Galois action: For $\gamma \in \mathrm{Gal}(F_{\mathrm{sep}}/F)$ and $g \in G_\alpha(F_{\mathrm{sep}})$, we set

$${}^\gamma g = a_\gamma \gamma(g) a_\gamma^{-1},$$

where juxtaposition denotes the usual action on $G(F_{\mathrm{sep}})$ and a is a 1-cocycle representing the class α. Such a group is known as a *strongly inner form of* G (continuing the assumption that G is simply connected), and its isomorphism class is independent of the choice of a.

Proof: Let E be an extension of F, and consider the diagram

$$\begin{array}{ccccc}
H^1(E((t)),G) & \xrightarrow{mr_G} & H^3(E((t)),(\mathbb{Q}/\mathbb{Z})'(2)) & \xrightarrow{\partial} & H^2(E,(\mathbb{Q}/\mathbb{Z})'(1)) \\
{\scriptstyle \tau_\alpha}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cdot - r_G(\alpha)}\downarrow & & \| \\
H^1(E((t)),G_\alpha) & \xrightarrow{mr_{G_\alpha}} & H^3(E((t)),(\mathbb{Q}/\mathbb{Z})'(2)) & \xrightarrow{\partial} & H^2(E,(\mathbb{Q}/\mathbb{Z})'(1)),
\end{array}$$

where τ_α is the isomorphism arising from a choice of some 1-cocycle a representing α. The left box commutes by [Gil00, p. 76, Lem. 7]. The right box commutes because ∂ is a group homomorphism and $\partial(r_G(\alpha)) = 0$.

Hence mr_G is ramified if and only if mr_{G_α} is. Since G and G_α are strongly inner forms of each other, they have the same Rost numbers; the displayed equation follows. ∎

3. $A_2 \subset D_4$

A simple group is said to be *trialitarian* if it is of type ${}^3D_4$ or ${}^6D_4$.

Lemma 3.1. (char $F \neq 2, 3$) *Let $L = F(\ell^{1/3})$ be a cubic field extension of F. Let G^q be the quasi-split trialitarian group over F associated with L. Then G^q contains a subgroup isomorphic to PGL_3 such that the induced diagram*

$$\begin{tikzcd} H^1(F, PGL_3) \arrow[r] \arrow[d] & H^3(F, \boldsymbol{\mu}_3^{\otimes 2}) \arrow[d] \\ H^1(F, G^q) \arrow[r, "r_{G^q}"] & H^3(F, (\mathbb{Q}/\mathbb{Z})'(2)) \end{tikzcd}$$

commutes up to sign, where the arrow on top is given by composing the connecting homomorphism $H^1(F, PGL_3) \to H^2(F, \boldsymbol{\mu}_3)$ with the map

$$\cdot \cup (\ell) \colon H^2(F, \boldsymbol{\mu}_3) \to H^3(F, \boldsymbol{\mu}_3^{\otimes 2}).$$

The reason that we may only claim that the diagram commutes up to sign is because the Rost invariant is only known up to sign. (The Rost invariant is canonically determined, but in some cases the Rost invariant of a particular cocycle is only known up to a scalar multiple which is relatively prime to the order of the Rost invariant.)

For $\rho\colon G \to H$ a map between simple simply connected algebraic groups over F, there is a positive integer n_ρ called the *Dynkin index* of ρ, see [Mer] or [Gar, 2.1] (where it is called the "Rost multiplier" of ρ). It was defined by Dynkin — without use of the Rost invariant, of course — in the case $G = SL_n$, see [Dyn57, p. 130]. One of the properties of n_ρ is that for E any extension of F, the composition

$$H^1(E, H) \xrightarrow{\rho} H^1(E, G) \xrightarrow{r_G} H^3(E, (\mathbb{Q}/\mathbb{Z})'(2))$$

is $n_\rho r_H$.

We will compute the Dynkin index in relatively easy situations. Let $\Lambda_{c,H}$ and $\Lambda_{c,G}$ denote the coroot lattices for H and G. The map $H \to G$ induces a map $\Lambda_{c,H} \to \Lambda_{c,G}$, and these lattices are endowed with unique minimal positive definite Weyl-invariant quadratic forms, which are 1 on short coroots. The restriction of the form q on $\Lambda_{c,G}$ to the image of a short coroot of H is n_ρ.

Proof of Lemma 3.1: The map $x \mapsto \mathrm{Tr}(x^2)$ defines a quadratic form on $M_3(F)$ and the group PGL_3 preserves the 8-dimensional subspace of trace zero elements. This gives rise to an embedding $PGL_3 \to SO_8$ which lifts to a map $PGL_3 \to \mathrm{Spin}_8$ [KMRT98, pp. 504, 505]. We can twist Spin_8 to obtain G^q. The subgroup PGL_3 of Spin_8 is preserved by this twist, and we obtain an embedding of PGL_3 in G^q.

The Springer construction [KMRT98, §38.A] gives an embedding of G^q in F_4 the split group of type F_4. By inspecting this embedding over L, we see that it has Dynkin index 1. Consequently, the Rost invariant r_{G^q} factors through F_4, so we can rewrite our desired composition as

$$H^1(F, PGL_3) \longrightarrow H^1(F, G^q) \longrightarrow H^1(F, F_4) \xrightarrow{r_{F_4}} H^3(F, \boldsymbol{\mu}_6^{\otimes 2}).$$

The set $H^1(F, F_4)$ classifies Albert F-algebras, and by [KMRT98, 39.9] the image of $[A] \in H^1(F, PGL_3)$ in $H^1(F, F_4)$ is the class of a certain kind of Albert algebra called a first Tits construction and denoted by $J(A, \ell)$. Then the Rost invariant is $[A] \cup (\ell)$ as desired by [Ros91] or [PR96]. □

4. ${}^3D_4, {}^6D_4 \subset {}^2E_6$

The main result of this section is to describe an embedding of isotropic but not quasi-split trialitarian groups into groups of type ${}^2E_6$. In this section, we assume that F has characteristic $\neq 2, 3$.

4.1. Related triples [KMRT98], [Gar98], [Gar01]**.** Write $\mathfrak{C}$ for the split Cayley algebra over F which has a canonical involution $\bar{\ }$ and a quadratic norm form $\mathfrak{n}$. Let $\star$ denote the (nonassociative, non-unital) multiplication on $\mathfrak{C}$ defined by $x \star y := \overline{x}\,\overline{y}$, where juxtaposition denotes the usual multiplication. We say that a similarity $t \in GL(\mathfrak{C})$ of $\mathfrak{n}$ with similarity factor $\mu(t)$ is *proper* if $\det t = \mu(t)^4$ (as opposed to $-\mu(t)^4$). The subgroup of $GL(\mathfrak{C})$ consisting of such similarities is denoted $GO(\mathfrak{C})^\circ$.

A triple $\underline{t} = (t_0, t_1, t_2)$ in $(GO(\mathfrak{C})^\circ)^{\times 3}$ is said to be *related* if

$$\mu(t_i)^{-1} t_i(x \star y) = t_{i+2}(x) \star t_{i+1}(y)$$

for all $i = 0, 1, 2$ and $x, y \in \mathfrak{C}$, with subscripts taken modulo 3. This defines a closed subgroup of $(GO(\mathfrak{C})^\circ)^{\times 3}$ which we call $\mathrm{Rel}\,(\mathfrak{C})$. For a related triple $\underline{t}$, it is always the case that $\mu(t_0)\mu(t_1)\mu(t_2) = 1$.

The simply connected split group Spin_8 of type D_4 is isomorphic to the intersection of $\mathrm{Rel}\,(\mathfrak{C})$ and $O(\mathfrak{C})^{\times 3}$ (where $O(\mathfrak{C})$ is the group of isometries of $\mathfrak{n}$) by [KMRT98, 35.7], and we identify the two groups by this isomorphism.

4.2. Albert algebras [KMRT98, Ch. IX]**.** The split Albert algebra J has as its underlying vector space the matrices in $M_3(\mathfrak{C})$ fixed by the conjugate transpose. With that in mind, we may write a general element of J as

$$\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix},$$

where $\varepsilon_i \in F$, $c_i \in \mathfrak{C}$, and the entries given as $\cdot$ are forced by symmetry. The algebra J is strictly power associative and hence endowed with a canonically determined norm form (normalized so that the identity has norm 1) N. For $\underline{t}$ a related triple, the linear transformation $g_{\underline{t}}$ of J given by

$$g_{\underline{t}} \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} := \begin{pmatrix} \mu(t_0)^{-1}\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \mu(t_1)^{-1}\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \mu(t_2)^{-1}e_2 \end{pmatrix}$$

is an isometry of the norm on J. This defines an injection of Spin_8 into the group $\mathrm{Inv}\,(J)$ of isometries of the norm.

4.3. Construction of a quasi-split ${}^2E_6$. The algebra J is also endowed with a linear trace map $\mathrm{Tr}\colon J \to F$ and the bilinear form T defined by $T(x, y) = \mathrm{Tr}(xy)$ is symmetric and nondegenerate. For each $f \in GL(J)$, there is a unique $f^\dagger \in GL(J)$ such that $T(f(x), f^\dagger(y)) = T(x, y)$ for all $x, y \in J$. The map $f \mapsto f^\dagger$ restricts to be an outer automorphism on $\mathrm{Inv}\,(J)$ defined over F. For ι the nontrivial F-automorphism of K, we define the group E_6^K to be $\mathrm{Inv}\,(J)$ with a twisted ι-action: For $f \in E_6^K(K)$, we have ${}^\iota f = \iota f^\dagger \iota$ where the action on the left is the action in E_6^K and juxtaposition denotes the usual action in $\mathrm{Inv}\,(J)$. This group is quasi-split of type ${}^2E_6$ [Gar01, 2.9(2)].

The norm $\mathfrak{n}$ induces an involution σ on $GL(\mathfrak{C})$ such that $\mathfrak{n}(f(x), y) = \mathfrak{n}(x, \sigma(f)y)$ for all $x, y \in \mathfrak{C}$ and $f \in GL(\mathfrak{C})$. We define $\sigma(\underline{t})$ to mean $(\sigma(t_0), \sigma(t_1), \sigma(t_2))$. Then

$g_{\underline{t}}^\dagger = g_{\sigma(\underline{t})^{-1}}$. A related triple $\underline{t}$ lies in Spin_8 if and only if $\sigma(\underline{t}) = \underline{t}^{-1}$, so the embedding of Spin_8 in E_6^K given by g is defined not just over K, but also over F.

4.4. Let L be a cubic extension of F endowed with a quaternion algebra Q whose corestriction down to F is split. By [Gar98, 4.7], there is a simply connected isotropic trialitarian group T over F whose Tits algebra is $M_4(Q)$. By [KMRT98, 43.9], there is a $b \in F^*$ such that $Q \otimes_L L(\sqrt{b})$ is split.

Lemma 4.5. ($\mathrm{char}\, F \neq 2, 3$) *Let L, Q, T, and b be as in* 4.4. *There exists a simply connected group ${}^2E_6$ of that type over F such that:*

1. *${}^2E_6$ has split Tits algebras.*
2. *${}^2E_6$ is of type ${}^1E_6$ over $F(\sqrt{b})$.*
3. *T is a subgroup of ${}^2E_6$ with Dynkin index* 1.

By [Gar98], every isotropic simply connected trialitarian group occurs as in 4.4.

Proof: If Q is split, then T is the quasi-split trialitarian group associated to the cubic extension L/F. So T is a subgroup with Dynkin index 1 of the group $\mathrm{Aut}(J)$ of automorphisms of J, which is the split group of type F_4 from the proof of Lemma 3.1. Since $\mathrm{Aut}(J)$ is the subgroup of $\mathrm{Inv}\,(J)$ consisting of the elements fixed by the outer automorphism $\dagger$, it is also a subgroup of the quasi-split group $E_6^{F(\sqrt{b})}$. This embedding also has Dynkin index 1 [Gar, 2.4], so we are done in this case.

We may assume that Q is not L-split. Since Q becomes split over $L(\sqrt{b})$, it is isomorphic to $(a, b)_L$ for some $a \in L^*$. Since $\mathrm{cor}_{L/F}[Q]$ is split by hypothesis and Brauer-equivalent to $(N_{L/F}(a), b)_F$, we have that $\alpha := N_{L/F}(a)$ is a norm from $F(\sqrt{b})$. For $a' := a^3\alpha^{-1}$, the algebra $(a, b)_L$ is isomorphic to $(a', b)_L$ and $N_{L/F}(a') = 1$. Replacing a with a', we may assume that $N_{L/F}(a) = 1$.

Set $P = L^c(\sqrt{b})$ for L^c the normal closure of L/F. Since Q is not split over L, it is not split over L^c by [Gar98, 3.2]. In particular, L^c does not contain a square root of b, so P is a quadratic extension of L^c.

To simplify our argument, we assume that L is not Galois over F, so $\mathrm{Gal}(L^c/F)$ is isomorphic to S_3. (The case where L is Galois over F is only easier.) Then, the group $\mathrm{Gal}(P/F)$ is isomorphic to $S_3 \times \boldsymbol{\mu}_2 \cong \mathbb{Z}/6 \rtimes \boldsymbol{\mu}_2$, where the factor of $\mathbb{Z}/6$ corresponds to the subgroup $\mathrm{Gal}(P/\Delta)$ for Δ the unique quadratic extension of F in L^c. We fix generators $\zeta := ((1\,2\,3), -1)$ (which generates the copy of $\mathbb{Z}/6$) and $\tau = ((2\,3), 1)$ (which generates the copy of $\boldsymbol{\mu}_2$ in $\mathbb{Z}/6 \rtimes \boldsymbol{\mu}_2$ corresponding to $\mathrm{Gal}(P/L(\sqrt{b}))$).

We construct the group ${}^2E_6$ by giving a 1-cocycle $z \in Z^1(P/F, E_6^K)$ for $K = F(\sqrt{b})$. Define a related triple $\underline{t} = (t_0, t_1, t_2)$ by setting $t_i = m_i P$, for

$$m_i = \mathrm{diag}\,(1, \rho^i(a), -\rho^i(a), \rho^{i+2}(a)^{-1}, \rho^{i+1}(a)^{-1}, -1, 1, \rho^i(a))$$

with $\rho := \zeta^2$, and P the matrix permuting the basis vectors as $(1\,2)(3\,6)(4\,5)(7\,8)$, for the basis of $\mathfrak{C}$ used in [Gar98] and [Gar]. This is indeed a related triple since $N_{L/F}(a) = 1$, see [Gar98, 1.5(3), 1.6, 1.8]. Also, define maps $r, \pi \in \mathrm{Inv}\,(J)(F)$ by

$$r\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_2 & c_1 & \cdot \\ \cdot & \varepsilon_0 & c_2 \\ c_0 & \cdot & \varepsilon_1 \end{pmatrix} \quad \text{and} \quad \pi\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_0 & \overline{c_1} & \cdot \\ \cdot & \varepsilon_2 & \overline{c_0} \\ \overline{c_2} & \cdot & \varepsilon_1 \end{pmatrix}.$$

Note that

$$r^\dagger = r, \quad \pi^\dagger = \pi, \quad \text{and} \quad r^{-1} g_{\underline{t}} r = g_{\rho(\underline{t})}.$$

We define z by setting

$$z_\zeta = g_{\underline{t}} r \quad \text{and} \quad z_\tau = \pi.$$

Observing that

$$\zeta g_{\underline{t}} \zeta^{-1} = g_{\rho(\sigma(\underline{t})^{-1})} = g_{\rho(\underline{t})^{-1}},$$

we have the formulas

$$(z_\zeta \zeta)^2 = r^2 \zeta^2 \quad \text{and} \quad (z_\zeta \zeta)^3 = g_{\underline{t}} \zeta^3.$$

Since $\pi\tau$ and $g_{\underline{t}}$ commute, it is easy to verify that z is a 1-cocycle in $Z^1(P/F, E_6^K)$. Set ${}^2E_6$ to be the twisted group $(E_6^K)_z$; it automatically satisfies (1) and (2).

Since the values of z normalize $\mathrm{Spin}_8 \subset E_6^K$, the group $(\mathrm{Spin}_8)_z$ is a subgroup of ${}^2E_6$ with Dynkin index 1. The restriction of z to Spin_8 is the descent given in [Gar98, 4.7] to construct T, i.e., $(\mathrm{Spin}_8)_z$ is isomorphic to T, hence (3). □

Remark 4.6. The isotropic group ${}^2E_6$ occurring in the preceding proposition is typically not quasi-split, even over L. This can be seen by examining the Rost invariant for $(z) \in H^1(P/L, E_6^K)$, which is typically nontrivial by [Gar, 6.7].

5. A construction

The purpose of this section is to construct a suitable cubic field extension so that we may apply 4.5. We suppose that the characteristic of F is $\neq 2, 3$. The following arguments are somewhat simpler than previously, thanks to suggestions by Adrian Wadsworth.

Lemma 5.1. (char $F \neq 2, 3$) *For $p, q \in F^*$, the ring $L = F(t)[x]/(x^3 + px + qt)$ is a cubic field extension of $F(t)$ which is not Galois over $F(t)$. There is a prolongation of the t-adic valuation on $F(t)$ to L which is unramified with residue degree* 1 *and with respect to which x has value* 1.

Proof: If L is not a field, then there is some $a \in F(t)$ such that $a^3 + pa + qt = 0$. Since a is integral over the UFD $F[t]$, it belongs to $F[t]$, so it makes sense to speak of the degree of a. In particular, at least two of the terms a^3, pa, and qt must have the same degree, which is also the maximum of the degrees. This implies that a cannot have positive degree. But then qt, with degree 1, is the unique term of maximal degree, which is a contradiction.

A similar argument shows that the discriminant $-4p^3 - 27q^2t^2$ of L is not a square in $F(t)$: Any square root $b \in F(t)$ of the discriminant would belong to $F[t]$ and have degree 1. Then the coefficient of t in b^2 would be nonzero. Thus L is not Galois over $F(t)$.

Hensel's Lemma gives that $x^3 + px + tq$ has a linear factor of the form $x - \pi$ in $F((t))[x]$, where π has value 1. The map $x \mapsto \pi$ gives an isomorphism of L with the subfield $F(t)(\pi)$ of $F((t))$, and the t-adic valuation obviously extends to L so that x has value 1. Since $F((t))$ is the completion of $F(t)$ with respect to the t-adic valuation and hence is unramified with residue degree 1, the claims about ramification and residue degree of our prolongation to L follow. □

Lemma 5.2. (char $F \neq 2, 3$) *Let $p, b \in F^*$ be such that the quaternion algebra $(p, b)_F$ is nonsplit. Let L be as in Lemma 5.1. Then the quaternion algebra $(x, b)_L$ is nonsplit and is not defined over $F(t)$.*

Proof: Since $N_{L/F(t)}(x) = -qt$, the corestriction of $(x,b)_L$ down to $F(t)$ is Brauer-equivalent to $(-qt,b)_{F(t)}$. This algebra is split if and only if the quadratic form $\langle 1,-b,qt\rangle$ is isotropic over $F(t)$. Over the completion $F((t))$, this form has residue forms $\langle 1,-b\rangle$ and $\langle q\rangle$. Since the algebra $(p,b)_F$ is nonsplit, the first form is anisotropic, hence $\langle 1,-b,qt\rangle$ is anisotropic over $F((t))$ by Springer's Theorem [Lam73, VI.1.9]. Thus $(-qt,b)_{F(t)}$ is nonsplit, and hence so is $(x,b)_L$.

For the sake of contradiction, we suppose that $(x,b)_L$ is defined over F. Since $\operatorname{cor}_{L/F(t)}\operatorname{res}_{L/F(t)}$ is multiplication by 3 on the Brauer group, $(x,b)_L$ is isomorphic to $(-qt,b)_L$. This implies that the algebra $(-xqt,b)_L$ is split. Since

$$-x(qt) = -x(-x^3-px) = x^4+px^2 \equiv x^2+p \mod L^{*2},$$

the algebra $(x^2+p,b)_L$ is split.

Let $\widehat{L}$ be a completion of L with respect to the prolongation of the t-adic valuation on $F(t)$ given by Lemma 5.1. The norm of $(x^2+p,b)_L$ is the form $\langle 1,-(x^2+p),-b,b(x^2+p)\rangle$ over L. Since x has value 1, over $\widehat{L}$ this form has one residue form $\langle 1,-p,-b,bp\rangle$ over the residue field F. This is the norm of the algebra $(p,b)_F$, which is anisotropic because the algebra is nonsplit. By Springer's Theorem, the norm of $(x^2+p,b)_L$ is anisotropic over $\widehat{L}$, hence the algebra is nonsplit, which contradicts our assumption that $(x,b)_L$ is defined over F. □

5.3. Application. We will make use of this construction as follows: Let G^q be a quasi-split simply connected group of type ${}^2E_6$ over F which becomes of type ${}^1E_6$ over $K = F(\sqrt{b})$.

Let $F_0 := F(p)$ for p an indeterminate. Since b is not a square in F, the quaternion algebra $(b,p)_{F_0}$ is nonsplit. Set $F_1 := F_0(t)$ and $L_1 := F_0(t)[x]/(x^3+px+t)$. Then L_1 is a cubic non-Galois extension of F_1 (by 5.1) and the algebra $(x,b)_{L_1}$ is nonsplit and not defined over F_1 (by 5.2). Set F' to be the function field of the Severi-Brauer variety of $(-t,b)_{F_1}$. Since F_1 is algebraically closed in F', $L' := L_1 \otimes_{F_1} F'$ is a field which is not Galois over F' and b is still a nonsquare in F'.

Since $(-t,b)_{F_1}$ is Brauer-equivalent to the corestriction of $(x,b)_{L_1}$, and $(-t,b)_{L'}$ is split, the corestriction of $(x,b)_{L'}$ down to F' is split. Thus there is a simply connected isotropic trialitarian group T over F' with Tits algebra $(x,b)_{L'}$. It is of type ${}^6D_4$ since L' is not Galois over F'. Since $(x,b)_{L_1}$ is not defined over F_1, it is not isomorphic to $(-t,b)_{L_1}$, hence $(x,b)_{L'}$ is not split. By 4.5, there is a simple simply connected group G' over L' of type ${}^2E_6$ which is a strongly inner form of $G^q_{L'}$ such that T is a subgroup of G' with Dynkin index 1.

6. Proof of the main theorem

Let G be a simple simply connected algebraic group over F such that $\operatorname{char} F$ is not a torsion prime for G.

6.1. Let $\alpha : H \to G$ be a homomorphism of simple simply connected algebraic groups with Dynkin index 1. If mr_H is ramified, then so is mr_G. If $n_G = n_H$ and $H^3_{\mathrm{nr}}(BH)_{\mathrm{norm}} = 0$, then $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = 0$.

Lemma 6.2. [Mer, 4.4] *If there exists an extension K of F such that $n_{G_K} = n_G$ and $H^3_{\mathrm{nr}}(B_K G)_{\mathrm{norm}} = 0$, then $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = 0$.*

Proof: If mr_G is unramified over F, then $mr_{G_K} = 0$. Since $n_{G_K} = n_G$, we have $mr_G = 0$. □

As a corollary, we obtain the following reduction:

Corollary 6.3. *Let G be a simple simply connected group over F and let G^q be the unique quasi-split inner form of G. Suppose that $n_G = n_{G^q}$. If there exists an extension F' of F such that $n_{G^q_{F'}} = n_{G^q}$ and $H^3_{\mathrm{nr}}(B_{F'}G^q)_{\mathrm{norm}} = 0$, then $H^3_{\mathrm{nr}}(B_F G)_{\mathrm{norm}} = 0$.*

Proof: The hypotheses on F' give $H^3_{\mathrm{nr}}(B_F G^q)_{\mathrm{norm}} = 0$ by the lemma. Let K be a generic quasi-splitting field for G as in [KR94]. Since $n_G = n_{G^q} = n_{G^q_K}$, we have $H^3_{\mathrm{nr}}(B_F G)_{\mathrm{norm}} = 0$ by the lemma. □

The remainder of this section is the proof of the Main Theorem 0.1, broken up by the type of the simple exceptional group G. We must show in all cases except for 6.6 below that every nontrivial multiple of the Rost invariant is ramified. By hypothesis, $\operatorname{char} F$ is not a torsion prime for G, and inspection of the table in 1.2 shows that the order n_G of the Rost invariant r_G is relatively prime to $\operatorname{char} F$. Consequently [MS83] the n_G-torsion of $H^3(F, (\mathbb{Q}/\mathbb{Z})'(2))$ (which contains the image of r_G) is identified with the subgroup $H^3(F, \boldsymbol{\mu}_{n_G}^{\otimes 2})$.

6.4. Types F_4 and G_2. Let G^d be the split group of the same type as G, where G has type F_4 or G_2. The Rost invariant (up to some invertible scalar multiple) has been explicitly computed for the split groups of type G_2 and F_4 [KMRT98, pp. 441, 533–537], and all nontrivial multiples of it are clearly ramified. Since the groups G^d and G have the same Rost number, $H^d_{\mathrm{nr}}(BG)_{\mathrm{norm}} = H^d_{\mathrm{nr}}(BG^d)_{\mathrm{norm}} = 0$.

6.5. Type ${}^{3,6}D_4$ with split Tits algebras. Let G be simply connected trialitarian with associated cubic extension L/F and quasi-split inner form G^q.

Let F' be the extension obtained from F by adjoining an indeterminate x and, if they are not already in F, a primitive cube root of unity ζ and a square root of the discriminant of the cubic extension L of F. By Corollary 6.3, it is sufficient to show that $mr_{G^q_{F'}}$ is ramified or trivial for all m. Note that $L' := L \otimes_F F'$ is a cubic Galois extension of F', hence by Kummer theory is of the form $L' = F'(\ell^{1/3})$ for some $\ell \in (F')^*$

If $mr_{G^q_{F'}}$ is nontrivial, it has order 2, 3, or 6. If it has order 2, it must equal $3r_{G^q_{F'}}$. Since $n_{G^q_{L'}} = 2$, we have $3r_{G^q_{L'}} = r_{G^q_{L'}}$, which is ramified [Mer, 8.7]. Hence $3r_{G^q_{F'}}$ is ramified. If $mr_{G^q_{F'}}$ has order 6, then $3mr_{G^q_{F'}}$ has order 2 and so is ramified, hence $mr_{G^q_{F'}}$ is ramified.

The remaining cases are $mr_{G^q_{F'}}$ of order 3, i.e., $m \equiv 2, 4 \mod 6$. As described in 3.1, PGL_3 is a subgroup $G^q_{F'}$. Let D the cyclic central simple algebra over $F'((t))$ determined by the extension $F'((t))(\sqrt[3]{t})$ and $x \in F'((t))^*/F'((t))^{*3}$. (See [KMRT98, §30.A] for information on cyclic algebras.) The image of $[D] \in H^1(F'((t)), PGL_3)$ under $mr_{G^q_{F'}}$ is up to sign $m[D] \cup (\ell) = m(t) \cup (x) \cup (\ell)$. This has residue $m(x) \cup (\ell)$. Since ℓ is not a cube in F' and $m \equiv 2, 4 \mod 6$, this class is nontrivial. Hence $mr_{G^q_{F'}}$ is ramified. This completes the case where G is trialitarian with split Tits algebras.

6.6. Type ${}^{3,6}D_4$ with nonsplit Tits algebra. Suppose that G is of type ${}^3D_4$ or ${}^6D_4$ with nonsplit Tits algebra, and let K be a generic quasi-splitting variety for

G. Since mr_{G_K} is ramified if it is nontrivial for $m \not\equiv 6 \mod 12$, the same is true of mr_G for such m.

We now focus on $6r_G$, which is nontrivial and whose image lies in $H^3(F, \boldsymbol{\mu}_2^{\otimes 2})$. Let L be a cubic extension of F associated with G. If G is of type ${}^6D_4$ (over F), then G_L is of type ${}^2D_4$, and $6r_{G_L} = 2r_{G_L}$ is ramified by [Mer, 8.3]. So we may assume that G is of type ${}^3D_4$. We will have proved that $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = \mathbb{Z}/2$ if we show that for every extension E of F, the composition

$$H^1(E((t)), G_{E((t))}) \xrightarrow{6r_{G_{E((t))}}} H^3(E((t)), \boldsymbol{\mu}_2^{\otimes 2}) \xrightarrow{\partial} H^2(E, \boldsymbol{\mu}_2) \tag{6.7}$$

is trivial.

If G is of type ${}^1D_4$ over E, then G_E is isomorphic to $\mathrm{Spin}(A, \sigma)$ for some central simple E-algebra A of degree 8 with orthogonal involution σ such that the even Clifford algebra $C_0(A, \sigma)$ is isomorphic to a direct sum $C_+ \times C_-$ of two central simple E-algebras of degree 8. If at least one of the algebras A, C_+, C_- is split, then $n_{G_E} = 2$, so $6r_{G_E} = 0$. If none of the three is split, then $6r_{G_E} = 2r_{G_E}$, and the composition 6.7 is 0 by [Mer, 8.5(2)].

Otherwise, G_E is of type ${}^3D_4$, i.e., $L_E := L \otimes_F E$ is a field. We have a commutative diagram

$$\begin{tikzcd}
H^1(E((t)), G_{E((t))}) \arrow[r, "6r_{G_{E((t))}}"] \arrow[d, "\mathrm{res}"'] & H^3(E((t)), \boldsymbol{\mu}_2^{\otimes 2}) \arrow[r, "\partial"] \arrow[d, "\mathrm{res}"'] & H^2(E, \boldsymbol{\mu}_2) \arrow[d, "\mathrm{res}_{L_E/E}"'] \\
H^1(L_E((t)), G_{L_E((t))}) \arrow[r, "6r_{G_{L_E((t))}}"] & H^3(L_E((t)), \boldsymbol{\mu}_2^{\otimes 2}) \arrow[r, "\partial"] & H^2(L_E, \boldsymbol{\mu}_2).
\end{tikzcd}$$

Since $G_{L_E((t))}$ is of type ${}^1D_4$, the composition of the two bottom arrows is 0, hence the image of $H^1(E((t)), G_{E((t))})$ in $H^3(L_E, \boldsymbol{\mu}_2)$ is 0. Since $\boldsymbol{\mu}_2$ is 2-torsion and $[L_E : E] = 3$, the map $\mathrm{res}_{L_E/E}$ is invertible, and so the composition (6.7), which is the top row of the diagram, is 0. This proves the Main Theorem for G trialitarian with nonsplit Tits algebras.

6.8. Type ${}^1E_6$. The split F_4 injects into the split group E_6^d of type E_6 with Rost number 1 [Gar, 3.4], and we have $n_{F_4} = n_{E_6^d} = 6$. Hence $H^3_{\mathrm{nr}}(BE_6^d)_{\mathrm{norm}} = 0$.

Since all groups of type ${}^1E_6$ have the same Rost number, we have $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = 0$ by 6.3.

6.9. Type ${}^2E_6$. Let G^q be the quasi-split inner form of the given group G of type ${}^2E_6$. Write $K = F(\sqrt{b})$ for the quadratic extension over which the groups are of type ${}^1E_6$. Let T, L', and F' be as in 5.3. We have $n_T = n_{G'} = 12$. By 6.6 and 6.1, we have $H^3_{\mathrm{nr}}(B_{F'}G')_{\mathrm{norm}} = 0$. Since $G^q_{F'}$ and G' are strongly inner forms over F', we have $H^3_{\mathrm{nr}}(B_{F'}G^q)_{\mathrm{norm}} = 0$ by 2.2. Hence $H^3_{\mathrm{nr}}(B_F G)_{\mathrm{norm}} = 0$ by Corollary 6.3.

6.10. Type E_7. Let G^q be the unique (quasi-)split inner form of our given group G of type E_7. Set $F' = F(x)$ and $K = F'(\sqrt{x})$. There is a quasi-split simply connected group E_6^K over F of type ${}^2E_6$ associated with the extension K/F'. This group injects into $G^q_{F'}$ with Dynkin index 1 [Gar, §3]. Since $n_{E_6^K} = n_{G^q_{F'}} = 12$, $H^3_{\mathrm{nr}}(B_{F'}G^q)_{\mathrm{norm}} = 0$ by 6.1 and 6.9. Then $H^3_{\mathrm{nr}}(B_F G)_{\mathrm{norm}} = 0$ by 6.3.

6.11. Type E_8. Again, let G^q be the unique (quasi-)split inner form of our given group G, which in this case is of type E_8.

For m an integer such that $1 \leq m < 60 = n_{G^q}$, we will show that mr_{G^q} is ramified.

Suppose first that $5mr_{G^q}$ is nonzero. The map $x \mapsto 5x$ sends the 60-torsion in $H^3(F, (\mathbb{Q}/\mathbb{Z})'(2))$ to the 12-torsion, i.e., it is a map $H^3(F, \boldsymbol{\mu}_{60}^{\otimes 2}) \to H^3(F, \boldsymbol{\mu}_{12}^{\otimes 2})$. Since the split group of type E_7 obviously lies in G^q with Dynkin index 1, $5r_{G^q}$ restricts to be $5r_{E_7}$, which also has order 12 and all of whose nontrivial multiples are ramified. Thus $5mr_{G^q}$ is ramified, hence mr_{G^q} is also.

Otherwise, $5mr_{G^q} = 0$, so $m = 12$, 24, 36, or 48. Let E be the extension of F generated by two indeterminates x and y and, if it is not already in F, a primitive 5th root of unity ζ. The cyclic algebra D given by the cyclic extension $E(x^{1/5})$ and the class of $y \in E^*/E^{*5}$ is an E-central division algebra.

If we remove the correct vertex from the extended Dynkin diagram of the split group E_8 of type E_8, we are left with a diagram of type $A_4 \times A_4$. By [Tit90, §1], this reflects the existence of an exact sequence

$$1 \to \boldsymbol{\mu}_5 \to SL_5 \times SL_5 \to H \to 1 \tag{6.12}$$

where the map $\boldsymbol{\mu}_5 \to SL_5 \times SL_5$ is given by $\zeta \mapsto (\zeta \operatorname{Id}, \zeta^2 \operatorname{Id})$ and H is a subgroup of E_8. Note that the first copy of SL_5 injects into H, hence E_8, and this injection has Dynkin index 1. As described in the [Gil01, §1], there is a class $z \in H^1(E, H)$ such that twisting (6.12) by z gives

$$1 \to \boldsymbol{\mu}_5 \to SL_1(D) \times SL_1(D') \to H_z \to 1,$$

where D' is Brauer-equivalent to $D^{\otimes 3}$. Thus $SL_1(D)$ is a subgroup of $H_z \subset (G^q)_z$ with Dynkin index 1.

Since D is nonsplit, $n_{SL_1(D)} = 5$ (note that characteristic $\neq 5$ by hypothesis), so $mr_{G^q_z}$ restricts to $mr_{SL_1(D)} = mr_{SL_1(D)}$. Since $SL_1(D)$ has Rost number 5, this is nonzero, and it is ramified by [Mer, 5.1]. Since all group of type E_8 are strongly inner forms of each other, the same conclusion holds for $mr_{G^q_E}$ by the Strongly Inner Lemma 2.2, and mr_{G^q} by 6.2.

Since $H^3_{\mathrm{nr}}(B_F G^q)_{\mathrm{norm}} = 0$, we have $H^3_{\mathrm{nr}}(B_F G)_{\mathrm{norm}} = 0$ for G of type E_8 by 6.3.

This completes the proof of the Main Theorem 0.1.

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UCLA Department of Mathematics, Los Angeles, CA 90095-1555
E-mail address: skip@member.ams.org