A SECOND DESCENT PROBLEM FOR QUADRATIC FORMS

BRUNO KAHN AHMED LAGHRIBI

ABSTRACT. Let F be a field of characteristic different from 2. We discuss a new descent problem for quadratic forms, complementing the one studied in [19] and [30]. More precisely, we conjecture that for any quadratic form q over F and any $\varphi \in \mathrm{Im}(W(F) \longrightarrow W(F(q)))$, there exists a quadratic form $\psi \in W(F)$ such that $\dim \psi \leq 2 \dim \varphi$ and $\varphi \sim \psi_{F(q)}$, where $F(q)$ is the function field of the projective quadric defined by $q = 0$. We prove this conjecture for $\dim \varphi \leq 3$ and any q, and get partial results for $\dim \varphi \in \{4, 5, 6\}$. We also give other related results.

CONTENTS

Date: January 25, 2002.
Keywords: quadratic form, Galois cohomology, rationality.

1. Introduction

Let F be a field of characteristic different from 2 and K/F an extension. Let φ be a quadratic form over K. We say that φ is *defined over* F if there exists a quadratic form ψ over F such that $\varphi \simeq \psi_K$, where $\simeq$ denotes isometry of quadratic forms.

1.1. **Problem.** *Under which conditions is φ defined over F?*

This problem is studied in [19] and [30] when K is the function field of a quadric: Conjecture 1 below predicts sufficient conditions for a positive answer in that case. In the present paper, our aim is to study a complementary problem, which we now explain.

If we replace $\simeq$ by $\sim$ (Witt equivalence), we say that φ is *defined over F up to Witt equivalence*: this means that $\varphi \in \operatorname{Im}(W(F) \longrightarrow W(K))$. Clearly, if φ is defined over F, it is defined over F up to Witt equivalence. We then may ask:

1.2. **Problem.** *Suppose that φ is defined over F up to Witt equivalence.*

(1) *What is the smallest dimension of an F-form ψ such that $\varphi \sim \psi_K$?*
(2) *Can one describe those ψ which have this smallest dimension?*

Conjecture 2 below tackles this issue again when K is the function field of a quadric; still in that case, Theorem 7.3 will provide a very detailed answer to Problem 1.2 for low-dimensional φs, and prove part of Conjecture 2 as a consequence.

More generally, suppose K/F finitely generated and regular: this means that K is the function field of a geometrically irreducible F-variety. The condition that $\varphi \in \operatorname{Im}(W(F) \longrightarrow W(K))$ implies that φ belongs to the *unramified Witt group* of K/F

$$W_{\mathrm{nr}}(K/F) = \operatorname{Ker}\left(W(K) \xrightarrow{(\partial_v^2)} \bigoplus_v W(K_v)\right)$$

where v runs through all discrete valuations of K which are trivial on F, K_v and ∂_v^2 denote respectively the residue field of v and the second residue homomorphism at v (associated to a local uniformiser). This condition is sufficient in the following cases, in which φ is even defined over F:

- K/F is purely transcendental.
- K/F is quadratic.
- K is the function field of a conic or a quadric defined by a 2-fold Pfister form ([4, Lemma 3.1], [5, Appendix], [37], [40]).

In general, however, the homomorphism $W(F) \to W_{\rm nr}(K/F)$ is not surjective (*cf.* Theorem 6.4 d)). When $K = F(q)$ is the function field of the projective quadric with equation $q = 0$, one has the following conjecture [19]:

Conjecture 1. *Let q be a quadratic form over F, $K = F(q)$ and φ a quadratic form over K such that:*

(1) $\varphi \in W_{\rm nr}(K/F)$,
(2) $\dim \varphi < \frac{1}{2} \dim q$.

Then φ is defined over F.

This conjecture was proved in certain cases where $\dim \varphi$ is small ([19] and [30]).

In this paper, we study the following complementary conjecture to Conjecture 1, and prove it in certain cases:

Conjecture 2. *Let q be a quadratic form over F, $K = F(q)$ and φ an anisotropic quadratic form over K. Suppose that $\varphi \in \operatorname{Im}(W(F) \to W(K))$. Then there exists a quadratic form ψ over F such that $\psi_K \sim \varphi$ and $\dim \psi \leq 2 \dim \varphi$.*

In order to state our results, it is convenient to introduce some notation. For (q, φ) as in Conjecture 2, define

$$C_F(q, \varphi) = \inf\{\dim \psi \mid \psi \in W(F) \text{ and } \varphi \sim \psi_{F(q)}\}.$$

Note that $C_F(q, \varphi)$ and $\dim \varphi$ have the same parity. Moreover, define:

$$\begin{aligned} C_F(q, n) &= \sup\{C_F(q, \varphi) \mid \dim \varphi = n\} \leq +\infty \\ C(m, n) &= \sup\{C_F(q, n) \mid F \text{ a field and } \dim q = m\} \leq +\infty \\ C(n) &= \sup\{C(m, n) \mid m \geq 2\} \leq +\infty. \end{aligned}$$

A reformulation of Conjecture 2 in terms of these constants is as follows:

1.3. **Reformulation.** *For any integer $n \geq 1$, one has $C(n) \leq 2n$. In particular, $C(n) \leq 2n - 1$ for n odd.*

(When n is odd, the inequality $C(n) \leq 2n-1$ follows from $C(n) \leq 2n$ and the fact that $C(n)$ has the same parity as n.)

These bounds are best possible: indeed, we shall show in Section 5:

Theorem 1. *For any $n \geq 1$, one has*

$$C(n) \geq C(4, n) \geq \begin{cases} 2n & \text{if } n \text{ is even} \\ 2n - 1 & \text{if } n \text{ is odd.} \end{cases}$$

The appearance of $C(4, n)$ in the formulation of this theorem suggests that the case $\dim q = 4$ is the most difficult to study: this is amply vindicated by our computations below.

Here are now the main results of this paper.

Theorem 2. *a) Conjecture 2 holds if*

(i) $\dim\varphi \leq 3$.
(ii) φ *is similar to a* 2*-fold Pfister form.*
(iii) φ *is a Pfister neighbour of dimension* 5.

b) For all $m \neq 4$, $C(m, 4) \leq 8$ *and* $C(m, 5) \leq 9$.
c) $C(4, 4) \leq 10$. *In particular,* $C(4) \leq 10$.

For $n = 5$ or 6, we also get the following partial results, which overlap with Theorem 2 b):

Theorem 3. *Let* q, K, φ *be as in Conjecture 2; let* D *be a central simple* F*-algebra such that* $[D_K] = c(\varphi)$.
a) If $\dim\varphi = 5$ *and* φ *is not a Pfister neighbour, Conjecture 2 holds, except perhaps when* $\dim q = 4$, $d_\pm q \neq 1$ *and* $\operatorname{ind}(D) = \operatorname{ind}(D \otimes C_0(q)) = 4$.
b) If $\dim\varphi = 6$ *and* φ *is an Albert form, Conjecture 2 holds and one even has* $C_F(q, \varphi) \leq 8$, *except perhaps in the same exceptional case as in a).*

Observe that in Theorem 2 and Theorem 3, the only instances where we cannot fully conclude are when $\dim q = 4$ (and $d_\pm q \neq 1$), confirming that this case is particularly difficult.

Theorems 2 and 3 follow from more precise results, which will be stated in Theorem 7.3. Roughly, we are able not only to prove the existence of the forms ψ appearing in Conjecture 2, but also to determine exactly those of smallest dimension, provided φ is not too complicated. (When φ is defined over F, ψ is usually unique, *cf.* [19, Lemma 3]. This is not the case in general, but all ψ of minimal dimension are at least of the same shape). It turns out that one is often in a "standard" situation, of the same type as the one in Proposition 4.5 below. Conversely, the fact that the exceptional case of Theorem 3 is the same as the one showing up in [15] is not a coincidence.

Unfortunately the proofs are not as simple as one might hope, and we are forced to go through tedious discussions involving cases and subcases. As in [19] for Conjecture 1, we hope that a more geometric understanding of Conjecture 2 will lead to a direct and general proof.

2. Notation and rappels

Most of the notations and definitions that we use are well-established (*cf.* inter alia [31], [42], [25, 26]). Let us only specify those which may not be standard:

(1) For $i \geq 0$, we denote by H^iF the Galois cohomology group $H^i(F, \mathbb{Z}/2)$.
(2) We denote by $d_\pm\varphi = ((-1)^{n(n-1)/2}\det\varphi) \in H^1F$ the *signed discriminant* of a quadratic form φ of dimension n (where $\det\varphi$ is the usual discriminant of φ), and by $c(q) \in H^2F$ its Clifford invariant. We also write $d_\pm = e^1$ and $c = e^2$.
(3) For $\varphi \in I^3F$, we denote by $e^3(\varphi) \in H^3F$ its Arason invariant [2].
(4) We denote by $D(\varphi)$ (*resp.* $G(\varphi)$) the set of values (*resp.* the group of similarity factors) of an anisotropic form φ.
(5) For two quadratic forms φ and ψ, we write $\varphi \sim \psi$ if $\varphi \perp -\psi$ is hyperbolic (Witt equivalence), and $\psi \leq \varphi$ if ψ is isometric to a sub-form of φ.
(6) We denote by φ_{an} the anisotropic part of a quadratic form φ.
(7) For $a_1, \dots, a_n \in F^*$, we denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$.
(8) We write $P_n(F)$ for the set of n-fold Pfister forms, $GP_n(F) = F^*P_n(F)$ and $GP(F) = \bigcup_{n\geq 1} GP_n(F)$.
(9) We shall usually abbreviate the expression *Pfister neighbour* [26, Def. 7.4] into *neighbour*.
(10) An *Albert form* is a 6-dimensional quadratic form with trivial signed discriminant. An Albert form may be isotropic.
(11) A *virtual Albert form* is an anisotropic 6-dimensional quadratic form which remains anisotropic on the quadratic extension given by its signed discriminant.
(12) For any quadratic form φ of dimension ≥ 2, we denote by X_φ the projective quadric of equation $\varphi = 0$ ($\dim X_\varphi = \dim\varphi - 2$) and by $F(\varphi)$ the function field of X_φ (if $\dim\varphi = 2$ and φ is isotropic, one therefore has $F(\varphi) = F \times F$).
(13) For any finite-dimensional central simple F-algebra A, we write $\mathrm{ind}(A)$ for the Schur index of A; $[A]$ for the class of A in H^2F; $SB(A)$ for the Severi-Brauer variety of A and $F(A)$ for the function field of $SB(A)$.
(14) If φ and φ' are two quadratic forms, we denote by $F(\varphi, \varphi')$ the function field of the product variety $X_\varphi \times_F X_{\varphi'}$. If A is a central simple F-algebra and φ is a quadratic form, we write $F(A, \varphi)$ for the function field of $SB(A) \times_F X_\varphi$. We have $F(\varphi, \varphi') =$

$F(\varphi)(\varphi') = F(\varphi')(\varphi)$, $F(A,\varphi) = F(A)(\varphi) = F(\varphi)(A)$. Similarly for more quadratic forms or more algebras.

(15) If φ and ψ are two anisotropic F-quadratic forms, we say that φ is *dominated by* ψ if $\psi_{F(\varphi)}$ is isotropic. Notation: $\varphi \preccurlyeq \psi$ or $\psi \succcurlyeq \varphi$. This is a preorder relation. One has: ($\varphi \leq a\psi$ for a scalar $a \in F^*$) $\Rightarrow$ ($\varphi \preccurlyeq \psi$).

(16) If $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$, we say that φ and ψ are *stably birationally equivalent*[1]. Notation: $\varphi \asymp \psi$. This is an equivalence relation.

(17) If φ is an F-quadratic form, we denote by $(\varphi_0, \varphi_1, \dots, \varphi_h)$ the sequence of its "higher kernel forms" in the sense of Knebusch [25, 26]. In particuliar, $\varphi_0 = \varphi_{\mathrm{an}}$ and $\varphi_1 = (\varphi_{F(\varphi_0)})_{\mathrm{an}}$. We write $(i_0(\varphi), i_1(\varphi), \dots, i_h(\varphi))$ for the sequence of the "higher Witt indices" of φ (this is sometimes called the splitting pattern of φ): in particular, $i_0(\varphi) = i_W(\varphi)$ (the classical Witt index) and $i_1(\varphi) = i_W(\varphi_{F(\varphi_0)})$. The integer h is the *height* of φ; if $\varphi \in IF$, the unique integer d such that $\varphi_{h-1} \in GP_d(F_{h-1})$ is the *degree* of φ, and the unique Pfister form which is determined by φ_{h-1} is called the *leading form* of φ. The height and degree are respectively denoted by $\mathrm{h}(\varphi)$ and $\deg(\varphi)$.

(18) [11, Def. 3.4] Let $1 \leq m < n$. We say that an anisotropic form φ of dimension 2^n is a *twisted* (n,m)*-fold Pfister form* if there exists $\sigma \in P_n(F) - \{0\}$ and $\pi \in P_m(F) - \{0\}$ such that $\varphi \sim \sigma \perp -\pi$. The form π is called the *twist* of φ. Notation: $\varphi \in P_{n,m}(F)$. We write $GP_{n,m}(F) = F^* P_{n,m}(F)$.

The Arason-Pfister *Hauptsatz* will be of constant use in the proofs. Recall its content:

2.1. **Theorem** (Arason-Pfister [3]). *If $\varphi \in I^n F$ is anisotropic, then $\dim \varphi \geq 2^n$. Moreover, if $\dim \varphi = 2^n$, then $\varphi \in GP_n(F)$.*

Recall also that, for $n \leq 3$, the cohomological invariant e^n induces an isomorphism

$$I^n F / I^{n+1} F \xrightarrow{\sim} H^n F \tag{2.1}$$

(Kummer for $n = 1$, Merkurjev [33] for $n = 2$, Rost [41] and Merkurjev-Suslin [35] for $n = 3$). We shall not need the fact that this result extends to all n in characteristic zero (Orlov-Vishik-Voevodsky [36]).

Finally, let us recall the statement of the index reduction theorems of Merkurjev [34], [45] and Schofield-van den Bergh [43]:

2.2. **Theorem.**

[1] Strictly speaking, this terminology should rather apply to X_φ and X_ψ.

(1) *Let q be an F-quadratic form, and D a division central F-algebra. Then* $\operatorname{ind}(D_{F(q)}) < \operatorname{ind}(D)$ *if and only if there exists a homomorphism $C_0(q) \to D$.*
(2) *Let D, A be two central simple F-algebras, and let $K = F(SB(A))$. Suppose that D is division. Then*

$$\operatorname{ind}(D_K) = \inf\{\operatorname{ind}(D \otimes_F A^{\otimes i}) \mid i \in \mathbb{Z}\}.$$

3. Cohomological kernels

Certain cohomological results will play a central rôle in the proofs. For any extension K/F and any integer $n \geq 0$, define

$$H^n(K/F) = \operatorname{Ker}(H^n F \to H^n K).$$

In degree 2, one has:

3.1. **Theorem.**
(1) *(Amitsur* [1]*) For any central simple F-algebra A of exponent 2, $H^2(F(A)/F) = \{0, [A]\}$.*
(2) *(Witt, Arason* [2]*) For any F-quadratic form q of dimension ≥ 4, one has $H^2(F(q)/F) = 0$, except if $q \in GP_2(F)$; in this case the kernel is generated by $e^2(q)$.*

In degree 3, there is a general theorem due to Peyre [39] concerning the function fields of projective homogeneous varieties. We shall only need special cases of this theorem, first of all for quadrics and Severi-Brauer varieties. In the first case, there is a previous, more precise result, due to Arason:

3.2. **Theorem** (Arason [2, Satz 5.6]). *Let q be an F-quadratic form of dimension ≥ 3, and $\alpha \in H^3(F(q)/F) - \{0\}$. Then there exists $\tau \in GP_3(F)$ such that*
(i) *$e^3(\tau) = \alpha$,*
(ii) *$q \leq \tau$.*

One deduces from this:

3.3. **Lemma.** *Let q be an anisotropic F-quadratic form of dimension 4 and discriminant d. Write $E = F(\sqrt{d})$ and let $N : E^* \to F^*$ be the norm map. Then*

$$H^3(F(q)/F) = \{e^3(\langle\langle N(x) \rangle\rangle \otimes q) \mid x \in E^*\}.$$

Proof. For $x \in E^*$, the form $\langle\langle N(x) \rangle\rangle \otimes q$ has trivial Clifford invariant, hence belongs to $GP_3(F)$. So $\{e^3(\langle\langle N(x) \rangle\rangle \otimes q) \mid x \in E^*\} \subset H^3(F(q)/F)$. Conversely, let $\alpha \in H^3(F(q)/F) - \{0\}$ and τ be as in the conclusion of Theorem 3.2. If one writes $\tau \simeq q \perp \psi$, ψ_E is similar to q_E, hence $\psi \simeq -aq$ for a suitable $a \in F^*$ by a theorem of Wadsworth [47]. But the form $q \perp -aq$ is in I^3F if and only if $c(q \perp -aq) = (a, d) = 0$, that is, $a \in N(E^*)$. ∎

3.4. **Lemma.** *Keep the same notation as in Lemma 3.3, and suppose $q \notin GP_2(F)$. Let ρ be a quadratic form such that $\rho_{F(q)} \sim q_{F(q)}$. Then $\rho \equiv N(x)q \pmod{I^4F}$ for some $x \in E^*$.*

Proof. Since $H^i(F(q)/F) = \{0\}$ for $i = 1, 2$, one has $q \perp -\rho \in I^3F$ and therefore $e^3(q \perp -\rho) \in H^3(F(q)/F)$. By Lemma 3.3, there exists $x \in E^*$ such that

$$q \perp -\rho \equiv \langle\langle N(x) \rangle\rangle \otimes q \pmod{I^4F}.$$

Hence the result. ∎

3.5. **Theorem** (Peyre [38]). *Let A be a central simple F-algebra of exponent 2. Then*

(1) *There is an injection* $\dfrac{H^3(F(A)/F)}{[A] \cdot H^1F} \hookrightarrow {}_2CH^2(SB(A))$, *where $CH^2(X)$ is the codimension 2 Chow group of a smooth variety X.*
(2) *For* $\operatorname{ind}(A) \leq 4$, ${}_2CH^2(SB(A)) = 0$.

3.6. **Theorem** (Karpenko [23]). *One also has ${}_2CH^2(SB(A)) = 0$ if A is a tensor product of three quaternion algebras.*

3.7. **Corollary.** $H^3(F(A)/F) = [A] \cdot H^1F$ *if A is a tensor product of at most three quaternion algebras.* ∎

We shall also need the case of a product of a quadric by a Severi-Brauer variety:

3.8. **Theorem** (Izhboldin-Karpenko [16, Prop. 2.1]). *Let A be a central simple F-algebra of exponent 2 and q an F-quadratic form of dimension ≥ 3. Then there is an isomorphism*

$$\frac{H^3(F(q,A)/F)}{H^3(F(q)/F) + H^3(F(A)/F)} \simeq \frac{CH^2(X_q \times SB(A))_{\mathrm{tors}}}{CH^2(X_q)_{\mathrm{tors}} + CH^2(SB(A))_{\mathrm{tors}}}.$$

Unfortunately, the results of the next theorem cannot always be derived directly from this general statement.

3.9. **Theorem.** *Let A and q be as in Theorem 3.8.*

(1) *(Izhboldin-Karpenko* [14, Theorem 4]*) Suppose that A is a division algebra of degree 8, and that* $\dim q \geq 5$*. If* $\mathrm{ind}(A_{F(q)}) < \mathrm{ind}(A)$*, then*
$$H^3(F(q,A)/F) = [A] \cdot H^1F.$$
(2) *(Izhboldin-Karpenko* [16, Th. 4.1 and Prop. 6.8]*) Suppose* $\mathrm{ind}(A) \leq 4$ *and* $\dim q = 4$*,* $d = d_\pm q \neq 1$*. Let* $E = F(\sqrt{d})$*. Then*
$$CH^2(X_q \times SB(A))_{\mathrm{tors}} = 0$$
and thus
$$H^3(F(q,A)/F) = H^3(F(q)/F) + H^3(F(A)/F)$$
except perhaps if $\mathrm{ind}(C(q_E) \otimes A_E) = 2$ *or* 4*. In the first case, the quotient* $\dfrac{H^3(F(q,A)/F)}{H^3(F(q)/F) + H^3(F(A)/F)}$ *is generated by* $e^3(\rho)$ *for any form* $\rho \in I^3F$ *of type*
$$\rho = kq + q'' + l\gamma$$
where γ is an Albert form verifying $c(\gamma) = [A]$*,* $k, l \in F^*$ *and* q'' *is a 4-dimensional form verifying* $d_\pm q'' = d$*,* $c(q''_E) = c(q_E) + [A_E]$*.*
(3) *(Izhboldin-Karpenko* [15, Theorem 3.1]*) Suppose that* $q = \langle -a, -b, ab, d\rangle$ *with* $d \neq 1$ *and that* $A = (a,b) \otimes_F (u,v) \otimes_F (d,s)$ *is a division algebra. Then*
$$H^3(F(q,A)/F) = H^3(F(q)/F) + [A] \cdot H^1F.$$

4. Conjugate forms

4.1**. Definition** ([26, Def. 8.7])**.** Two anisotropic F-forms φ and φ' are *conjugate* if $\dim\varphi = \dim\varphi'$ and $\varphi \perp -\varphi' \in GP(F)$. We denote this relation by $\varphi \approx \varphi'$.

If two forms are conjugate, their common dimension is a power of 2.

4.2**. Lemma.** *Let φ, ψ be two anisotropic F-forms, with $\varphi \not\approx \psi$ and $\dim\psi \leq \dim\varphi$. Suppose that $(\psi \perp -\varphi)_{F(\varphi)} \sim 0$. Then, either φ is a Pfister neighbour with complementary form ψ, or φ and ψ are conjugate.*

Proof. We have the inequality $\dim(\psi \perp -\varphi) - 2^{\deg(\psi\perp-\varphi)} < 2\dim\varphi$. Therefore, we are in the position to apply Fitzgerald's theorem [8, Th. 1.6], which states that $\psi \perp -\varphi \in GP(F) - \{0\}$. The conclusion follows immediately. ∎

4.3. **Proposition.** *a) For two anisotropic F-forms φ, φ' of dimension 2^n, the following conditions are equivalent:*

(i) $\varphi \approx \varphi'$.
(ii) $\varphi \perp -\varphi' \in I^{n+1}F$.
(iii) $(\varphi \perp -\varphi')_{F(\varphi)} \sim 0$.
(iv) $\varphi \asymp \varphi'$ *and* $\varphi_K \sim \varphi'_K$ *for any extension* K/F *such that* φ_K *and* φ'_K *are isotropic.*

b) The relation of conjugation is an equivalence relation.

Proof. a) (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) follows from the Hauptsatz. (iii) $\Rightarrow$ (i) follows from Lemma 4.2. (i) $\iff$ (iv) is [26, Th. 8.8].

b) follows from the equivalence between (i) and (ii) in a). ■

4.4. **Proposition.** *Let $\varphi \in GP_{n,n-1}(F)$, of twist π. For another F-quadratic form φ', the following conditions are equivalent:*

(i) $\varphi \approx \varphi'$.
(ii) $\varphi' \simeq a\varphi$ *with* $a \in G(\pi)$.

Proof. One has $\varphi \equiv -\pi \pmod{I^nF}$. (ii) $\Rightarrow$ (i) Since $\langle\langle a \rangle\rangle \otimes\pi \sim 0$, one has $\langle\langle a \rangle\rangle \otimes\varphi \in I^{n+1}F$, that is, $\varphi \perp -\varphi' \in I^{n+1}F$. By Proposition 4.3, one has $\varphi \approx \varphi'$. In the other direction, one has $\varphi' \simeq a\varphi$ by [12, Cor. 3.4, Th. 3.4] (see [27, Th. 3] for $n = 3$). Hence

$$\varphi \perp -\varphi' \equiv \langle\langle a \rangle\rangle \otimes\varphi \equiv -\langle\langle a \rangle\rangle \otimes\pi \pmod{I^{n+1}F}.$$

By the Hauptsatz, $\langle\langle a \rangle\rangle \otimes\pi \sim 0$. ■

4.5. **Proposition.** *Let q be an anisotropic F-quadratic form, $K = F(q)$ and $q_1 = (q_{F(q)})_{\mathrm{an}}$. Then,*

$$C_F(q, q_1) = \begin{cases} \dim q_1 & \text{if } q \text{ is a neighbour} \\ \dim q & \text{otherwise.} \end{cases}$$

In the first case, the complementary form of q is the unique F-form ψ of dimension $< \dim q$ such that $\psi_K \simeq -q_1$. In the second case, the set of those F-forms ψ such that $\dim\psi \le \dim q$ and $\psi_K \sim q_1$ is reduced to $\{q\}$ if $\dim q$ is not a power of 2, and equals the set of F-forms conjugate to q otherwise.

Proof. If q is a neighbour of complementary form ψ, then q_1 is defined over F by the form $-\psi$ by a theorem of Hoffmann [10, Theorem 1]. Moreover, let ψ' be another F-form such that $\psi'_K \simeq q_1$. Then $(\psi \perp \psi')_K \sim 0$; if $\psi \perp \psi' \nsim 0$, one has $\dim(\psi \perp \psi') \ge \dim q + \dim q_1$ by [19, Lemma 2]. This implies $\dim\psi' \ge \dim q$.

Suppose now that q is not a neighbour. Let ψ be an anisotropic F-form of minimal dimension such that $\psi_K \sim q_1$. One has $\dim\psi \leq \dim q$ and $(\psi \perp -q)_K \sim 0$. If $\dim q$ is not a power of 2, then $q \simeq \psi$ by Lemma 4.2. Otherwise, one reapplies Lemma 4.2 to deduce that ψ is conjugate to q; conversely, any F-form conjugate to q is a solution, thanks to the equivalence between (i) and (iii) in Proposition 4.3 a). ∎

5. Lower bounds

5.1. **Lemma.** *a) For any $m \geq 2$ and $n \geq 1$, $C(m,n) \geq n$.*
b) For any $n \geq 1$, $C(2,n) = C(3,n) = n$. If $q \in GP_2(F)$, then $C_F(q,n) = n$.
c) One has $C(m,n) = n$ for any $m > 2n$ as soon as Conjecture 1 is verified.

Proof. a) and c) are obvious; b) follows from the excellence of the function field of a quadratic form of dimension ≤ 3 or of a 2-fold Pfister form. ∎

5.2. **Proposition.** *Let $m \geq 2$ and $k, l \geq 1$ be integers. Then $C(m,k+l) \geq C(m,k) + C(m,l)$ and $C(k+l) \geq C(k) + C(l)$.*

Proof. Let q be an F-quadratic form of dimension m, $K = F(q)$ and $\varphi', \varphi'' \in \operatorname{Im}(W(F) \longrightarrow W(K))$ be anisotropic of respective dimensions k and l. Let $L = F((t))$ be the field of formal power series in t over F.

Set $\varphi = \varphi' \perp t\varphi''$. By Springer's theorem, φ is anisotropic. The statement will therefore follow from the inequality

$$C_L(q_L, \varphi) \geq C_F(q, \varphi') + C_F(q, \varphi''). \tag{5.1}$$

We have $\varphi \in \operatorname{Im}(W(L) \longrightarrow W(K \cdot L))$. Let $\eta \in W(L)$ be anisotropic and such that $\varphi \sim \eta_{K\cdot L}$. Write $\eta \simeq \eta' \perp t\eta''$ with $\eta', \eta'' \in W(F)$. Taking residue forms, we get:

$$\varphi' \sim \eta'_K \quad \text{and} \quad \varphi'' \sim \eta''_K.$$

The forms η' and η'' are anisotropic. Therefore

$$\dim \eta' \geq C_F(q, \varphi')$$
$$\dim \eta'' \geq C_F(q, \varphi'')$$

and

$$\dim \eta \geq C_F(q, \varphi') + C_F(q, \varphi'').$$

But this is true for any form η verifying $\varphi \sim \eta_{K\cdot L}$, hence (5.1). ∎

5.3**. Lemma.** *We have*
a) $C(m,1)=1$ *for any* $m \geq 2$.
b) $C(4,2) \geq 4$.

Proof. a) follows from [19, Theorem 2 (a)] for $m > 2$ and from Lemma 5.1 b) for $m = 2$. b) Let F, q be such that q is anisotropic over F, $\dim q = 4$ and $d_{\pm} q \neq 1$. Then q is not a neighbour: the statement therefore follows from Proposition 4.5. ∎

Proof of Theorem 1. For $n = 1$ the statement is trivial. Suppose $n \geq 2$. If n is even, write it $n = 2k$; the statement then follows from Lemme 5.3 iterated k times and Proposition 5.2. If n is odd, write it $n = n' + 1$; the statement follows from the even case and Proposition 5.2. ∎

6. The homomorphism $W(F) \to W_{\mathrm{nr}}(K/F)$

6.1**. Proposition.** *Let* q, q' *be two anisotropic* F*-quadratic forms of dimension* ≥ 3 *such that* $q \preccurlyeq q'$, *and let* $K = F(q), K' = F(q')$.
a) There is a commutative diagram

$$\begin{tikzcd} & W_{\mathrm{nr}}(K/F) \\ W(F) \arrow[ur] \arrow[dr] & \\ & W_{\mathrm{nr}}(K'/F). \arrow[uu] \end{tikzcd}$$

If moreover $q' \preccurlyeq q$, *the vertical arrow in this diagram is an isomorphism.*
b) Let $\varphi' \in W_{\mathrm{nr}}(K'/F)$, *and let* φ *be an anisotropic representative of the image of* φ' *in* $W_{\mathrm{nr}}(K/F)$ *by the map of a). Let* ψ *be an* F*-quadratic form such that* $\varphi \sim \psi_K$. *Then* $\varphi' \sim \psi_{K'}$ *in the following cases:*
(i) $q' \asymp q$ *(cf. Notation 2.16).*
(ii) *The following inequality holds:*

$$\dim \varphi + \dim \psi < 2(\dim q - i_1(q)).$$

Proof. a) can be proven as [20, Prop. 2.5 c)] by noting that the composite extension $K \cdot K'/K$ is purely transcendental.

b) Observe that $\dim \varphi \leq \dim \varphi'$. If $\varphi \sim \psi_K$, then $(\psi_{K'} \perp -\varphi')_{K \cdot K'}$ is hyperbolic. In the first case, the extension $K \cdot K'/K'$ is purely transcendental, hence $\psi_{K'} \perp -\varphi'$ is hyperbolic. In the second case this form is still hyperbolic, as one sees by applying [19, Lemma 2]. ∎

6.2. **Corollary.** *Conjectures 1 and 2 only depend on the stable birational equivalence class of q.* ■

6.3. **Remark.** The integer $\dim q - i_1(q) = \frac{1}{2}(\dim q + \dim q_1)$ is a stable birational invariant of q, by a result of Vishik and Karpenko [46, Cor. A.18], [24, Th. 8.1].

6.4. **Theorem.** *Let q be an F-quadratic form and $K = F(q)$.*
a) If $\dim q = 3$, the homomorphism $W(F) \to W_{\mathrm{nr}}(K/F)$ is surjective. This is also true if q is similar to a 2-fold Pfister form.
b) If F is of characteristic 0 and if q is a neighbour, the homomorphism $W(F) \to W_{\mathrm{nr}}(K/F)$ is surjective.
c) Suppose that q is an Albert form: $(q_K)_{\mathrm{an}}$ is therefore of the form $\alpha\tau$, with $\alpha \in K^*$ and $\tau \in P_2(K)$. Then $\tau \in W_{\mathrm{nr}}(K/F)$, but $\tau \notin \mathrm{Im}(W(F) \to W_{\mathrm{nr}}(K/F))$.

Proof. The first statement of a) is due to Parimala (*cf.* [4, Lemma 3.1]); the second one follows from Proposition 6.1 a). b) is [22, Th. 4]. Finally, c) is stated in [18, Remark 3 p. 249]: since this reference is sparing with details, let us give a proof for the reader's convenience.

Since $c(\tau) = c(q)_K$, τ is unramified (its residues are binary forms with trivial discriminants). It is now sufficient to show that $\varphi = \langle\langle \alpha \rangle\rangle \otimes \tau \notin \mathrm{Im}(W(F) \to W_{\mathrm{nr}}(K/F))$, since $\alpha\tau$ is in this image. Suppose that $\varphi \sim \psi_K$ for some F-form ψ. As $\varphi \in I^3K$ and $H^i(K/F) = \{0\}$ for $i = 1, 2$ (since $\dim q > 4$), necessarily $\psi \in I^3F$. Then $e^3(\varphi) = e^3(\psi)_K$. But this contradicts the fact that $e^3(\varphi) \in H^3K$ is not defined on F (*cf.* [18, proof of Th. 2 d), middle p. 249]). ■

6.5. **Question.** *For $K = F(q)$, is it true that $W(F) \to W_{\mathrm{nr}}(K/F)$ is surjective if $\dim q \in \{4, 5\}$?*

At least $W(F) \to \dfrac{W_{\mathrm{nr}}(K/F)}{I^5K \cap W_{\mathrm{nr}}(K/F)}$ is surjective in these cases: this follows from [20, Th. 4, 5 and 9] and [21, Th. 3].

7. Detailed statements of results

7.1. **Definition.** Let q, K, φ be as in Conjecture 2. We write

$$\mathrm{Desc}(\varphi) = \{\psi \in W(F) \mid \psi_K \sim \varphi\}$$
$$\mathrm{Desc}_0(\varphi) = \{\psi \in \mathrm{Desc}(\varphi) \mid \dim \psi = \dim \varphi\}.$$

An element of $\mathrm{Desc}(\varphi)$ is called a *Witt descent of* φ; an element of $\mathrm{Desc}_0(\varphi)$ is called a *descent of* φ.

By definition, φ is defined over F if and only if $\mathrm{Desc}_0(\varphi) \neq \emptyset$. Any element of $\mathrm{Desc}(\varphi)$ of minimal dimension is automatically anisotropic.

By Lemma 5.1 b), φ is always defined over F if $\dim q \leq 3$ or $q \in GP_2(F)$. This allows us to limit ourselves to $\dim q \geq 4$, $q \notin GP_2(F)$ to prove Theorems 2 and 3. *We shall make this assumption henceforth without further comments.*

7.2. **Lemma.** *Let $\psi, \psi' \in \mathrm{Desc}(\varphi)$. Then $d_\pm \psi = d_\pm \psi'$ and $c(\psi) = c(\psi')$.*

Proof. The form $\psi \perp -\psi'$ belongs to $W(K/F)$. Since the extension K/F is regular, the first claim is obvious. The assumption on q implies that $\mathrm{Br}(F) \to \mathrm{Br}(K)$ is injective (Theorem 3.1), hence the second claim. ∎

Let q, K, φ be as in Conjecture 2. We choose $\psi \in \mathrm{Desc}(\varphi)$ of minimal dimension. We denote by D the central division F-algebra such that $[D_K] = c(\varphi)$. Moreover, we set

$$L = F(D), n = \dim \varphi, d = d_\pm \psi, d' = d_\pm q,$$
$$q_1 = (q_{F(q)})_{\mathrm{an}}, E = F(\sqrt{d}) \text{ and } E' = F(\sqrt{d'}).$$

By Lemma 7.2, d and D are independent of the choice of ψ.

When $q \in GP_{2,1}(F)$ or $GP_{3,2}(F)$, we denote its twist by π. In the tableaux below, the expression "Other ψ" means the other members of $\mathrm{Desc}(\varphi)$ of minimal dimension.

7.3. **Theorem.** *Suppose that φ is not defined over F. Then*
a) One has $n > 1$.
b) For $n = 2$, we are in the following situation:

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	$q \in GP_{2,1}(F)$	$\varphi \simeq aq_1$, $a \in F^*$	aq	abq, $b \in G(\pi)$	4

c) For $n = 3$, we are in one of the following situations:

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	$q \in GP_{2,1}(F)$	$\varphi \simeq aq_1 \perp \langle c \rangle$, $a, c \in F^*$, $-ac \notin G(\pi)D(q)$	$aq \perp \langle c \rangle$	$abq \perp \langle c \rangle$, $b \in G(\pi)$	5
5	*not a neighbour*	$\varphi \simeq aq_1$, $a \in F^*$	aq	*no*	5

d) For $n = 4, \varphi \in GP_2(K)$, we are in one of the following situations:

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	$q \in GP_{2,1}(F)$	$\varphi \sim aq_1 \perp \theta_K$, $a \in F^*, \dim\theta = 4$	$aq \perp \theta$	*conjugate or Witt-equivalent to* $abq \perp \theta$, $b \in G(\pi)$	8 *if* $-a \notin G(\pi)D(q)D(\theta)$ 4 *or* 6 *else*
4 *or* 5	$q \leq \gamma$, γ *Albert anisotropic*	$\varphi \sim a\gamma_K$, $a \in F^*$	$a\gamma$	$ab\gamma$, $b \in G(\varphi)$	6
6	*Albert*	$\varphi \simeq aq_1$, $a \in F^*$	aq	*no*	6
5, 7 *or* 8	*not a neighbour,* $q \leq q'$, $q' \in GP_{3,2}(F)$ *anisotropic*	$\varphi \sim aq'_K$, $a \in F^*$, $\psi_L \notin I^4L$ *if* $\dim q = 5$	aq'	abq', $b \in G(\pi)$	8

e) For $n = 4$ and $\varphi \notin GP_2(K)$, we have $\mathrm{ind}(D_E) \in \{2, 4\}$*, and we are in one of the following situations:*

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	$q \in G_{2,1}F$,	$\mathrm{ind}(D_E) = 2$	$a\langle 1, -d\rangle \perp \tau$, *or* $a\langle 1, -d\rangle \perp bq \perp \delta$, $a, b \in F^*$, $\tau \in GP_2(F)$ $\dim\delta = 4$	?	≤ 10
4	$q \in GP_{2,1}(F)$, $q \leq \gamma$, γ *virtual Albert*	$\mathrm{ind}(D_E) = 4$, $\varphi \sim a\gamma_K$, $a \in F^*$	$a\gamma = aq \perp \theta$, $\dim\theta = 2$	$abq \perp \theta$, $b \in G(\pi)$	6
5	*not a neighbour*	$\varphi \simeq aq_1 \perp \langle b\rangle$, $a, b \in F^*$, $-ab \notin D(q)$	$aq \perp \langle b\rangle$	*no*	6
6	*neither Albert nor neighbour*	$\varphi \simeq aq_1$, $a \in F^*$	aq	*no*	6

f) For $n = 5$, φ a neighbour, we have $\varphi \simeq \tau \perp \langle d\rangle$ with $\tau \in GP_2(K)$. Then we are in one of the following situations[2]*:*

[2]In column 4 of rows 1 and 4, we only give those other forms ψ representing $d_\pm\varphi$.

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	$q \in GP_{2,1}(F)$	$\tau \sim aq_1 \perp \theta_K$, $a \in F^*, \dim \theta = 4$, $-a \notin G(\pi)D(q)D(\theta)$	$aq \perp \theta \perp \langle d\rangle$	$q' \perp \langle d\rangle$, $q' \approx abq \perp \theta$ *or* $q' \sim abq \perp \theta$, $b \in G(\pi)$	7 *or* 9
4 *or* 5	$\gamma := q \perp \xi$ *anisotropic Albert*	$\tau \sim a\gamma_K$, $a \in F^*$ $-ad \notin D(\gamma)$ *if* $\dim q = 5$; $-ad \notin D(bq \perp \xi)\ \forall b \in G(\pi)$ *if* $\dim q = 4$	$a\gamma \perp \langle d\rangle$	$abq \perp a\xi \perp \langle d\rangle$, $b \in G(\pi)$ *if* $\dim q = 4$; *no* *if* $\dim q = 5$	7
6	*Albert*	$\tau \simeq aq_1$, $-ad \notin D(q)$	$aq \perp \langle d\rangle$	*no*	7
5, 7 *or* 8	*not a neighbour,* $q \leq q'$, $q' \in GP_{3,2}(F)$ *anisotropic*	$\tau \sim aq'_K$, $a \in F^*$, $(\psi \perp \langle -d\rangle)_L \notin I^4L$ *if* $\dim q = 5$	$aq' \perp \langle d\rangle$	$q'' \perp \langle d\rangle$, $q'' \approx abq'$, $b \in G(\pi)$	7 *or* 9

g) For $n = 5$, φ not a Pfister neigbour, $\gamma = \varphi \perp \langle -d\rangle$ is an anisotropic Albert form. If $\operatorname{ind}(D) = 4$, let γ' be an Albert form such that $c(\gamma') = [D]$.

g.1) Suppose that γ is defined over F, and let $\gamma_0 \in \operatorname{Desc}_0(\gamma)$. Then:

(1) *$-d \notin D(\gamma_0)$, hence $\gamma_0 \perp \langle d\rangle$ is a descent of φ of minimal dimension, and $C_F(q,\varphi) = 7$.*
(2) *If $\dim q \geq 5$, then $\gamma_0 \perp \langle d\rangle$ is the unique descent of φ of minimal dimension.*
(3) *If $\dim q = 4$ and $d_\pm q \neq 1$, then a descent ψ of φ of minimal dimension satisfies $\gamma_0 \perp \langle d\rangle \simeq \psi$ or there exists $\rho \in GP_3(F) - \{0\}$ such that $q \preccurlyeq \rho$, $\rho \simeq \rho_1 \perp \rho_2$ and $\gamma_0 \perp \langle d\rangle \simeq \rho_1 \perp \rho_3$ with $\dim \rho_1 = 4$, and $\psi \simeq \rho_2 \perp -\rho_3$. In the case $d \in D(\psi)$, one has $\psi \simeq a\gamma_0 \perp \langle d\rangle$ for some $a \in F^*$.*

g.2) Suppose that γ is not defined over F. Then, we are in one of the following situations [3]:

[3] In column 4 we only give those other forms ψ representing $d_\pm\varphi$.

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	*not a neighbour*	$\mathrm{ind}(C_0(q)\otimes D) =$ $\mathrm{ind}(D) = 4$	?	?	?
4	$q \in GP_{2,1}F$	$\gamma \sim aq_1 \perp \theta_K$, $a \in F^*$, $\mathrm{ind}(D) = 4$, $\mathrm{ind}(C_0(q)\otimes D) = 2$, $c(\theta_{E'}) = c(q_{E'}) + [D_{E'}]$, $\dim\theta = 4$, $-a \notin G(\pi)D(q)D(\theta)$	$aq \perp \theta \perp \langle d\rangle$	$q' \perp \langle d\rangle$, $q' \approx acq \perp \theta$, $c \in G(\pi)$	7 *or* 9
4	$q \in G_{2,1}(F)$	$\gamma \simeq aq_1 \perp \theta_K$, $a \in F^*$, $\dim\theta = 4$ $\mathrm{ind}(C(aq \perp \theta)) = 8$	$aq \perp \theta \perp \langle d\rangle$	$q' \perp \langle d\rangle$, $q' \approx acq \perp \theta$, $c \in G(\pi)$	7 *or* 9
5 *to* 8	*not a neighbour*, $q \leq q'$, $q' \in I^2F$, $\dim q' = 8$, q_L *anisotropic* $\mathrm{ind}\, C(q') = 4$	$\gamma \sim aq'_K$, $a \in F^*$ $\mathrm{ind}(D) = 4$, $e^3(\psi \perp -\gamma')_L \neq 0$	$aq' \perp \langle d\rangle$,	$q'' \perp \langle d\rangle$, $q'' \approx aq'$	7 *or* 9
5 *to* 8	*not a neighbour*, $q \leq q'$, $q' \in I^2F$, $\dim q' = 8$, $\mathrm{ind}\, C(q') = 8$	$\gamma \sim aq'_K$, $a \in F^*$	$aq' \perp \langle d\rangle$,	$q'' \perp \langle d\rangle$, $q'' \approx aq'$	7 *or* 9

h) For $n = 6$, φ *an Albert form, we have* $\mathrm{ind}(D) \in \{4, 8\}$. *If* $\mathrm{ind}(D) = 4$, *let* γ *be an Albert form such that* $D \sim C(\gamma)$. *Then we are in one of the following situations:*

0	1	2	3	4	5
$\dim q$	conditions on q	conditions on φ	value of ψ	other ψ	$C_F(q,\varphi)$
4	*not a neighbour*	$\mathrm{ind}(C_0(q)\otimes D) =$ $\mathrm{ind}(D) = 4$	?	?	?
4	$q \in GP_{2,1}F$	$\varphi \sim aq_1 \perp bq''_K$ $a, b \in F^*$, $\mathrm{ind}(D) = 4$, $\mathrm{ind}(C(q_{E'})\otimes D_{E'}) = 2$, $c(q''_{E'}) = c(q_{E'}) + [D_{E'}]$, $\dim q'' = 4$, $-ab \notin G(\pi)D(q)D(q'')$	$aq \perp bq''$	*conjugate to* $acq \perp bq''$, $c \in G(\pi)$	8
4	$q \in G_{2,1}(F)$	$\varphi \simeq aq_1 \perp \theta_K$, $a \in F^*$, $\dim\theta = 4$, $\mathrm{ind}(C(aq \perp \theta)) = 8$	$aq \perp \theta$	*conjugate to* $abq \perp \theta$, $b \in G(\pi)$	8
5 *to* 8	*not a neighbour,* $q \le q'$, $q' \in I^2F$, $\dim q' = 8$, q_L *anisotropic,* $\mathrm{ind}(C(q')) = 4$	$\varphi \sim aq'_K$, $a \in F^*$ $e^3(\psi \perp -\gamma)_L \neq 0$	aq'	*conjugate to* aq'	8
5 *to* 8	*not a neighbour,* $q \le q'$, $q' \in I^2F$, $\dim q' = 8$, $\mathrm{ind}\, C(q') = 8$	$\varphi \sim aq'_K$, $a \in F^*$	aq'	*conjugate to* aq'	8

7.4. **Remarks.** a) In column 2 of row 4 in the tableau of g.2), the condition given on ψ amounts to saying that $\psi \not\equiv b\gamma' \pmod{I^4F}$ for any $b \in F^*$, thanks to Corollary 3.7. Moreover, this condition is independent of the choice of ψ, because for $\psi' \in \mathrm{Desc}(\varphi)$ we get $\psi \equiv \psi' \pmod{I^4F}$ since q is not a Pfister neighbour of dimension ≥ 5 and hence $H^i(K/F) = \{0\}$ for $i \leq 3$.
b) The same justification as in a) shows that the condition $\psi_L \notin I^4L$ in column 2 of row 4 in the tableau of d) is also independent of the choice of ψ.

For the proof of Theorem 7.3, we first establish a few more technical results in the next section. We then prove Theorem 7.3 in the following order, relatively to the type of φ: forms in $GP_2(K)$; Albert forms; 5-dimensional forms; and finally 4-dimensional forms not in $GP_2(K)$. Each case uses the results established for the previous cases.

8. More lemmas

The following lemma will be needed to justify the "uniqueness" statements (column 4) in the proof of Theorem 7.3.

8.1. **Lemma.** *Let q, K, φ be as in Conjecture 2, and let $\psi, \psi' \in \mathrm{Desc}(\varphi)$. Then*

(i) *If $\dim q \geq 5$ and q is not a neighbour, then $\psi \equiv \psi' \pmod{I^4F}$. If furthermore, $\dim \psi' \leq \dim \psi \leq 8$, then $\psi \approx \psi'$ or $\psi \simeq \psi'$ according as $\dim \psi = 8$ or not.*

(ii) *If $\dim q = 4$ and $d_{\pm} q \neq 1$, then $\psi \equiv \psi' \pmod{I^3F}$.*

Proof. From Theorems 3.1 and 3.2 we deduce that $H^i(K/F) = \{0\}$ for $i \leq 2$ (*resp.* for $i \leq 3$) when $\dim q = 4$ (*resp.* $\dim q \geq 5$). We finish the proof by using the bijectivity of e^2 and e^3, respectively. ∎

The following proposition is essentially contained in [27].

8.2. **Proposition.** *Let γ be a possibly isotropic Albert form over F, and let L denote the field $F(C(\gamma))$ (L is a generic splitting field for γ). Let ψ be a quadratic form such that $\psi \equiv \gamma \pmod{I^3F}$. Suppose that $\psi_L \in I^4L$. Then there exists a scalar $a \in F^*$ such that $\psi \equiv a\gamma \pmod{I^4F}$. If $\dim \psi < 10$, one has $\psi \sim a\gamma$.*

Proof. Consider the Arason invariant $e = e^3(\psi \perp -\gamma) \in H^3F$. By [27, corollaire 6], one has $e = (a) \cdot c(\gamma) = e^3(\langle a, -1\rangle \otimes \gamma)$ for some $a \in F^*$. Hence, by (2.1)

$$\psi \perp -\gamma \equiv a\gamma \perp -\gamma \pmod{I^4F}$$

and the claim follows. The last assertion follows from the Hauptsatz. ∎

8.3. **Proposition.** *Let K/F be a finitely generated extension and $\varphi \in \mathrm{Im}(W(F) \to W(K)) + I^nK$ be a form of dimension $< 2^{n-1}$. Then $\varphi \in W_{\mathrm{nr}}(K/F)$.*

Proof. Same as [19, Proposition 1 (a)]. ∎

The following technical corollary will be needed to handle the case where $\dim \varphi = 4$ in Theorem 7.3.

8.4. **Corollary.** *Let $\varphi \in I^3F$, q be a 4-dimensional form and $\tau \in GP_2(F)$. We assume $\varphi_{F(\tau,q)} \sim 0$. Then there exists a form θ of dimension 4 such that $c(q) \in \{c(\theta), c(\tau) + c(\theta)\}$ and*

(i) *If $c(q) = c(\theta)$, then $\pi := -q \perp \theta \in GP_3(F)$ and $\varphi \perp \pi \perp \tau \perp \alpha\tau \in I^4F$ for some $\alpha \in F^*$. In this case, $\pi_{F(q)} \sim 0$.*

(ii) *If $c(q) = c(\tau) + c(\theta)$, then $\varphi \perp \beta(-q \perp \theta) \perp \tau \in I^4F$ for some $\beta \in F^*$.*

Proof. By assumption one has $e^3(\varphi)_{F(\tau)} \in H^3(F(\tau, q)/F(\tau))$. By Lemma 3.3, $e^3(\varphi)_{F(\tau)} = e^3(-q \perp aq)$ for some $a \in F(\tau)^*$. By the bijectivity of e^3, one has $\varphi_{F(\tau)} \perp -q_{F(\tau)} \perp aq \in I^4F(\tau)$. Proposition

8.3 implies that $aq \in W_{\mathrm{nr}}(F(\tau)/F)$. By Theorem 6.4 b), $W(F) \to W_{\mathrm{nr}}(F(\tau)/F)$ is surjective; hence, by the excellence of $F(\tau)/F$, there exists $\theta \in W(F)$ of dimension 4 such that $aq \simeq \theta_{F(\tau)}$. Since $-q \perp aq \in GP_3(F(\tau))$ one gets $c(q)_{F(\tau)} = c(aq) = c(\theta)_{F(\tau)}$, hence $c(q) \in \{c(\theta), c(\tau) + c(\theta)\}$. We distinguish two cases:

(i) If $c(q) = c(\theta)$, then $\pi := -q \perp \theta \in GP_3(F)$ and $\pi_{F(q)} \sim 0$. Also, $e^3(\varphi) + e^3(-q \perp \theta) \in H^3(F(\tau)/F)$. By Theorem 3.2, there exists $\alpha \in F^*$ such that $e^3(\varphi) + e^3(-q \perp \theta) = e^3(\tau \perp \alpha\tau)$. So $\varphi \perp \pi \perp \tau \perp \alpha\tau \in I^4F$.

(ii) If $c(q) = c(\tau) + c(\theta)$. Then $-q \perp \theta \perp \tau \in I^3F$, hence $e^3(\varphi) + e^3(-q \perp \theta \perp \tau) \in H^3(F(\tau)/F)$. One reapplies Theorem 3.2 to find $\beta \in F^*$ such that $e^3(\varphi) + e^3(-q \perp \theta \perp \tau) = c(-q \perp \theta) \cdot (\beta)$ (because $c(\tau) = c(-q \perp \theta)$). So $\varphi \perp \beta(-q \perp \theta) \perp \tau \in I^4F$. ∎

The following proposition reinforces [19, Lemma 5].

8.5. **Proposition.** *Let D be a central division F-algebra. Let q be an F-form such that $5 \leq \dim q \leq 8$, and let $K = F(D)$. Consider the following conditions:*

(i) *q_K is an anisotropic neighbour, but q is not a neighbour.*

(ii) *There exists $q' \in I^2F$ of dimension 8 such that*
  a) *$q \leq q'$*
  b) *$c(q') = D$.*

Then (i) $\Rightarrow$ (ii), *and* (ii) $\Rightarrow$ (i) *if* $\mathrm{ind}(D) \geq 4$ *and* $\dim q \geq 6$.

Proof. (i) $\Rightarrow$ (ii): set $d = d_{\pm}q$ and $L = F(\sqrt{d_{\pm}q})$. We distinguish three cases:

- $\dim q = 5$. Since q is not a neighbour and q_K is a neighbour, we have $\mathrm{ind}(C_0(q)) = 4$ and $\mathrm{ind}(C_0(q)_K) \leq 2$ [26, Page 10]. By Theorem 2.2 (2), we have $\mathrm{ind}(C_0(q) \otimes D) = 2$, hence there exists $\tau = \langle 1 \rangle \perp \tau' \in P_2(F)$ such that $[D] = c(q) + c(\tau)$. We take $q' = q \perp d\tau'$.
- $\dim q = 6$. By assumption q_K is anisotropic, hence $d \neq 1$. Moreover, $q_{K \cdot L} \sim 0$, hence $c(q)_L \in H^2(K \cdot L/L) = \{0, [D_L]\}$. Since $q_L \not\sim 0$ (because q is not a neighbour [26, Page 10]), we have $c(q)_L \neq 0$. So $c(q)_L = [D_L]$, and thus $c(q) = [D] + (f, d)$ for some $f \in F^*$. We take $q' = q \perp -f\langle 1, -d \rangle$.
- $\dim q = 7$ or 8. By [26, Page 11], $c(q) \in H^2(K/F) = \{0, [D]\}$. Since q is not a neighbour, we have $c(q) = [D]$. We take $q' = q \perp \langle -d \rangle$ or q according as $\dim q = 7$ or 8.

(ii) $\Rightarrow$ (i): note that $q'_K \in I^3K$. By [27, Th. 4] (for $\mathrm{ind}(D) = 4$) or [6, Cor. 9.2] (for $\mathrm{ind}(D) = 8$), q'_K is anisotropic, so q_K is an anisotropic neighbour. If q were a neighbour of a 3-fold Pfister form π, then $\dim(q' \perp -a\pi)_{\mathrm{an}}$ would be ≤ 4 for some $a \in F^*$. This is impossible since $c(q' \perp -a\pi) = c(q')$ has index ≥ 4. ■

We shall also need the following proposition.

8.6. **Proposition.** *Let D be a central division F-algebra of exponent 2 and index 8, and let q be an F-form. Consider the following conditions:*

(i) $\mathrm{ind}(D_{F(q)}) < \mathrm{ind}(D)$.
(ii) *There exists a form $q' \in I^2F$ of dimension 8 such that $c(q') = [D]$ and $q \leq q'$.*

Then (ii) $\Rightarrow$ (i)*, and* (i) $\Rightarrow$ (ii) *if $q \notin GP_2(F)$ and q is not an Albert form.*

Proof. (ii) $\Rightarrow$ (i) is obvious. For (i) $\Rightarrow$ (ii), use Theorem 2.2 (1) and proceed as in the proof of [28, corollaire 3]. ■

8.7. **Remark.** In Proposition 8.6, condition (ii) cannot hold in the exceptional cases where $q \in GP_2(F)$ or q is an Albert form: otherwise, we would get a contradiction with the complementary form which would be in I^2F. In these cases however, one may find an q' as in (ii) which contains a codimension 1 sub-form of q.

9. Proof of Theorem 7.3: the cases $n = 1, 2, 3$

From now on, we keep the same notation as at the beginning of Section 7.

9.A. **The cases $n = 1, 2$.** The case $n = 1$ has been seen previously (Lemma 5.3 a)). For $n = 2$, by [19, Theorem 2 (b)], one has necessarily $\dim q = 4$, $d = d_\pm q = d_\pm \varphi$. Moreover, [19, Theorem 6] shows that one then has $\varphi \simeq aq_1$ for some $a \in F^*$, so we may choose $\psi = aq$. The uniqueness assertion follows from Propositions 4.5 and 4.4.

9.B. **The case $n = 3$.** According to [19, Theorem 2 (c)], we are in one of the following three cases:

(1) q is an Albert form.
(2) $\dim q = 5$, q not a neighbour.
(3) $\dim q = 4$, $d' \neq 1$.

In case (1), [19, Theorem 6] shows that $\varphi \simeq -d\tau'$, where $\tau = \langle 1 \rangle \perp \tau'$ is the leading form of q. Then $\varphi \perp \langle -d \rangle \simeq -d\tau$. But $\tau \notin \mathrm{Im}(W(F) \to W_{\mathrm{nr}}(K/F))$ by Theorem 6.4 c), a contradiction. This case is therefore impossible.

In case (2), we argue as in [19, Proof of Theorem 6]: the invariant $c(\varphi)$ is of the form $c'_{F(q)}$, where c' is of index 4. Merkurjev's index reduction theorem implies that $c' = c(q)$. But then $c(\varphi) = c(q_1)$, hence φ is similar to q_1 and $\varphi \simeq dd'q_1$. We may therefore choose $\psi = dd'q$. The uniqueness assertion follows from Proposition 4.5.

In case (3), the same argument gives that $c(q)_{E'} = c'_{E'}$. Then $c(\varphi)_{E'(q)} = c'_{E'(q)} = 0$ and $\varphi_{E'(q)}$ is isotropic. Therefore φ is of the form $\lambda\langle 1, -d'\rangle \perp \langle dd'\rangle$ for some $\lambda \in F(q)^*$. But then, $\lambda\langle 1, -d'\rangle \in \mathrm{Im}(W(F) \to W(F(q)))$. By the case $n = 2$, we therefore may choose $\psi = aq \perp \langle dd'\rangle$ for some $a \in F^*$. For uniqueness, let ψ' be of dimension 5 such that $\varphi \sim \psi'_K$. By Lemma 3.4 we get

$$\psi' \equiv aN(x)q \perp \langle dd'\rangle \pmod{I^4F}$$

for some $x \in E'^*$, hence $\psi' \simeq aN(x)q \perp \langle dd'\rangle$ by the Hauptsatz.

It follows that, in row 1 of the tableau, $C_F(q, \varphi) > 3$ if and only if the condition in column 2 is satisfied. ∎

10. Proof of Theorem 7.3: the case of a form similar to a 2-fold Pfister form

By [19, Theorems 2 and 6], φ is defined over F except perhaps in the following cases:

(1) $\dim q \leq 5$.
(2) q is an Albert form.
(3) $\dim q = 7$, $c(q)_K = c(\varphi)$.
(4) $\dim q = 8$, $q \in I^2F$, $c(q)_K = c(\varphi)$.

The cases (2) and (4) are handled in [19, Theorem 6]: in each of them, φ is necessarily of the form aq_1 for some $a \in F^*$. This yields row 4 of the tableau in dimension 8 as well as row 3, the uniqueness claim on ψ coming from Proposition 4.5 (and Proposition 4.4 in case (4)).

Observe that cases (3) and (4) are equivalent: indeed, in these two cases, $\mathrm{ind}\, c(q) = 2$ by [19, Lemma 4]. But a sub-form of dimension 7 in a form of dimension 8 of that type is stably birationally equivalent to it. Conversely, if q is anisotropic with $\dim q = 7$ and $\mathrm{ind}\, c(q) = 2$, then q cannot represent its discriminant $d_\pm q$, hence $q \perp \langle -d_\pm q\rangle$ is anisotropic of the above type. The equivalence between (3) and (4) now follows from Proposition 6.1 b).

From this, one derives easily row 4 of the tableau in dimension 7.

It remains to deal with the cases where $\dim q \leq 5$. If q is a Pfister neighbour, φ is defined over F by reduction to a 3-fold Pfister form (Proposition 6.1 b)). There are therefore two cases to examine:

(1a) $\dim q = 5$, q not a neighbour.
(1b) $\dim q = 4$, $d_\pm q \neq 1$.

In both cases, observe that $\operatorname{ind}(C(\psi)) \leq 4$. Since $\psi_L \in I^3L$, we have by the Hauptsatz

$$\psi_{K\cdot L} \sim \varphi_{K\cdot L} \sim 0 \tag{10.1}$$

In particular,

$$e^3(\psi_L) \in H^3(K \cdot L/L) \tag{10.2}$$

We have the following diagram of field extensions:

$$\begin{tikzcd} L \arrow[r, dash] \arrow[d, dash] & K \cdot L \arrow[d, dash] \\ F \arrow[r, dash] & K \end{tikzcd}$$

10.A. **The case** $\operatorname{ind}(C(\psi)) = 4$**.** Let γ be an anisotropic Albert form such that $c(\psi) = c(\gamma)$. Then $\psi \equiv \gamma \pmod{I^3F}$. The form γ_K is isotropic. It follows from [32] (*cf.* also Hoffmann's unpublished thesis) that q is similar to a sub-form of γ. Therefore q_L is isotropic since $\gamma_L \sim 0$. The extension $L \cdot K/L$ is therefore purely transcendental and $\psi_L \sim 0$ by (10.1). By Proposition 8.2, there exists $a \in F^*$ such that $\psi \equiv a\gamma \pmod{I^4F}$. Extending scalars to K, we conclude by the Hauptsatz that $\varphi \sim a\gamma_K$ since $\dim \varphi < 10$.

This justifies row 2 of the tableau, except for the uniqueness claim on ψ and the exact value of $C_F(q, \varphi)$. For another anisotropic Witt descent ψ' of φ, of dimension ≤ 6, we deduce by Lemma 8.1 that $\psi' \in I^2F$ and $c(\psi') = c(\gamma)$. Hence $\dim \psi' = 6$ since $\operatorname{ind}(C(\gamma)) = 4$. Moreover, ψ' is similar to γ. If $\psi' = ab\gamma$, one has $(\langle\langle b \rangle\rangle \otimes a\gamma)_K \sim 0$, which is equivalent to $b \in G(\varphi)$. Conversely, one does have $ab\gamma_K \sim \varphi$ for any b verifying this condition.

10.1**. Remark.** In case (1a), the map $H^3F \to H^3K$ is injective and one gets something slightly better: $b \in G(\varphi) \iff b \in G(\gamma)$.

10.B. **The case** $\operatorname{ind}(C(\psi)) = 2$**.** The reasoning is similar but more complicated. Of course, we assume that q is of type (1a) or (1b).

Let $\tau \in P_2(F)$ be such that $c(\psi) = c(\tau)$. Then $\psi \equiv \tau \pmod{I^3F}$. We shall distinguish the following cases:

(i) q_L is isotropic.
(ii) $\dim q = 5$ and q_L is not a neighbour.
(iii) $\dim q = 5$ and q_L is an anisotropic neighbour.
(iv) $\dim q = 4$.

10.B.1. Suppose that q is as in (i) or (ii). Then

$$H^3(K \cdot L/L) = 0.$$

From equation (10.2) and the bijectivity of e^3, we deduce that

$$\psi_L \in I^4 L.$$

By Proposition 8.2, there exists $a \in F^*$ such that $\psi \equiv a\tau \pmod{I^4 F}$, hence $\varphi \equiv a\tau_K \pmod{I^4 K}$. By the Hauptsatz, we therefore have $\varphi \simeq a\tau_K$ and φ is defined over F.

10.B.2. Suppose that q is as in (iii). By Proposition 8.5 there exists a form $q' \in I^2 F$ of dimension 8 such that $q \leq q'$ and $c(q') = c(\tau)$. Set

$$\xi = q' \perp -\tau.$$

Then $\xi \in I^3 F$. In particular, $\psi \perp -\xi \equiv \tau \pmod{I^3 F}$.

Note that q_L and q'_L are neighbours of the same 3-fold Pfister form, and $e^3(q'_L) = e^3(\xi)_L$. Hence, by Theorem 3.2 we get that $H^3(K \cdot L/L) = \{0, e^3(\xi)_L\}$.

• If $e^3(\psi_L) = 0$, then $\psi_L \in I^4 L$ and we conclude as in case (A) that φ is defined over F.

• If $e^3(\psi_L) = e^3(\xi)_L$, then $(\psi \perp -\xi)_L \in I^4 L$. Applying Proposition 8.2, we can find a scalar $a \in F^*$ such that $\psi \perp -\xi \equiv a\tau \pmod{I^4 F}$, or

$$\psi \equiv a(\xi \perp \tau) \equiv aq' \pmod{I^4 F}$$

since $\xi \in I^3 F$. The Hauptsatz then implies that

$$\varphi \sim aq'_K.$$

This justifies row 4 of the tableau in dimension 5, except for the uniqueness claim on ψ and the exact value of $C_F(q, \varphi)$.

Observe first that q' is anisotropic since q_L is anisotropic and q_L, q'_L are neighbours of the same 3-fold Pfister from. If $\psi' \in \mathrm{Desc}(\varphi)$ of dimension ≤ 8, then we deduce by Lemma 8.1 that $\psi' \equiv aq' \pmod{I^4 F}$. Hence we necessarily have $\dim \psi' = 8$ and $\psi' \approx aq'$. By Proposition 4.4, $\psi' \simeq baq'$ where $b \in G(\pi)$ and π is the twist of q'.

10.B.3. Suppose that q is as in (iv). We apply Corollary 8.4 to the form $\psi \perp -\tau$ to get the existence of a form θ of dimension 4 and $\pi \in GP_3(F)$ hyperbolic over K such that

$$(\psi \perp -\tau) \perp \pi \perp \tau \perp -\alpha\tau \in I^4 F$$

or

$$(\psi \perp -\tau) \perp -aq \perp -\theta \perp \tau \in I^4 F$$

for suitable scalars $\alpha, a \in F^*$. In both cases and after simplification, we extend scalars to K and apply the Hauptsatz to get $\alpha\tau_K \sim \varphi$ or

$(aq \perp \theta)_K \sim \varphi$. In the first case, φ is defined over F; in the second case, $C_F(q, \varphi) \leq 8$.

To finish the justification of row 1 of the tableau, we prove the uniqueness claim on ψ and the exact value of $C_F(q, \varphi)$. Let $\psi' \in \mathrm{Desc}(\varphi)$ of dimension ≤ 8. Then $(\psi' \perp -aq \perp -\theta)_K \sim 0$. By Lemma 3.4, we have

$$\psi' \equiv aN(x)q \perp \theta \pmod{I^4 F}$$

for some $x \in E'^*$. So ψ' is conjugate or Witt-equivalent to $aN(x)q \perp \theta$, according as $\dim \psi' = 8$ or not. Moreover, by the Hauptsatz $C_F(q, \varphi) \leq 6$ if and only if $-a \in G(\pi)D(q)D(\theta)$.

11. Proof of Theorem 7.3: the case of an Albert form

The method is parallel to that in the previous section. By [19, Theorem 2], φ is defined over F except perhaps in the following cases:

(1) $\dim q \leq 7$.
(2) $\dim q = 8$, $q \in I^2 F$, $c(q)_K = c(\varphi)$.

From these cases we exclude the following, after Proposition 6.1 b):

(3) q is a neighbour of a 3-fold Pfister form
(4) $\dim q = 7$, $\mathrm{ind}(C_0(q)) = 2$.

Indeed, in case (3) (*resp.* (4)), q is stably birationally equivalent to a 3-fold Pfister form (*resp.* to an 8-dimensonal form in $I^2 F$ with Clifford invariant of index 2).

Moreover, in case (2), φ is necessarily of the form aq_1 with $a \in F^*$ by [19, Theorem 6] (see also loc. cit., note bottom p. 153).

It therefore remains to handle the cases $\dim q \in \{4, 5, 6, 7\}$, q not a neighbour and not of type (4).

We have $\mathrm{ind}(C(\psi)) = 4$ or 8 since $\mathrm{ind}(C(\psi)_K) = 4$.

11.A. The case $\mathrm{ind}(C(\psi)) = 8$.

We first note that in this case, we have automatically $\dim \psi \geq 8$ since $\psi \in I^2 F$.

11.1. Lemma. *If* $\mathrm{ind}(C(\psi)) = 8$, *then* $C_F(q, \varphi) = 8$ *and* q *cannot be an Albert form.*

Proof. The index of $C(\psi)$ gets reduced by extension of scalars to K. Let δ be a form defined as follows: δ is an arbitrary 5-dimensional subform of q if q is an Albert form; $\delta = q$ otherwise. Since $F(q, \delta)/F(\delta)$ is purely transcendental, we have $\mathrm{ind}(C(\psi)_{F(\delta)}) \leq 4$. By Proposition 8.6, there exists $q' \in I^2 F$ of dimension 8 such that $\delta \leq q'$ and $c(\psi) = c(q')$. So, $\psi \perp -q' \in I^3 F$. Now, on the one hand

$$q'_{L(\delta)} \sim 0$$

because $q'_L \in GP_3(L)$ and $q'_{F(\delta)}$ is isotropic. On the other hand, by the Hauptsatz,

$$\psi_{L(q)} \sim 0$$

because $\varphi \sim \psi_{F(q)}$ and $\psi_L \in I^3L$. Therefore, $e^3(\psi \perp -q') \in H^3(L(q,\delta)/F)$. Note that

$$H^3(L(q,\delta)/F) \subseteq H^3(L(\delta)/F)$$

because $L(\delta,q)/L(\delta)$ is purely transcendental.

(A) Suppose that $\dim q \in \{5,6,7\}$. By Theorem 3.9 (1), we have $e^3(\psi \perp -q') = c(-q')(r)$ for some $r \in F^*$, hence $\psi \perp -rq' \in I^4F$. To finish, extend scalars to K and apply the Hauptsatz to get $\varphi \sim rq'_K$.

(B) Suppose $\dim q = 4$. Up to a scalar, we have $q = \langle -a,-b,ab,d_\pm q\rangle$. By the index reduction theorem of Merkurjev (Theorem 2.2 (1)), we have $[C(\psi)] = (a,b) + (d_\pm q, u) + (r,s)$ for suitable scalars $r,s,u \in F^*$. By Theorem 3.9 (3), there exists $x,y \in F^*$ such that $e^3(\psi \perp -q') = c(q')(x) + e^3(q \otimes \langle 1,-y\rangle)$ with $q \otimes \langle 1,-y\rangle \in GP_3(F)$. So, $\psi \perp -xq' \perp q \otimes \langle 1,-y\rangle \in I^4F$. Extend scalars to K and apply the Hauptsatz and the fact that $(q \otimes \langle 1,-y\rangle)_K \sim 0$ to get $\varphi \sim xq'_K$.

In both cases, we get an 8-dimensional form $q' \in I^2F$ such that $\delta \leq q'$, $\operatorname{ind}(C(q')) = 8$ and $\varphi \sim aq'_K$ for some scalar $a \in F^*$. Hence $C_F(q,\varphi) = 8$.

Since $\varphi \sim aq'_K$, the form q'_K is isotropic, hence q cannot be an Albert form otherwise q' would be isotropic by [27], [28], which is impossible because $\operatorname{ind}(C(q')) = 8$. ∎

Let us justify the uniqueness claims on ψ. Let $\psi' \in \operatorname{Desc}(\varphi)$ of dimension ≤ 8. It follows from Lemma 8.1 that $c(q') = c(\psi')$.

- If $\dim q \geq 5$, then by Lemma 8.1 again $aq' \approx \psi'$.
- If $\dim q = 4$, write $aq' = aq \perp \theta$ with $\dim\theta = 4$. By Lemma 3.4 (applied to $\rho = a(\psi' \perp -\theta)$), we then have $aN(y)q \perp \theta \equiv \psi'$ (mod I^4F) for some $y \in E'^*$, hence $aN(y)q \perp \theta \approx \psi'$ by Proposition 4.3 a).

This justifies rows 3 and 5 of the tableau.

11.B. **The case** $\operatorname{ind}(C(\psi)) = 4$.

11.2**. Lemma.** *Suppose* $\operatorname{ind}(C(\psi)) = 4$.
a) If $\dim q \geq 5$, *φ is defined over F unless q_L is an anisotropic neighbour. In the latter case,* $C_F(q,\varphi) \leq 8$.
b) If $\dim q = 4$, *then φ is defined over F unless* $\operatorname{ind}(C(q_{E'}) \otimes C(\psi)_{E'}) = 2$ *or* 4. *In the first case,* $C_F(q,\varphi) \leq 8$. *(In the second case, we cannot bound* $C_F(q,\varphi)$*.)*

Proof. Let γ be an anisotropic Albert form such that $c(\gamma) = c(\psi)$. In particular, $\psi \perp -\gamma \in I^3F$. We distinguish the following cases:

(i) $\dim q \geq 5$, q_L is not a neighbour or q_L is isotropic.
(ii) $\dim q \geq 5$, q_L is an anisotropic neighbour.
(iii) $\dim q = 4$.

By [17], $\varphi \simeq x\gamma_K$ for some $x \in K^*$. We have $\varphi_{K\cdot L} \sim \psi_{K\cdot L} \sim 0$ since $\gamma_L \sim 0$. Then

$$e^3(\psi \perp -\gamma) \in H^3(K \cdot L/F). \tag{11.1}$$

(A) Suppose that q satisfies (i). Then

$$H^3(K \cdot L/F) \subset H^3(L/F).$$

By Corollary 3.7, we have $e^3(\psi \perp -\gamma) = c(\gamma)(u)$ for some $u \in F^*$. So, $\psi \perp -u\gamma \in I^4F$. Extend scalars to K and apply the Hauptsatz to get $\varphi \sim u\gamma_K$. In this case, φ is defined over F.

(B) Suppose that q satisfies (ii). In particular, q is not an Albert form. By Proposition 8.5, there exists $q' \in I^2F$ of dimension 8 such that $q \leq q'$ and $c(q') = c(\gamma)$.

By equation (11.1), $e^3(\psi \perp -\gamma)_L \in \{0, e^3(q'_L)\}$ because q_L and q'_L are neighbours of the same 3-fold Pfister form.

• Suppose $e^3(\psi \perp -\gamma)_L = 0$. By corollary 3.7, there exists $v \in F^*$ such that $e^3(\psi \perp -\gamma) = c(\gamma)(v)$. So $\psi \perp -v\gamma \in I^4F$. Extend scalars to K and apply the Hauptsatz to get $\varphi \sim v\gamma_K$. In this case, φ is defined over F.

• Suppose $e^3(\psi \perp -\gamma)_L = e^3(q'_L)$. We obviously have $e^3(q'_L) = e^3(q' \perp \gamma)_L$. By Corollary 3.7, we have $e^3(\psi \perp -\gamma) + e^3(q' \perp \gamma) = c(q')(-a)$ for some $a \in F^*$. So $\psi \perp -aq' \in I^4F$. Extend scalars to K and apply the Hauptsatz to get $\varphi \sim aq'_K$.

(C) Suppose that q verifies (iii). If we are not in the exceptional cases of the lemma, we have (use Corollary 3.7, Theorem 3.9 (2), and Lemma 3.3):

$$e^3(\psi \perp -\gamma) = e^3(\langle\langle\, N(x) \,\rangle\rangle \otimes q) + e^3(\langle\langle\, a \,\rangle\rangle \otimes \gamma)$$

for some $x \in E'^*$ and $a \in F^*$, hence

$$\psi \perp -\gamma \equiv \langle\langle\, N(x) \,\rangle\rangle \otimes q \perp -\langle\langle\, a \,\rangle\rangle \otimes \gamma \pmod{I^4F}$$

or

$$\psi \equiv \langle\langle\, N(x) \,\rangle\rangle \otimes q \perp a\gamma \pmod{I^4F}.$$

Extending scalars to K and applying the Hauptsatz, we find that $\varphi \simeq a\gamma_K$. In this case, φ is defined over F.

If we are in case $\mathrm{ind}(C(q_{E'}) \otimes C(\psi)_{E'}) = 2$, the same references give this time:

$$e^3(\psi \perp -\gamma) = e^3(\langle\langle N(x) \rangle\rangle \otimes q) + e^3(\langle\langle f \rangle\rangle \otimes \gamma) + e^3(bq \perp q'' \perp c\gamma)$$

for q'' as in Theorem 3.9 (2) and some $x, f, b, c \in F^*$, hence

$$\psi \perp -\gamma \equiv \langle\langle N(x) \rangle\rangle \otimes q \perp - \langle\langle f \rangle\rangle \otimes \gamma \perp -fc(bq \perp q'' \perp c\gamma) \pmod{I^4F}$$

or

$$\psi \equiv \langle\langle N(x) \rangle\rangle \otimes q \perp -fc(bq \perp q'') \pmod{I^4F}$$

and, thanks to the Hauptsatz, $\varphi \sim -fc(bq \perp q'')_K$. So $C_F(q, \varphi) \leq 8$. Set $\alpha = -fcb$, $\beta = -fc$ and $\delta = \alpha q \perp \beta q''$. ∎

It remains to justify the uniqueness claim on ψ and the exact value of $C_F(q, \varphi)$. We keep the case distinction in the proof of Lemma 11.2.

(B) Let $\psi' \in \mathrm{Desc}(\varphi)$, with $\dim \psi' \leq 8$. It follows from Lemma 8.1 that $\psi' \equiv aq' \pmod{I^4F}$. We have $C_F(q, \varphi) = 8$, otherwise by the Hauptstaz aq' would be isotropic, hence also q_L, a contradiction. This justifies row 4 of the tableau.

(C) Let $\psi' \in \mathrm{Desc}(\varphi)$ of dimension ≤ 8. Then $(\delta \perp -\psi')_K \sim 0$. By Lemma 3.4, we have

$$\alpha N(y)q \perp \beta q'' \equiv \psi' \pmod{I^4F}$$

for some $y \in E'^*$. By the Hauptsatz, we have $\dim \psi' = 6$ if and only if $\alpha N(y)q \perp \beta q''$ is isotropic, that is, $-\alpha\beta \in G(\pi)D(q)D(q'')$, where π is the twist of q. This justifies row 2 of the tableau.

12. Proof of Theorem 7.3: the case of a 5-dimensional form

12.1. **Proposition.** *Let q, K, φ be as in Conjecture 2, with φ of dimension 5, and let $\psi \in \mathrm{Desc}(\varphi)$. If $\mathrm{ind}(C_0(\psi)) = \mathrm{ind}(C_0(q) \otimes C_0(\psi)) = 4$, we assume that $\dim q \neq 4$. Then $C_F(q, \varphi) \leq 9$. In particular, $C(m, 5) \leq 9$ for all $m \neq 4$.*

Proof. It will be divided into several cases.

12.A. φ is of one of the two following forms.

(1) φ is not a neighbour and $\gamma = \varphi \perp \langle -d \rangle$ is not defined over F.
(2) φ is a neighbour.

In case (1) and by the case of an Albert form, we deduce the existence of a form q' of dimension ≤ 8 such that $q \leq q'$ and $\gamma \sim aq'_K$ for some scalar $a \in F^*$. Then $\varphi \sim (aq' \perp \langle d \rangle)_K$ and $C_F(q, \varphi) \leq 9$. In particular, $C(m, 5) \leq 9$ for all $m \neq 4$.

In case (2) write $\varphi = \tau \perp \langle d\rangle$ with $\tau \in GP_2(K)$ and get back to the case of τ by applying Theorem 2.

Assume that φ is not defined over F. Notice that in case (2) τ is not defined over F.

Let us give a few justifications for column 4 of the tableaux in f) and g.2). First, observe that the rows of these tableaux, except for column 4, can be deduced respectively from those of the tableaux in d) and h).

12.A.1. *Case of the tableau in g.2).* Since γ is not defined over F, it follows from the tableau of an Albert form that q' is anisotropic of dimension 8. Let $\psi' \in \mathrm{Desc}(\varphi)$ of dimension ≤ 9.

(i) $\dim q \geq 5$: then φ is not defined over F, otherwise there would be ν of dimension 5 such that $\varphi \simeq \nu_K$. By Lemma 8.1, we would get $aq' \perp \langle d\rangle \perp -\nu \in I^4F$. By the Hauptsatz, q' would be isotropic, a contradiction. So $C_F(q,\varphi) = 7$ or 9. For the uniqueness claims, one reapplies Lemma 8.1 to get $aq' \perp \langle d\rangle \perp -\psi' \in I^4F$. If furthermore $d \in D(\psi')$ (as was supposed in the tableau in g.2)), then $\psi' = q'' \perp \langle d\rangle$ with $\dim q'' \leq 8$, and thus $aq' \perp -q'' \in I^4F$. In particular, q'' is anisotropic of dimension 8. Hence the uniqueness claim.

(ii) $\dim q = 4$: Set $aq' = aq \perp \theta$ with $\dim\theta = 4$. By Lemma 3.4, we have

$$aN(x)q \perp \theta \perp \langle d\rangle \perp -\psi' \in I^4F$$

for some $x \in E'^*$. The condition of column 2 implies that φ is not defined over F. Hence, $C_F(q,\varphi) = 7$ or 9. If furthermore $d \in D(\psi')$, then $\psi' = q' \perp \langle d\rangle$ with $\dim q' \leq 8$, and

$$aN(x)q \perp \theta \perp -q' \in I^4F$$

It is clear that the condition of column 2 implies that q' is anisotropic of dimension 8. The uniqueness claim is clear from Lemma 3.4.

This justifies column 4 in the tableau in g.2).

12.A.2. *Case of the tableau in f).* We have $H^i(K/F) = \{0\}$ for $i = 1, 2$ (*resp.* for $i \leq 3$) when $\dim q = 4$ (*resp.* $\dim q > 4$). Let $\psi' \in \mathrm{Desc}(\varphi)$ anisotropic of dimension ≤ 9.

(i) Case of row 1: we get by Lemma 3.4 that $abq \perp \theta \perp \langle d\rangle \perp -\psi' \in I^4F$ for some $b \in G(\pi)$. The condition of column 2 implies that $C_F(q,\varphi) = 7$ or 9. If furthermore $\psi' = \langle d\rangle \perp q'$, then $q' \approx abq \perp \theta$ or $q' \sim abq \perp \theta$ according as $\dim q' = 8$ or not.

(ii) Case of row 4: suppose that $\psi' = q'' \perp \langle d\rangle$. Since $H^i(K/F) = \{0\}$ for $i \leq 3$, we get $aq' \equiv q'' \pmod{I^4F}$. By the tableau in d) the form q' is anisotropic of dimension 8. Then, q'' is also anisotropic of dimension 8. In particular, $C_F(q,\varphi) = 7$ or 9. By Proposition 4.4 we get that $abq' \approx q''$ with $b \in G(\pi)$.

(iii) Case of row 3 and row 2 in dimension 5: we argue as in (ii). In this case, $C_F(q,\varphi) = 7$ by the condition in column 2.

(iv) Case of row 2 in dimension 4: write $\gamma \simeq q \perp \xi$. In this case one has $C_F(q,\varphi) \leq 7$. We get by Lemma 3.4

$$aN(y)q \perp a\xi \perp \langle d\rangle \perp -\psi' \in I^4F$$

for some $y \in E'^*$. By the Hauptsatz, we get $aN(y)q \perp a\xi \perp \langle d\rangle \simeq \psi'$. Now it is clear that the condition in the column 2 implies that $C_F(q,\varphi) = 7$.

12.B. φ **is not a neighbour and** $\mathrm{Desc}_0(\gamma) \neq \emptyset$**.** Let $\gamma_0 \in \mathrm{Desc}_0(\gamma)$. The justification of the statements (1) and (2) in g.2) are easy to make. For statement (3), let $\psi' \in \mathrm{Desc}(\varphi)$ be anisotropic of dimension 7. If $\gamma_0 \perp \langle d\rangle \perp -\psi' \not\sim 0$, then by a result of Fitzgerald there exists $\rho \in GP_3(F)$ such that $(\gamma_0 \perp \langle d\rangle \perp -\psi')_{\mathrm{an}} \simeq \rho$ and $q \preccurlyeq \rho$. Hence the claim since $i_W(\gamma_0 \perp \langle d\rangle \perp -\rho) = 4$. If $d \in D(\psi')$, then $\psi' \simeq \delta \perp \langle d\rangle$ for some Albert form δ. Since $c(\delta) = c(\gamma)$, we get by [17] that $\delta \simeq a\gamma$ for some $a \in F^*$.

13. Proof of Theorem 7.3: the case of a 4-dimensional form which is not a neighbour

By [19, Theorem 2], φ is defined over F except perhaps in the following cases:

(1) $\dim q = 4$ and $d_\pm q \neq 1$.
(2) $\dim q = 5$.
(3) $\dim q = 6$ and $d_\pm q \in \{1, d_\pm\varphi\}$.

Moreover, thanks to Proposition 6.1 b), φ is also defined over F if q is a neighbour of a 3-fold Pfister form.

Clearly, the signed discriminant d of φ is defined over F. Since $\varphi_{K\cdot E} \in GP_2(K\cdot E)$, we have $\mathrm{ind}(C(\psi)_{K\cdot E}) \leq 2$, hence $\mathrm{ind}(C(\psi)_E) \leq 4$. So there exists an Albert form γ and some $r \in F^*$ such that

$$c(\psi) = c(\gamma) + (r,d). \tag{13.1}$$

Let t be a variable over F, $\widetilde{F} = F((t))$ the field of formal power series in t over F, and $\widetilde{K} = \widetilde{F}(q)$. Let $D = C(\gamma)_{\widetilde{F}} \otimes_{\widetilde{F}} (rt,d)$, $M = \widetilde{F}(D)$ and $\varphi' = (\varphi \perp -t\langle 1,-d\rangle)_{\widetilde{K}}$, which is an anisotropic Albert form. We have the following diagram of field extensions:

$$\begin{array}{ccccc}
M & \text{———} & K\cdot M & \text{———} & \widetilde{K}\cdot M \\
| & & | & & | \\
F & \text{———} & K & \text{———} & \widetilde{K}
\end{array}$$

13.1. **Lemma.** *If* $\dim q \neq 4$*, then* $C_F(q,\varphi) \leq 6$.

Proof. We have $\varphi' \in \mathrm{Im}(W(\widetilde{F}) \longrightarrow W(\widetilde{K}))$. By Theorem 3, there exists $\delta \in W(\widetilde{F})$ anisotropic such that $\dim \delta \leq 8$ and $\varphi' \sim \delta_{\widetilde{K}}$. Write $\delta = \delta_1 \perp t\delta_2$ with $\delta_1, \delta_2 \in W(F)$ anisotropic. By Springer's Theorem, we get:

- $\varphi \sim (\delta_1)_K$,
- $\dim \delta_2 \geq 2$ because $d \neq 1$, hence $\dim \delta_1 \leq 6$.

Since φ is not defined over F, the form δ_1 is of dimension 6. Hence $(\delta_1)_K$ is isotropic. By the isotropy results in dimension 6 [9], [29], the form q cannot be an Albert form. Again by [9], [29] and as q is not a neighbour, we get $\delta_1 \simeq aq$ when $\dim q = 6$ (*resp.* $\delta_1 \simeq aq \perp \langle b\rangle$ when $\dim q = 5$) for some $a, b \in F^*$.

Uniqueness in dimension 6 follows from Propsoition 4.5. Concerning dimension 5, if ψ' is another anisotropic Witt descent of φ of dimension 6, then Lemma 8.1 and the Hauptsatz implie that $\psi' \simeq aq \perp \langle b\rangle$. Clearly, $-ab \notin D(q)$, because δ_1 is anisotropic.

This justifies rows 3 and 4 of the tableau.

13.2. **Lemma.** *If* $\dim q = 4$*, then* $C_F(q,\varphi) \leq 10$*. More precisely, with the same notation as at the beginning of this section, we have:*

1. *If* $\mathrm{ind}(C(\psi)_E) = 4$*, then* $C_F(q,\varphi) = 6$.
2. *If* $\mathrm{ind}(C(\psi)_E) = 2$*, then* ψ *is of one of the two following types:*
   - $\psi \simeq a\langle 1, -d\rangle \perp \tau$ *with* $a \in F^*$ *and* $\tau \in GP_2(F)$*,*
   - $\psi \simeq b\langle 1, -d\rangle \perp cq \perp \theta$ *for some* $b, c \in F^*$ *and* $\dim\theta = 4$.

Proof. Up to scaling we may write $q = \langle -\alpha, -\beta, \alpha\beta, d_{\pm}q\rangle$. We shall distinguish two cases:

(A) Suppose that $\mathrm{ind}(C(\psi)_E) = 4$. This amounts to say that γ_E is anisotropic. The algebra D is then a triquaternion division algebra [44, proposition 2.4]. The index of D gets reduced by extension of scalars to $\widetilde{K}$, since $c(\varphi') = [D_{\widetilde{K}}]$. By Proposition 8.6, there exists $\eta \in I^2\widetilde{F}$ of dimension 8 such that $q_{\widetilde{F}} \leq \eta$ and

$$c(\eta) = [D]. \tag{13.2}$$

By equations (13.1) and (13.2), we have

$$\psi_{\widetilde{F}} \perp -t\langle 1, -d\rangle \perp \eta \in I^3\widetilde{F}$$

hence

$$\varphi_{\widetilde{K}} \perp (-t\langle 1, -d\rangle \perp \eta)_{\widetilde{K}} \in I^3\widetilde{K}.$$

We have $\eta_{M(q)} \sim 0$, because $\eta_M \in GP_3(M)$ and $\eta_{\widetilde{F}(q)}$ is isotropic. By the Hauptsatz, we get

$$(\varphi \perp -t\langle 1, -d\rangle \perp \eta)_{M(q)} \sim 0. \tag{13.3}$$

So $e^3(\psi_{\widetilde{F}} \perp -t\langle 1,-d\rangle \perp \eta) \in H^3(M(q)/\widetilde{F})$. Since $q_{\widetilde{F}} \leq \eta$, We get by equation (13.2) that $[D] = [(d_\pm q, u)] + [(\alpha,\beta)] + [(r,s)]$ for some $r,s,u \in \widetilde{F}^*$. By Theorem 3.9 (3), the kernel

$$H^3(M(q)/\widetilde{F})$$

equals

$$c(\eta)\cdot H^1\widetilde{F} + \{e^3(q\otimes\langle 1,-y\rangle) \mid y \in \widetilde{F}^*, q\otimes\langle 1,-y\rangle \in GP_3(\widetilde{F})\}.$$

Let $a,b \in \widetilde{F}^*$ be such that

$$e^3(\psi_{\widetilde{F}} \perp -t\langle 1,-d\rangle \perp \eta) = e^3(\eta \perp a\eta) + e^3(q\otimes\langle 1,-b\rangle)$$

with $q\otimes\langle 1,-b\rangle \in GP_3(\widetilde{F})$. Then

$$\psi_{\widetilde{F}} \perp -t\langle 1,-d\rangle \perp -a\eta \perp q\otimes\langle 1,-b\rangle \in I^4\widetilde{F} \tag{13.4}$$

We have $(q\otimes\langle 1,-b\rangle)_{\widetilde{K}} \sim 0$. In (13.4), we extend scalars to $\widetilde{K}$ and apply the Hauptsatz to get

$$\varphi_{\widetilde{K}} \sim (a\eta \perp t\langle 1,-d\rangle)_{\widetilde{K}} \tag{13.5}$$

Set $a\eta = q' \perp tq''$ with $q',q'' \in W(F)$ anisotropic. We apply Springer's Theorem to (13.5) to deduce:

- $(q'' \perp \langle 1,-d\rangle)_K \sim 0$, hence necessarily $\dim q'' \geq 2$, because $d \neq 1$.
- $\varphi \sim q'_K$ and $\dim q' \leq 6$ by the first assertion.

As φ is not defined over F, necessarily $\dim q' = 6$, hence $\dim q'' = 2$. In particular, $C_F(q,\varphi) = 6$ and $d_\pm q' = d$.

Since $aq \leq q' \perp tq''$ and $\dim q'' < \dim q$, we deduce again from Springer's Theorem that, up to a square, $a \in F^*$ and $aq \leq q'$.

By Lemma 7.2 $c(q') = c(\psi)$. So $\operatorname{ind}(C(q')_E) = 4$, that is, q' is a virtual Albert form.

It remains to justify the uniqueness claims in this case. Let $\psi' \in \operatorname{Desc}(\varphi)$ of dimension 6. By Lemma 3.4

$$aN(x)q \perp \theta \equiv \psi' \pmod{I^4F}$$

for some $x \in E'^*$ and some form θ satisfying $q' \simeq aq \perp \theta$. By the Hauptsatz, $\psi' \simeq aN(x)q \perp \theta$. This justifies row 2 of the tableau.

(B) Suppose that $\operatorname{ind} C(\psi)_E = 2$. There exists $\tau = \langle 1\rangle \perp \tau' \in P_2(F)$ and $s \in F^*$ such that $c(\psi) = c(\tau) + (s,d)$. Consider the form $\mu = -s(\langle d\rangle \perp \tau')$.

We have $d_\pm\mu = d = d_\pm\varphi$ and $c(\mu) = c(\psi)$. In particular, $c(\mu)_K = c(\varphi)$. By [47], the form φ is similar to μ_K. Hence $K(\varphi) \simeq K(\mu)$.

We now repeat an argument from [18, Page 149]. Indeed, we have $\psi \perp -\mu \in I^3F$, hence

$$(\psi_K \perp -\mu_K)_{K(\mu)} \sim (\varphi \perp -\mu_K)_{K(\varphi)} \sim (\varphi_{K(\varphi)} \perp -\mu_{K(\mu)}) \in I^3K(\mu).$$

Since $\dim(\varphi_{K(\varphi)})_{\mathrm{an}}$ and $\dim(\mu_{K(\mu)})_{\mathrm{an}} \leq 2$, we have $\dim(\varphi_{K(\varphi)} \perp -\mu_{K(\mu)})_{\mathrm{an}} \leq 4$. By the Hauptsatz $(\psi \perp -\mu)_{K(\mu)} \sim 0$. Since $\mu_{F(\tau)}$ is isotropic, we get $(\psi \perp -\mu)_{K(\tau)} = 0$. By Corollary 8.4, there exists a 4-dimensional form ζ and $\pi \in GP_3(F)$ hyperbolic over K such that

$$(\psi \perp -s\tau \perp s\langle 1, -d\rangle) \perp \pi \perp \tau \perp u\tau \in I^4F$$

or

$$(\psi \perp -s\tau \perp s\langle 1, -d\rangle) \perp v(-q \perp \zeta) \perp \tau \in I^4F$$

for some $u, v \in F^*$. Observe that modulo I^4F, the two relations become

$$(\psi \perp -s\tau \perp s\langle 1, -d\rangle) \perp \pi \perp s(\tau \perp u\tau) \in I^4F$$

or

$$(\psi \perp -s\tau \perp s\langle 1, -d\rangle) \perp sv(-q \perp \zeta) \perp s\tau \in I^4F.$$

In these last two relations and after simplification, we extend scalars to K and apply the Hauptsatz to get $C_F(q,\varphi) \leq 6$ in the first case (because $\pi_K \sim 0$), and $C_F(q,\varphi) \leq 10$ in the second case. This justifies row 1 in the tableau. ∎

Still concerning row 1 of the tableau, we finally give certain cases where the form φ is defined over F.

13.3. **Proposition.** *Keep the same notation and hypotheses as in case (B) in the proof of Lemma 13.2. Then φ is defined over F when one of the following two conditions is satisfied:*

1. $\det q = d_{\pm}\varphi$
2. $\mathrm{ind}(C_0(\mu) \otimes C_0(q)) = 4$.

Proof. By assumption, we have (use [14, Corollary 2.13, Theorem 5.1, Theorem 5.8, Theorem 5.9])

$$H^3(K(\mu)/F) = H^3(K/F) + H^3(F(\mu)/F).$$

By Lemma 3.3, we have $e^3(\psi \perp -\mu) = e^3(\mu \perp -a\mu) + e^3(q \perp bq)$ for some $a, b \in F^*$ such that $\mu \perp -a\mu$, $q \perp bq \in GP_3(F)$. So $\psi \perp -a\mu \perp q \perp bq \in I^4F$. Since $(q \perp bq)_K \sim 0$, we extend scalars to K and apply the Hauptsatz to get $\varphi \simeq a\mu_K$.

References

[1] *S. A. Amitsur:* Generic splitting fields of central simple algebras, Ann. Math. **62.2** (1955), 8–43.

[2] *J. Kr. Arason:* Cohomologische Invarianten quadratischer Formen, J. Alg. **36** (1975), 448–491.

[3] *J. Kr. Arason, A. Pfister:* Beweis des Krull'schen Durchschnittsatzes für den Wittring, Invent. Math. **12** (1971), 173–176.

[4] *J.-L. Colliot-Thélène, R. Sujatha:* Unramified Witt groups of real anisotropic quadrics, *in*: W. Jacob and A. Rosenberg (ed.), K-theory and algebraic geometry: connections with quadratic forms and division algebras, Proceedings of Symposia in Pure Mathematics **58** (1995), 127–147.

[5] *R. Elman, T. Y. Lam, A. Wadsworth:* Amenable fields and Pfister extensions, Conference on Quadratic forms– 1976, ed. by G. Orzech, Queen's Papers in Pure and Appl. Math. **64**, Queen's Univ., Kingston, Ontario, 1977, 445–492.

[6] *H. Esnault, B. Kahn, M. Levine, E. Viehweg:* The Arason invariant and mod 2 algebraic cycles, J. Amer. Math. Soc. **11** (1998), 73–118.

[7] *R. W. Fitzgerald:* Function fields of quadratic forms, Math. Z. **178** (1981), 63–76.

[8] *R. W. Fitzgerald:* Witt kernels of function fields extensions, Pacific J. Math. **109** (1983), 89–106.

[9] *D. W. Hoffmann:* On 6-dimensional quadratic forms isotropic over the function field of a quadric, Comm. Alg. **22** (1994), 1999–2014.

[10] *D. W. Hoffmann:* Isotropy of quadratic forms over the function field of a quadric, Math. Z. **220** (1995), 461–476.

[11] *D. W. Hoffmann:* Twisted Pfister forms, Doc. Math. **1** (1996), 67–102.

[12] *D. W. Hoffmann:* On quadratic forms of height 2 and a theorem of Wadsworth, Trans. Amer. Math. Soc. **348** (1996), 3267–3281.

[13] *O. T. Izhboldin, N. A. Karpenko:* On the group $H^3(F(\psi, D)/F)$, Doc. Math. **2** (1997), 297–311.

[14] *O. T. Izhboldin, N. A. Karpenko:* Isotropy of 6-dimensional quadratic forms over function fields of quadrics, J. Alg. **209** (1998), 65–93.

[15] *O. T. Izhboldin, N. A. Karpenko:* Isotropy of virtual Albert forms over function fields of quadrics, Math. Nachr. **206** (1999), 111–122.

[16] *O. T. Izhboldin, N. A. Karpenko:* Isotropy of 8-dimensional quadratic forms over function fields of quadrics, Comm. Algebra **27** (1999), 1823–1841.

[17] *N. Jacobson:* Some applications of Jordan norms to involutorial simple associative algebras, Adv. in Math. **48** (1983), 149– 165.

[18] *B. Kahn:* Lower $\mathcal{H}$-cohomology of higher-dimensional quadrics, Arch. Math. (Basel) **65**, (1995), 244–250.

[19] *B. Kahn:* A descent problem for quadratic forms, Duke Math. J. **80.1** (1995), 139–159.

[20] *B. Kahn, M. Rost, R. Sujatha:* Unramified cohomology of quadrics, I, Amer. J. Math. **120** (1998), 841–891.

[21] *B. Kahn, R. Sujatha:* Unramified cohomology of quadrics, II, Duke Math. J. **106** (2001), 449–484.

[22] *B. Kahn, R. Sujatha:* Motivic cohomology and unramified cohomology of quadrics, J. Eur. Math. Soc. **2** (2000), 145–177.

[23] *N. Karpenko:* On topological filtration for triquaternion algebra, Nova J. of Math., Game Theory and Algebra **4** (1996), 161–167.
[24] *N. Karpenko:* On anisotropy of orthogonal involutions. J. Ramanujan Math. Soc. **15** (2000), 1–22.
[25] *M. Knebusch:* Generic splitting of quadratic forms I, Proc. London. Math. Soc. **33** (1976), 65–93.
[26] *M. Knebusch:* Generic splitting of quadratic forms II, Proc. London Math. Soc. **34** (1977), 1–31.
[27] *A. Laghribi:* Isotropie de certaines formes quadratiques de dimension 7 et 8 sur le corps des fonctions d'une quadrique, Duke Math. J. **85** (1996), 397–409.
[28] *A. Laghribi:* Formes quadratiques en 8 variables dont l'algèbre de Clifford est d'indice 8, K-Theory J. **12** (1997), 371–383.
[29] *A. Laghribi:* Formes quadratiques de dimension 6, Math. Nach. **204** (1999), 125–135.
[30] *A. Laghribi:* Sur le problème de descente des formes quadratiques, Arch. Math. (Basel) **73** (1999), 18–24.
[31] *T. Y. Lam:* The algebraic theory of quadratic forms, (2*nd* printing) Benjamin, New York, 1980.
[32] *D. Leep:* Function fields results, notes manuscrites prises par T. Y. Lam (1989).
[33] *A. S. Merkurjev:* The norm residue symbolof degree 2 (in Russian), Dokl. Akad. Nauk SSSR **261** (1981), 542–547. English translation: Soviet Math. Doklady **24** (1981), 546–551.
[34] *A. S. Merkurjev:* Simple algebras and quadratic forms, (in Russian), Izv. Akad. Nauk. SSSR **55** (1991), 218–224; English translation: Math. USSR-Izv. **38** (1992), 215–221.
[35] *A. S. Merkurjev, A. A. Suslin:* The norm residue homomorphism of degree 3, (in Russian), Izv. Akad. Nauk. SSSR Ser. mat. **54** (1990), 339–356. English translation: Math. USSR-Izv. **36** (1991), 349–368.
[36] *D. Orlov, A. Vishik, V. Voevodsky:* An exact sequence for Milnor's K-theory with applications to quadratic forms, preprint, 2000.
[37] *R. Parimala:* Witt groups of anisotropic conis, elliptic and hyperelliptic conics, J. Number Theory **28** (1988), 69–93.
[38] *E. Peyre:* Products of Severi–Brauer varieties and Galois cohomology, Proceedings of Symposia in Pure Mathematics, AMS Summer Research Institute, Santa Barbara, 1992. Volume **58.2**, 369–401.
[39] *E. Peyre:* Corps de fonctions de variétés homogènes et cohomologie galoisienne, C. R. Acad. Sci. Paris **321** (1995), 891–896.
[40] *M. Rost:* Quadratic forms isotropic over the function field of a conic, Math. Ann. **288** (1990), 511–513.
[41] *M. Rost:* Hilbert 90 for K_3 for degree-two extensions, preprint, 1986.
[42] *W. Scharlau:* Quadratic and Hermitian forms, Springer, Berlin, 1986.
[43] *A. Schofield, M. van den Bergh:* The index of a Brauer class on a Brauer-Severi variety, Trans. Amer. Math. Soc. ; Volume **333.2**, 1992, 729–739.
[44] *J.-P. Tignol:* Algèbres indécomposables d'exposant premier, Adv. Math. **65.3** (1987), 205–228.
[45] *J.-P. Tignol:* Réduction de l'indice d'une algèbre simple centrale sur le corps des fonctions d'une quadrique, Bull. Soc. Math. Belg. Sér. A **42** (1990), 735–745.

[46] *A. Vishik:* On the dimension of anisotropic forms in I^n, Max-Planck-Institut für Mathematik in Bonn, preprint MPI **2000-11** (2000), 1–41.

[47] *A. Wadsworth:* Similarity of quadratic forms and isomorphism of their function fields, Trans. Amer. Math. Soc. **208** (1975), 352–358.

Institut de Mathématiques de Jussieu, 175–179 rue du Chevaleret, 75013 Paris

E-mail address: kahn@math.jussieu.fr

Faculté des Sciences Jean Perrin, Rue Jean Souvraz - SP 18, F-62307 Lens

E-mail address: laghribi@euler.univ-artois.fr