COHOMOLOGY OF TORI OVER P-ADIC CURVES

CLAUS SCHEIDERER AND JOOST VAN HAMEL

Abstract. We establish a duality in the cohomology of arbitrary tori over smooth but not necessarily projective curves over a p-adic field. This generalises Lichtenbaum–Tate duality between the Picard group and the Brauer group of a smooth projective curve.

Introduction

Let C be a connected smooth projective curve over a p-adic field k, and let T_0 be a torus over the function field $K = k(C)$ of C. In order to relate the Galois cohomology of T_0 over K to the local cohomology groups of T_0 in the closed points of C (see [Sch2]), one needs a global duality theory for 'integral models' of T_0, namely tori over open subcurves of C. The aim of this paper is to set up such a theory.

The pattern of the result we seek is given by a theorem of Lichtenbaum [Li], which establishes a natural non-degenerate pairing

$$\mathrm{Pic}(C) \times \mathrm{Br}(C) \to \mathbb{Q}/\mathbb{Z}$$

('Lichtenbaum–Tate duality'). This can be seen as a pairing

$$H^1(C, \mathbb{G}_m) \times H^2(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$$

of étale cohomology groups.

In general, a K-torus T_0 does not necessarily have a model over C, but it always admits a model T over some open subset V of C. As is usual in cohomological duality over non-compact spaces, one has to work with a form of cohomology with compact supports. We define such a cohomology theory which is specially tailored for tori, and which we denote by $H^*_{cc}(V, T)$ (see Section 2). It is slightly different from the general notion of cohomology with compact supports in étale cohomology. We also need the (standard) notion of the *dual torus* T' of T (see Section 1.3). Our main result establishes for every integer q a natural pairing

$$H^q_{cc}(V, T) \times H^{3-q}(V, T') \to \mathbb{Q}/\mathbb{Z}$$

1991 *Mathematics Subject Classification.* Primary 14G20; secondary 14F22, 14L15, 11S25.

Key words and phrases. Algebraic tori, étale cohomology, algebraic curves, p-adic fields, duality theorems, Lichtenbaum–Tate duality, Poincaré duality, Brauer group, Picard group.

which is non-degenerate on the left if $q \neq 3$, and non-degenerate on the right if $q \neq 2$ (see Theorem 4.8).

We first prove our result in the case where T is the split torus $\mathbb{G}_{m,V}$. Then the general case is deduced from the split case by a descent argument. An important role is played by the perfect (cup product) pairings of the cohomology groups of the torsion groups,

$$H^q_c(V,\, {}_nT) \times H^{4-q}(V,\, {}_nT') \to \mathbb{Q}/\mathbb{Z}$$

(see Section 1.5) and by the Kummer exact sequences. In fact, not only the proof that our pairings are non-degenerate, but even their construction proceeds via these torsion pairings. For the construction, a more elegant approach in the spirit of [vH] would have been possible, but at the cost of employing much heavier technical machinery than the approach we take here. Moreover, and this was something of a surprise, the proof that our pairing is non-degenerate becomes essentially a formality, after embedding the pairing into the Kummer sequence. The crucial descent argument is in the proof of Proposition 4.5. Without the Kummer sequence, a descent argument seems much more cumbersome.

Here is an outline of the structure of the paper. In Section 1 we have gathered the known constructions and results we will need. This ranges from elementary observations regarding abelian groups to Poincaré duality in the étale cohomology of varieties over p-adic fields and Lichtenbaum–Tate duality. In Section 2 we define $H^*_{cc}(V,-)$, our modified cohomology of tori with compact supports, and study how it relates to the standard definition of étale cohomology with compact supports, $H^*_c(V,-)$. We also establish links between $H^*_{cc}(V,\mathbb{G}_m)$, motivic cohomology and generalised Jacobians. In Section 3 we construct the pairing $H^q_{cc}(V,\mathbb{G}_m) \times H^{3-q}(V,\mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$, using the Kummer sequences and the cup product pairing with coefficients in the nth roots of unity. To derive our duality in this case from Lichtenbaum–Tate duality, we use the localisation exact sequences (see Lemma 2.8). In Example 3.8 we also see where the degeneracies of the pairings pop up. In Section 4 we construct our pairings for arbitrary tori, and prove the duality result Theorem 4.8.

Acknowledgements. This paper results from a four-month visit of the second author to the Gerhard Mercator Universität in Duisburg, financed by the European Union in the framework of the network *K-Theory and Linear Algebraic Groups* (Contract ERB FMRX CT-97-0107). He would like to thank the institute for its hospitality and the European Union for the financial support.

1. Preliminaries

1.1. **Abelian groups.** Let M be an abelian group. Given $n \in \mathbb{N}$, we let ${}_nM$ resp. M/n be the kernel resp. the cokernel of $M \xrightarrow{n} M$, the multiplication by n map on M. We write $M^* := \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ for the group of characters of M. If M is a topological group, we write $M^\vee := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}/\mathbb{Z})$ for the group of continuous characters, where $\mathbb{Q}/\mathbb{Z}$ has the discrete topology. $M \mapsto M^\vee$ is an exact functor on the category of compact abelian groups.

For a prime ℓ we define $\mathrm{Tate}_\ell(M) := \varprojlim({}_{\ell^\nu}M)$; this is the *$\ell$-adic Tate module* of M. It is a torsion-free $\mathbb{Z}_\ell$-module. If ${}_\ell M$ is finite, then $\mathrm{Tate}_\ell(M)$ is free of finite rank. We let $\mathrm{Tate}(M) := \varprojlim({}_nM)$ be the *total Tate module* of M, and we have

$$\mathrm{Tate}(M) = \prod_\ell \mathrm{Tate}_\ell(M).$$

If ${}_\ell M$ is finite for every prime ℓ, then $\mathrm{Tate}(M)$ is a torsion-free profinite abelian group. The functor $\mathrm{Tate}(-)$ is left exact. If D is the largest divisible torsion subgroup of M, the inclusion $D \subset M$ induces an isomorphism $\mathrm{Tate}(D) \xrightarrow{\sim} \mathrm{Tate}(M)$.

The profinite completion of M is denoted by $\widehat{M}$ (or $M\widehat{\ }$). The group M is called *residually finite* if the natural map $M \to \widehat{M}$ is injective. There is a natural map $\varprojlim M/n \to \widehat{M}$. It is an isomorphism if M/ℓ is finite for every prime ℓ.

Let $D(M) := \ker(M \to \varprojlim M/n) = \bigcap_n nM$, the subgroup of divisible elements of M. We need the following elementary observation:

Lemma 1.1. *If $0 \to A \to B \to C$ is an exact sequence of abelian groups, and if for every prime ℓ the ℓ-primary torsion subgroup of C has finite exponent, then the sequence $0 \to D(A) \to D(B) \to D(C)$ is exact.*

Proof. Let $x \in A \cap D(B)$, and fix a prime ℓ. For every $n \in \mathbb{N}$ there is $y_n \in B$ with $\ell^n y_n = x$. By the hypothesis on C there exists $r \in \mathbb{N}$ with $\ell^r y_n \in A$ for every n. Thus $x \in \ell^m A$ for every $m \in \mathbb{N}$. □

Moreover, we need the following facts about profinite completions of locally compact groups:

Lemma 1.2. *Let A be a locally compact abelian group, and suppose that A contains a profinite open subgroup A_0 which is (topologically) finitely generated.*

(a) *Every subgroup of A of finite index is open in A.*
(b) *Assume further that the abelian group A/A_0 is finitely generated. Then the map $A \to \widehat{A}$ induces isomorphisms ${}_nA \xrightarrow{\sim} {}_n\widehat{A}$ and $A/n \xrightarrow{\sim} \widehat{A}/n$ for every $n \in \mathbb{N}$.*

Proof. The Sylow subgroups of A_0 are topologically finitely generated, and A_0 is their direct product. So (a) follows easily from the corresponding fact for finitely generated pro-p groups (see [Se2], I.4.2 ex. 6). As for (b), $A = A_1 \times F$ where F is free abelian (of finite rank) and A_1 is a profinite abelian group every finite index subgroup of which is open. Hence $\widehat{A} = A_1 \times \widehat{F}$, from which (b) is clear. $\square$

1.2. **Pairings.** Let A be a discrete torsion abelian group and B a profinite abelian group. Let $A \times B \to \mathbb{Q}/\mathbb{Z}$ be a continuous pairing (i.e., the annihilator of each $a \in A$ is open in B). The following lemma is a standard consequence of Pontryagin duality ([Po], see also [RiZa], Sec. 2.9, or [Wi], Sec. 6.4).

Lemma 1.3. *For a pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$ as above, we have that the induced map $A \to B^\vee$ is injective (resp., surjective, resp., bijective) if and only if the induced map $B \to A^*$ is surjective (resp., injective, resp., bijective).*

The pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$ is called *perfect* if the induced homomorphisms $A \to B^\vee$ and $B \to A^*$ are bijective. A pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$ between arbitrary abelian groups is said to be *non-degenerate on the left* (resp., *on the right*) if the induced homomorphism $A \to B^*$ (resp., $B \to A^*$) is injective. The pairing is called *non-degenerate* if it is so on the left and on the right.

Corollary 1.4. *A continuous pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$ between a discrete torsion abelian group A and a profinite abelian group B is perfect if and only if it is non-degenerate.*

Definition 1.5. Let I be a directed set. Let $\{A_i\}_{i\in I}$ be an inductive system of abelian groups and $\{B_i\}_{i\in I}$ a projective system of abelian groups, both indexed by I. A collection of pairings $\langle -,-\rangle_i \colon A_i \times B_i \to \mathbb{Q}/\mathbb{Z}$ indexed by $i \in I$ will be called a *system of pairings between* $\{A_i\}$ *and* $\{B_i\}$ if the pairings $\langle -,-\rangle_i$ are compatible with the transition maps.

Lemma 1.6. *Let I be a directed set. Let $\langle -,-\rangle_i$ $(i \in I)$ be a system of pairings between an inductive system of abelian groups $\{A_i\}_{i\in I}$ and a projective system of abelian groups $\{B_i\}_{i\in I}$.*

(a) *There is a unique continuous pairing $\langle -,-\rangle \colon (\varinjlim A_i) \times (\varprojlim B_i) \to \mathbb{Q}/\mathbb{Z}$ which is compatible with the pairings $\langle -,-\rangle_i$.*
(b) *If the groups A_i, B_i are finite and the pairings $\langle -,-\rangle_i$ are perfect $(i \in I)$, then the limit pairing (a) is a perfect pairing between the discrete torsion group $\varinjlim A_i$ and the profinite group $\varprojlim B_i$.*

Example 1.7. Any pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$ of discrete abelian groups gives rise to a continuous pairing

$$A_{\mathrm{tors}} \times \widehat{B} \to \mathbb{Q}/\mathbb{Z},$$

and also to a continuous pairing

$$(A \otimes \mathbb{Q}/\mathbb{Z}) \times \mathrm{Tate}(B) \to \mathbb{Q}/\mathbb{Z}$$

(which is the pairing $\varinjlim(A/n) \times \varprojlim({}_nB) \to \mathbb{Q}/\mathbb{Z}$).

Lemma 1.8. *Let $A \times B \to \mathbb{Q}/\mathbb{Z}$ be a perfect continuous pairing between a torsion abelian group A and a profinite abelian group B. Then for any $n \in \mathbb{N}$, the induced pairings*

$$({}_nA) \times B/n \to \mathbb{Q}/\mathbb{Z} \quad \textit{and} \quad A/n \times ({}_nB) \to \mathbb{Q}/\mathbb{Z}$$

are perfect.

Proof. Consider the multiplication by n exact sequences for A and B, and use exactness of the functors $(-)^*$ and $(-)^\vee$. □

Lemma 1.9. *Suppose we have two exact sequences $0 \to A_1 \to A_2 \to A_3 \to 0$ and $0 \leftarrow B_1 \leftarrow B_2 \leftarrow B_3 \leftarrow 0$ of abelian groups, and pairings $\beta_i \colon A_i \times B_i \to \mathbb{Q}/\mathbb{Z}$ ($i = 1, 2, 3$) that are compatible with these sequences. Suppose further that the middle pairing β_2 is non-degenerate.*

(a) *β_1 is non-degenerate on the left, and β_3 is non-degenerate on the right.*
(b) *β_1 is non-degenerate if and only if β_3 is non-degenerate.*

Proof. (a) is obvious, and (b) is immediate using the snake lemma. □

The proof of the following five lemma for pairings is immediate.

Lemma 1.10. *Suppose we have two exact sequences of abelian groups*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} A_5$$

and

$$B_1 \xleftarrow{g_1} B_2 \xleftarrow{g_2} B_3 \xleftarrow{g_3} B_4 \xleftarrow{g_4} B_5,$$

together with pairings $\beta_i \colon A_i \times B_i \to \mathbb{Q}/\mathbb{Z}$ ($i = 2, 3, 4$) that are compatible with the two sequences.

(a) *If β_4 is non-degenerate on the left and* $\ker(g_1)^\perp = \mathrm{im}(f_1)$, *then β_3 is non-degenerate on the left.*
(b) *If β_2 is non-degenerate on the right and* $\ker(f_4)^\perp = \mathrm{im}(g_4)$, *then β_3 is non-degenerate on the right.*

1.3. **Tori and their duals.** A general reference for tori over schemes is [SGA3], tome II. Given a scheme S, a torus over S (or: S-torus) is an S-group scheme T which locally with respect to the *fpqc* topology is isomorphic to $(\mathbb{G}_{m,S})^r$ for some integer $r \geq 0$. The S-torus T is said to be split if $T \cong (\mathbb{G}_{m,S})^r$ for some $r \geq 0$. If S is locally noetherian and normal, any S-torus is *isotrivial*, i.e., it splits over a finite étale covering S' of S (see [SGA3], Th. X.5.16). We will call such a covering $S' \to S$ a *trivialising covering* for T. If moreover S is connected, then S' can be taken to be finite and Galois over S.

Given an S-torus T, the *character group scheme* of T is $\mathrm{X}^*(T) = \mathcal{H}om_{S-gr}(T, \mathbb{G}_m)$. This is a *lattice group scheme* over S, by which we mean an S-group scheme which is locally constant free abelian of finite rank. We have $T = \mathcal{H}om_{S-gr}(\mathrm{X}^*(T), \mathbb{G}_m)$, and the functor $T \mapsto \mathrm{X}^*(T)$ defines a duality between the category of S-tori and the category of lattice group schemes over S.

Assume that S is locally noetherian, normal and connected. Then the category of S-tori is anti-equivalent to the category of isotrivial group schemes over S which are locally constant free abelian, by means of the functor X^*. We can also fix a geometric point $\bar{s}$ of S and identify the latter category with the category of $\pi_1(S, \bar{s})$-lattices (i.e., of discrete $\pi_1(S, \bar{s})$-modules which are free abelian of finite rank as abelian groups).

The *cocharacter group scheme* of T is $\mathrm{X}_*(T) = \mathcal{H}om_{S-gr}(\mathbb{G}_m, T)$. We have $T = \mathrm{X}_*(T) \otimes \mathbb{G}_m$. The natural pairing

$$\mathrm{X}^*(T) \times \mathrm{X}_*(T) \to \mathcal{H}om_{S-gr}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}_S$$

(of group schemes over S) identifies each of $\mathrm{X}^*(T)$ and $\mathrm{X}_*(T)$ with the dual lattice group scheme of the other.

The *dual torus* T' of T is defined to be

$$T' := \mathcal{H}om_{S-gr}(\mathrm{X}_*(T), \mathbb{G}_m) = \mathrm{X}^*(T) \otimes \mathbb{G}_m.$$

We have $\mathrm{X}^*(T') = \mathrm{X}_*(T)$ and $(T')' = T$ (canonical isomorphism).

Base change gives us for any morphism $f\colon X \to S$ a torus $T_X := T \times_S X$ on X, the pullback of T via f. Observe that in general T_X does not represent the étale sheaf f^*T (e.g., when f is the inclusion of a closed point on a curve). It does, however, if the map f is étale.

1.4. **Galois cohomology and duality.** Let F be a field and fix a separable closure F_s of F. By $G_F = \mathrm{Gal}(F_s/F)$ we denote the absolute Galois group of F. If M is a discrete G_F-module, Galois cohomology $H^q(G_F, M)$ is denoted by $H^q(F, M)$. For a group scheme G over F, we will mostly write $H^q(F, G)$ rather than $H^q(G_F, G(F_s))$. If M is a torsion module whose torsion is prime to $\mathrm{char}(F)$, then $M(i)$ is the i-th Tate twist of M, for $i \in \mathbb{Z}$.

Theorem 1.11 (Tate–Nakayama)**.** *Let T be a torus over a p-adic field k. The cup product pairing*

$$H^q(k, T) \times H^{2-q}(k, \mathrm{X}^*(T)) \to H^2(k, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z} \tag{1}$$

is non-degenerate for all $q \in \mathbb{Z}$. Moreover, it is a perfect pairing of finite groups for $q = 1$, and induces perfect pairings

$$H^0(k, T)^{\frown} \times H^2(k, \mathrm{X}^*(T)) \to \mathbb{Q}/\mathbb{Z}$$

for $q = 0$ and

$$H^0(k, \mathrm{X}^*(T))^{\frown} \times H^2(k, T) \to \mathbb{Q}/\mathbb{Z}$$

for $q = 2$, each between a profinite group and a discrete torsion group.

Proof. See [Se2], II.5.8, Thm. 6. (There the completion of $H^0(k,T)$ in the $q=0$ case is taken with respect to the system of open subgroups of finite index. By Lemma 1.2, the result is the same as the profinite completion of the abstract (discrete) group $H^0(k,T)$.) $\square$

1.5. **Poincaré duality for curves over p-adic fields.** By an algebraic variety over a field k, we mean a separable reduced k-scheme of finite type. A curve over k is a purely one-dimensional algebraic variety over k. Unless mentioned otherwise, all cohomology groups $H^*(-,-)$ will be étale cohomology. In view of the equivalence between the Galois cohomology and the étale cohomology of $\operatorname{Spec} k$, this is consistent with the notation in the previous section. Let k be a p-adic field, let C be a nonsingular projective curve over k (which in general we do not assume to be connected), let V be a dense open subscheme of C, and let $j\colon V \hookrightarrow C$ denote the inclusion. Given any étale sheaf F on V, the cohomology of F with compact supports is defined as

$$H^q_c(V,F) := H^q(C, j_!F),$$

where $j_!$ is the extension by zero. We have a trace map

$$\operatorname{tr}\colon\ H^4_c(V,\mathbb{Q}/\mathbb{Z}(2)) \to H^2(k,\mathbb{Q}/\mathbb{Z}(1)) = \mathbb{Q}/\mathbb{Z}$$

which is an isomorphism if (and, in fact, only if) V is connected. Indeed, $H^4_c(V,\mathbb{Q}/\mathbb{Z}(2)) = H^2\big(k,\, H^2_c(\overline{V},\mathbb{Q}/\mathbb{Z}(2))\big)$ by the Hochschild-Serre spectral sequence, and tr is the map induced by the trace $H^2_c(\overline{V},\mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{Q}/\mathbb{Z}(1)$ (see for example [Mi1],§V.2).

Theorem 1.12 (Poincaré duality)**.** *Let V be a nonsingular, not necessarily projective curve over a p-adic field k. Let M be a locally constant étale sheaf on V with finite stalks. Then the cup product pairing*

$$H^q_c(V,M) \times H^{4-q}(V,M^\vee(2)) \to H^4_c(V,\mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\operatorname{tr}} \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite abelian groups, for every $q \in \mathbb{Z}$.

Proof. This is a formal consequence of geometric Poincaré duality in étale cohomology, and Tate duality in the Galois cohomology of k with finite coefficients. See for example [Sa], Lemma 2.9 (or compare [Mi2], Th. II.7.6). $\square$

Corollary 1.13. *Let V be as above, and let T be a torus over V, with dual torus T'. Then the cup product pairing gives for any $q \in \mathbb{Z}$ and any $n \in \mathbb{Z}$ a perfect pairing*

$$H^q_c(V,{}_nT) \times H^{4-q}(V,{}_nT') \to H^4_c(V,\mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\operatorname{tr}} \mathbb{Q}/\mathbb{Z}$$

of finite abelian groups.

Proof. For the n-torsion subgroups we have ${}_nT = \mathcal{H}om(\mathrm{X}^*(T), \mu_n) = M^\vee(1)$ with $M := \mathrm{X}^*(T)/n$, and ${}_nT' = \mathrm{X}^*(T) \otimes \mu_n = M(1)$. Therefore, the perfectness for any $n \in \mathbb{N}$ is a particular case of Poincaré duality (Theorem 1.12). □

Note that for any torus T over V, the groups $H^q(V,T)/n$ are finite (for all $q \in \mathbb{Z}$, $n \in \mathbb{N}$), by the exact sequences $0 \to {}_nT \to T \xrightarrow{n} T \to 0$ and the finiteness of the groups $H^q(V, {}_nT)$. In particular, we have

$$H^q(V,T)\widehat{\ } = \varprojlim_n H^q(V,T)/n$$

for the profinite completions (cf. Section 1.1). We will use the notation

$$H^q(V, \mathrm{Tate}(T)) := \varprojlim_n H^q(V, {}_nT)$$

and

$$H^q_c(V, \mathrm{Tate}(T)) := \varprojlim_n H^q_c(V, {}_nT)$$

for all $q \in \mathbb{Z}$. These are profinite abelian groups.

Corollary 1.14. *Let T be a torus over V. The cup product pairings of Corollary* 1.13 *induce perfect pairings*

$$H^q_c(V, T_{\mathrm{tors}}) \times H^{4-q}(V, \mathrm{Tate}(T')) \to \mathbb{Q}/\mathbb{Z} \tag{2}$$

and

$$H^q_c(V, \mathrm{Tate}(T')) \times H^{4-q}(V, T_{\mathrm{tors}}) \to \mathbb{Q}/\mathbb{Z} \tag{3}$$

(in the sense of Section 1.2*) for any $q \in \mathbb{Z}$.*

Proof. Consider the pairings of sheaves

$${}_nT \times {}_nT' \to \mathbb{Z}/n(2) \hookrightarrow \mathbb{Q}/\mathbb{Z}(2)$$

constructed above for all $n \in \mathbb{N}$. It follows from the construction that for any positive integers m and n the natural maps ${}_nT \hookrightarrow {}_{mn}T$ and ${}_{mn}T' \twoheadrightarrow {}_nT'$ are adjoint with respect to these pairings. Hence the induced maps in cohomology are adjoint with respect to the cup product pairings, and the statement is an immediate consequence of Lemma 1.6 and Corollary 1.13. □

Let $\pi\colon W \to V$ be a finite flat morphism of nonsingular curves over k. For any smooth group scheme G over V we have trace maps

$$\pi_*\colon\ H^q(W, G_W) \to H^q(V, G)$$

and

$$\pi_*\colon\ H^q_c(W, G_W) \to H^q_c(V, G)$$

(see [SGA4], XVII.6.3). For M as in Theorem 1.12, these maps are adjoint to the pullback maps with respect to the cup product pairing. Namely, we have

$$\langle \pi_*\omega, \eta\rangle = \langle \omega, \pi^*\eta\rangle$$

for $\omega \in H_c^q(W, \pi^* M)$ and $\eta \in H^{4-q}(V, M^\vee(2))$, resp. for $\omega \in H^q(W, \pi^* M)$ and $\eta \in H_c^{4-q}(V, M^\vee(2))$ (see for example [SGA4.5], Dualité).

1.6. Lichtenbaum–Tate duality for curves over p-adic fields.

Theorem 1.15 (Lichtenbaum–Tate duality [Li]). *Let C be a smooth projective curve over a p-adic field k. For every $q \in \mathbb{Z}$ there is a natural non-degenerate pairing*

$$H^q(C, \mathbb{G}_m) \times H^{3-q}(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}. \tag{4}$$

The induced pairings

$$H^0(C, \mathbb{G}_m)^\frown \times H^3(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \tag{5}$$

and

$$H^1(C, \mathbb{G}_m)^\frown \times H^2(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \tag{6}$$

are perfect.

Proof. We may assume C to be geometrically connected. Consider the Hochschild–Serre spectral sequence

$$E_2^{ij} = H^i(k,\, H^j(\overline{C}, \mathbb{G}_m)) \Rightarrow H^{i+j}(C, \mathbb{G}_m). \tag{7}$$

Since $\mathrm{scd}(k) = 2$ and $H^j(\overline{C}, \mathbb{G}_m) = 0$ for $j \geq 2$, it follows that $H^q(C, \mathbb{G}_m) = 0$ for $q \geq 4$. So only the cases $q = 0$ and $q = 1$ need to be considered. For $q = 1$, our pairing is the pairing $\mathrm{Pic}(C) \times \mathrm{Br}(C) \to \mathbb{Q}/\mathbb{Z}$ constructed by Lichtenbaum in [Li]. He proves that $\mathrm{Br}(C) \to \mathrm{Pic}(C)^\vee$ is an isomorphism, where $\mathrm{Pic}(C)$ carries its natural locally compact topology (*loc. cit.*, Thm. 4, p. 131). Since $\mathrm{Pic}(C) \to \widehat{\mathrm{Pic}(C)}$ induces an isomorphism $\big(\widehat{\mathrm{Pic}(C)}\big)^\vee \xrightarrow{\sim} \big(\mathrm{Pic}(C)\big)^\vee$, this implies perfectness of the pairing $\widehat{\mathrm{Pic}(C)} \times \mathrm{Br}(C) \to \mathbb{Q}/\mathbb{Z}$, by Lemma 1.3.

For $q = 0$, the pairing (4) coincides with the obvious pairing $k^* \times H^2(k, \mathbb{Z}) \to \mathrm{Br}(k) = \mathbb{Q}/\mathbb{Z}$ from local class field theory, and the assertions are classical (it is the special case $T = \mathbb{G}_m$, $q = 0$ of Tate–Nakayama duality as treated in Theorem 1.11). To see that $H^3(C, \mathbb{G}_m) = H^2(k, \mathbb{Z})$, use the Hochschild–Serre spectral sequence (7) and the fact that $H^2(k, \mathrm{Pic}_0(\overline{C})) = 0$ (see [Se2], II.5.3). $\square$

Remarks 1.16.

1. For $q \geq 2$, $H^q(C, \mathbb{G}_m)$ is a (discrete) abelian torsion group, whereas for $q = 0$ and $q = 1$, it is an extension of a free abelian group of finite rank by a compact p-adic Lie group.

2. Lichtenbaum's construction of the pairing $\mathrm{Pic}(C) \times \mathrm{Br}(C) \to \mathbb{Q}/\mathbb{Z}$ is as follows. Suppose given a Weil divisor $D = \sum_P n_p \cdot P$ on C, representing a class $[D] \in \mathrm{Pic}(C)$, and a central simple $k(C)$-algebra A which is everywhere unramified on C, therefore representing a class

$[A] \in \mathrm{Br}(C)$. Let $A(P)$ denote the residue central simple algebra over $k(P)$, for P any closed point on C. Then

$$\langle [D], [A] \rangle = \sum_P n_P \cdot \mathrm{inv}_{k(P)} A(P),$$

where $\mathrm{inv}_K \colon \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ is the invariant from local class field theory, for K any local field.

3. For any abelian variety A over a p-adic field k, with dual abelian variety A', Tate established a natural pairing $H^0(k, A) \times H^1(k, A') \to \mathbb{Q}/\mathbb{Z}$ and proved that it is perfect [Ta]. Lichtenbaum's pairing is closely related to Tate's pairing for $A := J$, the Jacobian of the curve C. In fact, Lichtenbaum's proof proceeds by establishing this relation and then using perfectness of Tate's pairing. His conclusion also uses a theorem of Roquette [Ro] (to which Lichtenbaum gives a new proof), to the end that the order of $\ker\bigl(\mathrm{Br}(k) \to \mathrm{Br}(C)\bigr)$ is precisely the index of C.

Remark 1.17. When $\pi \colon \tilde{C} \to C$ is a finite covering of smooth projective curves, it follows from the construction that the trace map $\pi_* \colon H^*(\tilde{C}, \mathbb{G}_m) \to H^*(C, \mathbb{G}_m)$ is adjoint to the pullback map $\pi^* \colon H^*(C, \mathbb{G}_m) \to H^*(\tilde{C}, \mathbb{G}_m)$ with respect to Lichtenbaum's pairing.

1.7. **Relating Poincaré duality to Lichtenbaum–Tate duality.** We will now check that Lichtenbaum's pairing (4) is compatible with the cup product pairing for coefficients in the roots of unity. Let $n \in \mathbb{N}$ and $q \in \mathbb{Z}$, and consider the Kummer exact sequence

$$0 \to H^q(C, \mathbb{G}_m)/n \xrightarrow{\delta_n} H^{q+1}(C, \mu_n) \xrightarrow{\iota_n} {}_nH^{q+1}(C, \mathbb{G}_m) \to 0. \qquad (8)$$

We need the following observation.

Lemma 1.18. *Let C be a smooth projective curve over a p-adic field. Let m, $n \in \mathbb{N}$.*

(a) *The composite map*

$$H^q(C, \mu_m) \xrightarrow{\iota_m} H^q(C, \mathbb{G}_m) \xrightarrow{\delta_n} H^{q+1}(C, \mu_n)$$

coincides with the boundary map

$$\beta_{m,n} \colon H^q(C, \mu_m) \to H^{q+1}(C, \mu_n)$$

associated with the short exact sequence

$$0 \to \mu_n \to \mu_{mn} \xrightarrow{n} \mu_m \to 0.$$

(b) *The map $\beta_{m,n}$ is adjoint to the map $\beta_{n,m}$ with respect to the cup product pairings*

$$H^q(C, \mu_m) \times H^{4-q}(C, \mu_m) \to \mathbb{Q}/\mathbb{Z}$$

and

$$H^{q+1}(C, \mu_n) \times H^{3-q}(C, \mu_n) \to \mathbb{Q}/\mathbb{Z}.$$

We leave the proof as an exercise to the reader. □

Proposition 1.19. *Let C be a smooth projective curve over a p-adic field. Let $n \in \mathbb{N}$.*

(a) *Lichtenbaum's pairing* (4) *induces for every $q \in \mathbb{Z}$ a perfect pairing of finite groups*

$$H^q(C, \mathbb{G}_m)/n \times {}_nH^{3-q}(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}. \tag{9}$$

(b) *The mappings δ_n and ι_n in the Kummer sequence* (8) *are adjoint to each other with respect to the cup product pairings*

$$H^q(C, \mu_n) \times H^{4-q}(C, \mu_n) \to \mathbb{Q}/\mathbb{Z} \tag{10}$$

and the above pairings (9).

Proof. (a) Apply Lemma 1.8 to the pairings (5) and (6) in Theorem 1.15. Note that for $q = 0$ and $q = 1$, the group $A := H^q(C, \mathbb{G}_m)$ is of the type considered in Lemma 1.2(b). Therefore, the torsion and cotorsion of $\widehat{A}$ and of A are the same.

(b) We claim that

$$\langle \delta_n \alpha, \omega \rangle = \langle \alpha, \iota_n \omega \rangle$$

for $\alpha \in H^q(C, \mathbb{G}_m)/n$ and $\omega \in H^{3-q}(C, \mu_n)$. For $q = 0, 3$ this is essentially a formal consequence of the construction of Lichtenbaum's pairing, since the local class field pairing

$$H^0(k, \mathbb{G}_m) \times H^2(k, \mathbb{Z}) \to H^2(k, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

is the cup product.

For $q = 1, 2$ the situation is slightly more subtle. For $q = 1$ we may assume without loss of generality that α is the class $[P]$ of a closed point $P \in C$. It is a standard property of the cycle map that, given a diagram

$$\begin{tikzcd} P \arrow[rr, "i"] \arrow[dr, "\pi"'] & & C \arrow[dl, "\varphi"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

we have

$$\langle \delta_n[P], \omega \rangle = \pi_* i^* \omega$$

(cf. [SGA4.5], Cycle 2.3.1). We see from the construction of the pairing (4) (cf. Remark 1.16.2) that the right hand side of the equation coincides with $\langle [P], \iota_n \omega \rangle$, so we have proved the case $q = 1$.

The case $q = 2$ can be reduced to the case $q = 1$ by using that $H^2(C, \mathbb{G}_m)$ is a torsion group: Indeed, $\alpha = \iota_m \eta$ (modulo n) for some $m \in \mathbb{N}$ and $\eta \in H^2(C, \mu_m)$, so Lemma 1.18 implies that

$$\langle \delta_n \alpha,\, \omega \rangle = \langle \delta_n \circ \iota_m \, \eta,\, \omega \rangle = \langle \eta,\, \delta_m \circ \iota_n \, \omega \rangle,$$

and the case $q = 1$ gives us that

$$\langle \eta, \delta_m(\iota_n \omega) \rangle = \langle \iota_m \eta, \iota_n \omega \rangle = \langle \alpha, \iota_n \omega \rangle.$$

□

Taking the inverse and direct limit of the Kummer exact sequence, we get short exact sequences

$$0 \to H^q(C, \mathbb{G}_m)^\wedge \xrightarrow{\delta_{\mathrm{Tate}}} H^{q+1}(C, \widehat{\mathbb{Z}}(1)) \xrightarrow{\iota_{\mathrm{Tate}}} \mathrm{Tate}\big(H^{q+1}(C, \mathbb{G}_m)\big) \to 0$$

of profinite groups and

$$0 \leftarrow H^{3-q}(C, \mathbb{G}_m)_{\mathrm{tors}} \xleftarrow{\iota_\infty} H^{3-q}(C, \mathbb{Q}/\mathbb{Z}(1)) \xleftarrow{\delta_\infty} H^{2-q}(C, \mathbb{G}_m) \otimes \mathbb{Q}/\mathbb{Z} \leftarrow 0$$

of discrete torsion groups.

Corollary 1.20. *Let C be a smooth projective curve over a p-adic field. For every $q \in \mathbb{Z}$ we have the following diagrams with exact rows of compatible perfect pairings into $\mathbb{Q}/\mathbb{Z}$, which are induced by Lichtenbaum's pairing* (4) *and the cup product pairing:*

$$\begin{tikzcd}
0 \arrow[r] & H^q(C, \mathbb{G}_m)^\wedge \arrow[r, "\delta_{\mathrm{Tate}}"] & H^{q+1}(C, \widehat{\mathbb{Z}}(1)) \arrow[r, "\iota_{\mathrm{Tate}}"] & \mathrm{Tate}\left(H^{q+1}(C, \mathbb{G}_m)\right) \arrow[r] & 0 \\[-20pt]
 & \times & \times & \times & \\[-20pt]
0 & H^{3-q}(C, \mathbb{G}_m)_{\mathrm{tors}} \arrow[l] \arrow[dr] & H^{3-q}(C, \mathbb{Q}/\mathbb{Z}(1)) \arrow[l, "\iota_\infty"'] \arrow[d] & H^{2-q}(C, \mathbb{G}_m) \otimes \mathbb{Q}/\mathbb{Z} \arrow[l, "\delta_\infty"'] \arrow[dl] & 0 \arrow[l] \\
 & & \mathbb{Q}/\mathbb{Z} & &
\end{tikzcd}$$

Proof. The compatibility of the pairings follows from Proposition 1.19(b). By Theorem 1.15 (resp., Proposition 1.19(a)) and Lemma 1.6, the middle (resp., the left) column pairing is perfect. From Lemma 1.9(b) and Corollary 1.4 it follows that the right column pairing is perfect as well. □

2. Cohomology of tori with compact supports

2.1. **Introduction.** Let C be a (not necessarily smooth) projective curve over an arbitrary field F, let $j\colon V \to C$ be the inclusion of an open non-empty subset and let $i\colon Z \hookrightarrow C$ be the inclusion of its reduced closed complement.

Let T be a torus over V. Assume T extends to C, i.e., we have a torus S over C such that $T = S_V = j^*S$. We have two natural possibilities of defining cohomology of T with compact supports. The first is the standard sheaf-theoretic definition

$$H^q_c(V, T) := H^q(C, j_!T),$$

where $j_!T$ is the étale sheaf on C characterised by the short exact sequence

$$0 \to j_!T \to S \to i_*i^*S \to 0.$$

The second, more geometric definition is

$$H^q_{cc}(V, T) := H^q(C, \tilde{T}),$$

where $\tilde{T}$ is the étale sheaf on C defined by the short exact sequence

$$0 \to \tilde{T} \to S \to i_* S_Z \to 0$$

and S_Z is the pullback of S to Z (1.3). However, if T is an arbitrary torus on V which does not extend to a torus S on C, we cannot define $\tilde{T}$ and $H^*_{cc}(V,T)$ in this way.

2.2. **Definitions and basic properties.** Let F be a field and V a (reduced) curve over F. An open immersion $j\colon V \hookrightarrow C$ into a projective curve C will be called a *good compactification* of V if V is dense in C and $Z = C - V$ consists of nonsingular points of C. Every curve V has a good compactification that is unique up to isomorphism.

Definition 2.1. Let $j\colon V \to C$ be a good compactification of V, let $Z = C - V$ be the reduced complement and $i\colon Z \hookrightarrow C$ its inclusion in C. Let T be a torus on V.

(a) The étale sheaf $j_{!!}\mathbb{G}_m$ on C is defined by the exact sequence

$$0 \to j_{!!}\mathbb{G}_m \to \mathbb{G}_{m,C} \to i_*\mathbb{G}_{m,Z} \to 0. \tag{11}$$

(b) The étale sheaf $j_{!!}T$ on C is defined by $j_{!!}T := j_* \mathrm{X}_*(T) \otimes j_{!!}\mathbb{G}_m$.

(c) For $q \in \mathbb{Z}$ we define $H^q_{cc}(V,T) := H^q(C, j_{!!}T)$.

Remarks 2.2.

1. The stalk of the sheaf $j_* \mathrm{X}_*(T)$ at a geometric point $\overline{P}$ of Z is the subgroup of $\mathrm{X}_*(T)$ fixed by the monodromy action at $\overline{P}$. In particular, if T extends to a torus S on all of C, then $j_* \mathrm{X}_*(T) = \mathrm{X}_*(S)$, and thus $j_{!!}T$ coincides with the sheaf $\tilde{T}$ considered in 2.1 before. (Tensoring the sequence (11) with $j_* \mathrm{X}_*(T) = \mathrm{X}_*(S)$ gives the exact sequence $0 \to j_{!!}T \to S \to i_*S_Z \to 1$.) At the other extreme, if no subtorus $\neq \{1\}$ of T can be extended to any larger open subcurve of C, then $j_{!!}T = j_!T$.

2. The sheaf $j_{!!}\mathbb{G}_m$ appears under the name $\mathbb{G}_m(\mathfrak{m})$ in [SGA4], XVIII.1.5.8, where $\mathfrak{m}$ is the ideal defining the reduced closed subscheme Z of C. In [SGA4.5], Arcata VI, the notation ${}_Z\mathbb{G}_m$ is used.

Lemma 2.3. *Let $j\colon V \to C$ and T be as above. We have a natural injective map*

$$j_!T \to j_{!!}T$$

of sheaves on C which is an isomorphism on torsion prime to $\mathrm{char}(F)$. *The cokernel is a sheaf concentrated on Z, uniquely divisible if* $\mathrm{char}(F) = 0$.

Proof. The case $T = \mathbb{G}_m$ is clear, and it implies the general case since $j_!T = j_* \mathrm{X}_*(T) \otimes j_!\mathbb{G}_m$. □

Corollary 2.4. *Let $j\colon V \to C$ and T be as above, and assume* $\mathrm{char}(F) = 0$. *Then for $q \neq 1$, the natural map*

$$H^q_c(V,T) \to H^q_{cc}(V,T)$$

is an isomorphism. For $q = 1$ it is surjective, with a uniquely divisible kernel.

Proof. Follows from the previous lemma, together with the fact that $H^0_c(V,T) = H^0_{cc}(V,T) = 0$ if V is connected and $V \neq C$. □

Remark 2.5. Since $(j_{!!}T)|_V = T$, there are natural maps $H^q_{cc}(V,T) \to H^q(V,T)$, and the composition

$$H^q_c(V,T) \to H^q_{cc}(V,T) \to H^q(V,T)$$

is the canonical map for the usual definition of cohomology with compact supports.

Lemma 2.6 (Kummer sequence). *With notations as above we have, for any $n \in \mathbb{N}$ prime to $\operatorname{char}(F)$ and any $q \in \mathbb{Z}$, a short exact sequence*

$$0 \to H^q_{cc}(V,T)/n \to H^{q+1}_c(V, {}_nT) \to {}_nH^{q+1}_{cc}(V,T) \to 0. \tag{12}$$

Proof. Clear from the exact sequence of sheaves $0 \to j_!({}_nT) \to j_{!!}T \xrightarrow{n} j_{!!}T \to 0$. □

The following lemma, together with its corollary, is obvious:

Lemma 2.7. *Let V be a curve over a field F and T a torus over V. Let $U \subset V$ be a dense open subset such that $Y = V - U$ consists of regular points, and let $j'\colon U \to C$, $j\colon V \to C$ and $i\colon Y \hookrightarrow C$ be the inclusions into a good compactification C of V. Then we have a short exact sequence of sheaves on C*

$$0 \to j'_{!!}T_U \to j_{!!}T \to i_*T_Y \to 0.$$

Corollary 2.8 (Localisation). *With notations as above, we have a long exact sequence*

$$\cdots\ H^q_{cc}(U,T_U) \xrightarrow{j_*} H^q_{cc}(V,T) \to H^q(Y,T_Y)\ \cdots \tag{13}$$

where the second arrow is the composition $H^q_{cc}(V,T) \to H^q(V,T) \xrightarrow{i^*} H^q(Y,T_Y)$.

Example 2.9. When $V \neq \emptyset$ is open in a geometrically irreducible smooth projective curve C over F, and $Z = C - V$, we have exact sequences

$$0 \to F^* \to \bigoplus_{P \in Z} F(P)^* \to H^1_{cc}(V,\mathbb{G}_m) \to \operatorname{Pic}(C) \to 0 \tag{14}$$

and

$$0 \to H^2_{cc}(V,\mathbb{G}_m) \to \operatorname{Br}(C) \to \bigoplus_{P \in Z} \operatorname{Br} F(P) \to H^3_{cc}(V,\mathbb{G}_m) \to \cdots \tag{15}$$

Remarks 2.10.

1. When $\pi\colon V' \to V$ is a finite covering of curves over F and T is a torus on V, one defines pullback maps

$$\pi^*\colon\ H^q_{cc}(V,T) \to H^q_{cc}(V',T_{V'})$$

and trace maps

$$\pi_*\colon\ H^q_{cc}(V',T_{V'}) \to H^q_{cc}(V,T)$$

in the obvious way.

2. It is not hard to derive from the above a more explicit description of the groups $H^1_{cc}(V,\mathbb{G}_m)$ and $H^2_{cc}(V,\mathbb{G}_m)$ when V is smooth. To simplify notation we assume that V is connected. As before, let $V \hookrightarrow C$ be a smooth compactification and $Z = C - V$.

Denoting by $\mathrm{Div}(C)$ the group of divisors on C and by $\mathrm{div}\colon\ F(C)^* \to \mathrm{Div}(C)$ the divisor map, the group $H^1_{cc}(V,\mathbb{G}_m)$ is the cokernel of the map

$$F(C,Z)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(V),$$

where $F(C,Z)^* \subset F(C)^*$ is the subgroup consisting of all rational functions on C that take the value 1 on all points of Z (in particular, they do not have zeroes or poles in Z). In [SGA4.5], Arcata VI, the group $H^1_{cc}(V,\mathbb{G}_m)$ is denoted by $\mathrm{Pic}_Z(C)$. In terms of invertible sheaves it can be described as the group of isomorphism classes of pairs (L,τ), where L is an invertible sheaf on C and $\tau\colon\ L_Z \xrightarrow{\sim} \mathcal{O}_Z$ is a trivialisation of L at Z.

The group $H^2_{cc}(V,\mathbb{G}_m)$ is the subgroup of $\mathrm{Br}(C)$ of unramified central simple $F(C)$-algebras A for which the class of the residue central simple algebra $A(P)$ in $\mathrm{Br}(F(P))$ is trivial for every point $P \in Z$.

2.3. **Relations to generalised Jacobians and motivic cohomology.** Now assume C to be a geometrically connected smooth projective curve over F and $j\colon\ V \hookrightarrow C$ a dense open subset with (reduced) complement $Z = C - V$. Let $\varphi\colon\ C \to \operatorname{Spec} F$ be the structure morphism. As remarked before, the sheaf $j_{!!}\mathbb{G}_m$ already appears in [SGA4], Exp. XVIII, under the name $\mathbb{G}_m(\mathfrak{m})$, where $\mathfrak{m}$ is the ideal defining the reduced closed subscheme $Z \subset C$. In [SGA4], XVIII.1.6.3, it was observed that the higher direct image sheaf $R^1\varphi_* j_{!!}\mathbb{G}_m$ is representable by a group scheme locally of finite type over F (actually, representable not just on the small étale site, where we are working in the present paper, but already on the *fppf* site). This group scheme is denoted by $\mathrm{Pic}_{\mathfrak{m},C/F}$ in *loc. cit.*; it represents the *fppf* sheaf associated to the presheaf that sends an F-scheme S to the set of isomorphism classes of line bundles on $C \times S$ with a given trivialisation at $Z \times S$. In fact, when $V \neq C$, we see from the Hochschild–Serre spectral sequence that

$$H^1_{cc}(V,\mathbb{G}_m) = \mathrm{Pic}_{\mathfrak{m},C/F}(F).$$

The connected component of $\mathrm{Pic}_{\mathfrak{m},C/F}$ containing zero coincides with Rosenlicht's generalised Jacobian of C with modulus $\mathfrak{m}$ (see [Se1]); it is an extension of the Jacobian of the curve C by a torus (compare the exact sequence (14)).

The cohomology groups $H^*_{cc}(V, \mathbb{G}_m)$ are (in degrees ≤ 1) closely related to motivic cohomology. The localisation exact sequence for motivic cohomology $H^*_c(-, \mathbb{Z}(1))$ with compact supports and coefficients in $\mathbb{Z}(1)$ (see [FrVo], Sec. 9) and the fact that $H^*_c(X, \mathbb{Z}(1)) = H^*(X, \mathbb{Z}(1)) = H^*_{\mathrm{Zar}}(X, \mathbb{G}_m)$ when X is smooth and proper (see [Vo], Cor. 3.4.3) give us (for $V \neq C$) an exact sequence

$$0 \to F^* \to \Gamma(Z, \mathbb{G}_m) \to H^2_c(V, \mathbb{Z}(1)) \to \mathrm{Pic}(C) \to 0$$

similar to (14). So our group $H^1_{cc}(V, \mathbb{G}_m)$ should be motivic, i.e., naturally isomorphic to $H^2_c(V, \mathbb{Z}(1))$. We have not actually tried to construct the map that should give this isomorphism.

2.4. **Some properties over p-adic fields.** Let now V be a smooth connected curve over a p-adic field k and T a torus over V. From the Kummer sequence and the finiteness of the groups $H^q_c(V, {}_nT)$ we see that the groups $H^q_{cc}(V, T)/n$ and ${}_nH^q_{cc}(V, T)$ are finite. Therefore

$$H^q_{cc}(V, T)\widehat{\ } = \varprojlim_n H^q_{cc}(V, T)/n$$

for all $q \in \mathbb{Z}$ (cf. Section 1.1).

Lemma 2.11. *Let k be a p-adic field and G a commutative algebraic group over k. Then the group $G(k)$ is residually finite.*

Proof. Since $G(k)/n$ is finite for each $n \in \mathbb{N}$, we have to show that the subgroup $D(G(k))$ of $G(k)$ is trivial, cf. Section 1.1. This is the case if G is either finite, or a connected linear group, or an abelian variety. From this the general case follows, using Lemma 1.1. □

Proposition 2.12. *Let T be any torus on V. Then the groups $H^0(V, T)$, $H^0_{cc}(V, T)$ and $H^1_{cc}(V, T)$ are residually finite.*

Proof. The first two cases are clear. There is a finite Galois covering $W \to V$ such that the torus T_W on W splits. Writing Γ for the Galois group of this covering, we have an exact sequence

$$0 \to H^1(\Gamma, H^0_{cc}(W, T_W)) \to H^1_{cc}(V, T) \to H^1_{cc}(W, T_W).$$

The group $H^1_{cc}(W, T_W)$ is an extension of a free abelian group of finite type by the group $G(k)$ of k-points of a semi-abelian variety G over k, see Section 2.3. Therefore, $H^1_{cc}(W, T_W)$ is residually finite by Lemma 2.11. Using Lemma 1.1, it follows that $H^1_{cc}(V, T)$ is residually finite as well. □

In contrast, the group $H^1(V, T)$ need not be residually finite if V is affine. For an example see 3.8 below.

3. Lichtenbaum–Tate duality for open curves

In the following, let C be a smooth projective curve over a p-adic field k. Let $j\colon V \hookrightarrow C$ be a dense open subset, and let $i\colon Z \hookrightarrow C$ be the reduced complement.

Consider the localisation exact sequences

$$\cdots \to H^q_{cc}(V, \mathbb{G}_m) \xrightarrow{j_*} H^q(C, \mathbb{G}_m) \xrightarrow{i^*} H^q(Z, \mathbb{G}_m) \to \cdots \tag{16}$$

and

$$\cdots \leftarrow H^r(V, \mathbb{G}_m) \xleftarrow{j^*} H^r(C, \mathbb{G}_m) \xleftarrow{i_*} H^r_Z(C, \mathbb{G}_m) \leftarrow \cdots \tag{17}$$

Recall that $H^r_Z(C, \mathbb{G}_m) = H^{r-1}(Z, \mathbb{Z})$. We will define pairings

$$H^q_{cc}(V, \mathbb{G}_m) \times H^{3-q}(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \tag{18}$$

such that sequences (16) and (17) (for $r = 3 - q$) are adjoint to each other with respect to these pairings, the Lichtenbaum–Tate pairings (4) and the Tate–Nakayama pairings 1.11 (for T the direct image of $\mathbb{G}_{m,Z}$ to the base $\operatorname{Spec} k$).

These generalised Lichtenbaum pairings will be constructed from the cup product pairings (3) and (2), using the following two diagrams with exact rows (for $T = T' = \mathbb{G}_m$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^q_{cc}(V,T)^\wedge & \xrightarrow{\delta_{\mathrm{Tate}}} & H^{q+1}_c(V, \mathrm{Tate}(T)) & \xrightarrow{\iota_{\mathrm{Tate}}} & \mathrm{Tate}\left(H^{q+1}_{cc}(V,T)\right) & \longrightarrow & 0 \\
 & & & & \times & & & & \\
0 & \longleftarrow & H^{3-q}(V,T')_{\mathrm{tors}} & \xleftarrow{\iota_\infty} & H^{3-q}(V, T'_{\mathrm{tors}}) & \xleftarrow{\delta_\infty} & H^{2-q}(V,T') \otimes \mathbb{Q}/\mathbb{Z} & \longleftarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathbb{Q}/\mathbb{Z} & & & &
\end{array} \tag{19}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{q-1}_{cc}(V,T) \otimes \mathbb{Q}/\mathbb{Z} & \xrightarrow{\delta_\infty} & H^q_c(V, T_{\mathrm{tors}}) & \xrightarrow{\iota_\infty} & H^q_{cc}(V,T)_{\mathrm{tors}} & \longrightarrow & 0 \\
 & & & & \times & & & & \\
0 & \longleftarrow & \mathrm{Tate}\left(H^{4-q}(V,T')\right) & \xleftarrow{\iota_{\mathrm{Tate}}} & H^{4-q}(V, \mathrm{Tate}(T')) & \xleftarrow{\delta_{\mathrm{Tate}}} & H^{3-q}(V,T')^\wedge & \longleftarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathbb{Q}/\mathbb{Z} & & & &
\end{array} \tag{20}$$

The rows of these diagrams are obtained from the Kummer sequences (8) and (12) by taking inverse resp. direct limits. Note that the pairings in the middle columns are perfect (Corollary 1.14). We will denote the composite maps

$$H^q_{(cc)}(V,T) \to H^q_{(cc)}(V,T)^\wedge \xrightarrow{\delta_{\mathrm{Tate}}} H^{q+1}_{(c)}(V, \mathrm{Tate}(T))$$

by δ_{Tate} as well.

Lemma 3.1. *Let V be as above, and let T be a torus on V.*

(a) *For $q \geq 2$, the groups $H^q_{cc}(V,T)$ and $H^q(V,T)$ are torsion. In particular, $H^q_{cc}(V,T) \otimes \mathbb{Q}/\mathbb{Z} = H^q(V,T) \otimes \mathbb{Q}/\mathbb{Z} = 0$.*

(b) *For $q \geq 3$, the maps*

$$H^q_c(V, T_{\mathrm{tors}}) \xrightarrow{\iota_\infty} H^q_{cc}(V,T)$$

and

$$H^q(V, T_{\mathrm{tors}}) \xrightarrow{\iota_\infty} H^q(V,T)$$

are isomorphisms.

(c) $H^q(V,T) = H^q_{cc}(V,T) = 0$ *for* $q \geq 4$.

Proof. For any étale sheaf F on a curve X over a field, the groups $H^q(X,F)$ are torsion for $q \geq 2$ (see [Sch1] pp. 85-86). This proves (a), and (a) implies (b) by the Kummer exact sequences.

It follows from Poincaré duality (see Corollary 1.14) and from (b) that for (c) it is sufficient to show $H^0(V, \mathrm{Tate}(T)) = H^0_c(V, \mathrm{Tate}(T)) = 0$. Since T is split by a finite Galois covering $W \to V$, and since $H^0_{(c)}(V, \mathrm{Tate}(T))$ embeds into $H^0_{(c)}(W, \mathrm{Tate}(T))$, we can assume $T = \mathbb{G}_m$. In this case, both assertions are clear, since $\mathrm{Tate}(K^*) = \varprojlim_n \mu_n(K) = 0$ for any p-adic field K. □

We now return to the case of split tori ($T = T' = \mathbb{G}_m$):

Lemma 3.2. *Let V be as above. With respect to the cup product pairing, we have for all $q \in \mathbb{Z}$:*

(a) *The image of*

$$\delta_{\mathrm{Tate}} \colon H^{q-1}(V, \mathbb{G}_m)^\wedge \to H^q(V, \widehat{\mathbb{Z}}(1))$$

is orthogonal to the image of

$$\delta_\infty \colon H^{3-q}_{cc}(V, \mathbb{G}_m) \otimes \mathbb{Q}/\mathbb{Z} \to H^{4-q}_c(V, \mathbb{Q}/\mathbb{Z}(1)).$$

(b) *The image of*

$$\delta_{\mathrm{Tate}} \colon H^{q-1}_{cc}(V, \mathbb{G}_m)^\wedge \to H^q_c(V, \widehat{\mathbb{Z}}(1))$$

is orthogonal to the image of

$$\delta_\infty \colon H^{3-q}(V, \mathbb{G}_m) \otimes \mathbb{Q}/\mathbb{Z} \to H^{4-q}(V, \mathbb{Q}/\mathbb{Z}(1)).$$

Proof. It follows from Lemma 3.1 that for $q \leq 1$ or $q \geq 4$, the source of δ_∞ is zero, and that for $q = 3$, the source of δ_{Tate} contains a dense torsion subgroup, hence (a) and (b) hold when $q \neq 2$. (In fact, it can be checked that for $q = 3$ the source of δ_{Tate} is finite). Remains $q = 2$. Both assertions follow once we show for any $n \in \mathbb{N}$ that the images of the Kummer maps

$$\delta_n \colon H^1(V, \mathbb{G}_m) \to H^2(V, \mu_n)$$

and

$$\delta_n \colon H^1_{cc}(V, \mathbb{G}_m) \to H^2_c(V, \mu_n)$$

are orthogonal to each other. For $V = C$, this is true by Proposition 1.19(b). The case of general V follows from this, since the restriction map $H^1(C, \mathbb{G}_m) \to H^1(V, \mathbb{G}_m)$ is surjective and the canonical map $H^2_c(V, \mu_n) \to H^2(C, \mu_n)$ is adjoint to the restriction map $H^2(C, \mu_n) \to H^2(V, \mu_n)$ with respect to the cup product pairing. □

Lemma 3.3. *Let V be as above. The composite mapping*

$$\delta_{\mathrm{Tate}} \circ \iota_\infty \colon\ H^q_c(V, \mathbb{Q}/\mathbb{Z}(1)) \to H^{q+1}_c(V, \widehat{\mathbb{Z}}(1))$$

is adjoint to the composite mapping

$$\delta_{\mathrm{Tate}} \circ \iota_\infty \colon\ H^{3-q}(V, \mathbb{Q}/\mathbb{Z}(1)) \to H^{4-q}(V, \widehat{\mathbb{Z}}(1))$$

with respect to the cup product pairings (2) *and* (3)*.*

Proof. This follows from Lemma 1.18 by taking m, $n \to \infty$. □

Proposition 3.4. *Let V be as above. There is a unique family of pairings*

$$\langle -, - \rangle \colon\ H^q_{cc}(V, \mathbb{G}_m) \times H^{3-q}(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \tag{21}$$

($q \in \mathbb{Z}$) which is compatible

(i) *with the Tate–Nakayama pairing* 1.11 *(for split tori over p-adic fields) and the Lichtenbaum–Tate pairing* 1.15 *for nonsingular projective curves, via the localisation exact sequences* (16) *and* (17)*, and*

(ii) *with the cup product pairing* 1.12 *of torsion sheaves on V, via the Kummer exact sequences.*

Recall (Example 1.7) that the pairings (21) induce pairings between the first and third groups in the exact sequences (19), (20). By (ii) we mean that these exact sequences are compatible with these pairings and with the already constructed pairings between the middle groups.

Proof. We define the pairing by

$$\langle x, \iota_\infty y \rangle := \langle \delta_{\mathrm{Tate}} x, y \rangle$$

if $q = 0, 1$, and by

$$\langle \iota_\infty x, y \rangle := \langle x, \delta_{\mathrm{Tate}} y \rangle$$

if $q = 2, 3$. This gives well-defined pairings by Lemmas 3.1 and 3.2 and by the Kummer exact sequences (19), (20). Now (ii) follows from the construction and Lemma 3.3. Compatibility (i) follows from Proposition 1.19(b) and the fact that the localisation exact sequences with coefficients in $\mathbb{G}_m$ and coefficients in μ_n are compatible with the Kummer exact sequences for all $n \in \mathbb{N}$. □

Theorem 3.5 (Lichtenbaum–Tate duality for open curves)**.** *Let V be a nonsingular curve over a p-adic field. The pairing*

$$H^q_{cc}(V, \mathbb{G}_m) \times H^{3-q}(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$$

is non-degenerate on the left for $q \neq 3$, and non-degenerate on the right for $q \neq 2$. The induced pairings

$$H^0_{cc}(V, \mathbb{G}_m)^\frown \times\ H^3(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z},$$
$$H^1_{cc}(V, \mathbb{G}_m)^\frown \times\ H^2(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$$

and

$$H^2_{cc}(V, \mathbb{G}_m) \times H^1(V, \mathbb{G}_m)^\frown \to \mathbb{Q}/\mathbb{Z}$$

are perfect.

Proof. Using Corollary 1.4, the second assertion about the induced pairings follows from the first one and from the exact sequences (19), (20). The first assertion follows from the localisation exact sequences (16), (17) by Lemma 1.10, using the following observation. □

Lemma 3.6. *Let $r \in \mathbb{Z}$. The maps $H^r(C, \mathbb{G}_m) \xrightarrow{i^*} H^r(Z, \mathbb{G}_m)$ and $H^{3-r}(C, \mathbb{G}_m) \xleftarrow{i_*} H^{3-r}_Z(C, \mathbb{G}_m)$ are adjoint to each other with respect to the usual pairings. For $r \neq 2$, we have*

(a) $\ker(i_*)^\perp = \mathrm{im}(i^*)$

(b) $\ker(i^*)^\perp = \mathrm{im}(i_*)$

with respect to these pairings. For $r = 2$, the group $\ker(i^*)^\perp / \mathrm{im}(i_*)$ *is divisible.*

Proof. For $r = 1$ resp. $r = 3$, we have $H^r(Z, \mathbb{G}_m) = 0 = H^{3-r}_Z(C, \mathbb{G}_m)$, so (a) is trivial and (b) is equivalent to the nondegeneracy on the right of the Tate–Lichtenbaum pairing $H^r(C, \mathbb{G}_m) \times H^{3-r}(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$. For $r = 0$, the two maps are

$$i^* = \mathrm{incl}\colon\ k^* \to \bigoplus_{z \in Z} k(z)^*$$

and

$$i_* = \mathrm{cor}\colon\ \bigoplus_{z \in Z} H^2(k(z), \mathbb{Z}) \to H^2(k, \mathbb{Z}),$$

so both assertions follow from Nakayama-Tate duality for tori over k. For $r = 2$ we use that $H^1_Z(C, \mathbb{G}_m) = \bigoplus_{z \in Z} \mathbb{Z}$, $H^2_Z(C, \widehat{\mathbb{Z}}(1)) = \bigoplus_{z \in Z} \widehat{\mathbb{Z}}$ and $H^1(C, \mathbb{G}_m)$ injects into $H^2(C, \widehat{\mathbb{Z}}(1))$. The statement for $r = 2$ now follows easily from Poincaré duality with torsion coefficients and the fact that $\widehat{\mathbb{Z}}/\mathbb{Z}$ is divisible. □

Remarks 3.7.

1. If V is connected and affine (i.e., $V \neq C$), then for $q = 0$ we have the zero pairing: $H^0_{cc}(V, \mathbb{G}_m) = 0 = H^3(V, \mathbb{G}_m)$.

2. The (completed) pairing

$$H^3_{cc}(V, \mathbb{G}_m) \times H^0(V, \mathbb{G}_m)^\frown \to \mathbb{Q}/\mathbb{Z}$$

is non-degenerate on the right.

3. From the explicit descriptions of $H^q_{cc}(V, \mathbb{G}_m)$ and $H^q(V, \mathbb{G}_m)$ for $q = 1, 2$ (in terms of divisors and central simple algebras, cf.

Remark 2.10.2), one can give more concrete descriptions of our pairings in the spirit of Lichtenbaum's construction (cf. Remark 1.16).

The following example shows that the conditions $q \neq 3$ resp. $q \neq 2$ in the theorem cannot be avoided:

Example 3.8. Assume $V = C - \{P, Q\}$ with $P,\ Q \in C(k)$, let $x = [P] - [Q] \in \mathrm{Pic}_0(C)$, and assume that x generates an infinite subgroup $\langle x \rangle$ of $\mathrm{Pic}_0(C)$. Then $k[V]^* = k^*$. Let $G := \overline{\langle x \rangle}$, and consider the two exact sequences

$$\mathrm{Br}(C) \xrightarrow{i^*} \mathrm{Br}(k) \oplus \mathrm{Br}(k) \xrightarrow{\partial} H^3_{cc}(V, \mathbb{G}_m) \to H^3(C, \mathbb{G}_m)$$

and

$$\mathrm{Pic}(C) \leftarrow \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{0} k[V]^* \leftarrow k^*$$

(which are adjoint to each other). For all but finitely many primes q, the group G is q-divisible. Choose one such prime q. Let $\beta \in \mathrm{Br}(k)$ be a class of exponent q, and put $\alpha = \partial(\beta, 0)$. Then α lies in the left kernel of

$$H^3_{cc}(V, \mathbb{G}_m) \times H^0(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}.$$

But $\alpha \neq 0$. Indeed, otherwise there would be $\gamma \in \mathrm{Br}(C)$ with $(\beta, 0) = i^*(\gamma)$. But then $\langle \gamma, - \rangle$ would be a (continuous) character on G which sends x to β, hence which is annihilated by q. This would contradict the fact that G is q-divisible.

This shows that in general the pairing $H^3_{cc}(V, \mathbb{G}_m) \times H^0(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ (and therefore also its completion $H^3_{cc}(V, \mathbb{G}_m) \times H^0(V, \mathbb{G}_m)^\wedge \to \mathbb{Q}/\mathbb{Z}$) has a non-trivial left kernel.

The subgroup $G/\langle x \rangle$ of $\mathrm{Pic}(V)$ contains a subgroup isomorphic to $\mathbb{Z}_p/\mathbb{Z}$, and hence its Tate module is $\neq 0$. Therefore $\mathrm{Pic}(V)$ has a non-zero Tate module. Since in the pairing

$$\Big(H^2_{cc}(V, \mathbb{G}_m) \otimes \mathbb{Q}/\mathbb{Z}\Big) \times \mathrm{Tate}\big(H^1(V, \mathbb{G}_m)\big) \to \mathbb{Q}/\mathbb{Z}$$

the left hand group is zero ($H^2_{cc}(V, \mathbb{G}_m)$ is a torsion group), this shows that this pairing has a non-zero right kernel.

Also, $G/\langle x \rangle$ contains non-zero divisible elements, and hence $\mathrm{Pic}(V)$ is not residually finite. Therefore, the (non-completed) pairing

$$H^2_{cc}(V, \mathbb{G}_m) \times H^1(V, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$$

has a non-zero right kernel.

Corollary 3.9. *Let V be as above, let $q \in \mathbb{Z}$.*

(a) *The pairing*

$$\mathrm{Tate}\big(H^q_{cc}(V, \mathbb{G}_m)\big) \times \Big(H^{3-q}(V, \mathbb{G}_m) \otimes \mathbb{Q}/\mathbb{Z}\Big) \to \mathbb{Q}/\mathbb{Z}$$

is perfect if $q \neq 3$ (and always non-degenerate on the right).

(b) *The pairing*

$$\Big(H^q_{cc}(V,\mathbb{G}_m)\otimes\mathbb{Q}/\mathbb{Z}\Big)\times \mathrm{Tate}\big(H^{3-q}(V,\mathbb{G}_m)\big)\to\mathbb{Q}/\mathbb{Z}$$

is perfect if $q\neq 2$ (and always non-degenerate on the left).

Proof. Recall that for the pairings under consideration perfect is equivalent to non-degenerate (Corollary 1.4). By Lemma 1.9, this follows from sequences (19) for (a) (resp. (20) for (b)), using Theorem 3.5. □

4. Duality for general tori

Let V be a non-singular curve over the p-adic field k, and let T be an arbitrary torus on V. Recall that T' denote the dual torus of T. We are going to construct pairings

$$H^q_{cc}(V,T)\times H^{3-q}(V,T')\to\mathbb{Q}/\mathbb{Z}$$

($q\in\mathbb{Z}$) and to investigate their duality properties. For this, we keep referring to the Kummer exact sequences (19) and (20), together with the perfect pairings of their middle columns.

Lemma 4.1. *Let V and T be as above.*

(a) *The image of*

$$\delta_{\mathrm{Tate}}\colon\ H^{q-1}_{cc}(V,T)^\wedge\to H^q_c(V,\mathrm{Tate}(T))$$

is orthogonal to the image of

$$\delta_\infty\colon\ H^{3-q}(V,T')\otimes\mathbb{Q}/\mathbb{Z}\to H^{4-q}(V,T'_{\mathrm{tors}}).$$

(b) *The image of*

$$\delta_{\mathrm{Tate}}\colon\ H^{3-q}(V,T')^\wedge\to H^{4-q}(V,\mathrm{Tate}(T'))$$

is orthogonal to the image of

$$\delta_\infty\colon\ H^{q-1}_{cc}(V,T)\otimes\mathbb{Q}/\mathbb{Z}\to H^q_c(V,T_{\mathrm{tors}}).$$

Proof. (a) Let $\pi\colon W\to V$ be a finite étale map such that T splits over W. Consider the diagram

$$\begin{array}{ccccc}
H^{q-1}_{cc}(V,T)^\wedge & \times & \Big(H^{3-q}(V,T')\otimes\mathbb{Q}/\mathbb{Z}\Big) & \to & \mathbb{Q}/\mathbb{Z}\\
{\scriptstyle\pi^*}\big\downarrow & & \big\uparrow{\scriptstyle\pi_*} & & \\
H^{q-1}_{cc}(W,T_W)^\wedge & \times & \Big(H^{3-q}(W,T'_W)\otimes\mathbb{Q}/\mathbb{Z}\Big) & \to & \mathbb{Q}/\mathbb{Z},
\end{array}$$

where the pairings in both rows are defined by

$$(\alpha,\beta)\mapsto\delta_{\mathrm{Tate}}(\alpha)\cup\delta_\infty(\beta).$$

The maps π^* and π_* are compatible with these pairings. Since T_W is split, the lower pairing is zero by Lemma 3.2. Since

$$\pi_* \circ \pi^* \colon\ H^{3-q}(V, T') \otimes \mathbb{Q}/\mathbb{Z} \to H^{3-q}(V, T') \otimes \mathbb{Q}/\mathbb{Z}$$

is multiplication by the degree of π, and since the group $H^{3-q}(V, T') \otimes \mathbb{Q}/\mathbb{Z}$ is divisible, we see that the map π_* in the diagram is surjective. So the upper pairing is zero as well.

The proof of (b) goes as the proof of (a). □

Lemma 4.2. *Let V and T be as above. The composite mapping*

$$\delta_{\mathrm{Tate}} \circ \iota_\infty \colon\ H_c^q(V, T_{\mathrm{tors}}) \to H_c^{q+1}(V, \mathrm{Tate}(T))$$

is adjoint to the composite mapping

$$\delta_{\mathrm{Tate}} \circ \iota_\infty \colon\ H^{3-q}(V, T'_{\mathrm{tors}}) \to H^{4-q}(V, \mathrm{Tate}(T'))$$

with respect to the cup product pairings (2) *and* (3).

Proof. Compare Lemma 3.3. □

Proposition 4.3. *Let V and T be as above. For each $q \in \mathbb{Z}$ there is a unique pairing*

$$H_{cc}^q(V, T) \times H^{3-q}(V, T') \to \mathbb{Q}/\mathbb{Z} \tag{22}$$

which is compatible with the cup product pairings (2) *and* (3) *through the Kummer sequences.*

Proof. We define the pairing by

$$\langle x, \iota_\infty y\rangle := \langle \delta_{\mathrm{Tate}} x, y\rangle$$

if $q = 0, 1$ and by

$$\langle \iota_\infty x, y\rangle := \langle x, \delta_{\mathrm{Tate}} y\rangle$$

if $q = 2, 3$. This gives a well-defined pairing by Lemma 3.1 and Lemma 4.1. The compatibility with the cup product pairings through the Kummer sequences follows from the construction and Lemma 4.2. □

Remark 4.4. When $\pi \colon W \to V$ is a finite covering of smooth curves over k, the pairings just constructed are compatible with the pullback and trace maps induced by π:

$$\langle \alpha,\, \pi_*\gamma\rangle_V = \langle \pi^*\alpha,\, \gamma\rangle_W$$

holds for $\alpha \in H^q(V, T)$ and $\gamma \in H_{cc}^{3-q}(W, T'_W)$, resp. for $\alpha \in H_{cc}^q(V, T)$ and $\gamma \in H^{3-q}(W, T'_W)$.

Proposition 4.3 allows us to fill in our diagrams of pairings of Kummer sequences to get for every $q \in \mathbb{Z}$ the following diagrams of compatible pairings, with perfect pairings in the middle columns. Recall that $H^q(V, T) = H_{cc}^q(V, T) = 0$ for $q < 0$ or $q > 3$.

$$\begin{tikzcd}
0 \arrow[r] & H^q_{cc}(V,T)^\wedge \arrow[r, "\delta_{\mathrm{Tate}}"] & H_c^{q+1}(V,\mathrm{Tate}(T)) \arrow[r, "\iota_{\mathrm{Tate}}"] & \mathrm{Tate}\Big(H_{cc}^{q+1}(V,T)\Big) \arrow[r] & 0 \\[-15pt]
& \times & \times & \times & \\[-15pt]
0 & H^{3-q}(V,T')_{\mathrm{tors}} \arrow[l] \arrow[dr] & H^{3-q}(V,T'_{\mathrm{tors}}) \arrow[l, "\iota_\infty"'] \arrow[d] & H^{2-q}(V,T')\otimes\mathbb{Q}/\mathbb{Z} \arrow[l, "\delta_\infty"'] \arrow[dl] & 0 \arrow[l] \\
& & \mathbb{Q}/\mathbb{Z} & &
\end{tikzcd} \tag{23}$$

$$\begin{tikzcd}
0 \arrow[r] & H_{cc}^{q-1}(V,T)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r, "\delta_\infty"] & H^q_c(V,T_{\mathrm{tors}}) \arrow[r, "\iota_\infty"] & H^q_{cc}(V,T)_{\mathrm{tors}} \arrow[r] & 0 \\[-15pt]
& \times & \times & \times & \\[-15pt]
0 & \mathrm{Tate}\big(H^{4-q}(V,T')\big) \arrow[l] \arrow[dr] & H^{4-q}(V,\mathrm{Tate}(T')) \arrow[l, "\iota_{\mathrm{Tate}}"'] \arrow[d] & H^{3-q}(V,T')^\wedge \arrow[l, "\delta_{\mathrm{Tate}}"'] \arrow[dl] & 0 \arrow[l] \\
& & \mathbb{Q}/\mathbb{Z} & &
\end{tikzcd} \tag{24}$$

Proposition 4.5. *Let V and T be as above, let $q\in\mathbb{Z}$.*

(a) *The pairing*

$$\mathrm{Tate}\big(H^q_{cc}(V,T)\big)\times\Big(H^{3-q}(V,T')\otimes\mathbb{Q}/\mathbb{Z}\Big)\to\mathbb{Q}/\mathbb{Z}$$

is perfect if $q\neq 3$ (and always non-degenerate on the right).

(b) *The pairing*

$$\Big(H^q_{cc}(V,T)\otimes\mathbb{Q}/\mathbb{Z}\Big)\times\mathrm{Tate}\big(H^{3-q}(V,T')\big)\to\mathbb{Q}/\mathbb{Z}$$

is perfect if $q\neq 2$ (and always non-degenerate on the left).

Proof. We use the diagrams (23), (24). The kernels on the $-\otimes\mathbb{Q}/\mathbb{Z}$ side are trivial by Lemma 1.9. In order to prove the statement on the Tate side, we lift to a trivialising cover $\pi\colon W\to V$ for the torus T. In case (a) we get the diagram

$$\begin{tikzcd}
\mathrm{Tate}\big(H^q_{cc}(V,T)\big) \arrow[d, "\pi^*"'] & \times & \Big(H^{3-q}(V,T')\otimes\mathbb{Q}/\mathbb{Z}\Big) & \to & \mathbb{Q}/\mathbb{Z} \\
\mathrm{Tate}\big(H^q_{cc}(W,T_W)\big) & \times & \Big(H^{3-q}(W,T'_W)\otimes\mathbb{Q}/\mathbb{Z}\Big) \arrow[u, "\pi_*"] & \to & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}$$

Since T_W splits, the lower pairing is non-degenerate for $q\neq 3$, by Corollary 3.9. Since

$$\pi_*\circ\pi^*\colon\ \mathrm{Tate}\big(H^q_{cc}(V,T)\big)\to\mathrm{Tate}\big(H^q_{cc}(V,T)\big)$$

is multiplication by $\deg(\pi)$, and $\mathrm{Tate}\big(H^q_{cc}(V,T)\big)$ is torsion-free, it follows that the map π^* in the above diagram is injective. So for $q\neq 3$

the upper pairing is non-degenerate as well. The proof of the remaining part of (b) is exactly the same. □

Remark 4.6. Observe that the above, together with Lemma 3.1, implies that the Tate modules of $H^0(V,T)$, $H^0_{cc}(V,T)$ and $H^1_{cc}(V,T)$ are zero. This can also be derived directly from the basic properties of the groups involved (cf. Section 2.4). Note the contrast with the situation for $H^1(V,T)$ (cf. Example 3.8).

Corollary 4.7. *Let V and T be as above, let $q \in \mathbb{Z}$.*

(a) *The pairing*

$$H^q_{cc}(V,T)^\frown \times H^{3-q}(V,T')_{\mathrm{tors}} \to \mathbb{Q}/\mathbb{Z}$$

is perfect if $q \neq 2$ (and always non-degenerate on the left).

(b) *The pairing*

$$H^q_{cc}(V,T)_{\mathrm{tors}} \times H^{3-q}(V,T')^\frown \to \mathbb{Q}/\mathbb{Z}$$

is perfect if $q \neq 3$ (and always non-degenerate on the right).

Proof. Immediate from Proposition 4.5 and the Kummer exact sequences (23) (for (a)) and (24) (for (b)), by Lemma 1.9. □

Now we can state and prove our main theorem:

Theorem 4.8. *Let V be a non-singular curve over a p-adic field, and let T be a torus over V. The natural pairing*

$$H^q_{cc}(V,T) \times H^{3-q}(V,T') \to \mathbb{Q}/\mathbb{Z}$$

is non-degenerate on the left if $q \neq 3$, and non-degenerate on the right if $q \neq 2$. The induced pairings

$$H^0_{cc}(V,T)^\frown \times H^3(V,T') \to \mathbb{Q}/\mathbb{Z},$$

$$H^1_{cc}(V,T)^\frown \times H^2(V,T') \to \mathbb{Q}/\mathbb{Z}$$

and

$$H^2_{cc}(V,T) \times H^1(V,T')^\frown \to \mathbb{Q}/\mathbb{Z}$$

are perfect, while the pairing

$$H^3_{cc}(V,T) \times H^0(V,T')^\frown \to \mathbb{Q}/\mathbb{Z}$$

is non-degenerate on the right.

Proof. Corollary 4.7 already contains the statements about the induced pairings. The first part of the theorem follows from these facts, using that each of the groups $H^0_{cc}(V,T)$, $H^0(V,T)$ and $H^1_{cc}(V,T)$ is residually finite (see Section 2.4). □

Remark 4.9. The comparison between $H^*_{cc}(V,T)$ and $H^*_c(V,T)$ (Lemma 2.4) gives a similar result for the groups $H^*_c(V,T)$, which is slightly weaker in degree $q = 1$.

References

[SGA3] M. Demazure, A. Grothendieck: *Schémas en Groupes (SGA 3)*, tomes I–III, Lect. Notes Math. **151**, **152**, **153**, Springer, Berlin, 1970.

[SGA4] M. Artin, A. Grothendieck et al: *Théorie des Topos et Cohomologie Étale des Schemas (SGA 4)*, tomes I–III, Lect. Notes Math. **269**, **270**, **305**, Springer, Berlin, 1972, 1973.

[SGA4.5] P. Deligne: *Cohomologie Étale (SGA* $4\frac{1}{2}$*)*, Lect. Notes Math. **569**, Springer, Berlin, 1977.

[FrVo] E. M. Friedlander, V. Voevodsky: Bivariant cycle cohomology. In: *Cycles, Transfers, and Motivic Homology Theories*, Ann. Math. Studies **143**, Princeton University Press, Princeton, N.J., 2000, pp. 138–187.

[Li] S. Lichtenbaum: Duality theorems for curves over p-adic fields. Invent. math. **7**, 120–136 (1969).

[Mi1] J. S. Milne: *Étale Cohomology.* Princeton University Press, Princeton, N.J., 1980.

[Mi2] J. S. Milne: *Arithmetic Duality Theorems.* Academic Press, Boston, Mass., 1986.

[Po] L. S. Pontryagin: *Topological Groups.* Gordon and Breach, New York, 1966.

[RiZa] L. Ribes, P. Zalesskii: *Profinite groups.* Springer, Berlin, 2000.

[Ro] P. Roquette: Splitting of algebras by function fields of one variable. Nagoya Math. J. **17**, 625–642 (1966).

[Sa] S. Saito: A global duality theorem for varieties over global fields. In: *Algebraic K-theory: Connections with Geometry and Topology* (J. F. Jardine, V. P. Snaith, eds.), Lake Louise (1987), Kluwer, 1989.

[Sch1] C. Scheiderer: *Real and Étale Cohomology.* Lect. Notes Math. **1588**, Springer, Berlin, 1994.

[Sch2] C. Scheiderer: Local-global principles for Galois cohomology of tori over p-adic curves. In preparation.

[Se1] J.-P. Serre: *Groupes algébriques et corps de classes.* Hermann, Paris, 1959.

[Se2] J.-P. Serre: *Cohomologie galoisienne.* Lect. Notes Math. **5**, 5ème édition, Springer, Berlin, 1994.

[Ta] J. Tate: WC-groups over p-adic fields. Sém. Bourbaki no. 156, Déc. 1956, 13 pp.

[vH] J. van Hamel: Lichtenbaum–Tate duality for varieties over p-adic fields, preprint.

[Vo] V. Voevodsky: Triangulated categories of motives over a field. In: *Cycles, Transfers, and Motivic Homology Theories*, Ann. Math. Studies **143**, Princeton University Press, Princeton, N.J., 2000, pp. 188–238.

[Wi] J. S. Wilson: *Profinite Groups.* Clarendon Press, Oxford, 1998.

Institut für Mathematik, Fakultät 4, Universität Duisburg, 47048 Duisburg, Germany

E-mail address: `claus@uni-duisburg.de`

School of Mathematics and Statistics, F07, University of Sydney, NSW 2006, Australia

E-mail address: `vanhamel@member.ams.org`