THE DISCRIMINANT OF A SYMPLECTIC INVOLUTION

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Abstract. An invariant for symplectic involutions on central simple algebras of degree divisible by 4 over fields of characteristic different from 2 is defined on the basis of Rost's cohomological invariant of degree 3 for torsors under symplectic groups. We relate this invariant to trace forms and show how its triviality yields a decomposability criterion for algebras with symplectic involution.

1. Introduction and Statement of Results

In contrast with orthogonal involutions, for which invariants corresponding to the discriminant and Clifford algebras of quadratic forms are defined, no "classical" invariant is known for symplectic involutions on central simple algebras, besides the signature (see [6, (11.10)]). Using the cohomological invariant of degree 3 defined by Rost for torsors under simply connected absolutely simple linear algebraic groups, we introduce an invariant of symplectic involutions on central simple algeras of degree multiple of 4 with values in the third Galois cohomology group of the center with coefficients $\{\pm 1\}$ and give an alternative description in terms of trace forms. We call this invariant *discriminant* since it is the first nontrivial invariant, and because it is directly linked to the discriminant of hermitian forms, see Example 2. Even though its definition is elementary, Rost's computation of the invariants of torsors under symplectic groups is needed to prove that there is no other cohomological invariant of degree 3 and to establish the relation with trace forms. In the final section, we prove that symplectic involutions of trivial discriminant on central simple algebras of degree 8 and index 4 afford a special type of decomposition. In a sequel to this paper, the discriminant is used to give examples of non R-trivial adjoint symplectic groups of even index.

1.1. **Definition of the discriminant.** Throughout this paper, F denotes a field of characteristic different from 2. Let A be a finite-dimensional central simple F-algebra, and $\theta\colon A \to A$ be an anti-automorphism of order 2. We recall that θ is called a *symplectic involution* on A if, after scalar extension to a splitting field, θ is adjoint to an alternating form, see [6, (2.5)]. From now on, we suppose the involution θ is of this type. In this case, the degree $\deg A$ is necessarily an even integer $n = 2m$.

The symplectic group $\mathbf{Sp}(A,\theta)$ is the group scheme over F defined by

$$\mathbf{Sp}(A,\theta)(E) = \{x \in A \otimes_F E \mid \theta(x)x = 1\}$$

for any commutative F-algebra E.

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Let $\mathrm{Sym}(A,\theta)$ be the F-vector space of elements in A fixed by θ,

$$\mathrm{Sym}(A,\theta) = \{x \in A \mid \theta(x) = x\}.$$

We denote by $\mathrm{Sym}(A,\theta)^\times$ the set of units in $\mathrm{Sym}(A,\theta)$,

$$\mathrm{Sym}(A,\theta)^\times = \mathrm{Sym}(A,\theta) \cap A^\times.$$

We recall that the *pfaffian reduced norm* is the homogeneous polynomial function of degree m

$$\mathrm{Nrp}_\theta \colon \; \mathrm{Sym}(A,\theta) \to F$$

uniquely determined by the following conditions:

$$\mathrm{Nrp}_\theta(1) = 1 \qquad \text{and} \qquad \mathrm{Nrp}_\theta(x)^2 = \mathrm{Nrd}_A(x) \qquad \text{for } x \in \mathrm{Sym}(A,\theta),$$

see [6, (2.10)].

The cohomology set $H^1\big(F, \mathbf{Sp}(A,\theta)\big)$ can be represented as

$$H^1\big(F, \mathbf{Sp}(A,\theta)\big) \simeq \mathrm{Sym}(A,\theta)^\times/{\sim} \tag{1}$$

where $\sim$ is the equivalence relation defined by $x \sim y$ if and only if there exists $u \in A^\times$ such that $y = ux\theta(u)$, see [6, (29.24)].

Let $\mathbf{G_m}$ be the multiplicative group. The Kummer exact sequence

$$1 \to \mu_2 \to \mathbf{G_m} \xrightarrow{2} \mathbf{G_m} \to 1$$

allows us to identify the cohomology sets $H^1(F,\mu_2)$ and $H^2(F,\mu_2)$ respectively with the quotient $F^\times/F^{\times 2}$ and with the 2-torsion subgroup of the Brauer group. For all $x \in F^\times$, we denote by $(x)_2 \in H^1(F,\mu_2)$ the cohomology class associated to $xF^{\times 2}$. Similarly, we denote by $[A] \in H^2(F,\mu_2)$ the cohomology class associated to the Brauer class of A. We define

$$\Delta_\theta \colon \; \mathrm{Sym}(A,\theta)^\times \to H^3(F,\mu_2)$$

as the map given by the cup-product

$$\Delta_\theta(s) = \big(\mathrm{Nrp}_\theta(s)\big)_2 \cup [A].$$

It follows from the properties of Nrp_θ (see the proof of Proposition 1 below) that Δ_θ is well defined on the set of equivalence classes under the relation $\sim$. The induced map on the quotient can be interpreted under the bijection (1) as the Rost invariant of $H^1\big(F, \mathbf{Sp}(A,\theta)\big)$, see [6, p. 440].

Since Nrp_θ is homogeneous of degree m, we obtain, for $\alpha \in F^\times$ and $s \in \mathrm{Sym}(A,\theta)^\times$, the following relation

$$\Delta_\theta(\alpha s) = \big(\alpha^m \,\mathrm{Nrp}_\theta(s)\big)_2 \cup [A] = \begin{cases} \Delta_\theta(s) & \text{if } m \text{ is even,} \\ \Delta_\theta(s) + (\alpha)_2 \cup [A] & \text{if } m \text{ is odd.} \end{cases} \tag{2}$$

Therefore, if m is even one can define a relative invariant for symplectic involutions on A as follows:

Definition. Let A be a central simple algebra over F of degree $n = 2m \equiv 0 \bmod 4$. Let θ and σ be symplectic involutions on A. There exists (see [6, (2.7)]) $s \in \mathrm{Sym}(A,\theta)^\times$ such that

$$\sigma = \mathrm{Int}(s) \circ \theta$$

where $\mathrm{Int}(s)$ denotes the inner automorphism associated with s,

$$\mathrm{Int}(s)(x) = sxs^{-1} \qquad \text{for } x \in A.$$

The element s is uniquely determined up to multiplication by an element of $F^\times$. By (2), it follows that $\Delta_\theta(s) \in H^3(F,\mu_2)$ only depends on σ, since m is even. We call this element the *discriminant* of σ with respect to θ and denote it by $\Delta_\theta(\sigma)$. Thus,

$$\Delta_\theta(\sigma) = \big(\mathrm{Nrp}_\theta(s)\big)_2 \cup [A] \in H^3(F,\mu_2).$$

In the case $m = 2$, an analogue of this invariant has been studied in [6, §16.B], where it is denoted by $j_\theta(\sigma)$. Theorem (16.19) of [6] shows that this invariant classifies, up to conjugation, symplectic involutions on a central simple algebra of degree 4.

In section 2 we establish the following elementary result:

1. Proposition. (a) *The discriminant $\Delta_\theta(\sigma)$ only depends on the conjugacy classes of θ and σ, namely, if u, $v \in A^\times$ and*

$$\theta' = \mathrm{Int}(u) \circ \theta \circ \mathrm{Int}(u)^{-1}, \qquad \sigma' = \mathrm{Int}(v) \circ \sigma \circ \mathrm{Int}(v)^{-1},$$

then

$$\Delta_{\theta'}(\sigma') = \Delta_\theta(\sigma).$$

In particular, if σ and θ are conjugate, then $\Delta_\theta(\sigma) = 0$.

(b) *Let ρ, σ and θ be symplectic involutions on A; then*

$$\Delta_\rho(\sigma) = \Delta_\rho(\theta) + \Delta_\theta(\sigma) \qquad \text{and} \qquad \Delta_\theta(\sigma) = \Delta_\sigma(\theta).$$

If the Schur index $\operatorname{ind} A$ divides $\frac{1}{2}\deg A$, i.e., if $A \simeq M_2(A_0)$ for some central simple F-algebra A_0, then A carries hyperbolic symplectic involutions, such as $\gamma\otimes\theta_0$, where γ is the (unique) symplectic involution on $M_2(F)$ and θ_0 is an arbitrary orthogonal involution on A_0. Since all hyperbolic involutions are pairwise conjugate, we may set $\Delta = \Delta_\theta$ for any hyperbolic symplectic involution θ.

2. Example. Consider the algebra $A = \mathrm{End}_Q V$, where Q is a quaternion division F-algebra and V is an m-dimensional Q-vector space. Symplectic involutions on A are then adjoint to hermitian forms on V with respect to the conjugation involution on Q. Suppose that m is even and let σ be the involution adjoint to a fixed hermitian form h on V. Let

$$h = \langle \alpha_1, \dots, \alpha_m \rangle$$

be the diagonalization of h relative to some orthogonal basis e of V (α_1, ..., $\alpha_m \in F^\times$), then

$$\Delta(\sigma) = \big((-1)^{m/2}\alpha_1 \cdots \alpha_m\big)_2 \cup [Q].$$

Indeed, let θ be the hyperbolic involution adjoint to the hermitian form with diagonalization $\langle 1, -1, \dots, 1, -1\rangle$ relative to the basis e. Then, identifying A with $M_m(Q)$ by e, we get

$$\sigma = \mathrm{Int}\,\mathrm{diag}(\alpha_1, -\alpha_2, \dots, \alpha_{m-1}, -\alpha_m) \circ \theta,$$

and we can compute $\Delta(\sigma) = \Delta_\theta(\sigma)$ by Lemma 10(e) below.

Note that if $V_0 \subset V$ is the F-subspace spanned by e, then $A = (\mathrm{End}_F V_0) \otimes Q$, and we obtain a decomposition $\sigma = \sigma_0 \otimes \gamma$ where σ_0 is the involution adjoint to the bilinear form on V_0 with diagonalization $\langle \alpha_1, \dots, \alpha_m\rangle$ relative to e, and γ is the canonical (conjugation) involution on Q.

This example can be slightly generalized:

3. Example. Consider the algebra $A = A_0 \otimes_F Q$ where Q is a quaternion F-algebra and A_0 is a central simple F-algebra. Let σ_0 be an orthogonal involution on A_0, γ the canonical involution on Q, and

$$\sigma = \sigma_0 \otimes \gamma.$$

Suppose that $\operatorname{ind} A_0$ divides $\frac{1}{2}\deg A_0$. Then

$$\Delta(\sigma) = (\operatorname{disc}\sigma_0)_2 \cup [Q], \tag{3}$$

where $\operatorname{disc}\sigma_0 \in F^\times/F^{\times 2}$ is the discriminant of the orthogonal involution σ_0 (see [6, §7]). Indeed, let θ_0 be a hyperbolic orthogonal involution on A_0 and let $x_0 \in \operatorname{Sym}(A_0,\theta_0)^\times$ be such that $\sigma_0 = \operatorname{Int}(x_0)\circ\theta_0$. The involution $\theta = \theta_0\otimes\gamma$ is hyperbolic, and we have $\sigma = \operatorname{Int}(x_0\otimes 1)\circ\theta$, so that

$$\Delta(\sigma) = \big(\operatorname{Nrp}_\theta(x_0\otimes 1)\big)_2 \cup [A].$$

Thus, by Lemma 10(d), $\operatorname{Nrp}_\theta(x_0\otimes 1) = \operatorname{Nrd}_{A_0}(x_0)$. Equation (3) follows, since $\operatorname{disc}\sigma_0$ is represented by $\operatorname{Nrd}_{A_0}(x_0)$, and $\big(\operatorname{Nrd}_{A_0}(x_0)\big)_2 \cup [A_0] = 0$.

1.2. **Trace forms.** Let A be an arbitrary central simple F-algebra. For every involution $\sigma : A \to A$, the associated trace form $T_\sigma\colon\ A \to F$ is defined as follows:

$$T_\sigma(x) = \operatorname{Trd}_A\big(\sigma(x)x\big)$$

where Trd_A denotes the reduced trace. Denote by T_σ^+ the restriction of T_σ to $\operatorname{Sym}(A,\sigma)$; this form can also be seen as the restriction to $\operatorname{Sym}(A,\sigma)$ of the form $T_A\colon\ A \to F$ defined by

$$T_A(x) = \operatorname{Trd}_A(x^2).$$

As is the case with involutions of the other types (see [6, §11]), the discriminant of symplectic involutions can be expressed in terms of trace forms; indeed we have the following result:

4. Theorem. *Let A be a central simple algebra over F and let θ and σ be symplectic involutions on A. The class in the Witt ring WF of the difference $T_\sigma^+ - T_\theta^+$ lies in the third power of the fundamental ideal, namely*

$$T_\sigma^+ - T_\theta^+ \in I^3F.$$

Moreover, if $e_3\colon\ I^3F \to H^3(F,\mu_2)$ denotes the Arason invariant, we obtain

$$e_3(T_\sigma^+ - T_\theta^+) = \begin{cases} \Delta_\theta(\sigma) & \text{if } \deg A \equiv 0 \mod 4,\\ 0 & \text{if } \deg A \equiv 2 \mod 4.\end{cases}$$

A proof of this result is given in section 3 below. For the trace forms T_σ, we have the following result:

5. Corollary. *Keeping the notation of the previous theorem, we have $T_\sigma - T_\theta \in I^4F$ and*

$$e_4(T_\sigma - T_\theta) = \begin{cases} (-1)_2 \cup \Delta_\theta(\sigma) & \text{if } \deg A \equiv 0 \mod 4,\\ 0 & \text{if } \deg A \equiv 2 \mod 4,\end{cases}$$

where $e_4\colon\ I^4F \to H^4(F,\mu_2)$ denotes the degree 4 invariant.

Proof. Let T_σ^- be the restriction of T_σ (or of $-T_A$) to the space of skew-symmetric elements in A. We have

$$T_\sigma = T_\sigma^+ + T_\sigma^- \qquad \text{and} \qquad T_A = T_\sigma^+ - T_\sigma^-,$$

so that $T_\sigma = 2T_\sigma^+ - T_A$. Similarly, $T_\theta = 2T_\theta^+ - T_A$, so that

$$T_\sigma - T_\theta = 2(T_\sigma^+ - T_\theta^+),$$

hence the corollary is a direct consequence of the previous theorem. □

In the special case where θ is hyperbolic we get:

6. Proposition. *Suppose $A = M_2(A_0)$ for some central simple F-algebra A_0, and let θ be a hyperbolic symplectic involution on A. Then T_θ^+ is Witt-equivalent to $\langle 2\rangle \cdot T_{A_0}$, and T_θ is hyperbolic. If $\deg A \equiv 2 \bmod 4$, then A is split, hence every symplectic involution on A is hyperbolic. If $\deg A \equiv 0 \bmod 4$, then, for any symplectic involution σ on A, we have $T_\sigma \in I^4F$ and*

$$e_4(T_\sigma) = (-1)_2 \cup \Delta(\sigma).$$

1.3. **Decomposability of symplectic involutions.** Section 4 below will be devoted to the relations between the discriminant and the decomposability of symplectic involutions as tensor products of involutions defined on subalgebras. Our main result is concerned with degree 8 algebras with index dividing 4. Such algebras can be written in the form $A = M_2(A_0)$, where A_0 is a central simple algebra of degree 4, hence they carry hyperbolic symplectic involutions. The case $\operatorname{ind} A = 1$ is trivial, since every symplectic involution on a split algebra is hyperbolic, and is omitted in the following theorem.

7. Theorem. *Let A be a central simple F-algebra of degree 8 having index 2 or 4. For any symplectic involution σ on A, there is a quaternion subalgebra $Q \subset A$ such that*

(1) *Q is stable under σ, and σ restricts to the conjugation involution γ on Q,*
(2) *the index of the centralizer A_0 of Q in A is $\frac{1}{2} \operatorname{ind} A$.*

We thus have

$$A = A_0 \otimes_F Q \qquad \textit{and} \qquad \sigma = \sigma_0 \otimes \gamma,$$

where σ_0 is the restriction of σ to A_0 (which is an orthogonal involution on A_0).

When $\operatorname{ind} A = 2$, this theorem is easily proved and can be readily generalized to any degree, see Example 2 or [1, Proposition 3.4]. The case $\operatorname{ind} A = 4$ is treated in Section 4.

Theorem 7 shows that the discriminant of a symplectic involution σ on a central simple F-algebra of degree 8 and index 2 or 4 can be computed as in Example 3 above. The following theorem gives a necessary and sufficient condition for the discriminant to be trivial.

8. Theorem. *Let A be a central simple F-algebra of degree 8 with index dividing 4. For any symplectic involution σ on A, $\Delta(\sigma) = 0$ if and only if σ leaves invariant a subalgebra $A_1 \subset A$ isomorphic to $M_2(F)$ and restricts to an orthogonal involution on A_1.*

A proof is given in Section 4.

When the condition of Theorem 8 holds, the centralizer A_1' of A_1 in A is also invariant under σ, and the restriction of σ to A_1' must be symplectic. As any symplectic involution on a degree 4 algebra decomposes as a tensor product of involutions on quaternion subalgebras (see [6, (16.16)]), it follows that $A_1' = A_2 \otimes A_3$ for some quaternion subalgebras A_2, $A_3 \subset A_1'$ invariant under σ. Therefore, Theorem 8 can be reformulated as follows:

9. Corollary. *Let A be a central simple F-algebra of degree 8 with index dividing 4. Then, for every symplectic involution σ on A, we have $\Delta(\sigma) = 0$ if and only if there exist quaternion subalgebras A_1, A_2, $A_3 \subset A$, with $A_1 \simeq M_2(F)$, and orthogonal involutions σ_1 and σ_2 on A_1 and A_2 respectively, such that*

$$A = A_1 \otimes A_2 \otimes A_3 \qquad \textit{and} \qquad \sigma = \sigma_1 \otimes \sigma_2 \otimes \gamma_3$$

where γ_3 denotes the canonical involution on A_3.

2. Discriminants and pfaffian Norms

The goal of this section is to prove Proposition 1. Throughout the section, A denotes a central simple F-algebra of degree $n = 2m$.

10. Lemma. *Let σ and θ be symplectic involutions on A and let $s \in \mathrm{Sym}(A,\theta)^\times$ be an element such that $\sigma = \mathrm{Int}(s) \circ \theta$. Then,*

(a) *for every $x \in \mathrm{Sym}(A,\sigma) \cap \mathrm{Sym}(A,\theta)$,*
$$\mathrm{Nrp}_\sigma(x) = \mathrm{Nrp}_\theta(x);$$

(b) *for every $x \in \mathrm{Sym}(A,\theta)$, the product sx lies in $\mathrm{Sym}(A,\sigma)$ and*
$$\mathrm{Nrp}_\sigma(sx) = \mathrm{Nrp}_\theta(s)\,\mathrm{Nrp}_\theta(x);$$

(c) *for every $x \in \mathrm{Sym}(A,\theta)^\times$,*
$$\mathrm{Nrp}_\theta(x^{-1}) = \mathrm{Nrp}_\theta(x)^{-1}.$$

(d) *Suppose $A = A_1 \otimes_F A_2$ for some central simple F-algebras A_1, $A_2 \subset A$ of degree $n_1 = 2m_1$ and $n_2 = 2m_2$ respectively; if $x_1 \in A_1$ and $x_2 \in A_2$ are such that $x_1 \otimes x_2 \in \mathrm{Sym}(A,\theta)$, then*
$$\mathrm{Nrp}_\theta(x_1 \otimes x_2) = \mathrm{Nrd}_{A_1}(x_1)^{m_2}\,\mathrm{Nrd}_{A_2}(x_2)^{m_1}.$$

(e) *Suppose $A = M_r(A_0)$ and $\theta\big((a_{ij})_{1\le i,j\le r}\big) = \big(\theta_0(a_{ij})\big)^t_{1\le i,j\le r}$ for some symplectic involution θ_0 on the central simple F-algebra A_0. For the diagonal matrix $x = \mathrm{diag}(x_1,\dots,x_r)$ with $x_i \in \mathrm{Sym}(A_0,\theta_0)$ for $i = 1,\dots,r$, we have*
$$\mathrm{Nrp}_\theta(x) = \mathrm{Nrp}_{\theta_0}(x_1)\cdots\mathrm{Nrp}_{\theta_0}(x_r).$$

Proof. (a) Let t be an indeterminate over F. Define
$$\mathrm{Prp}_{\sigma,x}(t) = \mathrm{Nrp}_\sigma(t-x) \in F[t], \qquad \mathrm{Prp}_{\theta,x}(t) = \mathrm{Nrp}_\theta(t-x) \in F[t].$$

Those polynomials, called *pfaffian characteristic polynomials* in [6, p. 19], are monic and satisfy
$$\mathrm{Prp}^2_{\sigma,x} = \mathrm{Pcrd}_{A,x} = \mathrm{Prp}^2_{\theta,x},$$

where $\mathrm{Pcrd}_{A,x}(t) = \mathrm{Nrd}_{A(t)}(t-x)$ is the reduced characteristic polynomial of x. Therefore, $\mathrm{Prp}_{\sigma,x}(t) = \mathrm{Prp}_{\theta,x}(t)$, and evaluation at $t=0$ yields $\mathrm{Nrp}_\sigma(x) = \mathrm{Nrp}_\theta(x)$.

(b) Straightforward calculations show that $\sigma(sx) = sx$ if $\theta(x) = x$. Let us consider the two sides of the equality we aim to prove as polynomial functions of x. The squares of the two sides are equal since the reduced norm is multiplicative, hence they are equal up to sign. Moreover, they are equal and nonzero for $x = 1$ in view of part (a). Hence, they are equal for all x.

(c) We apply (b) with $x = s^{-1}$ and use the relation $\mathrm{Nrp}_\sigma(1) = 1$.

(d) By taking the square root on both sides of the equation

$$\mathrm{Pcrd}_{A,x_1\otimes x_2} = (\mathrm{Pcrd}_{A_1,x_1})^{n_2}(\mathrm{Pcrd}_{A_2,x_2})^{n_1},$$

we obtain

$$\mathrm{Prp}_{\theta,x_1\otimes x_2} = (\mathrm{Pcrd}_{A_1,x_1})^{m_2}(\mathrm{Pcrd}_{A_2,x_2})^{m_1}.$$

The property follows by considering the constant terms.

(e) As in the preceding case, the property follows by extracting the monic square root of each side of the equation

$$\mathrm{Pcrd}_{A,x} = \mathrm{Pcrd}_{A_0,x_1}\cdots\mathrm{Pcrd}_{A_0,x_r}.$$

□

Proposition 1 easily follows from the lemma above. Indeed, if $\sigma = \mathrm{Int}(s)\circ\theta$, so that $\Delta_\theta(\sigma) = \big(\mathrm{Nrp}_\theta(s)\big)_2 \cup [A]$, then $\theta = \mathrm{Int}(s^{-1})\circ\sigma$ and hence $\Delta_\sigma(\theta) = \big(\mathrm{Nrp}_\sigma(s^{-1})\big)_2 \cup [A]$. Now Lemma 10 shows that

$$\mathrm{Nrp}_\sigma(s^{-1}) = \mathrm{Nrp}_\theta(s)^{-1},$$

hence $\big(\mathrm{Nrp}_\sigma(s^{-1})\big)_2 = \big(\mathrm{Nrp}_\theta(s)\big)_2$, and so $\Delta_\sigma(\theta) = \Delta_\theta(\sigma)$. If ρ is another symplectic involution, and if $t \in \mathrm{Sym}(A,\rho)$ is such that $\theta = \mathrm{Int}(t)\circ\rho$, then $\sigma = \mathrm{Int}(st)\circ\rho$. Part (b) of Lemma 10 yields

$$\mathrm{Nrp}_\sigma(st) = \mathrm{Nrp}_\theta(s)\,\mathrm{Nrp}_\theta(t).$$

Moreover, part (a) shows that $\mathrm{Nrp}_\sigma(st) = \mathrm{Nrp}_\rho(st)$ and $\mathrm{Nrp}_\theta(t) = \mathrm{Nrp}_\rho(t)$. Therefore, the preceding equality can be written as

$$\mathrm{Nrp}_\rho(st) = \mathrm{Nrp}_\theta(s)\,\mathrm{Nrp}_\rho(t).$$

It follows that

$$\Delta_\rho(\sigma) = \big(\mathrm{Nrp}_\rho(st)\big)_2 \cup [A] = \big(\mathrm{Nrp}_\theta(s)\big)_2 \cup [A] + \big(\mathrm{Nrp}_\rho(t)\big)_2 \cup [A] = \Delta_\theta(\sigma) + \Delta_\rho(\theta),$$

which completes the proof of part (b) of Proposition 1.

Let now $v \in A^\times$ and $\sigma' = \mathrm{Int}(v)\circ\sigma\circ\mathrm{Int}(v)^{-1}$, so that $\sigma' = \mathrm{Int}\big(vs\theta(v)\big)\circ\theta$. Then,

$$\Delta_\theta(\sigma') = \big(\mathrm{Nrp}_\theta(vs\theta(v))\big)_2 \cup [A].$$

By [6, (2.13)], $\mathrm{Nrp}_\theta\big(vs\theta(v)\big) = \mathrm{Nrd}_A(v)\,\mathrm{Nrp}_\theta(s)$. Since $\big(\mathrm{Nrd}_A(v)\big)_2 \cup [A] = 0$, it follows that

$$\Delta_\theta(\sigma') = \Delta_\theta(\sigma).$$

Similarly, if θ' is a symplectic involution conjugate to θ, then $\Delta_{\sigma'}(\theta') = \Delta_{\sigma'}(\theta)$. Now, part (b) of Proposition 10 shows that $\Delta_{\theta'}(\sigma') = \Delta_{\sigma'}(\theta')$ and $\Delta_\theta(\sigma') = \Delta_{\sigma'}(\theta)$. Therefore,

$$\Delta_{\theta'}(\sigma') = \Delta_\theta(\sigma').$$

We already observed that $\Delta_\theta(\sigma') = \Delta_\theta(\sigma)$, hence

$$\Delta_{\theta'}(\sigma') = \Delta_\theta(\sigma)$$

and the proof of Proposition 10 is complete.

3. Discriminant and trace form

In this section we prove Theorem 4 and Proposition 6.

Let Fields_F be the category of fields containing F and let G be any algebraic group over F. We consider the functor

$$H^1(G)\colon\ \mathsf{Fields}_F \to \mathsf{Sets}^*$$

where Sets^* denotes the category of pointed sets, associating to every $L \in \mathsf{Fields}_F$ the Galois cohomology set $H^1(L,G)$.

Similarly, the "cohomological cycle module" $M = \bigoplus_{d\geq 0} H^d\bigl(\bullet, \mathbb{Q}/\mathbb{Z}(d-1)\bigr)$, introduced by Rost in [9], yields a sequence of functors

$$M_d = H^d\bigl(\bullet, \mathbb{Q}/\mathbb{Z}(d-1)\bigr)\colon\ \mathsf{Fields}_F \to \mathsf{Sets}^*\,.$$

The natural transformations $H^1(G) \to M_d$ are called *cohomological invariants of degree d* in Merkurjev's paper [7]. These invariants form a group denoted $\mathrm{Inv}^d(H^1(G), M)$. (The group $\mathrm{Inv}^d(H^1(G), M)$ is denoted by $\mathrm{Inv}^d\bigl(G, \mathbb{Q}/\mathbb{Z}(d-1)\bigr)$ in [6, §31.B].)

Let now A be a central simple F-algebra of degree $n = 2m$ and let θ be a symplectic involution on A. We take for G the group $\mathbf{GSp}(A,\theta)$ of symplectic similitudes; this is the algebraic group scheme defined by

$$\mathbf{GSp}(A,\theta)(E) = \{g \in A\otimes_F E \mid \theta(g)g \in E^\times\}$$

for any commutative F-algebra E. The set $H^1\bigl(L, \mathbf{GSp}(A,\theta)\bigr)$ is in one to one correspondence with the set of conjugacy classes of symplectic involutions defined on $A_L = A\otimes_F L$, the class of θ being the distinguished one (see [6, (29.23)]). The following proposition shows that for symplectic involutions there is no (nontrivial) cohomological invariant of degree 1 or 2.

11. Proposition. *Suppose $M = \bigoplus_{d\geq 0} H^d\bigl(\bullet, \mathbb{Q}/\mathbb{Z}(d-1)\bigr)$. If the algebra A is split, we get $H^1\bigl(L, \mathbf{GSp}(A,\theta)\bigr) = 1$, for every $L \in \mathsf{Fields}_F$, so that*

$$\mathrm{Inv}^d\bigl(H^1\bigl(\mathbf{GSp}(A,\theta)\bigr), M\bigr) = 0 \qquad \textit{for all } d.$$

If A is not split, we have

$$\mathrm{Inv}^d(H^1\bigl(\mathbf{GSp}(A,\theta)\bigr), M\bigr) = 0 \qquad \textit{for } d = 1,\,2$$

and

$$\mathrm{Inv}^3(H^1\bigl(\mathbf{GSp}(A,\theta)\bigr), M\bigr) = \begin{cases} 0 & \textit{if } \deg A \equiv 2 \mod 4,\\ \mathbb{Z}/2\mathbb{Z} & \textit{if } \deg A \equiv 0 \mod 4.\end{cases}$$

Proof. Every symplectic involution on a split algebra is hyperbolic. Therefore, when A is split, $H^1\bigl(L, \mathbf{GSp}(A,\theta)\bigr) = 1$ for all $L \in \mathsf{Fields}_F$.

For the rest of the proof, we may thus assume that A is not split. Let

$$\mu\colon\ \mathbf{GSp}(A,\theta) \to \mathbf{G_m}$$

be the homomorphism which associates to each similitude g its multiplier $\mu(g) = \theta(g)g$. The cohomology sequence induced by the exact sequence

$$1 \to \mathbf{Sp}(A,\theta) \to \mathbf{GSp}(A,\theta) \xrightarrow{\mu} \mathbf{G_m} \to 1$$

yields for every $L \in \mathit{Fields}_F$ the exact sequence

$$L^\times \to H^1\big(L, \mathbf{Sp}(A,\theta)\big) \to H^1\big(L, \mathbf{GSp}(A,\theta)\big) \to 1$$

since $H^1(L, \mathbf{G_m}) = 1$ by Hilbert's Theorem 90. Therefore, for every d, we have an exact sequence

$$0 \to \mathrm{Inv}^d\big(H^1\big(\mathbf{GSp}(A,\theta)\big), M\big) \to \mathrm{Inv}^d\big(H^1\big(\mathbf{Sp}(A,\theta)\big), M\big) \to \mathrm{Inv}^d(\mathbf{G_m}, M).$$

For $d = 1$ or 2, we obtain, by [6, (31.15)], $\mathrm{Inv}^d\big(H^1\big(\mathbf{Sp}(A,\theta)\big), M\big) = 0$ and hence

$$\mathrm{Inv}^d\big(H^1\big(\mathbf{GSp}(A,\theta)\big), M\big) = 0.$$

The group $\mathrm{Inv}^3\big(H^1\big(\mathbf{Sp}(A,\theta)\big), M\big)$ is of order 2, the nontrivial element being the Rost invariant Δ_θ defined in the introduction. Equation (2) shows that this invariant is zero in $\mathrm{Inv}^3(\mathbf{G_m}, M)$ if and only if $\deg A \equiv 2 \bmod 4$. □

When $\deg A \equiv 0 \bmod 4$, the unique nontrivial invariant of degree 3 is the discriminant. Our next goal is to give an explicit description of this invariant in terms of trace forms.

Let $T^+_\theta\colon \mathrm{Sym}(A,\theta) \to F$ be the quadratic form

$$T^+_\theta(x) = \mathrm{Trd}_A(\theta(x)x) = \mathrm{Trd}_A(x^2).$$

This forms only depends, up to isometry, on the conjugacy class of θ since, if $\theta' = \mathrm{Int}(v) \circ \theta \circ \mathrm{Int}(v)^{-1}$ for some $v \in A^\times$, then $\mathrm{Int}(v)$ defines an isometry between T^+_θ and $T^+_{\theta'}$. Consider $L \in \mathit{Fields}_F$. The map sending every symplectic involution $\sigma\colon A_L \to A_L$ to the discriminant

$$\mathrm{disc}(T^+_\sigma - T^+_\theta) \in L^\times/L^{\times 2} = H^1(L, \mu_2)$$

defines a cohomological invariant $H^1\big(\mathbf{GSp}(A,\theta)\big) \to H^1(\mu_2)$. By Proposition 11, this invariant is trivial, hence $T^+_\sigma - T^+_\theta \in I^2L$. Similarly, the map sending every symplectic involution σ to the Witt (–Clifford) invariant

$$e_2(T^+_\sigma - T^+_\theta) \in H^2(L, \mu_2)$$

defines a cohomological invariant of degree 2. Again, by Proposition 11, we get $e_2(T^+_\sigma - T^+_\theta) = 0$, and hence $T^+_\sigma - T^+_\theta \in I^3L$ using Merkurjev's theorem. This proves the first part of Theorem 4. Note that the equality $e_2(T^+_\sigma - T^+_\theta) = 0$ can also be derived from Quéguiner's explicit calculation of the Hasse invariant of trace forms in [8, p. 307].

Consider the map associating to every symplectic involution $\sigma\colon A_L \to A_L$ the Arason invariant

$$e_3(T^+_\sigma - T^+_\theta) \in H^3(L, \mu_2).$$

Using Proposition 11, we see that this invariant is trivial either if $\deg A \equiv 2 \bmod 4$ or if A is split. We claim that it coincides with the discriminant $\Delta_\theta(\sigma)$ if A is nonsplit and $\deg A \equiv 0 \bmod 4$. To prove this, it suffices to show that it is nontrivial because, by Proposition 11, there is an unique nontrivial invariant in $\mathrm{Inv}^3\big(H^1\big(\mathbf{GSp}(A,\theta)\big), M\big)$. Therefore, our goal is to find a field $L \in \mathit{Fields}_F$ and two symplectic involutions σ_1, σ_2 on A_L such that $e_3(T^+_{\sigma_1} - T^+_{\sigma_2}) \neq 0$. If θ is any involution on A, the equality

$$e_3(T^+_{\sigma_1} - T^+_\theta) = e_3(T^+_{\sigma_1} - T^+_{\sigma_2}) + e_3(T^+_{\sigma_2} - T^+_\theta)$$

shows that at least one of the terms $e_3(T^+_{\sigma_1} - T^+_\theta)$ and $e_3(T^+_{\sigma_2} - T^+_\theta)$ is nonzero, hence the invariant $\sigma \mapsto e_3(T^+_\sigma - T^+_\theta)$ is nontrivial.

12. Lemma. *Let A be a central simple F-algebra of exponent 2. There exists a field $L \in \mathsf{Fields}_F$ such that* $\operatorname{ind} A_L = 2$.

Proof. We may take for L the fonction field of a suitable generalized Severi–Brauer variety (cf [2]). Alternatively, we may consider a multiquadratic extension M/F of minimal degree splitting A (such extensions exist, because A is Brauer–equivalent to a tensor product of quaternion algebras by a theorem of Merkurjev) and take for L a proper maximal subfield of M. The algebra A_L is not split, because M is of minimal degree, and it is of index 2 because it is split by the quadratic extension M/L. □

Thus we may suppose, after possibly replacing F by a suitable extension, that $\operatorname{ind} A = 2$ i.e. that A is Brauer–equivalent to a quaternion division algebra Q over F. Then, denoting by V an F-vector space of dimension m, we obtain

$$A \simeq Q \otimes_F \operatorname{End}_F V.$$

For the rest of this section, we fix an isomorphism identifying A with $Q \otimes \operatorname{End}_F V$. Let b be a symmetric non-degenerate bilinear form on V. The symmetric square S^2V and the exterior square $\bigwedge^2 V$ are endowed with symmetric bilinear forms b^{S^2} and $b^{\wedge 2}$ respectively, defined by

$$b^{S^2}(x_1 \cdot x_2, y_1 \cdot y_2) = b(x_1, y_1)b(x_2, y_2) + b(x_1, y_2)b(x_2, y_1)$$

and

$$b^{\wedge 2}(x_1 \wedge x_2, y_1 \wedge y_2) = b(x_1, y_1)b(x_2, y_2) - b(x_1, y_2)b(x_2, y_1).$$

13. Lemma. *Let $Q = (\alpha, \beta)_F$. On $A = Q \otimes_F \operatorname{End}_F V$, consider the symplectic involution $\sigma = \gamma \otimes \operatorname{ad}_b$, where γ is the quaternion conjugation on Q and ad_b is the (orthogonal) involution adjoint to b. Then, the bilinear form $B^+_\sigma(x, y) = \operatorname{Trd}_A(xy)$ on $\operatorname{Sym}(A, \sigma)$ (which is the polar form of the quadratic form T^+_σ) decomposes as an orthogonal sum*

$$B^+_\sigma = b^{S^2} \perp \langle -\alpha, -\beta, \alpha\beta \rangle \cdot b^{\wedge 2}.$$

Proof. Let $\operatorname{Skew}(\operatorname{End}_F V, \operatorname{ad}_b)$ be the F-vector space of endomorphisms f of V such that $\operatorname{ad}_b(f) = -f$, and let Q^0 be the F–vector space of pure quaternions in Q. Straightforward calculation shows that the decomposition

$$\operatorname{Sym}(A, \sigma) = \big(F \otimes \operatorname{Sym}(\operatorname{End}_F V, \operatorname{ad}_b)\big) \oplus \big(Q^0 \otimes \operatorname{Skew}(\operatorname{End}_F V, \operatorname{ad}_b)\big)$$

is orthogonal with respect to the form B^+_σ. Let B^+_b and B^-_b be the restrictions of the bilinear trace form $B(f, g) = \operatorname{tr}(fg)$ to $\operatorname{Sym}(\operatorname{End}_F V, \operatorname{ad}_b)$ and $\operatorname{Skew}(\operatorname{End}_F V, \operatorname{ad}_b)$ respectively. The decomposition above yields

$$B^+_\sigma = \langle 2 \rangle \cdot B^+_b \perp \langle 2\alpha, 2\beta, -2\alpha\beta \rangle \cdot B^-_b.$$

The lemma follows, since, by [6, (11.4)] $B^+_b \simeq \frac{1}{2} b^{S^2}$ and $B^-_b \simeq -\frac{1}{2} b^{\wedge 2}$. □

If $b = \langle a_1, \dots, a_m\rangle$ is a diagonalization of b, it is easily verified that

$$b^{S^2} \simeq m\langle 2\rangle \perp \big(\perp_{i<j} \langle a_i a_j\rangle\big) \qquad \text{and} \qquad b^{\wedge 2} \simeq \perp_{i<j} \langle a_i a_j\rangle$$

(cf [6, p. 135]). The formula of the preceding lemma can then be written as

$$B^+_\sigma \simeq m\langle 2\rangle \perp \langle 1, -\alpha, -\beta, \alpha\beta\rangle \cdot b^{\wedge 2},$$

hence, in terms of quadratic forms,

$$T^+_\sigma = m\langle 2\rangle + n_Q \cdot q^{\wedge 2}$$

where n_Q denotes the norm form of Q and $q^{\wedge 2}$ is the quadratic form defined by $q^{\wedge 2}(x) := b^{\wedge 2}(x, x)$.

Let b_1 and b_2 be two nonsingular symmetric bilinear forms on V, and let

$$\sigma_1 = \gamma \otimes \mathrm{ad}_{b_1}, \qquad \sigma_2 = \gamma \otimes \mathrm{ad}_{b_2}$$

be the symplectic involutions on $A = Q \otimes \mathrm{End}_F V$ constructed as in Lemma 13. Observe that $T^+_{\sigma_1} - T^+_{\sigma_2} = n_Q \cdot (q_1^{\wedge 2} - q_2^{\wedge 2})$, hence

$$e_3(T^+_{\sigma_1} - T^+_{\sigma_2}) = [Q] \cup \mathrm{disc}(q_1^{\wedge 2} - q_2^{\wedge 2}). \tag{4}$$

Explicit calculation shows that

$$\mathrm{disc}(q_1^{\wedge 2} - q_2^{\wedge 2}) = \det b_1^{\wedge 2} \cdot \det b_2^{\wedge 2} = (\det b_1 \cdot \det b_2)^{m-1}. \tag{5}$$

Adjoining an indeterminate to F if necessary, we may assume that there exists an element $t \in F^\times$ not belonging to $\mathrm{Nrd}(Q)$. By a theorem of Merkurjev, this element satisfies $[Q] \cup (t)_2 \neq 0$. It is easy to find two bilinear forms b_1 and b_2 on V such that $\det b_1 \cdot \det b_2 = tF^{\times 2}$. Since m is even, it follows from (4) and (5) that the corresponding involutions σ_1 and σ_2 satisfy

$$e_3(T^+_{\sigma_1} - T^+_{\sigma_2}) \neq 0.$$

This completes the proof of Theorem 4.

We now turn to Proposition 6 and assume $A = M_2(F) \otimes_F A_0$. Since all hyperbolic involutions are conjugate, we may assume moreover $\theta = \gamma \otimes \theta_0$ for some orthogonal involution θ_0 on A_0, where γ is the unique symplectic involution on $M_2(F)$ (which is hyperbolic). As in Lemma 13, we have an orthogonal decomposition

$$\mathrm{Sym}(A, \theta) = \big(F \otimes \mathrm{Sym}(A_0, \theta_0)\big) \oplus \big(\mathrm{Skew}(M_2(F), \gamma) \otimes \mathrm{Skew}(A_0, \theta_0)\big)$$

which yields

$$T^+_\sigma = \langle 2\rangle \cdot T^+_{\theta_0} \perp \langle 2, -2, -2\rangle \cdot T^-_{\theta_0}.$$

Therefore, T^+_σ is Witt-equivalent to $\langle 2\rangle \cdot (T^+_{\theta_0} - T^-_{\theta_0}) = \langle 2\rangle \cdot T_{A_0}$. Since the adjoint involution to T_θ is $\theta \otimes \theta$, by [6, (11.1)], it is clear that T_θ is hyperbolic when θ is hyperbolic. If $\deg A \equiv 2 \bmod 4$, then $\deg A_0$ is odd, hence A_0 is split. Therefore, A is also split. The other statements in Proposition 6 follow from Corollary 5.

4. Discriminant and decomposability of involutions

Our first goal in this section is to give a proof of Theorem 7. As observed in Section 1.3, the theorem is easy if $\mathrm{ind}\, A = 2$. Therefore, we assume $\mathrm{ind}\, A = 4$. We may then represent A as

$$A = \mathrm{End}_D V$$

where D is a division algebra of degree 4 and V is a 2-dimensional D-vector space. Let θ_0 be an arbitrary symplectic involution on D. The involution σ is adjoint to a hermitian form h on V (with respect to θ_0). Using an orthogonal basis of V relative to h, we may identify

$$A = M_2(D) \qquad \text{and} \qquad \sigma = \operatorname{Int}\operatorname{diag}(u_1, u_2) \circ \hat{\theta}_0$$

for some u_1, $u_2 \in \operatorname{Sym}(D, \theta_0)^\times$, where

$$\hat{\theta}_0\big((a_{ij})_{1\le i,j\le 2}\big) = \big(\theta_0(a_{ij})\big)^t_{1\le i,j\le 2},$$

i.e., $\hat{\theta}_0 = t\otimes\theta$ on $A = M_2(F)\otimes_F D$. Substituting $\operatorname{Int}(u_1)\circ\theta_0$ for θ_0, we may assume $u_1 = 1$. By [6, (16.16)], we may find a decomposition of D into a tensor product of quaternion subalgebras stable under θ_0,

$$D = Q_1 \otimes_F Q, \qquad \theta_0 = \theta_1 \otimes \gamma$$

where θ_1 is an orthogonal involution on Q_1 and γ is the canonical involution on Q. Moreover, we may assume $u_2 \in Q_1$. Then

$$\sigma = \operatorname{Int}\operatorname{diag}(1, u_2) \circ (t \otimes \theta_1 \otimes \gamma) = \sigma_0 \otimes \gamma$$

with $\sigma_0 = \operatorname{Int}\operatorname{diag}(1, u_2) \circ t \otimes \theta_1$ on $M_2(F) \otimes Q_1$. Theorem 7 is thus proved. Note that the quaternion algebra Q is not uniquely determined by [6, (16.16)].

Let us now prove Theorem 8, starting with the following general remark:

14. Lemma. *Let σ and θ be two symplectic involutions on a central simple algebra A having degree a multiple of 4. Assume that the algebra A decomposes as $A = A_1 \otimes A_2$ for some subalgebras A_1, $A_2 \subset A$ and, furthermore, that there exist involutions σ_1, θ_1 on A_1 and σ_2, θ_2 on A_2 such that $\sigma = \sigma_1 \otimes \sigma_2$ and $\theta = \theta_1 \otimes \theta_2$. Moreover assume* $\deg A_1 \equiv 2 \bmod 4$ *and* $\deg A_2 \equiv 0 \bmod 4$. *Then, if σ_1 and θ_1 are of orthogonal type, $\Delta_\theta(\sigma) = 0$.*

Proof. Consider $u_1 \in \operatorname{Sym}(A_1, \theta_1)^\times$ and $u_2 \in \operatorname{Sym}(A_2, \theta_2)^\times$ such that

$$\sigma_1 = \operatorname{Int}(u_1) \circ \theta_1 \qquad \text{and} \qquad \sigma_2 = \operatorname{Int}(u_2) \circ \theta_2,$$

hence

$$\sigma = \operatorname{Int}(u_1 \otimes u_2) \circ \theta.$$

Then $\Delta_\theta(\sigma) = \big(\operatorname{Nrp}_\theta(u_1 \otimes u_2)\big)_2 \cup [A]$, and, by Lemma 10(d),

$$\operatorname{Nrp}_\theta(u_1 \otimes u_2) = \operatorname{Nrd}_{A_1}(u_1)^{\frac{1}{2}\deg A_2} \operatorname{Nrd}_{A_2}(u_2)^{\frac{1}{2}\deg A_1}. \tag{6}$$

Since $\deg A_2 \equiv 0 \bmod 4$, the first factor is a square. Moreover, since σ_1 and θ_1 are of orthogonal type, σ_2 and θ_2 must be of symplectic type. Hence,

$$\operatorname{Nrd}_{A_2}(u_2) = \operatorname{Nrp}_{\theta_2}(u_2)^2.$$

Therefore, equation (6) shows that $\operatorname{Nrp}_\theta(u_1 \otimes u_2) \in F^{\times 2}$, so that $\Delta_\theta(\sigma) = 0$. □

Even in the case $\deg A = 8$, the converse of Lemma 14 does not hold, since there are examples of algebras with involution which do not contain any invariant quaternion subalgebra on which the restriction of the involution is of orthogonal type. Indeed, arguing as in the proof of Corollary 9 (see section 1.3), we see that if θ is a symplectic involution on a central simple algebra A of degree 8, and if $A_1 \subset A$ is a quaternion subalgebra on which θ restricts to an orthogonal involution θ_1, then A and θ decompose in the form

$$A = A_1 \otimes A_2 \otimes A_3 \qquad \text{and} \qquad \theta = \theta_1 \otimes \theta_2 \otimes \gamma_3,$$

where A_1, A_2 and A_3 are quaternion algebras, θ_1 and θ_2 are orthogonal involutions on A_1 and A_2 respectively, and γ_3 is the canonical involution on A_3. This implies, in particular, that the signature of θ with respect to every ordering of the field F is either 0 or 8. For example, if $F = \mathbb{R}$ is the field of real numbers and θ is the involution adjoint to the hermitian form $\langle 1, 1, 1, -1\rangle$ on the usual quaternion algebra $\mathbb{H}$, then $\operatorname{sgn}\theta = 4$, and so $A = M_4(\mathbb{H})$ has no quaternion subalgebras on which θ restricts to an orthogonal involution. Therefore, even though $\Delta_\theta(\theta) = 0$, there is no decomposition as in Lemma 14. (See Example 15 for a subtler example where the converse of Lemma 14 fails.)

Returning to the proof of Theorem 8, we suppose until the end of this section that A is a central simple F-algebra of degree 8, with index dividing 4. Let σ be a symplectic involution on A, and suppose $A_1 \simeq M_2(F)$ is an invariant subalgebra on which the restriction of σ is an orthogonal involution. In this situation, we have a decomposition $A = A_1 \otimes A_1'$, where A_1' denotes the centralizer of A_1, and $\sigma = \sigma_1 \otimes \sigma_1'$, where σ_1 and σ_1' are the restrictions of σ to A_1 and A_1' respectively. As $A_1 \simeq M_2(F)$, we can find a hyperbolic symplectic involution θ_1 on A_1 and set

$$\theta = \theta_1 \otimes \sigma_1'.$$

The involution θ is hyperbolic of symplectic type and, by Lemma 14, we have

$$\Delta(\sigma) = \Delta_\theta(\sigma) = 0.$$

Conversely, let σ be a symplectic involution on A such that $\Delta(\sigma) = 0$. To prove that σ leaves invariant a subalgebra of A isomorphic to $M_2(F)$ on which it restricts to an orthogonal involution, we consider separately various cases, depending on the index of A. If A is split, every symplectic involution is hyperbolic and the property is a consequence of [1, Theorem 2.2]. If $\operatorname{ind} A = 2$, we can always represent A in the form

$$A = \operatorname{End}_Q V$$

where Q is a quaternion algebra and V is a 4-dimensional vector space over Q. The involution σ is then adjoint to a hermitian form h on V (with respect to the canonical involution γ on Q). Let e be an orthogonal basis for h. Since h is determined by σ up to a factor in $F^\times$, we may assume that the diagonalization of h with respect to the basis e is $\langle 1, \alpha_1, \alpha_2, \alpha_3\rangle$ with α_1, α_2, $\alpha_3 \in F^\times$. Let $V_0 \subset V$ be the F-subspace with basis e. We have $V = V_0 \otimes_F Q$ and

$$A = (\operatorname{End}_F V_0) \otimes_F Q, \qquad \sigma = \sigma_0 \otimes \gamma$$

where σ_0 is the orthogonal involution on $\operatorname{End}_F V_0$ adjoint to the bilinear symmetric form $\langle 1, \alpha_1, \alpha_2, \alpha_3\rangle$. As in Example 2, $\Delta(\sigma) = (\alpha_1\alpha_2\alpha_3)_2 \cup [Q]$. Therefore, the condition $\Delta(\sigma) = 0$ implies, by a theorem of Merkurjev, that $\alpha_1\alpha_2\alpha_3 \in \operatorname{Nrd}_Q(Q^\times)$. Changing basis if necessary, we may assume that $\alpha_3 = \alpha_1\alpha_2$. Then

$$\langle 1, \alpha_1, \alpha_2, \alpha_3\rangle = \langle 1, \alpha_1\rangle \otimes \langle 1, \alpha_2\rangle.$$

This implies $\sigma_0 = \sigma_1 \otimes \sigma_2$ on $\operatorname{End}_F V_0 \simeq M_2(F) \otimes M_2(F)$, where σ_1 and σ_2 are the involutions adjoint to the bilinear forms $\langle 1, \alpha_1\rangle$ and $\langle 1, \alpha_2\rangle$, respectively. This proves the theorem in this case.

Finally, suppose $\operatorname{ind} A = 4$. As in the proof of Theorem 7, given at the beginning of this section, we may then represent A as $A = \operatorname{End}_D V$ where D is a division algebra of degree 4 and V is a 2-dimensional D-vector space. For the rest of the proof, we use the same notation as in the proof of Theorem 7. We may thus assume $A = M_2(D) = M_2(F)\otimes_F D$ and $\sigma = \operatorname{Int}\operatorname{diag}(1,u_2)\circ\hat\theta_0$ for some symplectic involution θ_0 on D and $\hat\theta_0 = t\otimes\theta_0$. The involution $\theta = \operatorname{Int}\operatorname{diag}(1,-1)\circ\hat\theta_0$ is hyperbolic, and $\sigma = \operatorname{Int}\operatorname{diag}(1,-u_2)\circ\theta$. By Lemma 10(e), we have

$$\operatorname{Nrp}_\theta\bigl(\operatorname{diag}(1,-u_2)\bigr) = \operatorname{Nrp}_{\theta_0}(-u_2) = \operatorname{Nrp}_{\theta_0}(u_2),$$

hence

$$\Delta(\sigma) = \Delta_\theta(\sigma) = \bigl(\operatorname{Nrp}_{\theta_0}(u_2)\bigr)_2 \cup [D].$$

Therefore, the condition $\Delta(\sigma) = 0$ implies by [6, (16.19)] that the involution $\operatorname{Int}(u_2)\circ\theta_0$ on D is conjugate to θ_0. We may then find $v\in D^\times$ such that

$$\operatorname{Int}(u_2)\circ\theta_0 = \operatorname{Int}(v)\circ\theta_0\circ\operatorname{Int}(v)^{-1} = \operatorname{Int}\bigl(v\theta_0(v)\bigr)\circ\theta_0,$$

hence $u_2 = v\theta_0(v)\lambda$ for some $\lambda\in F^\times$. The involution θ is conjugate to

$$\operatorname{Int}\operatorname{diag}(1,v)^{-1}\circ\theta\circ\operatorname{Int}\operatorname{diag}(1,v) = \operatorname{Int}\operatorname{diag}(1,\lambda)\circ\hat\theta_0,$$

which restricts to $\operatorname{Int}\operatorname{diag}(1,\lambda)\circ t$ on $M_2(F)\subset M_2(D)$. Therefore, θ leaves the subalgebra $\operatorname{diag}(1,v)M_2(F)\operatorname{diag}(1,v^{-1})$ invariant and restricts to an orthogonal involution on that subalgebra. The proof of Theorem 8 is thus complete.

15. Example. The following is another example where the converse of Lemma 14 fails. Consider three quadratic extensions K_1, K_2, K_3 of a field k,

$$K_i = k(\sqrt{a_i}) \qquad \text{for some } a_i\in k,$$

such that $K_1\otimes_k K_2\otimes_k K_3$ is a field, and let $F = k(x_1,x_2,x_3)$ be the field of rational fractions in three indeterminates over k. For $i=1$, 2, 3, consider K_i as a subfield of the quaternion algebra $A_i = (a_i,x_i)_F$. On the tensor product

$$A = A_1\otimes_F A_2\otimes_F A_3,$$

consider the symplectic involution

$$\theta = \theta_1\otimes\theta_2\otimes\gamma_3,$$

where γ_3 is the conjugation involution on A_3 and θ_1 (resp. θ_2) is an orthogonal involution on A_1 (resp. A_2) which is the identity on K_1 (resp. K_2).

Let $\lambda\in N_{K_1/k}(K_1^\times)\cap N_{K_2/k}(K_2^\times)$. By a well-known property of biquadratic extensions (see for instance [4, 2.13]), we may find $u\in K_1\otimes_k K_2$ and $v\in k^\times$ such that

$$\lambda = v^2 N_{K_1\otimes_k K_2/k}(u).$$

Viewing $u = u\otimes 1$ in A, we let

$$\sigma = \operatorname{Int}(u)\circ\theta.$$

By Lemma 10(d), we have $\operatorname{Nrp}_\theta(u) = \operatorname{Nrd}_{A_1\otimes_F A_2}(u) = N_{K_1\otimes_k K_2/k}(u)$, hence

$$\Delta_\theta(\sigma) = (\lambda)\cup[A].$$

Since λ is a norm from K_1 and K_2, hence a reduced norm from A_1 and A_2, it follows that

$$\Delta_\theta(\sigma) = (\lambda)\cup[A_3].$$

Therefore, $\Delta_\theta(\sigma) = 0$ if and only if $\lambda\in N_{K_1/k}(K_1^\times)\cap N_{K_2/k}(K_2^\times)\cap N_{K_3/k}(K_3^\times)$.

Now, suppose $\sigma = \operatorname{Int}(w)\circ\sigma'\circ\operatorname{Int}(w)^{-1}$ for some involution σ' leaving A_1 invariant. Then $\sigma' = \operatorname{Int}\bigl(w^{-1}u\theta(w)^{-1}\bigr)\circ\theta$, and the proof of Lemma 14 shows that

$\operatorname{Nrp}_\theta\bigl(w^{-1}u\theta(w)^{-1}\bigr) \in F^{\times 2}$. Since $\operatorname{Nrp}_\theta\bigl(w^{-1}u\theta(w)^{-1}\bigr) = \operatorname{Nrp}_\theta(u)\operatorname{Nrd}_A(w)^{-1} = \lambda v^{-2}\operatorname{Nrd}_A(w)^{-1}$, it follows that

$$\lambda \in F^{\times 2}\cdot \operatorname{Nrd}_A(A^\times).$$

By Proposition 9 of [3], we then have

$$\lambda \in k^{\times 2}\cdot N_{M/k}(M^\times) \qquad \text{with } M = K_1\otimes_k K_2\otimes_k K_3.$$

Therefore, examples of triquadratic extensions $M = K_1\otimes_k K_2\otimes_k K_3/k$ such that

$$N_{K_1/k}(K_1^\times)\cap N_{K_2/k}(K_2^\times)\cap N_{K_3/k}(K_3^\times) \neq k^{\times 2}\cdot N_{M/k}(M^\times)$$

yield examples of involutions σ for which $\Delta_\theta(\sigma) = 0$ even though σ is not conjugate to an involution leaving A_1 invariant. Triquadratic extensions of this type were constructed in [10] (see also [5, Proposition 3]).

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