THE INVARIANT e^0

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Abstract. The image of the group homomorphism $e^0_- : W^-(\mathbb{P}^1\times\mathbb{P}^1) \to E(\mathbb{P}^1\times\mathbb{P}^1)$ from the Witt group of skew symmetric bilinear forms over a product of two projective lines into the subquotient $E(\mathbb{P}^1\times\mathbb{P}^1)$ of the Grothendieck group $K_0(\mathbb{P}^1\times\mathbb{P}^1)$ is equal to $\mathbb{Z}_2 H\boxtimes H$.

Introduction

Let X be a scheme with the trivial vector bundle $\mathcal{O}_X$. If M is a vector bundle on X then its dual bundle is

$$M^\wedge = \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X).$$

Let $K_0(X)$ denote the Grothendieck ring of X. The dualization functor $^\wedge$ induces an involution $^\wedge : K_0(X) \to K_0(X)$ given by the formula

$$[M] \longmapsto [M^\wedge].$$

Consider now the following complex

$$\cdots \xrightarrow{1-^\wedge} K_0(X) \xrightarrow{1+^\wedge} K_0(X) \xrightarrow{1-^\wedge} K_0(X) \xrightarrow{1+^\wedge} K_0(X) \xrightarrow{1-^\wedge} \cdots$$

and define $EX = \ker(1-^\wedge)/\mathrm{im}(1+^\wedge)$.

A pair (M,β) is said to be a symmetric (skew symmetric) bilinear space over X if M is a vector bundle on X and $\beta : M \to M^\wedge$ is an isomorphism such that $\beta = \beta^\wedge$ ($\beta = -\beta^\wedge$). Let $W(X)$ (respectively $W^-(X)$) be a Witt ring of symmetric (resp.Witt group of skew symmetric) bilinear spaces over X.

M. Szyjewski introduced the invariant $e^0_+ : W(X) \to EX$

$$e^0_+(M,\beta) = [M]\mathrm{mod}(\mathrm{im}(1+^\wedge))$$

which is a natural generalization of the dimension index.

We will focus our attention on the case when $X = \mathbb{P}^1\times\mathbb{P}^1$ is the product of two projective lines over a fixed field k. Let p_1 and p_2 be the projections onto $\mathbb{P}^1$. For given two vector bundles F and G on $\mathbb{P}^1$, $F\boxtimes G = p_1^*([F])\otimes p_2^*([G])$. We have

$$E(\mathbb{P}^1\times\mathbb{P}^1) = \mathbb{Z}_2[\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}] \oplus \mathbb{Z}_2 H\boxtimes H,$$

where $H = [\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}(-1)_{\mathbb{P}^1}] \in K_0(\mathbb{P}^1)$ and

$$\mathrm{im}(e^0_+) = \mathbb{Z}_2[\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}].$$

(see [Sz, 2.1.6, p. 458])

One can now consider $e^0_- : W^-(\mathbb{P}^1\times\mathbb{P}^1) \to E(\mathbb{P}^1\times\mathbb{P}^1)$. The main purpose of this paper is to show that $\mathrm{im}(e^0_-) = \mathbb{Z}_2 H\boxtimes H$.

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The image of e^0_-.

Theorem. *Let* $e^0_- : W^-(\mathbb{P}^1 \times \mathbb{P}^1) \to E(\mathbb{P}^1 \times \mathbb{P}^1)$*. Then* $im(e^0_-) = \mathbb{Z}_2 H \boxtimes H$.

Proof.
First we show that

$$\operatorname{im}(e^0_-) \subseteq \mathbb{Z}_2 H \boxtimes H.$$

Choose any $(M, \beta) \in W^-(\mathbb{P}^1 \times \mathbb{P}^1)$. The elements 1, $p_1^*([\mathcal{O}(-1)])$, $p_2^*([\mathcal{O}(-1)])$, $[\mathcal{O}(-1)]$ form a basis of the free abelian group $K_0(\mathbb{P}^1 \times \mathbb{P}^1)$. Thus

$$[M] = a + bp_1^*([\mathcal{O}(-1)]) + cp_2^*([\mathcal{O}(-1)]) + d[\mathcal{O}(-1)],$$

for some $a, b, c, d \in \mathbb{Z}$. We find the coefficients of $[M^\wedge]$ with respect to the chosen basis:

$$\begin{aligned}
[M^\wedge] &= a + bp_1^*([\mathcal{O}(1)]) + cp_2^*([\mathcal{O}(1)]) + d[\mathcal{O}(1)] \\
&= a + b(2 - p_1^*([\mathcal{O}(-1)])) + c(2 - p_2^*([\mathcal{O}(-1)])) \\
&\quad + d(4 - 2p_1^*([\mathcal{O}(-1)]) - 2p_2^*([\mathcal{O}(-1)]) + [\mathcal{O}(-1)]) \\
&= a + 2b + 2c + 4d + (-2d - b)p_1^*([\mathcal{O}(-1)]) \\
&\quad + (-2d - c)p_2^*([\mathcal{O}(-1)]) + d[\mathcal{O}(-1)]
\end{aligned}$$

Since $[M] = [M^\wedge]$ we get the system of three linear equations

$$\begin{cases} a = a + 2b + 2c + 4d \\ b = -2d - b \\ c = -2d - c \end{cases}.$$

Hence $c = b = -d$ and

$$\begin{aligned}
[M] &= a + bp_1^*([\mathcal{O}(-1)]) + bp_2^*([\mathcal{O}(-1)]) - b[\mathcal{O}(-1)] \\
&= a + b - b(1 - p_1^*([\mathcal{O}(-1)]) - p_2^*([\mathcal{O}(-1)]) + [\mathcal{O}(-1)]) \\
&= a + b - bH \boxtimes H \\
&= \dim(M) - bH \boxtimes H.
\end{aligned}$$

Since (M, β) is nonsingular its dimension has to be an even number, so

$$e^0_-(M, \beta) = b(\text{mod}2)H \boxtimes H.$$

Now we prove that $\operatorname{im}(e^0_-) \supseteq \mathbb{Z}_2 H \boxtimes H$. It is enough to show that there exists a bilinear skew symmetric form (P, β) such that $e^0_-(\ P, \beta) = H \boxtimes H$. Let $\mathbb{P}^1 \times \mathbb{P}^1$ be given in $\mathbb{P}^3 = \operatorname{Proj} k[x, y, z, t]$ by the equation $xy = zt$. Consider the open cover of $\mathbb{P}^1 \times \mathbb{P}^1$ consisting of the sets $D(x), D(y), D(z), D(t)$ (for any homogenous polynomial $f \in k[x, y, z, t]$, $D(f) = \{p \in \operatorname{Proj} k[x, y, z, t] : f \notin p\}$). Denote $M = p_2^*(\mathcal{O}(1)) \oplus p_1^*(\mathcal{O}(1)) \oplus \mathcal{O}$. Then M is given by the following transition maps

$$\chi^\#_{y,x} = \begin{bmatrix} \frac{x}{z} & 0 & 0 \\ 0 & \frac{x}{t} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \chi^\#_{z,x} = \begin{bmatrix} \frac{x}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \chi^\#_{t,x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{t} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One may notice that

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{y}{x} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x}{z} & 0 & 0 \\ 0 & \frac{x}{t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{z}{y} & 0 & 1 \\ \frac{t}{y} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{x}{y} & 0 & \frac{x}{z} \\ 0 & \frac{x}{t} & -\frac{x}{z} \\ 0 & 0 & -\frac{y}{z} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{y}{x} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{x}{z} & 1 & 0 \\ \frac{y}{z} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{y}{x} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x}{t} & 1 & 0 \\ 1 & 0 & 0 \\ \frac{y}{t} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{t} & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{y}{x} & 1 \end{bmatrix},$$

so there is an exact sequence

$$0 \to \mathcal{O}(1) \to M \to P \to 0$$

where P be is the subbundle given by transitions:

$$\pi^{\#}_{y,x} = \begin{bmatrix} \frac{x}{t} & -\frac{x}{z} \\ 0 & -\frac{y}{z} \end{bmatrix}, \qquad \pi^{\#}_{z,x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \pi^{\#}_{t,x} = \begin{bmatrix} -1 & 0 \\ -\frac{y}{x} & 1 \end{bmatrix}.$$

Since $\overset{2}{\Lambda}P \cong \mathcal{O}$ (notice that $\det \pi^{\#}_{y,x} = \det \pi^{\#}_{t,x} = -1$ and $\det \pi^{\#}_{z,x} = 1$), there exists a skew symmetric nonsingular bilinear form

$$P \times P \to \overset{2}{\Lambda}P \cong \mathcal{O}.$$

Moreover

$$[p_2^*(\mathcal{O}(1))] + [p_1^*(\mathcal{O}(1))] + [\mathcal{O}] = [\mathcal{O}(1)] + [P],$$

thus the class $[P]$ in the group $E(\mathbb{P}^1 \times \mathbb{P}^1)$ is

$$\begin{aligned} [P] &= [p_2^*(\mathcal{O}(1))] + [p_1^*(\mathcal{O}(1))] + [\mathcal{O}] - [\mathcal{O}(1)] \\ &= 2[\mathcal{O}] - ([\mathcal{O}] - [p_1^*(\mathcal{O}(1))] - [p_2^*(\mathcal{O}(1))] + [\mathcal{O}(1)]) \\ &= H \boxtimes H. \end{aligned}$$

References

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