UNIPOTENT SUBGROUPS OF REDUCTIVE GROUPS IN CHARACTERISTIC $p > 0$

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Abstract. — We prove two conjectures of Tits on the unipotent elements of semisimple algebraic groups defined over a field with positive characteristic.

Summary

1. — Introduction

1.1. — Good finite subgroups. Let k be a field of characteristic $p \geq 0$, and then k_s be a separable closure of k. Let G/k be a reductive connected algebraic group. For the notions on algebraic groups used here, we refer to the paper by Borel and Tits [BT1] and the book of Demazure and Gabriel [DG]. If G is semisimple and absolutely almost simple, we denote by $S(G)$ the finite set of torsion primes of G as defined by Serre in [Se2]. In the case of G simply connected, the the Dynkin index will be denoted by d_G (cf. [LS], § 2). Note taht the prime factors of d_G lie in $S(G)$. We recall Tits' definition of good unipotent elements.

Definition 1 [T2]. — *Let u be a unipotent element of $G(k)$. The element u is k–good if u lies in the unipotent radical of a k–parabolic subgroup of G, otherwise u is k–bad.*

In characteristic 0, all unipotent elements are k–good, so the definition is relevant only if $\mathrm{char}(k) = p > 0$; in that case, a unipotent element u has finite order $q = p^r$ and can be viewed as a morphism $u^\sharp : \mathbb{Z}/q\mathbb{Z} \to G$ sending 1 to u. This viewpoint leads to the following definition of goodness for morphisms $M \to G$ where M/k is a finite étale group.

Definition 1'. — *Let M/k be a finite étale group and $\phi : M/k \to G$ be a morphism. The morphism ϕ is k–good if $\phi(M(k_s))$ lies in the radical of a k–parabolic subgroup of G, otherwise ϕ is k–bad.*

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If ϕ is injective, we also say that M is a k–good subgroup of G. We say that $\phi : M \to G$ is unipotent if $\phi(M(k_s))$ consists of unipotent elements. Let us prove that the two definitions are compatible.

Lemma 1. — *Assume* $\mathrm{char}(k) = p > 0$*. Let* u *be a unipotent element of order* q *and* $u^\sharp : \mathbb{Z}/q\mathbb{Z} \to G$ *be the associated morphism. Then* u *is* k*–good if and only if* $u^\sharp$ *is* k*–good.*

Proof : If u is k–good, it is obvious that the morphism $u^\sharp$ is k–good. Conversely, assume that the morphism $u^\sharp : \mathbb{Z}/q\mathbb{Z} \to G$ is k–good. Then $u = u^\sharp(1)$ lies in the radical $R(P)$ of a k–parabolic subgroup P of G, which is an extension of a k–torus S by the unipotent radical R_uP of P, i.e. we have an exact sequence $1 \to R_u(P) \to R(P) \to S \to 1$. As $S(k)$ has no element of order p, u lies in $R_u(P)(k)$, and u is k–good. □

Definition 2 [T2]. — *Let* α *be a* k*–automorphism of* G *of finite order. The morphism* α *is* k*–isotropic if* α *normalizes some proper* k*–parabolic subgroup of* G*, otherwise* ϕ *is* k*–anisotropic.*

As above, we can extend this definition to isotropic morphisms $M \to \mathrm{Aut}(G)$ which will yield the previous one in the case $M = \mathbb{Z}/n\mathbb{Z}$.

Definition 2'. — *Let* $\phi \in \mathrm{Hom}_{k-gr}(M, \mathrm{Aut}(G))$*. The morphism* ϕ *is* k*–isotropic if* $\phi(M(k_s))$ *normalizes some proper* k*–parabolic subgroup of* G*, otherwise* ϕ *is* k*–anisotropic.*

We say that $\phi \in \mathrm{Hom}_{k-gr}(M, G)$ is k–isotropic if the composite $Ad\circ\phi : M \to G$ is k–isotropic. Obviously, k–anisotropic implies k–bad.

1.2. — The conjectures of Tits. Our first result is the proof of the following conjecture (see Theorem 2 of § 2).

Conjecture 1 [T3]. — *Assume* $p > 0$*,* $[k : k^p] \leq p$ *and that* G/k *is semisimple and simply connected. Then every unipotent subgroup of* $G(k)$ *(i.e a subgroup consisting of unipotent elements) is* k*–embeddable into the unipotent radical of a* k*-parabolic subgroup of* G*.*

The case $p = 2$ is due to Tits [T2, § 4.5]. Further known cases are when p is not a torsion prime of G, and types A_n and C_n (*loc. cit.*, § 3.5 and 4.4). In conjecture 1, the condition $[k : k^p] \leq p$ is necessary as the following theorem of Tits shows.

Theorem 1 [T4, prop. S1] and [T5, th. 7]. — *Assume that* G *is split simply connected and almost simple. If* $[k : k^p] \geq p^2$ *and* p *divides* d_G*, then the group* $G(k)$ *contains a* k*–bad unipotent element of order* p*.* □

Our proof of conjecture 1 is based on the reduction to a problem in Galois cohomology for the group $G_{k((t))}$, which is done in Proposition 3 (§ 2.2), and on known cases of Serre's conjecture II in Galois cohomology [Gi1]. Our second result is the proof of the following conjecture (see Theorem 3 of § 4), which is related to the shape of conjugacy classes of k–anisotropic automorphisms.

Conjecture 2 [T2]. — *Assume* $p > 0$ *and that* G/k *is split and almost simple. Let* α *be a* k*–anisotropic automorphism of order* p*. Then* α *normalizes a maximal* k*–split torus of* G*.*

Let us recall first the known cases handled by Tits.

1) $p = 2$, all types,
2) $p = 3$, G of type D_4 or E_6, using a triality argument due to Harder [H],
3) type A_n.

It turns out that conjecture 2 and a large part of the present paper make sense in a broader setting, including the characteristic zero case. To this end, we formulate the following conjecture, which extends the preceeding one.

Conjecture 2'. — *Assume G/k is split almost simple. Let l be a prime of $S(G)$. We assume that $l = \mathrm{char}(k)$ or either that k contains a primitive l–root of unity. Let α be a k–anisotropic automorphism of order l. Then α normalizes a maximal k–split torus of G.*

Such anisotropic automorphisms are called *special* automorphisms. In section 4, we prove conjecture 2' except for the E_8 case. However, the case of E_8 and $l = p = 5$ is proven in [Gi3]. Tits' original conjecture is thus true.

The proof of conjecture 2' requires several steps. First we work in characteristic zero and prove by a case-by-case analysis conjecture 2' (except for the E_8 case) by using methods inspired by the proof of Hasse principle (§ 4.2). The tame case $l \neq p$ is then not hard and we can concentrate on the wild case $l = p = \mathrm{char}(k)$. We consider a complete discrete valuation ring A with residue field k and fraction field F_A of characteristic zero. We remark that special elements lift in characteristic zero, i.e. if $\alpha \in \mathrm{Aut}(G)(k)$ is a special automorphism of order p, then there exists $\widetilde{\alpha} \in \mathrm{Aut}(G)(A)$ of order p specializing to α. We show that anisotropic automorphisms lift in characteristic 0 (§ 4.3). The proof concludes by a specialization argument.

1.3. — Complements on finite subgroups. In this section, we remark on the extension of some known results to our present setting. The argument as used in the proof of proposition 3.2 of [T2] yields in fact the following.

Proposition 1. — *Let $\phi \in \mathrm{Hom}_{k-gr}(M, G)$, and let P/k be a k–parabolic subgroup normalized by ϕ. Let L be a Levi subgroup of P defined over k, so that $P = R_u(P) \rtimes L$ and let ϕ' be the composition of $M \to P \to L$.*

a) The following are equivalent :

i) ϕ is k–good in G

ii) ϕ' is k–good in G

iii) ϕ' is k–good in L.

b) If P is minimal among all k–parabolic subgroups normalized by ϕ, then ϕ' is k–anisotropic in L. □

Similarly, we can extend lemma 3.5 of [T2].

Lemma 2. — *Let H/k be a reductive subgroup of G/k. Let $\phi : M \to \mathrm{Aut}(G, H) \subset \mathrm{Aut}(G)$ be a k–anisotropic morphism. Then the morphism $\phi : M \to \mathrm{Aut}(G, H) \to \mathrm{Aut}(H)$ is k–anisotropic.* □

We also recall :

Proposition 2 ([BT2], prop. 3.6). — *Let $U \subset G(k)$ be a unipotent subgroup such that every element of U is k_s–embeddable in a Borel subgroup. Then U is k–embeddable in the unipotent radical of a k–parabolic subgroup of G.* □

In particular and in contrast to the tame case, a unipotent element u is k–good iff it is k_s–good.

Lemma 3. — *Let $g \in G(k)$ be an element of finite order invertible in k. Then g lies in a maximal k–torus of G. In particular g is k_s–good.*

Proof : Let $g \in G(k)$ be an element of finite order invertible in k. Then g is semisimple and g lies in some maximal torus T/k. As $T \times_k k_s$ is a split maximal torus, g is k_s–good. □

More generally, if $\mathrm{char}(k) = 0$, Borel-Mostow's theorem asserts that an automorphism of G of finite order normalizes a maximal torus of G [BM] (see also [P]).

2. — Finite unipotent subgroups and Galois cohomology

2.1. — A Galois cohomology class. Let us set $K = k((t))$ and denote by K_{mod} a maximal tamely ramified extension of K. Let $I = I_w \rtimes \mu_m$ be a k–étale group, semi–direct product of μ_m ($m \in k^\times$) by a constant p–group I_w. Throughout this paper, we assume that

there exists a totally ramified field extension L/K such that $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ *is an I–torsor.*

This hypothesis is satisfied in the two extreme cases $I = \mu_m$ (obviously) and $I = I_w$ since the maximal pro–p quotient of $\mathrm{Gal}(K_s/K)$ is a free pro–p group [Se1, § II.2.2, Cor. 1]. We denote by $\widetilde{L} = L.K(\mu_m)$, which is a Galois extension of K with Galois group $\widetilde{I}$. So, there is a diagram of field extensions

$$\begin{tikzcd}[arrows=dash]
 & \widetilde{L} & \\
L \arrow[ur] & & K(\mu_m), \arrow[ul] \\
 & L \cap K(\mu_m) \arrow[ul] \arrow[ur] & \\
 & K \arrow[u] &
\end{tikzcd}$$

and the residue field of $\widetilde{L}$ is $k(\mu_m)$. We denote by $f_\theta \in Z^1(\widetilde{I}, I(\widetilde{L}))$ the 1–cocycle defined by the I–torsor $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ and by $\theta \in H^1(K, I)$ its class. For simplicity, the reader can think of $I = \mathbb{Z}/p\mathbb{Z}$; this case, which is of main interest for us, was treated in [Gi2].

We consider the set $\mathrm{Hom}_{k-gr}(I, G)$ and pick $\phi \in \mathrm{Hom}_{k-gr}(I, G)$. The key point, due to Serre, is to associate to $\phi : I \to G$ the 1–cocycle

$$f_\phi = \phi_*(f_\theta) \quad \text{in} \quad Z^1(\widetilde{I}, G(\widetilde{L}))$$

and the cohomology class

$$\gamma(\phi) = \phi_*(\theta) = [f_\phi] \quad \text{in} \quad H^1(K, G).$$

Remark 1 : We shall use functoriality for a surjective base change $\lambda : I' \to I$ and an extension $L'/L/K$ of monodromy group I' : if there exists θ' such that $\lambda_*(\theta') = \theta$, then $\phi_*(\theta)_{L'} = \lambda^*(\phi \circ \lambda)$ in $H^1(L', G)$.

2.2. — How to see if a finite unipotent subgroup is k–good

PROPOSITION 3. — *a) If ϕ is k–good, then $\gamma(\phi) = 1$ in $H^1(K, G)$.*

b) Assume that ϕ is unipotent. Then the following assertions are equivalent :
i) ϕ is k–good,
ii) $\gamma(\phi) = 1$ in $H^1(K, G)$,
iii) $\gamma(\phi)_{K_{mod}} = 1 \in H^1(K_{mod}, G)$.

Proof : *a*) : Let P/k be a parabolic subgroup of G/k such that the radical $R(P)$ contains $\mathrm{Im}(\phi)$. The group $R(P)$ is split and we know that $H^1(K, R(P)) = 1$. Thus $\gamma(\phi) = 1$.

b) : As we consider unipotent morphisms, we can assume that $I = I_w$, i.e. I is a finite p–group. There is only the implication *iii*) $\Longrightarrow$ *i*) to prove. By hypothesis, there exists a tamely ramified extension K'/K of valuation ring O'/O and residue extension k'/k such that $\gamma(\phi)_{K'} = 1$ in $H^1(K', G)$. One can assume $K' = k'((t'))$, $(t')^e = t$ and $(e, p) = 1$, and that $G_{k'}$ is split. Let B/k' be a Borel subgroup of G/k'. We consider the extension $L' = L.K'/K'$, which is Galois of group $I = \mathrm{Gal}(L'/K')$ and valuation ring $O_{L'}$. Let us denote by $X/k' = G/B$ the variety of Borel subgroups of G and by $\pi : G/k' \to X/k'$ the canonical map. This variety is projective, the morphism π is smooth and one knows that the map on k'–points $G(k') \to X(k')$ is onto. Using Hensel's lemma and the valuative criterion of properness, one obtains easily that the map $\pi : G(O_{L'}) \to X(O_{L'})$ is onto and the map $G(O_{L'})/B(O_{L'}) \xrightarrow{\sim} G(k')/B(k')$ is bijective. We have the following commutative diagram of pointed sets (cf. [Se1], § 1.5.4, prop. 36)

$$\begin{array}{ccccc}
\Big[G(O_{L'})/B(O_{L'})\Big]^I & \to & H^1(I, B(O_{L'})) & \to & H^1(I, G(O_{L'})) \\
\downarrow \wr & & \downarrow & & \downarrow \\
\Big[G(L')/B(L')\Big]^I & \to & H^1(I, B(L')) & \to & H^1(I, G(L')).
\end{array}$$

Taking the fixed points of the twisted I–sets by the cocycle f_ϕ, one gets

$$\Big[{}_{f_\phi}\Big(G(O_{L'})/B(O_{L'})\Big)\Big]^I \xrightarrow{\sim} \Big[{}_{f_\phi}\Big(G(L')/B(L')\Big)\Big]^I,$$

so the class $[f_\phi] \in H^1(I, G(O_{L'}))$ has a reduction in $H^1(I, B(O_{L'}))$. Thus there exists $h \in G(O_{L'})$ such that $h\phi(\sigma)({}^\sigma h^{-1}) \in B(O_{L'})$ for any $\sigma \in I$. Let $\overline{h}$ be the image of h in $G(k')$. Then $\overline{h}\phi(\sigma)(\overline{h})^{-1} \in B(k')$ for any $\sigma \in I$ (we used here that L/K is totally ramified) and ϕ is k'–good. By Proposition 2, because k'/k is separable, ϕ is k–good. ∎

The wild inertia group $\mathrm{Gal}(K_s/K_{mod})$ is a pro–p group. If p is a good prime for G, i.e. if p is not a torsion prime of G, we know that $H^1(K_{mod}, G) = 1$ [Se2, th 4"]. The last proposition together with Proposition 2 gives then another proof of Tits' result [T2, cor. 2.6].

Corollary 1. — *Assume* $p = \mathrm{char}(k) \notin S(G)$. *Then any unipotent subgroup* U *of* $G(k)$ *is* k*–embeddable in the unipotent radical of a* k*–parabolic subgroup of* G.

2.3. — The case $[k : k^p] \leq p$

Theorem 2. — *Assume* $\mathrm{char}(k) = p > 0$, $[k : k^p] \leq p$ *and that* G *is semisimple simply connected. Then every unipotent subgroup of* $G(k)$ *is* k*–embeddable in the unipotent radical of a* k*-parabolic subgroup of* G.

We recall Kato's definition [K] of the p–dimension $dim_p(k)$ of k by means of cohomology groups $H^i_p(k)$. Let Ω_k be the k-vector space of the 1–differential forms of the $\mathbb{Z}$-algebra k. For any nonnegative integer i, we set $\Omega^i_k = \bigwedge^i \Omega_k$ and the exterior differential d maps Ω^i_k to Ω^{i-1}_k. There exists an unique additive p–linear application $\gamma : \Omega^i_k \to \Omega^i_k / d\Omega^{i-1}_k$ such that $\gamma(x\omega) = x^p w$

for any logarithmic differential form $\omega = dy_1/y_1 \wedge \cdots \wedge dy_i/y_i$. The operator ω is inverse of the Cartier operator. We set

$$H_p^{i+1}(k) = Coker\Big(\gamma - 1 : \Omega_k^i \to \Omega_k^i / d\Omega_k^{i-1}\Big).$$

If $[k : k^p] = \infty$, one sets $\dim_p(k) = \infty$. If $[k : k^p] = p^r$, one sets :
- $\dim_p(k) = r$ if $H_p^{r+1}(k') = 0$ for any finite extension k'/k,
- $\dim_p(k) = r + 1$ otherwise.

Proof of Theorem 2 : Assume $[k : k^p] \leq p$. By Proposition 2, one can assume that $k = k_s$, that G is split and almost simple and that U is generated by a unipotent element of order q. One has ${}_pBr(k') = H_p^2(k') = 0$ for any finite extension k'/k because k is separably closed, so $\dim_p(k) \leq 1$. Then $\dim_p(K) \leq 2$ by [K, corollary to th. 3] where $K = k((t))$ as in § 2. By definition, one has $H_p^3(K') = 0$ for any finite field extension K'/K. As the wild inertia group $\mathrm{Gal}(K_s/K_{mod})$ is a pro–p group, the main result of [Gi1] gives then

$$H^1(K_{mod}, G) = 1.$$

Then Proposition 3.b shows that u is k–good. □

2.4. — Bad unipotents and torsors on the affine line. Raghunathan has conjectured that if G is semisimple simply connected, then G–torsors on the affine line $\mathbb{A}_k^1$ are constant, i.e. come from G–torsors on $\mathrm{Spec}(k)$ [R, p. 189]. It turns out that bad unipotent elements yield counterexamples to this conjecture.

PROPOSITION 4. — *Assume that G/k admits a bad unipotent element u of order p. Let $\mathcal{P} : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ be the Artin–Schreier covering, which is Galois of group $\mathbb{F}_p$ and defined by $t = \mathcal{P}(x) = x^p - x$. Let us define the 1–cocycle $h = (h_\sigma)_{\sigma \in \mathbb{F}_p}$ for this covering by*

$$h_\sigma = u^\sigma \in G(k[x]) \ \ (\sigma \in \mathbb{F}_p)$$

Then the cocycle h defines a G–torsor on $\mathbb{A}_k^1$ which is not isomorphic to a constant torsor.

Proof : One can assume $k = k_s$. So $H^1(k, G) = 1$ and we have to show that the torsor defined by h is not trivial. We lift our cocycle h in $G\big(k((\frac{1}{x}))\big)$ and we observe that $k(x)/k(t)$ is wildly ramified at ∞. As u is a bad unipotent element, Proposition 3.b shows that $[h]_{k((\frac{1}{t}))}$ is not trivial in $H^1(k((\frac{1}{t})), G)$, and a fortiori $[h]$ is not trivial in $H^1(\mathbb{A}_k^1, G)$. □

Remark 2 : Theorem 1 (§ I.2) yields non constant G–torsors on the affine line for G split semisimple simply connected provided p divides d_G and $[k : k^p] \geq p^2$.

3. — Conjugacy classes of k–anisotropic automorphisms of split groups

From now on, we assume that G/k is split and semisimple. So G is the extension of $\mathbb{Z}$ to k of the corresponding Chevalley group scheme $G_{/\,\mathrm{Spec}(\mathbb{Z})}$. For any ring A, we denote by simplicity $G \times_{\mathbb{Z}} A = G_J \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$.

The goal of this section is to give a powerful cohomological criterion analogous to Proposition 3 for distinguishing conjugacy classes of k–anisotropic automorphisms. We denote by $T/\mathbb{Z}$ a maximal split torus of $G/\mathbb{Z}$ and by $B/\mathbb{Z}$ a Borel subgroup containing T. Let $J \subset \mathrm{Aut}(G, B, T)$ be a subgroup and let us consider the semi-direct product $G_J/\mathbb{Z} = G/\mathbb{Z} \rtimes J$. We have a natural map $G_J/\mathbb{Z} \to \mathrm{Aut}(G)/\mathbb{Z}$ with kernel the center of $G/\mathbb{Z}$ and we note that if $G/\mathbb{Z}$ is adjoint and $J = \mathrm{Aut}(G, B, T)$, one has $G_J/\mathbb{Z} = \mathrm{Aut}(G)/\mathbb{Z}$. Let us introduce the following notation :
- $W = N_G(T)/T$ and $W_J = N_{G_J}(T)/T = W \rtimes J$ (the Weyl groups),

- $\widehat{T} = \mathrm{Hom}_{\mathbb{Z}-gr}(T, \mathbb{G}_m)$, $\widehat{T}^0 = \mathrm{Hom}_{\mathbb{Z}-gr}(\mathbb{G}_m, T)$.

Proposition 5. — *Let* $\phi \in \mathrm{Hom}_{gr}\big(I, G_J(k)\big)$.

a) The following assertions are equivalent

i) ϕ *is* k*–anisotropic,*

ii) $\Big[{}_{f_\phi} G_J(O_{\widetilde{L}})\Big]^{\widetilde{I}} = \big({}_{f_\phi} G_J\big)(K)$,

iii) the group ${}_{f_\phi}G_J/K$ *is* K*–anisotropic.*

b) Assume ϕ *is* k*–anisotropic. Let* $\phi' : I \to G_J$ *be another group morphism. The following assertions are equivalent :*

i) ϕ' *is conjugate to* ϕ *under* $G_J(k)$*,*

ii) $\gamma(\phi) = \gamma(\phi')$ *in* $H^1(K, G_J)$.

c) Let k'/k *be a Galois extension such that* $\phi/k' : I \to G_J$ *is a* k'*–anisotropic morphism. With the notation* $Z_G(\phi) = Z_G(\mathrm{Im}(\phi))$*, one has an injection*

$$H^1(k'/k, Z_{G_J}(\phi)) \hookrightarrow H^1(K.k'/K, {}_{f_\phi} G_J).$$

Proof : Let us begin by the second assertion.

$b)$ The implication $i) \Longrightarrow ii)$ is obvious. Conversely, assume that $\gamma(\phi) = \gamma(\phi')$ in $H^1(\widetilde{L}/K, G)$. Then there exists $g \in G(L)$ such that

$$g^{-1}\phi(\sigma)\, {}^\sigma g = \phi'(\sigma) \qquad (\sigma \in \widetilde{I}).$$

Let $\mathcal{B}$ be the Bruhat–Tits building of the group $G_{\widetilde{L}}$ [BrT,§ I.7.4] which is a metric space. This building is equipped with an (isometric) left action of the group $G_J(\widetilde{L})$ and an action of the Galois group $\widetilde{I} = \mathrm{Gal}(\widetilde{L}/K)$ denoted by $x \mapsto {}^\sigma x$ for any $\sigma \in \widetilde{I}$. We denote by c the center of the building, i.e. the vertex of $\mathcal{B}$ fixed by $G(O_{\widetilde{L}})$. One defines a twisted action of $\widetilde{I}$ on $\mathcal{B}$ by

$$\sigma.x = \phi(\sigma).{}^\sigma x \qquad (\sigma \in \widetilde{I}).$$

This action is isometric and simplicial (i.e. maps a facet on a facet). Clearly, because $\mathrm{Im}(\phi) \subset G_J(k)$, the center c is fixed under the twisted action of $\widetilde{I}$. Due to

$$\sigma.(g.c) = \phi(\sigma)({}^\sigma g).c = g\phi'(\sigma).c = g.c,$$

we see that $g.c$ is also fixed by $\widetilde{I}$ for the twisted action. Assume that $g.c \neq c$. As the segment $[c, g.c]$ is fixed pointwise by $\widetilde{I}$, there exists a facet of $\mathcal{B}$ containing strictly c which is stabilized by the twisted action of $\widetilde{I}$. Hence there exists (by the isomorphim between the link of c in $\mathcal{B}$ and the spherical building of $G_J/k(\mu_m)$) a facet of the spherical building of $G_J/k(\mu_m)$ which is stabilized by $\widetilde{I}$. In other words, there exists a proper k–parabolic subgroup P of G such that

$$\Big[{}_{f_\phi}\Big(G_J(k(\mu_m))/N_{G_J}(P)(k(\mu_m))\Big)\Big]^{\widetilde{I}} \neq \emptyset, \ \ i.e$$

$$\Big\{[\varphi] \in G_J(k(\mu_m))/N_{G_J}(P)(k(\mu_m)) \ \ | \ \ \varphi^{-1}\,\mathrm{Im}(\phi)\varphi \in N_{G_J}(P)(k(\mu_m))\Big\}^{\mathrm{Gal}(k(\mu_m)/k)} \neq \emptyset,$$

which means that ϕ normalizes the k–parabolic subgroup $\varphi(P)$. This contradicts the anisotropy of ϕ. We conclude that $g.c = c$, i.e. $g \in G_J(O_L)$. Reducing the identity $g^{-1}\phi(\sigma)({}^\sigma g) = \phi'(\sigma)$ to $G_J(k(\mu_m))$, one gets $\overline{g}^{-1}\phi\overline{g} = \phi'$ with $\overline{g} \in G(k)$.

a) *i)* $\Longrightarrow$ *ii)* : Assume ϕ is k–anisotropic. The same argument as before with $\phi' = \phi$ shows that

$$\big({}_{f_\phi}G_J\big)(K) = \{g \in G_J(\widetilde{L}) \ \mid\ \phi(s)({}^{\sigma}g)\phi(\sigma)^{-1} = g \ \forall \sigma \in \widetilde{I}\} \subset G_J(O_{\widetilde{L}}),$$

so $\big({}_{f_\phi}G_J\big)(K) = \Big[{}_{f_\phi}G_J(O_{\widetilde{L}})\Big]^{\widetilde{I}}$.

ii) $\Longrightarrow$ *iii)* : As $\big({}_{f_\phi}G\big)(K) \subset G_J(O_{\widetilde{L}})$ is bounded, the group ${}_{f_\phi}G/K$ does not contain any non trivial K–split torus, hence it is K–anisotropic.

iii) $\Longrightarrow$ *i)* : We prove *not i)* $\Longrightarrow$ not *iii)*. Assume ϕ is k–isotropic, i.e. there exists a k–parabolic subgroup P such that $\mathrm{Im}(\phi) \subset N_{G_J}(P)(k)$. Then the class $\gamma(\phi)$ in $H^1(K, G_J)$ comes from $H^1(K, N_{G_J}(P))$, and the twisted group ${}_{f_\phi}G/K$ is isotropic.

c) We set $K' = K \otimes_k k'$, $O' = L \otimes_k k'$, etc.. The first assertion shows that $\big({}_{f_\phi}G_J\big)(K') = \Big[{}_{f_\phi}G_J(O_{\widetilde{L}'})\Big]^{\widetilde{I}}$. But the map of $\mathrm{Gal}(k'/k)$ groups $Z_{G_J}(\phi)(k') \to \Big[{}_{f_\phi}G_J(O_{\widetilde{L}'})\Big]^{\widetilde{I}}$ is split, so the map $H^1(k'/k, Z_{G_J}(\phi)) \to H^1(K.k'/K, {}_{f_\phi}G_J)$ is injective. $\square$

First, we give the following corollary of the proposition and of Lemma 3 (§ 1.3).

Corollary 2. — *Assume that the adjoint group G_{ad} of G is simple. Assume m is an integer such that k contains a primitive m-th root of unity and such that there exists a k–anisotropic automorphism of order m. Then $m \in S(G)$.*

Proof : One can assume that $G = G_{ad}$. Let α be a k–anisotropic automorphism of order m and assume that $m \notin S(G)$. The exact sequence $1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$ and the fact that $\mathrm{Out}(G)$ is a finite group such that the prime divisors of $\sharp\,\mathrm{Out}(G)$ are bad primes shows that α is an inner automorphism. By Lemma 3, there exists a maximal torus S which contains α. The choice of a primitive m–root of unity ζ induces an isomorphim $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \mu_m$; taking $L = k(\sqrt[m]{t})/K$ and $\theta = (t) \in H^1(K, \mu_m)$ as in section 2, we see that $\gamma(\alpha) \in \mathrm{Im}\big(H^1(L/K, S) \to H^1(L/K, G)\big)$. But $H^1(L/K, S)$ is an $S(G)$-primary torsion group, so $H^1(L/K, S) = 1$ and $\gamma(\alpha) = 1$. By Proposition 5.a, α is then k–isotropic. Contradiction. $\square$

Questions : Assume $p > 0$. We lack an analogue of Proposition 2 (§ 3.1) for k–anisotropic automorphisms of order p. More precisely, we ask the following :

1) Let α be a k–anisotropic automorphism of G of order p. Is α_{k_s} k_s–anisotropic ?
2) Let α, α' be k–anisotropic automorphisms of G of order p such that α and α' are conjugate under $\mathrm{Aut}(G)(k_s)$. Are α and α' conjugate under $\mathrm{Aut}(G)(k)$?

4. — Anisotropic automorphisms are special

In this section, we prove conjecture 2'; as mentionned earlier, this proves Tits original conjecture is proven as the case E_8, $p = 5$ has already been done in [Gi3].

Theorem 3. — *Conjecture 2' is true for all types excepting possibly case E_8, $l = 5 \neq p$.*

The steps have been described in section 1.2 and we take into account cases done by Tits in the wild case. Their proof works also in the tame case, which is much simpler.

4.1. — Special automorphisms

Definition 3. — *A k–automorphism f of a torus S/k is anisotropic if the k–group S^f of fixed points of S by f is finite.*

Definition 3'. — *a) An element w of W_J is special if the automorphism $w : T \to T$ defined by $t \mapsto w.t$ is anisotropic.*

b) An element of $G_J(k)$ of order n is special (relative to the split torus T) if it lies in $N_{G_J}(T)(k)$ and its image in W_J is a special element of order n.

Some special elements of W_J are listed in [Sp] for any simple type.

Lemma 4. — *Let w be a special element of W_J of order n.*

a) The homomorphism $N(w) = 1 + w + ... + w^{n-1} : T \to T$ is trivial.

b) Let g be a special element relative to T mapping to w. Then for any $\tau \in T(k)$, τg is a special element of $G_J(k)$ of order n and is $N_G(T)(\overline{k})$–conjugate to g.

c) Let g be an element in $N_G(T)(k)$ such that g is k–anisotropic in G. Then g acts anisotropically on T. Moreover, if $\mathrm{char}(k) = p > 0$ and g has order p^r, then g is special (relative to T).

d) Assume that n is odd. Let A be a complete discrete valuation ring with residue field k and fraction field F_A. Let g be a special element of $G_J(k)$ (relative to T) mapping to w. Then there exists $\widetilde{g} \in N_{G_J \times_{\mathbb{Z}} A}(T)(A)$ of order n mapping to g in $G_J(k)$ such that $\widetilde{g}$ is a special element of $G_J(F_A)$ (relative to T) and mapping to w in W_J.

Proof : *a*) As $(1-w).N(w) = 0$ and $1-w$ is an invertible automorphism of $\widehat{T} \otimes_{\mathbb{Z}} \mathbb{Q}$, then $N(w) = 0$ as an automorphism of $\widehat{T} \otimes_{\mathbb{Z}} \mathbb{Q}$, hence $N(w) : T \to T$ is trivial.

b) Let us denote by w the image of g in W_J. We have to show that τg has order n ; it is given by the computation

$$(\tau g)^n = \tau \times g\tau g^{-1} \times g^2\tau g^{-2} \cdots \times g^{n-1}\tau g^{(-n+1)} = \tau \times (w.\tau) \times \cdots \times (w^{n-1}.\tau) = N(w).\tau = 1.$$

So the order of τg divides n and as w has order n, the order of τg is exactly n. We consider the equation

$$\tau g = xgx^{-1} \qquad (x \in T(\overline{k})),$$

or equivalently

$$\tau = xgx^{-1}g^{-1} = x \times w.x^{-1} \qquad (x \in T(\overline{k})).$$

As the group morphism $T \to T$ given by $t \mapsto t \times w.t^{-1}$ has finite kernel, it is surjective, so there exists $x \in T(\overline{k})$ such that $\tau g = xgx^{-1}$.

c) Let g be an element of $N_{G_J}(T)(k)$ such that g is k–anisotropic. Assume that g acts isotropically on T, then $T^{\langle g \rangle} = \mathrm{Ker}(1 - g : T \to T)$ has dimension greater than 1, so there exists a k–split subtorus S of T such that $g \in Z_G(S)(k)$. As $Z_G(S)$ is a k–Levi subgroup of a proper k–parabolic subgroup, g is k–isotropic. So g acts anisotropically on T. Assume moreover that g has order p^r ; we have to show that the image of g in W_J, say w_0, has order p^r. If not, $g^{p^{r-1}}$ is an element of order p of $T(k)$, and as $\mu_p(k) = 1$, this leads to a contradiction.

d) As G_J is a Chevalley group defined over $\mathbb{Z}$, the sequence $1 \to T \to N_{G_J}(T_J) \to W_J \to 1$ can be defined over $\mathbb{Z}$, and as $H^1_{\acute{e}t}(\mathbb{Z}, T) \xrightarrow{\sim} \big(\mathrm{Pic}(\mathbb{Z})\big)^{rank(G)} = 1$, one has the exact sequence

$$1 \to T(\mathbb{Z}) \to N_{G_J}(T_J)(\mathbb{Z}) \to W_J \to 1$$

and an isomorphism $T(\mathbb{Z}) \approx (\mathbb{Z}/2\mathbb{Z})^{rank(G)}$ (see [T1]). So the class of the preceeding extension is killed by 2. Let $g \in N_{G_J}(T_J)(k)$ be of odd order n. Then there exists $g_0 \in N_{G_J}(T_J)(\mathbb{Z})$ of order n with same image in W_J as g. There thus exists $\tau \in T(k)$ such that $\alpha = \tau\overline{\alpha}_0$, where $\overline{\alpha}_0$ is the reduction in k of α_0 via the map $N_{G_J}(T_J)(\mathbb{Z}) \to N_{G_J}(T_J)(A) \to N_{G_J}(T_J)(k)$. As $T \times_{\mathbb{Z}} A$ is a smooth group scheme, the map $T(A) \to T(k)$ is surjective by Hensel's lemma. We pick $\widetilde{\tau} \in T(A)$ mapping to τ. The element $\widetilde{g} := \widetilde{\tau}\alpha_0$ of $N_{G_J}(T_J)(A)$ lifts $g = \tau\overline{\alpha}_0$, maps to w in W_J and by assertion *b*), $\widetilde{g}$ is a special element of $G_J(F_A)$ relative to T. □

4.2. — Triality

In this section, we handle the case $l = 3$ assuming then that k contains a primitive third root of unity. First, let us remark that we can assume that G is adjoint, so the group $\mathrm{Aut}(G) = G \rtimes \mathrm{Aut}(G, B, T)$ is of the type we studied before. Let $\alpha \in \mathrm{Aut}(G)(k)$ be a k–anisotropic automorphism of order 3. We shall use the following triality argument.

Lemma 5. — *Assume that G has type D_4 (resp. F_4, E_6, E_7, E_8). Then α normalizes a k–Levi subgroup of a k–parabolic subgroup of G of type A_2 (resp. A_2, $A_2 \times A_2$, E_6, E_7).*

Proof : Let P be a k–parabolic subgroup of G such that P_{red} has type A_3 (resp. C_3, D_5, E_6, E_7). Then $\mathrm{codim}_G(P) = 6$ (resp. $15, 16, 27, 57$). Let us define the k–group

$$C = P \cap \alpha(P) \cap \alpha^2(P).$$

Then $codim_G(C) \leq 3 \times codim_G(P)$, so $dim_G(C) \geq 28 - 10$ (resp. 7, 30, 52, 77) (cf. [PR] p. 380). We claim that C is a k–Levi subgroup of some k–parabolic subgroup of G. Let Q be a k–parabolic subgroup such that $C \subset Q \subset P$ and minimal for this property. Then the k–parabolic subgroup $R_u(Q).\big(Q \cap \alpha(Q)\big)$ contains C and is contained in Q, so $R_u(Q).\big(Q \cap \alpha(Q)\big) = Q$ and $M := Q \cap \alpha(Q)$ is a k–Levi subgroup of G ([BT1], proposition 4.10). But $C = M \cap \alpha^2(Q)$ is a k–parabolic subgroup of M ; if $C \neq M$, then α normalizes the split k–unipotent group $R_u(C)$, and by Proposition 3.1 of [BT2], α normalizes a k–proper parabolic subgroup of G, which contradicts the assumption of anisotropy. So C is a k–Levi subgroup of Q/k and the restriction of α is still k–anisotropic (Lemma 2). As $dim_k(C) \geq 10$, the only possibility is that C has type A_2. Other cases are considered on a case-by-case basis using Corollary 2 and we leave the details of the argument to the reader. □

Let us recall that Theorem 3 is true for A_2 and prove it inductively in the cases considered in Lemma 5. Lemma 5 gives a k–Levi subgroup C of a k–parabolic proper subgroup of G normalized by α. So the connected center $Z_G(C)^0$ is a k–split torus normalized by α and, by induction, the derived group $\mathcal{D}C$ contains a k–split maximal torus S normalized by α. It follows that $Z_G(C)^0.S$ is a maximal k-split torus of C (and G) normalized by α.

4.3. — Lifting in characteristic 0.

Lemma 6. — *Assume $p = \mathrm{char}(k) > 0$. Let A be a complete discrete valuation ring with residue field k and fraction field F_A. Let $\phi : I \to G_J$ be a group homomorphism which lifts to $\widetilde{\phi} : I \to G_J \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$.*

a) If $\widetilde{\phi} : I \to G_J \times_{\mathbb{Z}} F_A$ is a F_A–isotropic morphism, then ϕ is a k–isotropic morphism.

b) Assume ϕ is k–anisotropic. If $\widetilde{\phi}$ normalizes some maximal F_A–split torus of G_{J,F_A}, then ϕ normalizes some maximal k–split torus of G_J/k. Moreover, let $\phi' : I \to G_J$ be a group homomorphism which lifts to $\widetilde{\phi} : I \to G_J \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$ such that $\widetilde{\phi}$ and $\widetilde{\phi}'$ are conjugate under $G_J(F_A)$. Then ϕ and ϕ' are conjugate under $G_J(k)$ and ϕ' normalizes some maximal k–split torus of G_J/k.

Proof : a) Assume that $\widetilde{\phi}$ is a k–isotropic element of $G_J(F_A)$. This means that there exists a standard parabolic subgroup $P/\mathbb{Z}$ of $G/\mathbb{Z}$ such that I (through $\widetilde{\phi}$) has a fixed point in $X(F_A)$, where $X = G_J/N_{G_J}(P)$ is the projective $\mathbb{Z}$–scheme of parabolic subgroups of G_J of the same type as P. As $X(A) = X(F_A)$, then I has a fixed point in $X(k)$, and ϕ is k–isotropic.
b) Assume that ϕ is k–anisotropic and that $\widetilde{\phi}$ normalizes a F_A–maximal split torus, i.e. there exists $g \in G_J(F_A)$ such that

$$(*) \qquad g^{-1}\widetilde{\phi}(\sigma)g = \widetilde{f}(\sigma) \in N_{G_J}(T)(F_A) \ \ (\sigma \in I),$$

where $\widetilde{f} : I \to N_{G_J}(T)(F_A)$ is a group homomorphism. We consider the Bruhat–Tits building $\mathcal{B}$ of the group G_{J,F_A} with center c, which is the vertex stabilized by $G_J(A)$. We denote by $\mathcal{B}^I$ the fixed point of X by I (through $\widetilde{\phi}$), which is convex and contains c. The torus T defines an apartment $\mathcal{A} = \widehat{T} \otimes_{\mathbb{Z}} \mathbb{R} \subset X$. From $(*)$, we get $g.\mathcal{A} \subset \mathcal{B}^I$. Let us denote by $\pi(c)$ the projection of c on the apartment $g.\mathcal{A}$.

Claim $\pi(c) = g.c = c$: We write $c = g.a$ with $a \in \mathcal{A}$. For any $\sigma \in I$, one has

$$g.a = \phi(\sigma) g.a = g\widetilde{f}(\sigma).a,$$

so $\widetilde{f}(\sigma).a = a$. As I acts on $\mathcal{A}$ trough $\widetilde{f}$, we get $a \in \big(\widehat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R}\big)^I = \mathcal{A}^I$. But $\big(\widehat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R}\big)^I = \big(\widehat{T}^0\big)^I \otimes_{\mathbb{Z}} \mathbb{R}$, so if $a \neq c$, it means then $\widetilde{f}$ is k–isotropic, which gives a contradiction to the anisotropy of ϕ by a). We conclude that $\pi(c) = g.c$.

As $\mathcal{B}^I$ is convex, the segment $[c, \pi(c)]$ is fixed pointwise by α. If $c \neq \pi(c)$, there exists a facet F of $\mathcal{B}$ strictly containing c which is stabilized by I. By the isomorphism between the link of c and the spherical building of G_J/k, there exists a proper k–parabolic subgroup P of G such that $\phi(P) = P$, which again gives a contradiction to the anisotropy of ϕ. We conclude that $\pi(c) = c$, i.e. $c \in g.\mathcal{A}$. So $g.c = c$, i.e. $g \in G_J(A)$ and $\widetilde{f}(\sigma) \in N_{G_J}(T)(A)$. Reducing $(*)$ in k, it yields $\overline{g}^{-1}\phi(\sigma)\overline{g} \in N_{G_J}(T)(k)$, so ϕ normalizes the maximal k–split torus $\overline{g}.T$. One can then assume that ϕ and $\widetilde{\phi}$ normalizes T.

Now, let $\phi' : I \to G_J$ be a group homomorphism which lifts to $\widetilde{\phi} : I \to G_J \times_{\mathbb{Z}} A$ and such that $\widetilde{\phi}$ and $\widetilde{\phi}'$ are conjugate under $G_J(F_A)$. Let $g \in G_J(F_A)$ be an element such that $\widetilde{\phi}' = g\widetilde{\phi}g^{-1}$. Then $\widetilde{\phi}'$ normalizes the F_A–torus gTg^{-1}. But $\widetilde{\phi}$ is anisotropic by assertion a), so $\widetilde{\phi}'$ is anisotropic and the proof of assertion b) shows that $g \in G_J(A)$. We have then $\widetilde{\phi}' = g\widetilde{\phi}g^{-1}$ with $g \in G_J(A)$, so $\phi = \overline{g}\phi(\overline{g})^{-1}$ with $\overline{g} \in G_J(k)$, and ϕ' normalizes the maximal k–split torus $\overline{g}T\overline{g}^{-1}$. □

Lemma 7. — *Let $\phi \in \mathrm{Hom}_{gr}(I, G_J(k))$ be a k–anisotropic morphism. Then there exists $g \in G_J(k_s)$ such that the following hold*

i) $g^{-1}\,\mathrm{Im}(\phi) g \subset N_{G_J}(T)(k_s)$,

ii) $g^{-1}\,{}^s g \subset N_{G_J}(T)(k_s)\ \ \forall s \in \mathrm{Gal}(k_s/k)$.

The torus $g.T$ is defined over k and normalized by ϕ. Moreover, in the case I is cyclic, one can replace condition i) by

i′) $g^{-1}\,\mathrm{Im}(\phi) g \in N_{G_J}(T)(k)$,

Remark 3 : Lemma 7 yields conjecture 2 of Tits by an uniform argument in the case k is separably closed.

Proof of Lemma 7 : It is well–known that the map

$$H^1(K, N_{G_J}(T)) \to H^1(K, G_J)$$

is surjective. In other words, there exists a finite Galois extension M/K such that $K \subset L \subset M \subset K_s$, an element $g \in G_J(M)$ and a cocycle $h \in Z^1\big(\mathrm{Gal}(M/K), N_{G_J}(T)(K_s)\big)$ such that

$$(*) \qquad g^{-1}\phi(s)\,{}^s g = h_s \qquad (s \in \mathrm{Gal}(M/K)).$$

Let us denote by K'/K the maximal unramified subextension of M, by O' its valuation ring and by k' its residue field Set $L' = L.K$. We may assume $K' = k'((t))$.

$$\begin{array}{ccccc} & & M & & \\ & & | & & \\ & & L' & & \\ & / & & \backslash & \\ K' & & & & L \\ & \backslash & & / & \\ & & K & & \end{array}$$

Let $\mathcal{B}$ be the Bruhat–Tits building of the group G_M [BrT1, § I.7.4]. This building is equipped with an action of the group $G_J(M)$ and an action of the Galois group $\mathrm{Gal}(M/K)$ denoted by $x \mapsto {}^s x$ for any $s \in \mathrm{Gal}(M/K)$. We denote by c the center of the building, i.e. the vertex of $\mathcal{B}$ fixed by $G_J(O_M)$, where O_M is the valuation ring of M. The torus $T \times_k M$ defines an apartment $\mathcal{A} = \widehat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathcal{B}$, which contains the center c, and which is fixed pointwise by $\mathrm{Gal}(M/K)$. One defines a twisted action of $\mathrm{Gal}(M/K)$ on $\mathcal{B}$ by

$$s.x = \phi(s).{}^s x \qquad (s \in \mathrm{Gal}(M/K)).$$

This action is isometric and simplicial (i.e. maps a facet on a facet). Clearly, since $\mathrm{Im}(\phi) \subset G_J(k')$, the center c is fixed under the twisted action of $\mathrm{Gal}(M/K)$. We claim that the apartment $g\mathcal{A}$ is stabilised by $\mathrm{Gal}(M/K)$ for the twisted action. This follows from the following computation; for any $x = g.a \in \mathcal{A}$, one has

$$s.x = \phi(s).{}^s(g.a) = g.(h_s.a) \in g\mathcal{A}.$$

Let us denote by $\pi(c) = g.a$ the projection of c to the apartment $g\mathcal{A}$.

Claim : $\pi(c) = g.c = c$: As c is fixed under the twisted action of $\mathrm{Gal}(L/K)$, the projection $\pi(c)$ is also fixed. Moreover (observe ${}^s a = a$), one has $\pi(c) = g.a = s.(g.a) = \phi(s).{}^s g.{}^s a = g h_s.a \qquad (s \in \mathrm{Gal}(L/K))$, so

$$a = h_s.a \qquad (s \in \mathrm{Gal}(M/K)).$$

Now, we consider the maximal torus

$${}_nT/K \subset {}_nG/K \approx {}_{f_\phi}G/K.$$

The group ${}_nG/K \approx_{f_\phi} G/K$ is anisotropic by Proposition 5, so the torus ${}_nT$ is anisotropic and $0 = \widehat{{}_nT}^0(K) \otimes_{\mathbb{Z}} \mathbb{R} = \{x \in \mathcal{A} \mid h_s.x = x\}$. One deduces that $a = c$, i.e. $\pi(c) = h.c$.

Let us assume that $\pi(c) \neq c$. Then the segment $[c, \pi(c)]$ is fixed pointwise by the twisted action of $\mathrm{Gal}(M/K)$. So there exists a facet of $\mathcal{B}$ containing strictly c which is stabilized by the twisted action of $\mathrm{Gal}(M/K)$ and there exists (by the isomorphim between the link of c in $\mathcal{B}$ and the spherical building of G/k) a facet of the spherical building of G/k which is pointwise fixed by $\mathrm{Gal}(M/K)$. In other words, there exists a proper k–parabolic subgroup of G such that

$$\Big[{}_{f_\phi}\Big(G_J(k')/N_{G_J}(P)(k')\Big)\Big]^{\mathrm{Gal}(M/K)} \neq \emptyset.$$

The group $\mathrm{Gal}(M/K)$ acts on the previous set through $\mathrm{Gal}(L'/K) = I \times \mathrm{Gal}(k'/k)$. One has

$$\Big[{}_{f_\phi}\Big(G_J(k')/N_{G_J}(P)(k')\Big)\Big]^I = \{[d] \in G_J(k')/N_{G_J}(P)(k') \ \mid \ d^{-1}\,\mathrm{Im}(\phi) d \in N_{G_J}(P)(k')\},$$

$$\Big[{}_{f_\phi}\Big(G_J(k')/N_{G_J}(P)(k')\Big)\Big]^{\mathrm{Gal}(k'/k)}$$
$$= \{[d] \in G_J(k')/N_{G_J}(P)(k') \ \mid \ d^{-1}({}^s d) \in N_{G_J}(P)(k') \ \ \forall s \in \mathrm{Gal}(k'/k)\}.$$

Let $d \in N_{G_J}(P)(k')$ be a coset in the intersection of the two previous sets and let us denote by $z \in Z^1(k'/k, N_{G_J}(P))$ the cocycle defined by $z_s = d^{-1}({}^s d)$. The element d gives rise to a trivialization $\beta_d : {}_z G_J \xrightarrow{\sim} G_J$ such that $\mathrm{Im}(\phi) \subset \beta_d({}_z N_{G_J}(P))$, which gives a contradiction to the anisotropy of ϕ. We conclude that $\pi(c) = c$. Let us denote by $a \mapsto \overline{a}$ the map $G_J(O_M) \to G_J(k')$. The identity $(*)$ holds in $G_J(O_M)$ and replacing d by $\overline{d} \in G_J(k')$, one can assume that $g \in G_J(k') \subset G_J(L')$, that $L' = M$ and $h_s \in G_J(k') \subset G_J(L')$ for every $s \in \mathrm{Gal}(L'/K)$. So we get *i)* and *ii)*.

Assume now that I is cyclic with a generator σ. Identity $(*)$ for σ yields

$$(**) \qquad g^{-1}\phi(\sigma) g = h_\sigma \ \in \ N_{G_J}(T)(k').$$

Since $\mathrm{Gal}(k'/k)$ acts trivially on the Weyl group W_J, the map $s \mapsto h_\sigma^{-1}({}^s h_\sigma)$ defines a $\mathrm{Gal}(k'/k)$–cocycle in $T(k')$ and by Hilbert's theorem 90, there exists $\tau \in T(k')$ such that $h_\sigma^{-1}({}^s h_\sigma) = \tau^{-1}({}^s\tau)$. Replacing g by $g\tau$ in $(**)$, one can assume that $h_\sigma \in N_{G_J}(k)$ and taking $h = h_\sigma$, the proposition is proved. ∎

Lemma 8. — *Assume that $p = \mathrm{char}(k)$ is an odd prime ; let A be a complete discrete valuation ring with residue field k and fraction field F_A. Let H/k be a k–form of G_J, $\mathfrak{H}$ a A–form of $G_J \times_{\mathbb{Z}} A$ with special fiber H/k Let S be a maximal k–torus of H, and let $\mathfrak{S}$ be a maximal A–torus of $\mathfrak{H}$ with special fiber S. Let $h \in N_H(S)(k)$ be an element of finite order $q = p^r$. Then h lifts to an element $\widetilde{h} \in N_{\mathfrak{H}}(\mathfrak{S})(A)$ of order q.*

Remark 4 : Such an $\mathfrak{H}$ exists by Hensel isomorphism $H^1_{ét}(A, \mathrm{Aut}(G_J)) \xrightarrow{\sim} H^1(k, \mathrm{Aut}(G_J))$ and such an A–torus $\mathfrak{S}$ exists by [SGA3, exp. XV, § 8].

Proof : Let A_s/A be the étale extension associated to k_s/k. We remark first that we can assume that h generates H/H^0. Let us denote by w the image of h in $(N_H(S)/S)(k)$, which has order q.

First case : w acts anisotropically on $S \times_k k_s$: First, remark that the lifting assumption in $N_{\mathfrak{H}}(\mathfrak{S})(A_s)$ is satisfied by Lemma 4.d (§ 4.1, we use here that p is odd). Let us denote the Weyl group scheme $N_{\mathfrak{H}}(\mathfrak{S})/\mathfrak{S}$ by $W(\mathfrak{S})/A$; that is a twisted finite group, so isomorphic to $N_G(S)/S$. We also denote by $\langle w \rangle$ the cyclic subgroup of $W(\mathfrak{S})$ generated by w and by N_w/A the preimage of $\langle w \rangle$ in $N_{\mathfrak{H}}(\mathfrak{S})$. By Hensel's lemma, one has the following exact commutative diagram of pointed sets

$$\begin{array}{ccccccccc}
1 & \to & \mathfrak{S}(A) & \to & N_w(A) & \to & \langle w \rangle & \to & H^1(A, \mathfrak{S}) \\
 & & \downarrow & & \downarrow & & \| & & \downarrow \wr \\
1 & \to & S(k) & \to & N_w(k) & \to & \langle w \rangle & \to & H^1(k, S),
\end{array}$$

and

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{S} & = & \mathbb{S} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathfrak{S}(A) & \longrightarrow & N_w(A) & \longrightarrow & \langle w \rangle & \longrightarrow & 1 \\
 & & {\scriptstyle sp}\Big\downarrow & & {\scriptstyle sp}\Big\downarrow & & \Big\| & & \\
1 & \longrightarrow & S(k) & \longrightarrow & N_w(k) & \longrightarrow & \langle w \rangle & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

where $\mathbb{S} := \mathrm{Ker}\big(\mathfrak{S}(A) \to S(k)\big)$. So, the obstruction to lifting h to an element of order q in $N_{\mathfrak{H}}(\mathfrak{S})(A)$ is the class, say η in $H^2(\langle h \rangle, \mathbb{S})$, of the restriction to the cyclic group $\langle h \rangle$ of the vertical extension. We put $\mu/A = \mathrm{Ker}(1 - w : \mathfrak{S} \to \mathfrak{S})$ which is a finite A–multiplicative group, and we have

$$H^2(\langle h \rangle, \mathbb{S}) = \big(\mathbb{S}\big)^{\langle w \rangle} / N(w).\mathbb{S} = \mathrm{Ker}\big(\mu(A) \to \mu(k)\big),$$

because $N(w) = 0$ (Lemma 4.a). The same approach for $k = k_s$ gives an obstruction η_s in $\mu(A_s)$ which vanishes. As $\mu(A) \subset \mu(A_s)$, one has $\eta = 0$, and h lifts to $N_{\mathfrak{H}}(\mathfrak{S})(A)$.

Second case : h acts isotropically on $S \times_k k_s$: This means that $\dim(S^{\langle h \rangle}) \geq 1$, i.e. there exists a non–trivial subtorus S_0 of S such that $h \in Z_H(S_0)(k)$. We have $Z_{H^0}(S_0) = Z_H(S_0)^0$. As h generates H/H^0, we have an exact sequence $1 \to Z_{H^0}(S_0) \to Z_H(S_0) \to H/H^0 \to 1$ which is split by h. The element h acts trivially on the coradical torus (cf. [SGA3], exp. XXII, § 6.2) $C =: corad(Z_{H^0}(S_0))$ of $Z_{H^0}(S_0)$, and it defines an extension of the canonical morphism $Z_{H^0}(S_0) \to C$ to a morphism $Z_H(S_0)^0 \to C$. So one can define the group H'/k by the following diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & C & = & C & & & & \\
 & & \Big\uparrow & & \Big\uparrow & & & & \\
1 & \longrightarrow & Z_{H^0}(S_0) & \longrightarrow & Z_H(S_0) & \longrightarrow & H/H^0 & \longrightarrow & 1 \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\| & & \\
1 & \longrightarrow & DZ_{H^0}(S_0) & \longrightarrow & H' & \longrightarrow & H/H^0 & \longrightarrow & 1. \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

So H' is a k–form of some $G'_{J'}$ and $h \in H(k)$ because $C(k)$ has no k–points of p–power order. Denoting $S' = \mathrm{Ker}(S \to C) \subset H'$, h acts anisotropically on the maximal torus $S' \times_k k_s$ of H'. Defining $\mathfrak{H}'/A$ and $\mathfrak{S}'/A$ by Hensel's lift, the first case shows that h lifts in $N_{\mathfrak{H}'}(\mathfrak{S}')(A) = N_{\mathfrak{H}}(\mathfrak{S})(A) \cap \mathfrak{H}'(A)$, so h lifts in $N_{\mathfrak{H}}(\mathfrak{S})(A)$. ∎

4.4. — End of the proof of Theorem 3

Recall that $\mathrm{char}(k) = p > 0$ and we consider a k–anisotropic automorphism α of prime order l of G. We can assume that l is odd. Let A be a complete discrete valuation ring with residue field k and with fraction field F_A such that F_A contains a primitive p–root of unity. First, we show that α lifts to $\mathrm{Aut}(G)(A)$, i.e. there exists $\widetilde{\alpha} \in \mathrm{Aut}(G)(F_A)$ of order l lifting α. If $l = p$, this is a consequence of Lemmas 7 and 8. If $l \neq p$, then $\mathbb{Z}/l\mathbb{Z}$ is a finite group of multiplicative type; as $\mathrm{Aut}(G)$ is smooth, the Grothendieck rigidity theorem [SGA3, exp. VI, Cor. 7.3] says that the map

$$\mathrm{Hom}_{A-gr}\big(\mathbb{Z}/l\mathbb{Z}, \mathrm{Aut}(G)\big) \to \mathrm{Hom}_{k-gr}\big(\mathbb{Z}/l\mathbb{Z}, \mathrm{Aut}(G)\big)$$

is surjective, so α lifts in characteristic zero. By Lemma 6.a, $\widetilde{\alpha}$ is an F_A–anisotropic automorphism of G/F_A, the characteristic zero case shows that $\widetilde{\alpha}$ is special and finally Lemma 6.b shows that α is special.

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