AN INVARIANT OF SIMPLE ALGEBRAIC GROUPS

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Abstract. The Rost invariant associated with a simple simply connected algebraic group G is used to define an invariant of strongly inner forms of G. This invariant takes values in a quotient of $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$. It is used to answer a question of Serre about groups of type E_6 and to prove a generalization of Gille's splitting criterion for groups of type E_6 and E_7.

For G a semisimple algebraic group over a field k with Dynkin diagram Δ, there is a k-map $\mathrm{Aut}(G) \to \mathrm{Aut}(\Delta)$ and it is natural to ask: What is the image of $\mathrm{Aut}(G)(k) \to \mathrm{Aut}(\Delta)(k)$? This question is easily reduced to the case where G is simply connected, so we assume henceforth that G is such. The Tits algebras (see §2 and 3.3) of G provide an obvious obstruction to the surjectivity of the map, and we say that G is *flayed* if they are the only obstruction (see §3).

For flayed simple groups, we can use Rost's invariant for G-torsors to give an invariant a_G of groups which are strongly inner forms of G, see §5. This invariant has already been constructed by ad hoc means in some special cases (see 5.3 and 5.4). As an application, we obtain an isomorphism criterion which generalizes a result of Gille's (see 7.2).

Notation and conventions. For a field k, we write k_{sep} and k_{alg} for separable and algebraic closures respectively and $\mathrm{Gal}(k)$ for the Galois group of k_{sep}/k. Typically we make no restrictions on the characteristic of k.

We say that an algebraic group is *simple* if it is $\neq 1$, is connected, and has no nontrivial connected normal subgroups over an algebraic closure. (Such groups are sometimes called "absolutely almost simple".)

1. Background on flat cohomology

In order to obtain characteristic-free results, we use flat cohomology as in [Car66] and [Wat79] instead of Galois cohomology. Let G be an affine group scheme over a field k, or more generally an fpqc sheaf in the sense of [Wat79, 15.6]. We set $H^0(k, G) = G(k)$ and

$$Z^1(k, G) = \{g \in G(\otimes_k^2 k_{\mathrm{alg}}) \mid (d^0 g)(d^2 g) = d^1 g\},$$

where $d^i : \otimes_k^n k_{\mathrm{alg}} \to \otimes_k^{n+1} k_{\mathrm{alg}}$ is the map which inserts a 1 after the i-th place. Two 1-cocycles (= elements of $Z^1(k, G)$) g, g' are *cohomologous* if there is some $h \in G(k_{\mathrm{alg}})$ such that $g' = (d^0 h) g (d^1 h)^{-1}$ in $G(\otimes_k^2 k_{\mathrm{alg}})$. This defines an equivalence relation on $Z^1(k, G)$, and the quotient set is denoted $H^1(k, G)$. It has base point the class of the 1-cocycle $g = 1$. If G is abelian, one can define $H^q(k, G)$ for all $q \geq 0$ using Čech cohomology, see [Wat79, p. 139] for a concrete definition.

If G is smooth, then $H^q(k, G)$ is canonically identified with the Galois cohomology set $H^q(\mathrm{Gal}(k_{\mathrm{sep}}/k), G(k_{\mathrm{sep}}))$ by [Wat79, 17.8]. Consequently, if you wish to avoid

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flat cohomology, you may skip to the beginning of §2; all of the results will still hold if one adds some minor restrictions on the characteristic. For example, $H^1(k, \boldsymbol{\mu}_n) = k^*/k^{*n}$ in all characteristics, but for Galois cohomology this is only true when n is not divisible by char(k).

The most useful results in Galois cohomology from §I.5 of Serre's classic [Ser02] all hold for flat cohomology. Presumably this can be seen using the massive machinery of [Gir71], but we view these results as consequences of direct cocycle computations as in [Ser02]. Details are left to the reader.

If $1 \to F \to G \to H \to 1$ is an exact sequence of algebraic affine group schemes over k, Waterhouse [Wat79, 18.1] proves the existence of the corresponding exact sequence in cohomology ending with $\cdots \to H^1(k, G) \to H^1(k, H)$. If additionally F is central in G, this sequence may be extended so as to end with

$$\cdots \to H^1(k, H) \xrightarrow{\delta^1} H^2(k, F)$$

by setting

$$\delta^1(h) = (d^0 g)(d^2 g)(d^1 g)^{-1}$$

for $g \in G(\otimes_k^2 k_{\mathrm{alg}})$ any inverse image of $h \in Z^1(k, H)$. (Such a g exists because the map $G(\otimes_k^2 k_{\mathrm{alg}}) \to H(\otimes_k^2 k_{\mathrm{alg}})$ is surjective, see [Sha64, pp. 418–420].)

1.1. Change of base point. A 1-cocycle $\gamma \in Z^1(k, G)$ acts by conjugation on G, and so defines a 1-cocycle $\mathrm{Int}\,(\gamma) \in Z^1(k, \mathrm{Aut}(G))$. The set $H^1(k, \mathrm{Aut}(G))$ classifies k_{alg}/k-forms of G, and we write G_γ for the group corresponding to the class of $\mathrm{Int}\,(\gamma)$. There is a k_{alg}-isomorphism

$$\alpha_\gamma \colon G_\gamma \times k_{\mathrm{alg}} \xrightarrow{\sim} G \times k_{\mathrm{alg}}.$$

The map $\tau_\gamma \colon H^1(k, G_\gamma) \to H^1(k, G)$ given by $\tau_\gamma(g') = \gamma\, \alpha_\gamma(g')$ is a bijection which maps the base point in $H^1(k, G_\gamma)$ to the class of $\gamma \in H^1(k, G)$.

1.2. Action by the center. Let Z denote the (schematic) center of the affine group scheme G over k. For 1-cocycles $\zeta \in Z^1(k, Z)$ and $\gamma \in Z^1(k, G)$, we define $\zeta \cdot \gamma$ to be the product $\zeta\gamma \in G(k_{\mathrm{alg}} \otimes k_{\mathrm{alg}})$. This is a 1-cocycle with values in G and its equivalence class in $H^1(k, G)$ depends only on the equivalence classes of ζ and γ in $H^1(k, Z)$ and $H^1(k, G)$ respectively. Hence $\cdot$ defines an action of the group $H^1(k, Z)$ on the pointed set $H^1(k, G)$.

Now fix $\alpha \in Z^1(k, G)$. Since $\mathrm{Int}\,(\alpha)$ restricts to the identity on Z, the center of G_α is canonically identified with the center Z of G. Moreover, the map τ_α is compatible with $\cdot$ in the sense that

$$\zeta \cdot \tau_\alpha(\gamma) = \tau_\alpha(\zeta \cdot \gamma)$$

for every $\zeta \in H^1(k, Z)$ and $\gamma \in H^1(k, G)$.

1.3. One really wants to twist exact sequences of the form $1 \to F \to G \to H \to 1$ where F is central in G. Let η be a 1-cocycle with values in H and let $\gamma \in G(\otimes_k^2 k_{\mathrm{alg}})$ be any inverse image of η. This γ may not be a 1-cocycle, but $\mathrm{Int}\,(\gamma)$ is a 1-cocycle with values in $\mathrm{Aut}(G)$. This gives a "twisted" exact sequence $1 \to F \to G_\gamma \to H_\eta \to 1$. The two short exact sequences give exact sequences in cohomology such that the following diagram is commutative:

$$\begin{tikzcd} H^1(k,H) \arrow[r, "\delta^1"] & H^2(k,F) \\ H^1(k,H_\eta) \arrow[u, "\tau_\eta"] \arrow[r, "\delta^1_\eta"] & H^2(k,F) \arrow[u, "?+\delta^1(\eta)"'] \end{tikzcd}$$

Here δ^1_η denotes the connecting map coming from the twisted sequence. The proof is the same as for Galois cohomology.

2. The Tits class

In this section and for the rest of the paper G denotes a simply connected semisimple algebraic group over k.

The set $H^1(k, \mathrm{Aut}(G))$ classifies k-forms of G, i.e., k-groups G' which are isomorphic to G over k_{alg}. The natural map $H^1(k, \mathrm{Aut}(G)^\circ) \to H^1(k, \mathrm{Aut}(G))$ need not be injective, but there is a unique element ν_G which maps to the class of the unique quasi-split inner form of G in $H^1(F, \mathrm{Aut}(G))$. (This is standard, see for example [KMRT98, 31.6] or [MPW96, p. 531].)

The exact sequence

$$1 \to Z \to G \to \mathrm{Aut}(G)^\circ \to 1 \tag{2.1}$$

induces an exact sequence

$$H^1(k,Z) \to H^1(k,G) \to H^1(k,\mathrm{Aut}(G)^\circ) \xrightarrow{\delta^1} H^2(k,Z). \tag{2.2}$$

The *Tits class* t_G of G is defined to be

$$t_G = -\delta^1(\nu_G) \qquad \text{in } H^2(k,Z).$$

2.3. The map $\mathrm{Aut}(\Delta) \to \mathrm{Aut}(Z)$. Let Λ be a lattice of weights and let Λ_r be the root lattice for G with respect to some maximal torus. For Z the (schematic) center of G, Λ/Λ_r is canonically identified with the Cartier dual $Z^* = \mathrm{Hom}(Z, \mathbb{G}_m)$ of Z. This gives rise to a commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(G) \arrow[r] \arrow[d] & \mathrm{Aut}(Z) \arrow[d, equal] \\ \mathrm{Aut}(\Delta) \arrow[r] & \mathrm{Aut}(Z^*), \end{tikzcd} \tag{2.4}$$

where the left-hand arrow is the "$*$-action" from [Tit66b]. Consequently, we have an action of $\mathrm{Aut}(\Delta)$ on Z.

Since G is simply connected, the sequence

$$1 \to \mathrm{Aut}(G)^\circ \to \mathrm{Aut}(G) \to \mathrm{Aut}(\Delta) \to 1 \tag{2.5}$$

is exact. This gives a map $\mathrm{Aut}(\Delta)(k) \to H^1(k, \mathrm{Aut}(G)^\circ)$.

Lemma 2.6. *Fix $\pi \in \mathrm{Aut}(\Delta)(k)$ and let γ be a 1-cocycle representing the image of π in $H^1(k, \mathrm{Aut}(G)^\circ)$. Then $\pi^2(t_{G_\gamma}) = t_G$.*

Since γ takes values in $\mathrm{Aut}(G)^\circ$, the centers of G and G_γ are canonically identified, so it is sensical to compare $\pi^2(t_{G_\gamma}) \in H^2(k, Z(G_\gamma))$ with $t_G \in H^2(k, Z(G))$.

Proof. Note that $\mathrm{Aut}(G)^\circ$ and $\mathrm{Aut}(G)$ are smooth, so we may view their corresponding H^1's as Galois cohomology. The group G_γ has the same k_{sep}-points as G, but a different Galois action $\circ$ given by $\sigma \circ g = \gamma_\sigma \sigma g$ for $g \in G(k_{\mathrm{sep}})$, $\sigma \in \mathrm{Gal}(k)$, and where juxtaposition denotes the usual Galois action on G.

Since γ has trivial image in $H^1(k, \mathrm{Aut}(G))$, there is some $f \in \mathrm{Aut}(G)(k_{\mathrm{sep}})$ such that $\gamma_\sigma = f^{-1}\, {}^\sigma\! f$ for every $\sigma \in \mathrm{Gal}(k)$. This map f gives a k-isomorphism $G_\gamma \xrightarrow{\sim} G$.

Sequence (2.1) gives a commutative diagram

$$\begin{array}{ccc} H^1(k, \mathrm{Aut}(G_\gamma)^\circ) & \xrightarrow{\delta^1_\gamma} & H^2(k, Z) \\ {\scriptstyle f^1}\downarrow & & {\scriptstyle f^2}\downarrow \\ H^1(k, \mathrm{Aut}(G)^\circ) & \xrightarrow{\delta^1} & H^2(k, Z). \end{array}$$

Let $\eta \in Z^1(k, \mathrm{Aut}(G_\gamma)^\circ)$ be a 1-cocycle representing ν_{G_γ}. Then $f(\eta)$ is a 1-cocycle in $Z^1(F, \mathrm{Aut}(G)^\circ)$. Moreover, f is a k-isomorphism $f : (G_\gamma)_\eta \xrightarrow{\sim} G_{f(\eta)}$. Since $(G_\gamma)_\eta$ is k-quasi-split, we have $f^1(\nu_{G_\gamma}) = \nu_G$. The commutativity of the diagram gives $f^2(t_{G_\gamma}) = t_G$. □

A standard twisting argument also gives:

Lemma 2.7. *Let γ be a 1-cocycle in $H^1(k, \mathrm{Aut}(G)^\circ)$. Then $t_{G_\gamma} = t_G + \delta^1(\gamma)$.* □

3. Flayed groups

Recall that G denotes a semisimple simply connected group.

The exactness of (2.5) and Lemma 2.6 immediately give:

Proposition 3.1. *Let π be in $\mathrm{Aut}(\Delta)(k)$. If π is in the image of $\mathrm{Aut}(G)(k)$, then $\pi^2(t_G) = t_G$.* □

Definition 3.2. We say that G is *flayed* if every $\pi \in \mathrm{Aut}(\Delta)(k)$ such that $\pi^2(t_G) = t_G$ lies in the image of $\mathrm{Aut}(G)(k)$ (that is, if the converse to Prop. 3.1 holds for every $\pi \in \mathrm{Aut}(\Delta)(k)$).

3.3. One can still make a characteristic-free definition of "flayed" without resorting to flat cohomology. Let λ be a dominant weight of G. The absolute Galois group acts on the cone of dominant weights Λ^+, and we write $k(\lambda)$ for the extension of F corresponding to the stabilizer of λ by Galois theory. Over $k(\lambda)$, there is a unique simple $k(\lambda)$-representation $(V_\lambda, \rho_\lambda)$ of G such that $(V_\lambda, \rho_\lambda) \otimes k_{\mathrm{sep}}$ contains the simple k_{sep}-representation with highest weight λ as a direct summand [Tit71].

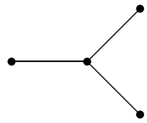
The group $\mathrm{Aut}(\Delta)$ acts on Λ^+ in a manner compatible with the Galois action. Since π is k-defined, we have $k(\lambda) = k(\pi(\lambda))$ for all $\lambda \in \Lambda^+$. The condition "$\pi^2(t_G) = t_G$" is equivalent to:

$$\boxed{\begin{array}{l}\mathrm{End}_G(V_\lambda, \rho_\lambda) \text{ is isomorphic to } \mathrm{End}_G(V_{\pi(\lambda)}, \rho_{\pi(\lambda)}) \text{ as } k(\lambda)\text{-algebras} \\ \text{for every } \lambda \in \Lambda^+.\end{array}}$$

These endomorphism algebras are the *Tits algebras* of G, up to Brauer equivalence.

Example 3.4. Quasi-split simple simply connected groups are always flayed.

Example 3.5 (${}^6D_4$)**.** For G simple of type ${}^6D_4$, the Galois group acts as S_3 on the diagram



Thus $\mathrm{Aut}(\Delta)(k) = 1$, and G is (trivially) flayed.

Example 3.6. For G simple, the material in [KMRT98] easily gives that G is flayed except perhaps when G is of type ${}^2A_n$ for $n + 1 \equiv 0 \pmod 4$; is of type ${}^2D_n$ for $n \geq 4$; or is a strongly inner form of a quasi-split group of type ${}^3D_4$ or ${}^2E_6$.

Example 3.7 (isotropic D_4)**.** If G is a simple isotropic group of type D_4 over a field of characteristic $\neq 2$, it is flayed. When G is of classical (i.e., ${}^1D_4$ or ${}^2D_4$) type, it is isomorphic to $\mathrm{Spin}(A, \sigma)$ for A a central simple algebra of degree 8 and σ an isotropic orthogonal involution. It is easy to see that G is flayed, except when G is of type ${}^2D_4$ and A is split by the quadratic extension k' which makes G of type ${}^1D_4$. In that case, $\mathrm{Aut}(\Delta)(k)$ has a unique nontrivial element, which has order 2; an outer automorphism of G of order 2 may be constructed using [MT95, §1.4.2].

Otherwise G is trialitarian, i.e., of type ${}^3D_4$ or ${}^3D_6$. If the Tits class t_G is 0, then G is quasi-split [Gar98, 5.6], hence flayed. If t_G is nonzero, the only element of $\mathrm{Aut}(\Delta)$ which preserves t_G is the identity, and G is again flayed.

Proposition 3.8. *Let G be a semisimple simply connected algebraic group over k. Then G is flayed if and only if the sequence*

$$H^1(k, Z) \to H^1(k, G) \to H^1(k, \mathrm{Aut}(G))$$

is exact.

Proof. Suppose first that G is flayed and let $\hat{\gamma} \in H^1(k, G)$ lie in the kernel. Write γ for its image in $H^1(k, \mathrm{Aut}(G)^\circ)$. Since $\hat{\gamma}$ is in the kernel, γ is the image of some $\pi \in \mathrm{Aut}(\Delta)(k)$. Moreover, G_γ is isomorphic to G, hence $\pi^2(t_G) = \pi^2(t_{G_\gamma}) = t_G$ by Lemma 2.6. Since G is flayed, γ is trivial in $H^1(k, \mathrm{Aut}(G)^\circ)$, hence $\hat{\gamma}$ lies in the image of $H^1(k, Z)$ by the exactness of (2.2). Since the displayed sequence is always a 0-sequence, we have proved that it is exact.

Conversely, suppose that the sequence is exact and take $\pi \in \mathrm{Aut}(\Delta)(k)$ such that $\pi^2(t_G) = t_G$. For γ a 1-cocycle representing the image of π in $H^1(k, \mathrm{Aut}(G)^\circ)$, we have $t_{G_\gamma} = t_G$ by Lemma 2.6, hence $\delta^1(\gamma) = 0$ by Lemma 2.7. That is, γ is the image of some $\hat{\gamma}$ in $H^1(k, G)$. The image of $\hat{\gamma}$ in $H^1(k, \mathrm{Aut}(G))$ is the same as the image of π under the composition

$$\mathrm{Aut}(\Delta)(k) \to H^1(k, \mathrm{Aut}(G)^\circ) \to H^1(k, \mathrm{Aut}(G)),$$

which is trivial. By exactness, $\hat{\gamma}$ lies in the image of $H^1(k, Z)$, hence γ is trivial in $H^1(k, \mathrm{Aut}(G)^\circ)$ and π is in the image of $\mathrm{Aut}(G)(k) \to \mathrm{Aut}(\Delta)(k)$. $\square$

4. The Rost invariant

Recall that $\mathbb{Z}/m\mathbb{Z}(2)$ is the $\mathrm{Gal}(k)$-module $\boldsymbol{\mu}_m^{\otimes 2}$ if m is not divisible by $\mathrm{char}(k)$ and is defined in terms of Milnor K-theory (or Witt vectors) otherwise, see [Mer03, App. A]. The colimit $\varinjlim \mathbb{Z}/m\mathbb{Z}(2)$ is denoted by $\mathbb{Q}/\mathbb{Z}(2)$, and we write

$$H^3(k) = H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \quad \text{and} \quad H^3(k, m) = H^3(k, \mathbb{Z}/m\mathbb{Z}(2)).$$

These are abelian groups and $H^3(k,m)$ is identified with the m-torsion in $H^3(k)$.

For G a simple simply connected algebraic group, Rost constructed a canonical morphism of functors $H^1(*,G) \to H^3(*)$. It is known as the *Rost invariant*, and we denote it by r_G. The image of $r_G(k)$ is always a torsion subgroup of $H^3(k)$ of exponent dividing a natural number n_G, the *Dynkin index* of G. These numbers and details of the definition of the Rost invariant may be found in [Mer03].

A fundamental property of the Rost invariant is its compatibility with twisting:

Proposition 4.1. (See [Gil00, p. 76, Lem. 7] or [MPT01]) *For $\alpha \in Z^1(k,Z)$, the diagram*

$$\begin{tikzcd} H^1(k,G_\alpha) \arrow[r, "\sim", "\tau_\alpha"'] \arrow[d, "r_{G_\alpha}"'] & H^1(k,G) \arrow[d, "r_G"] \\ H^3(k) \arrow[r, "?+r_G(\alpha)"] & H^3(k) \end{tikzcd}$$

commutes. □

For Z the center of G, we write r_G also for the composition

$$H^1(*,Z) \to H^1(*,G) \xrightarrow{r_G} H^3(*).$$

Corollary 4.2. *For every $\zeta \in H^1(k,Z)$ and $\gamma \in H^1(k,G)$, we have*

$$r_G(\zeta \cdot \alpha) = r_G(\zeta) + r_G(\alpha)$$

Proof. Pick a 1-cocycle $z \in Z^1(k,Z)$ representing ζ. We have a diagram

$$\begin{tikzcd} H^1(k,G) \arrow[r, equal] \arrow[d, "r_G(k)"'] & H^1(k,G_z) \arrow[r, "\sim", "\tau_z"'] \arrow[d, "r_{G_z}(k)"'] & H^1(k,G) \arrow[d, "r_G(k)"] \\ H^3(k) \arrow[r, equal] & H^3(k) \arrow[r, "?+r_G(\zeta)"] & H^3(k). \end{tikzcd}$$

The groups G and G_z are smooth, so we may view their H^1's as Galois cohomology. The right-hand square commutes by Prop. 4.1. The left-hand square commutes because the Rost invariant is canonical. Hence the whole diagram commutes. The composition in the top row maps $\gamma \mapsto \zeta \cdot \gamma$ (1.2), so the commutativity gives the desired formula. □

The proof is the same as that for [Gar01, 7.1], but we have included it here for the convenience of the reader. This result is stronger, since the group $H^1(k,Z)$ carries more information when Z is not smooth.

Corollary 4.3. *Fix $\alpha \in Z^1(k,G)$. Write Z for the (canonically identified) centers of G and G_α. The maps r_G and r_{G_α} agree on $H^1(k,Z)$.*

Proof. Fix $\zeta \in H^1(k,Z)$. By the preceding corollary,

$$r_G(\zeta) = r_G(\zeta \cdot \alpha) - r_G(\alpha).$$

By 1.2, we have $\zeta \cdot \alpha = \tau_\alpha(\zeta \cdot 1) = \tau_\alpha(\zeta)$, hence

$$r_G(\zeta) = r_G(\tau_\alpha(\zeta)) - r_G(\alpha) = r_{G_\alpha}(\zeta),$$

where the last equality is by Proposition 4.1. □

4.4. We say that the Rost invariant *has central kernel* (over k) if the kernel of $r_G(k)$ is contained in the image of $H^1(k, Z) \to H^1(k, G)$. In that case, Corollary 4.2 gives: *If $\gamma \in H^1(k, G)$ has $r_G(\gamma) \in r_G(H^1(k, Z))$, then γ is in the image of $H^1(k, Z)$.*

Example 4.5. Let q denote the 6-dimensional unit quadratic form $\langle 1, 1, 1, 1, 1, 1\rangle$ over $\mathbb{R}$, and write C for the kernel of $\mathrm{Spin}(q) \to SO(q)$. The kernel of the Rost invariant $r_{\mathrm{Spin}(q)}$ is the image of $H^1(\mathbb{R}, C)$ by [Gar01, 1.2], hence it is central. On the other hand, the map $SO(q)(\mathbb{R}) \to H^1(\mathbb{R}, C) \cong \mathbb{R}^*/\mathbb{R}^{*2}$ is the spinor norm, which is not surjective. Thus the kernel of the Rost invariant is not trivial, but it is central.

5. The invariant a_G

Recall that for an algebraic group G', the set $H^1(k, \mathrm{Aut}(G))$ classifies k-forms of G'. We say that G' is a *strongly inner form* of G if G lies in the image of the map $H^1(k, G') \to H^1(k, \mathrm{Aut}(G'))$. This defines an equivalence relation on the class of simply connected semisimple groups.

For α an inverse image of G in $H^1(k, G')$, we define

$$a_G(G') = -r_{G'}(\alpha) \quad \text{in} \quad \frac{H^3(k)}{r_G(H^1(k, Z))}. \tag{5.1}$$

The denominator $r_G(H^1(k, Z))$ is a group by 4.2.

If we identify G with G'_α, then G' is isomorphic to G twisted by $\beta = \tau_\alpha^{-1}(1)$, where τ_α denotes the isomorphism $H^1(k, G) \xrightarrow{\sim} H^1(k, G')$. By Proposition 4.1, $r_G(\beta) = -r_{G'}(\alpha)$.

Theorem 5.2. *Let G be a simple, simply connected, and flayed algebraic group. The element $a_G(G')$ in* (5.1) *is well-defined (i.e., does not depend on the choice of α). This defines a map*

$$a_G\colon \boxed{\begin{array}{l}\text{isomorphism classes of strongly}\\ \text{inner forms of } G\end{array}} \longrightarrow \frac{H^3(k)}{r_G(H^1(k, Z))}.$$

The Rost invariant of G has central kernel if and only if the kernel of a_G is $\{G\}$.

This sort of invariant is not really new. An analogue of it can be found in [BP98, p. 664] where the Rost invariant is used to give an invariant of hermitian forms. The construction is the same, except that here we push r_G forward along $G \to \mathrm{Aut}(G)$ instead of, for example, along $\mathrm{Spin}(h) \to O(h)$.

Proof. Let α, α' be two inverse images of G in $H^1(k, G')$. Then $\tau_\alpha^{-1}(\alpha')$ and $\tau_\alpha^{-1}(\alpha) = 1$ in $H^1(k, G)$ have trivial image in $H^1(k, \mathrm{Aut}(G))$. Since G is flayed, they lie in the same $H^1(k, Z)$-orbit. Action by the center is compatible with inner twists by 1.2, so α and α' are in the same $H^1(k, Z)$-orbit. By Proposition 4.1, $r_{G'}(\alpha)$ and $r_{G'}(\alpha')$ differ by an element of $r_{G'}(H^1(k, Z))$. But this last group is equal to $r_G(H^1(k, Z))$ by Corollary 4.3, so $a_G(G')$ is well-defined.

Suppose that r_G has central kernel. If G' lies in the kernel of a_G, i.e., $r_G(\beta)$ is in $r_G(H^1(k, Z))$, then β is in the image of $H^1(k, Z)$ by 4.4.

Suppose finally that the kernel of a_G is $\{G\}$. If β is in the kernel of r_G, then $a_G(G') = 0$ for G' isomorphic to G_β. Prop. 3.8 gives that β is in the image of $H^1(k, Z)$. □

Example 5.3 (Groups of type C_{even})**.** Let G be simply connected of type C_n for n even. The Rost invariant restricts to be trivial on the center of G and $n_G = 2$, hence a_G takes values in $H^3(k, 2)$. The invariant a_G was constructed in this case in [KMRT98, pp. 440, 441] and was studied in the case where $\text{char}(k) \neq 2$ in [BMT02].

Example 5.4 (Relative primality)**.** We define the *exponent* of a finite abelian affine group scheme H in the obvious way: it is the smallest natural number (if such exists) such that the kernel of multiplication by n on H is all of H. For example, the exponent of $\boldsymbol{\mu}_n$ is n regardless of the characteristic of k.

Suppose that G is simple and flayed, and let n_Z denote the exponent of the center Z. Suppose also that n_G factors as $n_Z \cdot m$ for m a natural number relatively prime to n_Z. There is a canonical factorization

$$H^3(k, n_G) = H^3(k, n_Z) \times H^3(k, m).$$

Since $r_G(H^1(k, Z))$ is n_Z-torsion, a_G induces a map

$$(a_G)_m \colon \boxed{\begin{array}{l}\text{isomorphism classes of strongly}\\ \text{inner forms of } G\end{array}} \longrightarrow H^3(k, m).$$

This occurs, for example, when G is of type A_{n-1} for n odd, in which case $m = 2$ and $(a_G)_2$ was constructed in [KMRT98, pp. 438, 439]. Other groups for which such a "reduced" invariant exists are those of type D_4 with $m = 3$; E_6 with $m = 2$ or 4; and E_7 with $m = 3$.

If G, G', and G'' are simple algebraic groups which are strongly inner forms of each other and G and G' are flayed, then the method used in the proof of 5.2 immediately gives:

$$a_G(G'') = a_G(G') + a_{G'}(G''). \tag{5.5}$$

In the special case where the groups are of type C_n for n even and $\text{char}(k) \neq 2$, this was proved in [BMT02, Prop. 1b].

6. An application to groups of type E_6

In this section, we assume that k has characteristic $\neq 2, 3$.

Tits gave a construction which takes an Albert k-algebra and a quadratic étale k-algebra and produces a Lie algebra of type E_6, see [Tit66a] or [Jac71]. In terms of Galois cohomology, there are compatible maps

$$F_4 \to E_6 \quad \text{and} \quad F_4 \times \mathbb{Z}/2 \to \text{Aut}(E_6) \cong \text{Aut}(E_6)^\circ \rtimes \mathbb{Z}/2$$

where F_4 and E_6 denote the split simply connected groups of those types, and Tits' construction corresponds to the induced map

$$H^1(k, F_4 \times \mathbb{Z}/2) \to H^1(k, \text{Aut}(E_6)).$$

Serre asked: *Is Tits' construction complete up to odd degree extensions?* That is: Let γ be in $H^1(k, \text{Aut}(E_6))$. Is it necessarily true that there exists an extension L of k of odd degree such that $\text{res}_{L/k}(\gamma)$ lies in the image of $H^1(L, F_4 \times \mathbb{Z}/2) \to H^1(L, \text{Aut}(E_6))$?

Since the image of F_4 lies in the simply connected group E_6, every group arising from the Tits construction has trivial Tits class. However, for every group of type E_6 over k, there is some odd-degree extension of k which kills the Tits class. (This requires a small argument for groups of type ${}^2E_6$.) Therefore, the Tits class does not present an obstruction to Tits' construction being complete, and to answer Serre's question we may focus on groups with trivial Tits class.

Let k be a field (of characteristic $\neq 2, 3$) with a quadratic extension K and associated quasi-split simply connected group G of type ${}^2E_6$. As described in Example 5.4, there is an invariant $(a_G)_4$ of strongly inner forms of G with values in $H^3(k, 4)$.

Let β be a class in $H^1(k, \mathrm{Aut}(G))$ arising from Tits' construction. Inspecting the construction, we find that $(a_G)_4(G_\beta)$ is 2-torsion, i.e., lies in the subgroup $H^3(k, 2)$ of $H^3(k, 4)$, and is a symbol in $H^3(k, 2)$. (These statements are true because the same statements hold for the Rost invariant of F_4-torsors.)

By modifying the construction given in [Fer69, pp. 64, 65] or by looking in [Che02], one can find a particular field k with quadratic extension K and a class γ in $H^1(K/k, G)$ such that $(a_G)_4(G_\gamma)$ also lies in $H^3(k, 2) \subseteq H^3(k, 4)$ but is not a symbol.

Non-symbols in $H^3(k, 2)$ remain non-symbols after an odd-degree extension [Ros99]. Hence G_γ provides an example of a group of type ${}^2E_6$ which does not arise from Tits' construction, even after a field extension of odd degree.

7. The isomorphism criterion

Lemma 7.1. *Let G be a simple, simply connected, and flayed algebraic group over k. For E a finite extension of k, the kernel of the natural map*

$$\frac{H^3(k)}{r_G(H^1(k, Z))} \longrightarrow \frac{H^3(E)}{r_G(H^1(E, Z))}.$$

is $[E : k]^2$-torsion.

Proof. Let $\rho \in H^3(k)$ lie in the kernel of the map, i.e., there is $\zeta \in H^1(E, Z)$ such that $\mathrm{res}_{E/k}(\rho) = r_{G_E}(\zeta)$. Set $\zeta_0 = \mathrm{cor}_{E/k}(\zeta)$ in $H^1(k, Z)$. (The corestriction may be defined for flat cohomology in a manner entirely analogous to that for Galois cohomology. For example, we have that $\mathrm{cor}_{E/k} \circ \mathrm{res}_{E/k}$ is multiplication by $[E : k]$.) Since the Rost invariant is compatible with scalar extension, we have:

$$\mathrm{res}_{E/k}\, r_G(\zeta_0) = r_{G_E}(\mathrm{res}_{E/k}\, \mathrm{cor}_{E/k}\, \zeta).$$

The term $\mathrm{res}_{E/k}\, \mathrm{cor}_{E/k}\, \zeta$ is a sum of Galois conjugates of multiples of ζ. Since the Rost invariant is functorial, it is compatible with the Galois action, hence

$$\mathrm{res}_{E/k}\, r_G(\zeta_0) = \mathrm{res}_{E/k}\, \mathrm{cor}_{E/k}\, r_{G_E}(\zeta) = [E : k]\, \mathrm{res}_{E/k}(\rho),$$

and

$$[E : k]^2 \rho = \mathrm{cor}_{E/k}[E : k]\, \mathrm{res}_{E/k}(\rho) = \mathrm{cor}_{E/k}\, \mathrm{res}_{E/k}\, r_G(\zeta_0) = [E : k] r_G(\zeta_0).$$

This last term is in $r_G(H^1(k, Z))$, which proves the claim. □

For G a simple algebraic group, recall the set $S(G)$ of primes associated with G from [Ser95, §2.2]:

type of G	elements of $S(G)$
A_n	2 and prime divisors of $n + 1$
B_n, C_n, D_n ($n \geq 5$), G_2	2
D_4, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

Theorem 7.2. *Let G be a simple, simply connected, and flayed group over k such that the Rost invariant r_G has central kernel. Let G' be a strongly inner form of G defined over k such that there exist finite extensions $E_1, \dots, E_r$ of k such that*

(1) *G and G' are isomorphic over E_i for all i; and*
(2) *$\gcd\{[E_1 : k], \dots, [E_r : k]\}$ is not divisible by any prime in $S(G)$.*

Then G and G' are k-isomorphic.

Proof. Hypotheses (1) and (2) combined with 7.1 give that $a_G(G')$ is ℓ-torsion for some natural number ℓ not divisible by any prime in $S(G)$.

However, $a_G(G')$ is n_G-torsion, and every prime factor of n_G is in $S(G)$. Therefore, $a_G(G')$ has order dividing $\gcd(n_G, \ell) = 1$. Hence G' is in the kernel of a_G, and the conclusion follows by 5.2. □

Gille had previously obtained this conclusion for the cases where G is split of type E_6 or E_7 [Gil97, p. 116, Thm. C]. We get his result as a consequence of the general fact that the Rost invariant has trivial kernel for such groups [Gar01]. Note that our result here is somewhat broader than Gille's. For example, it also applies to quasi-split groups of type ${}^2E_6$ and D_4.

Conversely, Chernousov [Che02] used Gille's result to prove that the Rost invariant has trivial kernel for quasi-split groups of type E_6 and E_7.

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