ESSENTIAL DIMENSION OF QUADRICS

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To the memory of Oleg Izhboldin

Abstract. Let X be an anisotropic projective quadric over a field F of characteristic not 2. The essential dimension $\dim_{es}(X)$ of X, as defined by Oleg Izhboldin, is

$$\dim_{es}(X) = \dim(X) - i(X) + 1 ,$$

where $i(X)$ is the first Witt index of X (i.e., the Witt index of X over its own function field).

Let Y be a complete (possibly singular) algebraic variety over F with all closed points of even degree and such that Y has a closed point of odd degree over $F(X)$. Our main theorem states that $\dim_{es}(X) \leq \dim(Y)$ and that in the case $\dim_{es}(X) = \dim(Y)$ the quadric $X_{F(Y)}$ is isotropic.

Applying the main theorem to a projective quadric Y, we get a proof of Izhboldin's conjecture stated as follows: if an anisotropic quadric Y becomes isotropic over $F(X)$, then $\dim_{es}(X) \leq \dim_{es}(Y)$, and the equality holds if and only if X is isotropic over $F(Y)$.

Let (V, φ) be a non-degenerate quadratic form over a field F of characteristic not 2 and let $X = Q(\varphi)$ be the quadric hypersurface given by the equation $\varphi(x) = 0$ in the projective space $\mathbb{P}(V)$. We say that the quadric X is *anisotropic* if φ is an anisotropic quadratic form. By Springer's theorem, every closed point of an anisotropic quadric X has even degree. Is it possible to compress X rationally, i.e., to find a rational morphism $X \to Y$ to a variety Y of smaller dimension with all closed points of even degree?

The quadratic form φ is isotropic over the function field $F(X)$, hence, by the general theory of quadratic forms, $\varphi_{F(X)}$ is isomorphic to $\psi \perp k\mathbb{H}$ for some anisotropic quadratic form ψ and some $k \geq 1$, where $\mathbb{H}$ stays for the hyperbolic plane. The number k is called the *first Witt index* of φ (or X), and we denote it by $i(\varphi)$ (or $i(X)$). Let $V' \subset V$ be a subspace of codimension $i(X) - 1$. Since $V' \otimes F(X)$ intersects nontrivially a totally isotropic subspace of $V \otimes F(X)$, the anisotropic quadric $X' = Q(\varphi|_{V'})$ becomes isotropic over $F(X)$, i.e., X compresses to the subvariety X' of dimension $\dim(X) - i(X) + 1$. The latter integer is denoted $\dim_{es}(X)$ and called the *essential dimension* of X.

We prove in the paper (Theorem 3.1) that an anisotropic quadric X cannot be compressed to a variety Y of dimension smaller than $\dim_{es}(X)$ with all

Date: 11 June 2002.

Key words and phrases. Quadratic forms, first Witt index, complete varieties, Chow groups, correspondences. *2000 Mathematical Subject Classifications:* 11E04; 14C25.

The second author was supported in part by NSF Grant #0098111.

closed points of even degree. Moreover, if there is a rational morphism $X \to Y$ with $\dim(Y) = \dim_{es}(X)$, then there is a rational morphism $Y \to X$, i.e., X is isotropic over $F(Y)$.

Applying the main theorem to a projective quadric Y, we get a proof of Izhboldin's conjecture (Theorem 4.1) stated as follows: if an anisotropic quadric Y becomes isotropic over $F(X)$, then $\dim_{es}(X) \le \dim_{es}(Y)$, and the equality holds if and only if X is isotropic over $F(Y)$.

A *field* in the paper is an arbitrary field of characteristic not 2 (the characteristic restriction is important only there where quadratic forms are involved). By *scheme* we mean a separated scheme of finite type over a field, and by *variety* an integral scheme. We write $\mathrm{CH}_d(Y)$ for the d-th *Chow group* which is the group of rational equivalence classes of dimension d algebraic cycles on the scheme Y.

1. First Witt index of generic subforms

We are going to determine the first Witt index of certain subforms of a given anisotropic quadratic form. These subforms are generic in a sense (living over certain purely transcendental extensions of the base field), at least their first Witt indices turn out to be the minimal possible ones. The construction of these subforms is borrowed from [5, proof of lemma 7.9] (where a different property of these subforms is studied).

We recall that the first Witt index of an anisotropic quadratic F-form φ coincides with the minimal positive Witt index of φ_E, when E runs over all field extension of F. In particular, $i(\varphi) \le i(\varphi_L)$ for any extension L/F such that φ_L is anisotropic.

Proposition 1.1. *Let φ be an anisotropic quadratic F-form, and let n be an integer such that $0 \le n \le \dim\varphi - 2$. There exists a purely transcendental field extension $\tilde{F}/F$ and an n-codimensional subform $\psi \subset \varphi_{\tilde{F}}$ such that*

$$i(\psi) = \begin{cases} i(\varphi) - n, & \text{if } n < i(\varphi); \\ 1, & \text{if } n \ge i(\varphi). \end{cases}$$

Proof. It suffices to give a proof for $n = 1$. Let t be an indeterminate. We consider the quadratic $F(t)$-form $\varphi_{F(t)} \perp \langle -t\rangle$ and construct $\tilde{F}$ as its function field. The field extension $\tilde{F}/F$ is clearly a purely transcendental one. Moreover, the anisotropic form $\varphi_{\tilde{F}}$ represents t, therefore $\varphi_{\tilde{F}} \simeq \psi \perp \langle t\rangle$ for certain 1-codimensional subform $\psi \subset \varphi_{\tilde{F}}$.

We are going to determine the first Witt index of ψ. First of all we clearly have: $i(\psi) \ge i(\varphi) - 1$ and $i(\psi) \ge 1$. Moreover, we have the following isomorphisms of $\tilde{F}(\varphi)$-forms (we omit the subscript $\tilde{F}(\varphi)$ in the formula):

$$\psi \perp \mathbb{H} \simeq \psi \perp \langle t\rangle \perp \langle -t\rangle \simeq \varphi \perp \langle -t\rangle \simeq \varphi' \perp \langle -t\rangle \perp i\mathbb{H}$$

where $i = i(\varphi)$ and φ' is the anisotropic part of the form $\varphi_{F(\varphi)}$. Cancelling one copy of $\mathbb{H}$, we get $\psi \simeq \varphi' \perp \langle -t\rangle \perp (i-1)\mathbb{H}$ over $\tilde{F}(\varphi)$. Note that the $\tilde{F}(\varphi)$-form $\varphi' \perp \langle -t\rangle$ is anisotropic because the field extension $\tilde{F}(\varphi)/F(\varphi)(t)$ is purely transcendental (by the reason that the $F(\varphi)(t)$-form $\varphi\perp\langle -t\rangle$ is isotropic). Therefore the Witt index of $\psi_{\tilde{F}(\varphi)}$ is $i-1$. If $i-1$ is positive, then $i(\psi)\leq i-1$ and we are done in this case. Otherwise $i(\psi)\leq i(\psi_{\tilde{F}(\varphi)})=i(\varphi'_{\tilde{F}(\varphi)}\perp\langle -t\rangle)$. Since the form φ' remains anisotropic over the function field of the latter form (which is a purely transcendental extension of $\tilde{F}(\varphi)$), the latter first Witt index is equal to 1. □

Remark 1.2. In the case $i(\varphi)>1$, the first Witt index of *every* 1-codimensional subform is known to be $i(\varphi)-1$. This is a result due to A. Vishik which we do not use in this paper.

2. Correspondences

Let X and Y be schemes over a field F. Suppose that X is equidimensional and set $d=\dim(X)$. A *correspondence from X to Y*, denoted $\alpha\colon X\rightsquigarrow Y$, is an element $\alpha\in \mathrm{CH}_d(X\times Y)$. A correspondence α is called *prime* if α is represented by a prime (irreducible) cycle. Every correspondence is the sum of prime correspondences.

Let $\alpha\colon X\rightsquigarrow Y$ be a correspondence. Assume that X is a variety and Y is complete. The projection morphism $p\colon X\times Y\to X$ is proper and hence the push-forward homomorphism

$$p_*\colon \mathrm{CH}_d(X\times Y)\to \mathrm{CH}_d(X)=\mathbb{Z}\cdot[X]$$

is defined [1, § 1.4]. The number $\deg(\alpha)\in\mathbb{Z}$ such that $p_*(\alpha)=\deg(\alpha)\cdot[X]$ is called the *degree* of α. Clearly, $\deg(\alpha+\beta)=\deg(\alpha)+\deg(\beta)$ for every two correspondences $\alpha,\beta\colon X\rightsquigarrow Y$.

A correspondence $\alpha\colon \operatorname{Spec}F\to Y$ is represented by a 0-cycle z on Y. We set $\deg(z)=\deg(\alpha)$. This coincides with the usual notion of degree for 0-cycles as defined in [1, def. 1.4].

The image of a correspondence $\alpha\colon X\rightsquigarrow Y$ under the pull-back homomorphism

$$\mathrm{CH}_d(X\times Y)\to \mathrm{CH}_0(Y_{F(X)})$$

with respect to the flat morphism $Y_{F(X)}\to X\times Y$ is represented by a 0-cycle on $Y_{F(X)}$. The degree of this cycle is equal to $\deg(\alpha)$ (see [8, lemma 1.4]).

Lemma 2.1. *Let $\tilde{F}/F$ be a purely transcendental field extension. Then*

$$\deg \mathrm{CH}_0(Y)=\deg \mathrm{CH}_0(Y_{\tilde{F}})\ .$$

Proof. It suffices to consider the case where $\tilde{F}$ is the function field of the affine line $\mathbb{A}^1$. The statement follows from the fact that the restriction homomorphism $\mathrm{CH}_*(Y)\to \mathrm{CH}_*(Y_{F(\mathbb{A}^1)})$ is surjective (cf. [7, proof of prop. 3.12]) as the composite of the surjections

$$\mathrm{CH}_*(Y)\to \mathrm{CH}_{*+1}(Y\times\mathbb{A}^1)\quad\text{and}\quad \mathrm{CH}_{*+1}(Y\times\mathbb{A}^1)\to \mathrm{CH}_*(Y_{F(\mathbb{A}^1)})$$

(for the surjectivity of the first map see [1, prop. 1.9]). □

A rational morphism $X \to Y$ defines a degree 1 prime correspondence $X \rightsquigarrow Y$ as the closure of its graph. Moreover, there are natural bijections between the sets of:

0) rational morphisms $X \to Y$;
1) degree 1 prime cycles on $X \times Y$;
2) rational points of $Y_{F(X)}$.

Similarly, the following two sets are naturally bijective for every $r > 0$:

1) degree r prime cycles on $X \times Y$;
2) closed points of $Y_{F(X)}$ of degree r.

Let $g : Y \to Y'$ be a morphism of complete schemes. The image β of a correspondence $\alpha\colon X \rightsquigarrow Y$ under the push-forward homomorphism

$$(\mathrm{id}_X \times g)_*\colon \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_d(X \times Y')$$

is a correspondence from X to Y'. The following statement is a consequence of functoriality of the push-forward homomorphisms:

Lemma 2.2. $\deg(\beta) = \deg(\alpha)$. □

Let $X' \subset X$ be a closed subvariety such that the embedding $i\colon X' \hookrightarrow X$ is regular of codimension r [1, B.7.1]. Then the embedding $i \times \mathrm{id}_Y : X' \times Y \hookrightarrow X \times Y$ is also regular of codimension r, hence the pull-back homomorphism

$$(i \times \mathrm{id}_Y)^*\colon \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_{d-r}(X' \times Y)$$

is defined [1, § 6]. The pull-back γ of the correspondence α is a correspondence from X' to Y.

Lemma 2.3. $\deg(\gamma) = \deg(\alpha)$. □

Proof. The statement follows from the commutativity of the diagram [1, th. 6.2]:

$$\begin{tikzcd}
\mathrm{CH}_d(X \times Y) \arrow[r, "(i\times\mathrm{id}_Y)^*"] \arrow[d, "p_*"'] & \mathrm{CH}_{d-r}(X' \times Y) \arrow[d, "p'_*"] \\
\mathrm{CH}_d(X) \arrow[r, "i^*"] & \mathrm{CH}_{d-r}(X'),
\end{tikzcd}$$

where p and p' are the projections. □

Let $\alpha\colon X \rightsquigarrow Y$ be a correspondence between schemes of dimension d. We write α^t for the element in $\mathrm{CH}_d(Y \times X)$ corresponding to α under the exchange isomorphism $X \times Y \simeq Y \times X$. The correspondence $\alpha^t : Y \rightsquigarrow X$ is called the *transpose* of α.

3. Main theorem

In this section X is an anisotropic projective quadric over a field F. We recall that the essential dimension $\dim_{es}(X)$ of X is defined as the integer $\dim(X) - i(X) + 1$.

Theorem 3.1. *Let X be an anisotropic projective F-quadric and let Y be a complete F-variety with all closed points of even degree. Suppose Y has a closed point of odd degree over $F(X)$. Then*

(1) $\dim_{es}(X) \le \dim(Y)$*;*

(2) *if, moreover,* $\dim_{es}(X) = \dim(Y)$*, then X is isotropic over $F(Y)$.*

Proof. A closed point of Y over $F(X)$ of odd degree gives rise to a prime correspondence $\alpha\colon X \rightsquigarrow Y$ of odd degree. By Springer's theorem, to prove the statement **(2)** it is sufficient to find an odd degree correspondence $Y \rightsquigarrow X$.

Assume first that $i(X) = 1$, so that $\dim_{es}(X) = \dim(X)$. We prove both statements simultaneously by induction on $n = \dim(X) + \dim(Y)$.

If $n = 0$, i.e., X and Y are of dimension zero, we have $X = \operatorname{Spec} K$ and $Y = \operatorname{Spec} L$, where K and L are field extensions of F with $[K : F] = 2$ and $[L : F]$ even. Taking the push-forward to $\operatorname{Spec} F$ of the correspondence α we get the formula

$$[K : F] \cdot \deg(\alpha) = [L : F] \cdot \deg(\alpha^t).$$

Since $\deg(\alpha)$ is odd, $\alpha^t\colon Y \rightsquigarrow X$ is a correspondence of odd degree.

Assume that $n > 0$ and let d be the dimension of X. We are going to prove **(2)**, so that we have $\dim(Y) = d > 0$. It is sufficient to show that $\deg(\alpha^t)$ is odd. Assume that the degree of α^t is even. Let $x \in X$ be a closed point of degree 2. Since the degree of the correspondence $Y \times x\colon Y \rightsquigarrow X$ is 2 and the degree of $x \times Y\colon X \rightsquigarrow Y$ is zero, we can modify α by an appropriate multiple of $x \times Y$ and therefore assume that $\deg(\alpha)$ is odd and $\deg(\alpha^t) = 0$. Hence the degree of the pull-back of α^t on $X_{F(Y)}$ is zero. By [7, prop. 2.6] or [9], the degree homomorphism

$$\deg\colon \mathrm{CH}_0(X_{F(Y)}) \to \mathbb{Z}$$

is injective. Therefore there is a nonempty open subset $U \subset Y$ such that the restriction of α on $X \times U$ is trivial. Write Y' for the reduced scheme $X \setminus U$, $i\colon X \times Y' \to X \times Y$ and $j\colon X \times U \to X \times Y$ for the closed and open embeddings respectively. The sequence

$$\mathrm{CH}_d(X \times Y') \xrightarrow{i_*} \mathrm{CH}_d(X \times Y) \xrightarrow{j^*} \mathrm{CH}_d(X \times U)$$

is exact [1, prop. 1.8]. Hence there exists $\alpha' \in \mathrm{CH}_d(X \times Y')$ such that $i_*(\alpha') = \alpha$. We can view α' as a correspondence $X \rightsquigarrow Y'$. By Lemma 2.2, $\deg(\alpha') = \deg(\alpha)$, hence $\deg(\alpha')$ is odd. Since α' is a sum of prime correspondences, we can find a prime correspondence $\beta : X \rightsquigarrow Y'$ of odd degree, i.e., Y' has a closed point of odd degree over $F(X)$. The class β is represented by a prime cycle, hence we may assume that Y' is irreducible. Since $\dim Y' < \dim X$, by induction hypothesis, we get a contradiction with the statement **(1)**.

In order to prove **(1)** assume that $\dim(Y) < \dim(X)$. Let $Z \subset X \times Y$ be a prime cycle representing α. Since $\deg(\alpha)$ is odd, the field extension $F(X) \hookrightarrow F(Z)$ is of odd degree. The restriction of the projection $X \times Y \to Y$ gives a proper morphism $Z \to Y$. Replacing Y by the image of this morphism, we come to the situation where $Z \to Y$ is a surjection and so, the function field $F(Z)$ is a field extension of $F(Y)$.

In view of Proposition 1.1, extending the scalars to a purely transcendental extension $\tilde{F}$ of F, we can find a subquadric X' of X of the same dimension as Y having $i(X') = 1$. We note that according to Lemma 2.1, the hypothesis on X and Y is still satisfied over $\tilde{F}$. By Lemma 2.3, the pull-back of α with respect to the regular embedding $X' \times Y \hookrightarrow X \times Y$ produces an odd degree correspondence $X' \rightsquigarrow Y$. Since $\dim(X') < \dim(X)$, by the induction hypothesis, the statement **(2)** holds for X' and Y, that is, there exists an odd degree (in fact, of degree 1) correspondence $\beta \colon Y \rightsquigarrow X'$. We compose β with the embedding $X' \hookrightarrow X$ to produce an odd degree (in fact, of the same degree as β) correspondence $\gamma : Y \rightsquigarrow X$ (Lemma 2.2). We may assume that γ is prime. Let $T \subset Y \times X$ be a prime cycle representing γ. Since the degree of γ is odd, the projection $T \to Y$ is surjective, so that $F(T)$ is a field extension of $F(Y)$ of odd degree.

Using the odd degree prime correspondences $\alpha \colon X \rightsquigarrow Y$ and $\gamma : Y \rightsquigarrow X$, we are going to construct an odd degree correspondence $\delta \colon X \rightsquigarrow X$ with even $\deg(\delta^t)$ getting this way a contradiction with

Theorem 3.2 ([8, th. 6.4])**.** *Let X be an anisotropic quadric with $i(X) = 1$. Then for every correspondence δ on $X \times X$, one has* $\deg(\delta) \equiv \deg(\delta^t) \pmod 2$.

Note that in the case where Y is regular we can simply take δ as the composite of the correspondences α and γ (cf. [8, proof of prop. 7.1]).

Lemma 3.3. *Let $F \hookrightarrow L$ and $F \hookrightarrow E$ be two field extensions with odd degree $[L : F]$. Then there is a field K and field extensions $L \hookrightarrow K$ and $E \hookrightarrow K$ such that $[K : E]$ is odd.*

Proof. We may assume that L is generated over F by one element, say θ. Let $f \in F[t]$ be the minimal polynomial of θ (of odd degree). Choose an odd degree irreducible polynomial $g \in E[t]$ dividing f and set $K = E[t]/gE[t]$. $\square$

By Lemma 3.3 applied to the field extensions $F(T)$ and $F(Z)$ of $F(Y)$, we can find a field extension K of $F(T)$ and $F(Z)$ such that $[K : F(Z)]$ is odd. Let a variety S over F be a projective model of the field extension K/F. Replacing S by the closure of the graph of the rational morphism $S \to Z \times T$, we come to the situation where the rational morphisms $S \to Z$ and $S \to T$ are regular. Let f be the composite of $S \to Z$ with $Z \to X$ and g be the composite of $S \to T$ and $T \to X$. We write δ for the correspondence $X \rightsquigarrow X$ given by the image of the morphism $(f, g) \colon S \to X \times X$. The degree

$$\deg(\delta) = [F(S) : F(X)] = [F(S) : F(Z)] \cdot [F(Z) : F(X)]$$

is odd and the degree of the transpose of δ is zero since g is not surjective as $\dim T = \dim Y < \dim X$, a contradiction.

We have proven Theorem 3.1 in the case $i(X) = 1$. Consider now the general case (the first Witt index of X is arbitrary). Let X' be a subquadric of X with $\dim(X') = \dim_{es}(X') = \dim_{es}(X)$ which we may find after extending the scalars to a purely transcendental extension according to Proposition 1.1. By Lemma 2.3, the pull-back $\beta \colon X' \rightsquigarrow Y$ of α with respect to the embedding of X' into X is an odd degree correspondence. Therefore $\dim X' \leq \dim Y$ by the first part of the proof. If $\dim X' = \dim Y$, then again by the first part of the proof, X' and hence X have rational points over $F(Y)$. $\square$

Remark 3.4. For X and Y as in part **(2)** of Theorem 3.1, assume additionally that $\dim(X) = \dim_{es}(X)$, i.e., $i(X) = 1$. In the proof of Theorem 3.1, it is shown that $\deg(\alpha^t)$ is odd for every odd degree correspondence $\alpha\colon X \rightsquigarrow Y$.

We have also the following more precise version of Theorem 3.1:

Corollary 3.5. *Let X and Y be as in Theorem 3.1. Then there exists a closed subvariety $Y' \subset Y$ such that*

(i) $\dim(Y') = \dim_{es}(X)$;
(ii) $Y'_{F(X)}$ *possesses a closed point of odd degree;*
(iii) $X_{F(Y')}$ *is isotropic.*

Proof. Let $X' \subset X$ be a subquadric with $\dim(X') = \dim_{es}(X)$. Then, by Theorem 4.1, $\dim_{es}(X') = \dim(X')$. An odd degree closed point on $Y_{F(X)}$ gives an odd degree correspondence $X \rightsquigarrow Y$ which in turn gives an odd degree correspondence $X' \rightsquigarrow Y$. We may assume that the latter correspondence is prime and take a a prime cycle $Z \subset X' \times Y$ representing it. We define Y' as the image of the proper morphism $Z \to Y$. Clearly, $\dim(Y') \leq \dim(Z) = \dim(X') = \dim_{es}(X)$. On the other hand, Z gives an odd degree correspondence $X' \rightsquigarrow Y'$, therefore $\dim(Y') \geq \dim(X')$ by Theorem 3.1, and condition (i) of Corollary 3.5 is satisfied. Moreover, $Y'_{F(X')}$ has a closed point of odd degree. Since the field $F(X \times X')$ is purely transcendental over $F(X')$ as well as over $F(X)$, Lemma 2.1 shows that there is an odd degree closed point on $Y'_{F(X)}$, that is, the condition (ii) of Corollary 3.5 is satisfied. Finally the quadric $X'_{F(Y')}$ is isotropic by Theorem 3.1; therefore $X_{F(Y')}$ is isotropic. $\square$

Example 3.6. Let the dimension of X be equal to $2^n - 1$ for some n. Since there exists a field extension E/F such that X_E is given by an anisotropic Pfister neighbor ([2, th. 2]), the first Witt index of X is 1 by Theorem 4.1, that is, $\dim_{es}(X) = \dim(X)$. The first part of Theorem 3.1 is therefore a generalization (to the case of arbitrary $\dim(X)$ and of arbitrary complete Y) of [2, th. 1], while the second part generalizes [4, th. 0.2].

4. Application to the algebraic theory of quadratic forms

Now we apply Theorem 3.1 to a special (but may be the most interesting) case where the variety Y is also a projective quadric:

Theorem 4.1. *Let X and Y be anisotropic quadrics over F and suppose that Y is isotropic over $F(X)$. Then*

(1) $\dim_{es}(X) \leq \dim_{es}(Y)$;
(2) *moreover, the equality* $\dim_{es}(X) = \dim_{es}(Y)$ *holds if and only if X is isotropic over $F(Y)$.*

Proof. Let us choose a subquadric $Y' \subset Y$ with $\dim(Y') = \dim_{es}(Y)$ (we can do it over a pure transcendental extension of the base field by Proposition 1.1). Since Y' becomes isotropic over $F(Y)$ and Y is isotropic over $F(X)$, Y' is isotropic over $F(X)$. According to Theorem 3.1, $\dim_{es}(X) \leq \dim(Y')$. Moreover, in the case of equality, X is isotropic over $F(Y')$ and hence over $F(Y)$. Conversely, if X is isotropic over $F(Y)$, interchanging the roles of X and Y, we get as above the inequality $\dim_{es}(Y) \leq \dim_{es}(X)$, hence the equality holds. □

Example 4.2. If some anisotropic 11-dimensional quadratic form is not a Pfister neighbor, then its first Witt index is 1 (this is a result due to B. Kahn with an elementary proof given in [3]). Therefore, we recover a theorem of O. Izhboldin ([6, th. 5.3]) stating that an anisotropic 10-dimensional quadratic form remains anisotropic over the function field of any quadratic form of dimension > 10, if this second form is not a 4-fold Pfister neighbor.

Example 4.3. Similarly, if some anisotropic 13-dimensional quadratic form is not a Pfister neighbor, then its first Witt index is 1. Therefore, an anisotropic 12-dimensional quadratic form remains anisotropic over the function field of any quadratic form of dimension > 12, if this second form is not a 4-fold Pfister neighbor. This result is new.

Example 4.4. Coming back to the 11-dimensional forms, we also see, that if some anisotropic 11-dimensional form ψ becomes isotropic over the function field of another 11-dimensional form φ and φ is not a Pfister neighbor, then φ is isotropic over the function field of ψ. In the situation where the Schur index of the even Clifford algebra of ψ is at least 16, it can be then shown that φ is similar to ψ. This result is of particular interest in view of attempts to construct a field with the u-invariant 11.

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