ON THE FIRST WITT INDEX OF QUADRATIC FORMS

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ABSTRACT. We prove Hoffmann's conjecture determining the possible values of the first Witt index of anisotropic quadratic forms of any given dimension. The proof makes use of the Steenrod type operations on the modulo 2 Chow groups constructed by P. Brosnan.

Let F be a field of characteristic $\neq 2$. For an anisotropic quadratic form ϕ over F with $\dim(\phi) \geq 2$, the first Witt index $i_1(\phi)$ is the Witt index (i.e., the dimension of a maximal totally isotropic subspace) of the form $\phi_{F(\phi)}$ over the function field $F(\phi) = F(X_\phi)$ of the projective quadric given by ϕ. Clearly, this $i_1(\phi)$ is the minimal positive Witt index of ϕ_E, when E runs over all field extension of F.

We are going to prove the following

Conjecture 0.1 (Hoffmann). *Let us write the integer* $\dim\phi - 1$ *as a sum of powers of 2:*

$$\dim(\phi) - 1 = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$$

with $0 \leq n_1 < n_2 < \cdots < n_r$. *Then the integer* $i_1(\phi) - 1$ *is a partial sum of this sum (including the empty one and not including the whole sum):*

$$i_1(\phi) - 1 = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$$

for some $0 \leq s \leq r - 1$.

We remark that for any given $n \geq 2$, all the values of $i_1(\phi)$, prescribed by Conjecture 0.1 for forms ϕ with $\dim(\phi) = n$, are really possible, that is, do occur for suitable ϕ over suitable F. To see it, we take a field k with an anisotropic r-fold Pfister form $\langle\langle a_1, \ldots, a_r \rangle\rangle$. Then we take indeterminates $t_0, \ldots, t_m$, set $F = k(t_0, \ldots, t_m)$, and notice that the first index of the quadratic F-form $q = \langle\langle a_1, \ldots, a_r \rangle\rangle \otimes \langle t_0, \ldots, t_m \rangle$ is equal to 2^r. Therefore, by [4, lemma 7.3], for every $j = 0, 1, \ldots, 2^r - 1$, the first Witt index of certain j-codimensional subform $\phi \subset q_{\tilde{F}}$ over certain purely transcendental field extension $\tilde{F}/F$ is $2^r - j$ (while $\dim(\phi) - i_1(\phi)$ is still $2^r m$). [1]

We prove Conjecture 0.1 in §3. In §1 and §2 we introduce two different tools needed in the proof.

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[1] By a theorem of A. Vishik, one may even take $\tilde{F} = F$ with any j-codimensional subform $\phi \subset q$.

All the time we are working with Chow group $\mathrm{CH}^*(Y)$ of various equidimensional F-varieties (in fact, Y is always a projective quadric or a product of projective quadrics here). We underline that this Chow group $\mathrm{CH}^*(Y)$ is the *modulo 2* Chow group. It is graded in the usual way: $\mathrm{CH}^*(Y) = \bigoplus_{i=0}^{d} \mathrm{CH}^i(Y)$ ($d = \dim Y$), where $\mathrm{CH}^i(Y)$ is the group of classes of cycles of codimension i. Sometimes, the lower indices $\mathrm{CH}_i(Y) = \mathrm{CH}^{d-i}(Y)$, indicating the dimension of cycles, are used. If we consider $\mathrm{CH}^*(Y)$ as a usual group (or ring), not a graded one, in a place, we write simply $\mathrm{CH}(Y)$, omitting $*$ in the notation.

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1. Vishik's principle

In this section we formulate and prove an extremely useful observation, due to A. Vishik, concerning correspondences on projective quadrics. We refer to this as to Vishik's principle. Since originally neither the formulation nor the proof are not in the elementary terms used here, we give a different formulation and a complete proof. We restrict ourself to the case of odd-dimensional quadrics: this is enough for our purposes and slightly simplifies the consideration.

Let X be the projective quadric over F given by an anisotropic quadratic form ϕ. We set $d = \dim(X)$ (that is, $d = \dim(\phi) - 2$) and assume that this integer is odd. Therefore, over an algebraic closure $\bar{F}$ of F, the variety $\bar{X} = X_{\bar{F}}$ is cellular with one cell in each dimension. Moreover, writing $e^i \in \mathrm{CH}^i(\bar{X})$ ($i = 0, 1, \ldots, d$) for the classes of the closures of the cells, we have: for $i < d/2$, the element e^i is represented by an i-codimensional linear section of $\bar{X}$ (and therefore is defined over F), while the element $e_i = e^{d-i}$ is represented by an i-dimensional linear subspace of $\bar{X}$. In particular, the product $e^1 \cdot e^i$ is e^{i+1} for $i \notin \{(d-1)/2,\ d\}$ and is 0 otherwise.

It follows that the exterior products $e^i \times e^j$ form a basis of the Chow group $\mathrm{CH}^*(\bar{X} \times \bar{X})$ (considered as a vector space over the field $\mathbb{Z}/2\mathbb{Z}$). For any $\alpha \in \mathrm{CH}^*(X \times X)$ we define $\alpha_{ij} \in \{0,\ 1\}$ as the coefficients in the representation of $\bar{\alpha} = \alpha_{\bar{F}} \in \mathrm{CH}^*(\bar{X} \times \bar{X})$ as a linear combination of the basis elements, so that we have

$$\bar{\alpha} = \sum_{i,j} \alpha_{ij}(e^i \times e^j)\ .$$

Clearly, for a homogeneous $\alpha \in \mathrm{CH}^n(X \times X)$ the coefficient α_{ij} can be non-zero only if $j = n - i$. For such α, we set $\alpha_i = \alpha_{i\, n-i}$.

Proposition 1.1 (Vishik's principle)**.** *Setting $i_1 = i_1(\phi)$, for every element $\alpha \in \mathrm{CH}^d(X \times X)$ and every $i = 0, 1, \ldots, i_1 - 1$, we have:*

$$\alpha_i = \alpha_{d-i_1+1+i}\ .$$

Proof. Let us first consider the case of $i_1 = 1$. Here we have only one equality to prove, namely, $\alpha_0 = \alpha_d$. Since α_0 coincides with the *index* (sometimes also called *degree*, [3, def. of example 16.1.4]) of the correspondence α, while α_d is the index of the transposition of α, the equality needed is given by [4, th. 6.4].

In the general case (i_1 is arbitrary), let us take a subquadric $X' \subset X$ of codimension $i_1 - 1$ such that $i_1(X') = 1$. We can certainly find such a subquadric, at least if we extend the scalars up to a purely transcendental extension of the base field, according to [4, lemma 7.3] (in fact, by a stronger [4, cor. 8.3], which we do not really need here, any subquadric $X' \subset X$ of codimension $i_1 - 1$ has this property).

Since the Witt index of the quadratic form $\phi_{F(X)}$ is greater that i, the quadric $X_{F(X')}$ contains a linear subspace of dimension i. Its closure in $X' \times X$ gives an element $\beta \in \mathrm{CH}^{d-i}(X' \times X)$ (this β is a preimage of the class of the linear subspace under the surjective flat pull-back $\mathrm{CH}^{d-i}(X' \times X) \twoheadrightarrow \mathrm{CH}^{d-i}(X_{F(X')})$ with respect to the evident injective morphism of schemes $X_{F(X')} \to X' \times X$). We may consider β as a correspondence from X' to X (see [3, §16.1]). Note that the dimension of the cycle representing β is not the same as the dimension of the graphs of morphisms $X' \to X$ (which is equal to $\dim X'$), that is, the *degree* ([3, example 16.1.1]) of β is not 0. More precisely, $\deg \beta = -i$.

Let now $Y \subset X$ be a subquadric of codimension i. Then the quadric $X'_{F(Y)}$ has a rational point, and its closure in $Y \times X'$ gives an element $\gamma \in \mathrm{CH}^{d+i-i_1+1}(X \times X')$ (the push-forward of the element in $\mathrm{CH}^{d-i_1+1}(Y \times X')$). This γ, viewed as a correspondence from X back to X', is of degree i. The composition ([3, def. 16.1.1]) $\gamma \circ \alpha \circ \beta$ of the three correspondences (note that the degree of the correspondence α is 0) is a correspondence from X' to X' of degree $-i+0+i=0$. It is then an exercise on the calculus of correspondences (note that all the computations needed can be done over $\bar{F}$) to check that the index of this correspondence is α_i, while the index of its transposition is a multiple of α_{d-i_1+1+i}. Thus we are done by [4, th. 6.4]. $\square$

2. Brosnan's construction

Here we recall the construction and basic properties of the cohomological Steenrod type operations on the modulo 2 Chow groups given in [1]. Certainly, the idea that it could be possible and useful to construct in an algebraic context something similar to the well-known topological Steenrod operations, is due to Voevodsky. Brosnan's construction described bellow is completely independent of and much simpler than those of Voevodsky, however.

Let X be a smooth quasi-projective scheme over a field F. Let G be the group of 2-roots of unity viewed as an algebraic group over F. Since $\operatorname{char} F \neq 2$ as everywhere in the paper, G is identified with the finite group $\mathbb{Z}/2\mathbb{Z}$ and therefore acts on the product $X \times X$ by the factors exchange. For any n-codimensional cycle α on X, the product $\alpha \times \alpha$ is a $2n$-codimensional cycle on $X \times X$ invariant with respect to the action of G. Therefore, $\alpha \times \alpha$ determines an element of the equivariant Chow group $\mathrm{CH}^{2n}_G(X \times X)$ as defined in [2]. The map to $\mathrm{CH}^{2n}_G(X \times X)$ one obtains this way factors through the rational equivalence ([1, prop. 4.2]) and therefore gives a map $\mathrm{CH}^n(X) \to \mathrm{CH}^{2n}_G(X \times X)$.

Furthermore, the diagonal imbedding $\delta : X \to X \times X$ is equivariant with respect to the trivial action of G on X, giving a pull-back of the equivariant Chow groups:

$$\delta^* \colon \mathrm{CH}^{2n}_G(X \times X) \to \mathrm{CH}^{2n}_G(X)\ .$$

Finally, the equivariant graded Chow ring $\mathrm{CH}^*_G(X)$ is identified with the graded ring $\mathrm{CH}^*(X)[t]$ of polynomials in one variable t over the usual modulo 2 Chow ring $\mathrm{CH}^*(X)$, where t is of degree 1 ([1, th. 6.1]). In particular,

$$\delta^*(\alpha \times \alpha) = \sum_{i \geq 0} S^{n-i}(\alpha) \cdot t^i$$

with some $S^i(\alpha) \in \mathrm{CH}^{n+i}(X)$.

The maps $S^i \colon \mathrm{CH}^*(X) \to \mathrm{CH}^{*+i}(X)$ with $i \geq 0$ obtained this way, one calls them *Steenrod operations*; their sum (which is in fact finite because $S^i = 0$ for $i > \dim X$)

$$S = S_X = S^0 + S^1 + \cdots \colon \mathrm{CH}(X) \to \mathrm{CH}(X)$$

is the *total Steenrod operation* (we omit the $*$ in the notation of the Chow group to notify that S is not homogeneous). They have the following basic properties (see [1] for the proofs): for any smooth quasi-projective F-scheme X, the total operation $S \colon \mathrm{CH}(X) \to \mathrm{CH}(X)$ is a (non-homogeneous) ring homomorphism such that for every morphism $f \colon Y \to X$ of smooth quasi-projective F-schemes and for every field extension E/F, the squares

$$\begin{array}{ccc} \mathrm{CH}(Y) & \xrightarrow{S_Y} & \mathrm{CH}(Y) \\ \uparrow{\scriptstyle f^*} & & {\scriptstyle f^*}\uparrow \\ \mathrm{CH}(X) & \xrightarrow{S_X} & \mathrm{CH}(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{CH}(X_E) & \xrightarrow{S_{X_E}} & \mathrm{CH}(X_E) \\ \uparrow{\scriptstyle \mathrm{res}_{E/F}} & & {\scriptstyle \mathrm{res}_{E/F}}\uparrow \\ \mathrm{CH}(X) & \xrightarrow{S_X} & \mathrm{CH}(X) \end{array}$$

are commutative. Moreover, the restriction $S^i|_{\mathrm{CH}^n(X)}$ is 0 for $n < i$ and the squaring for $n = i$; finally S^0 is the identity.

Since S is a ring homomorphism commuting with the pull-backs, one has the **Cartan formula:** $S_{X_1 \times X_2}(\alpha_1 \times \alpha_2) = S_{X_1}(\alpha_1) \times S_{X_2}(\alpha_2)$ for any $\alpha_i \in \mathrm{CH}(X_i)$ with X_i smooth quasi-projective.

Also, the total Steenrod operation satisfies the following **Riemann-Roch** type formula:

$$f_*\big(S_Y(\alpha) \cdot c(-T_Y)\big) = S_X\big(f_*(\alpha)\big) \cdot c(-T_X)$$

(in other words, S corrected by $c(-T)$ this way, commutes with the push-forwards) for any proper $f \colon Y \to X$ and any $\alpha \in \mathrm{CH}(Y)$, where $f_* \colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ is the push-forward, c is the total Chern class, T_X is the tangent bundle of X, and $c(-T_X) = c^{-1}(T_X)$ (the expression $-T_X$ has a sense if one considers T_X as the element of $K_0(X)$). This formula is proved in [1]. Also it follows from the previous formulated properties of S by the general Riemann-Roch theorem of Panin-Smirnov.

A particular case of the Riemann-Roch formula (f is a closed imbedding and α is the class of Y itself) is the **Wu formula:** for every closed imbedding

$i\colon Y \hookrightarrow X$ with smooth quasi-projective X and Y, one has $S([Y]) = i_*\big(c(N)\big)$, where N is the normal bundle of the imbedding i.

3. Hoffmann's conjecture

Conjecture 0.1 is proved after some preliminary computations.

Lemma 3.1. *Let X be a smooth projective quadric of dimension d and let $i: X \hookrightarrow P$ its imbedding in the $(d+1)$-dimensional projective space P. We write $e \in \mathrm{CH}^1(X)$ for the class of a hyperplane section of X, that is, $e = i^*(h)$ with $h \in \mathrm{CH}^1(P)$ being the class of a hyperplane in P. Finally, we write T_X for the tangent bundle of X. Then the total Chern class $c(T_X)$ in the modulo 2 Chow group $\mathrm{CH}(X)$ is equal to $(1+e)^{d+2}$.*

Proof. The exact sequence of vector X-bundles

$$0 \to T_X \to i^*(T_P) \to i^*(\mathcal{O}_P(2)) \to 0$$

gives the equality $c(T_X)\cdot i^*\big(c(\mathcal{O}_P(2))\big) = i^*\big(c(T_P)\big)$. Since $c(\mathcal{O}_P(2)) = 1+2h = 1$ (we are working with the *modulo 2* Chow groups) and $c(T_P) = (1+h)^{d+2}$, we get $c(T_X) = (1+e)^{d+2}$. □

Corollary 3.2. *Let X be a smooth isotropic projective quadric of dimension d with the Witt index $i_W(X) > n$ for some $n \geq 0$. Let $i\colon P \hookrightarrow X$ be the class of an n-dimensional projective space lying on X (P is given by an $(n+1)$-dimensional totally isotropic subspace of the quadratic space defining X). Then the total Chern class $c(N)$ of the normal bundle N of the imbedding $P \subset X$ is equal to $(1+h)^{d+1-n}$, where $h \in \mathrm{CH}^1(P)$ is the class of a hyperplane.*

Proof. The exact sequence of vector P-bundles ([3, B.7.2])

$$0 \to T_P \to i^*(T_X) \to N \to 0$$

gives the equality $c(N) = i^*\big(c(T_X)\big)\cdot c(T_P)^{-1}$. Since $i^*(e) = h$, we get by Lemma 3.1 that $i^*(c(T_X))$ is equal to $(1+h)^{d+2}$. Since $c(T_P) = (1+h)^{n+1}$, we get the equality required. □

Corollary 3.3. *In the situation of Corollary 3.2, one has*

$$S([P]) = [P]\cdot(1+e)^{d+1-n}\ ,$$

where $[P] \in \mathrm{CH}_n(X)$ is the class of P, while $S\colon \mathrm{CH}(X) \to \mathrm{CH}(X)$ is the total Steenrod operation.

Proof. By the Wu formula (see §2), we have $S([P]) = i_*(c(N))$. Using Corollary 3.2 we get

$$i_*(c(N)) = i_*((1+h)^{d+1-n}) = (i_* \circ i^*)(1+e)^{d+1-n}\ .$$

Since the composition $i_* \circ i^*$ coincides with the multiplication by $[P]$, the desired formula follows. □

Proof of Conjecture 0.1. Let ϕ be an anisotropic quadratic form giving a counterexample to Conjecture 0.1. Then for the integer r such that the difference $\dim(\phi)-i_1(\phi)$ is divisible by 2^r and not divisible by 2^{r+1}, one has $i_1(\phi) \geq 2^r+1$. Extending the scalars to certain purely transcendental extension and replacing ϕ by its certain subform of codimension $i_1(\phi) - (2^r + 1)$, we can come to the situation where $i_1(\phi)$ is precisely $2^r + 1$ ([4, lemma 7.3]). Note that the difference $\dim(\phi) - i_1(\phi)$ is still the same.

Let X be the projective quadric given by ϕ. We set $d = \dim(X)$ (that is, $d = \dim(\phi) - 2$) and notice that this integer is odd. Therefore, we are in the situation of §1 and for every homogeneous $\alpha \in \mathrm{CH}^*(X \times X)$ the coefficients $\alpha_i \in \mathbb{Z}/2\mathbb{Z}$ are defined.

Since the Witt index of the quadratic form $\phi_{F(X)}$ is 2^r+1, the quadric $X_{F(X)}$ contains a linear subspace of dimension 2^r. Its closure (or the transposition of the closure) in $X \times X$ gives an element $\alpha \in \mathrm{CH}^{d-2^r}(X \times X)$ with $\alpha_{d-2^r} = 1$ (using Proposition 1.1, it can be easily seen that $\alpha_0 = 1$ as well; we do not need this statement however). Let us check that $\alpha_1 = \cdots = \alpha_{2^r} = 0$ for this α (in fact, the "symmetric" coefficients, namely, $\alpha_{d-2^r-1}, \dots, \alpha_{d-2^{r+1}}$ are 0 as well, but we do not need this statement).

The product $\beta = \alpha \cdot (e^0 \times e^{2^r})$ (where we consider e^0 and e^{2^r} as elements of $\mathrm{CH}^{2^r}(X)$ what is possible because $2^r < i_1(\phi) \leq \dim(\phi)/2$ whereby $2^r < d/2$) is an element of $\mathrm{CH}^d(X \times X)$ with $\beta_i = \alpha_i$ for $i = 0, 1, \dots, d - 2^r$ and $\beta_i = 0$ for $i = d - 2^r + 1, \dots, d$. Since $\beta_i = 0$ for $i = d - 2^r + 1, \dots, d$ and $i_1(\phi) = 2^r + 1$, Proposition 1.1 says that $\beta_1 = \cdots = \beta_{2^r} = 0$. Since $\alpha_i = \beta_i$ for $i = 0, 1, \dots, d - 2^r$, we get that $\alpha_1 = \cdots = \alpha_{2^r} = 0$.

Let us keep in mind the last relation and consider the element

$$\gamma = S^{2^r}(\alpha) \in \mathrm{CH}^d(X \times X) \,,$$

where S^{2^r} is the Steenrod operation. We are going to show that $\gamma_{2^r} = 0$ (in fact we will even see that $\gamma_1 = \cdots = \gamma_{2^r} = 0$). By the Cartan formula (see §2), one has $S(e^a \times e^b) = S(e^a) \times S(e^b)$ for any a, b. Since the non-zero graded components of the elements $S(e^a), S(e^b) \in \mathrm{CH}^*(\bar{X})$ are in codimensions $\geq a$ and $\geq b$ respectively, the element $S(e^a \times e^b)$ is a linear combination of the elements $e^i \times e^j$ only with $i \geq a$ and $j \geq b$. In particular, for every $i = 2^r + 1, \dots, d - 2^r$, the element $S^{2^r}(e^i \times e^{d-2^r-i}) \in \mathrm{CH}^d(\bar{X} \times \bar{X})$ is a linear combination of the elements $e^j \times e^{d-j}$ with $j > 2^r$. Besides, since S is the identity on CH^0 and in particular on $e^0 \in \mathrm{CH}^0(\bar{X})$, the element $S^{2^r}(e^0 \times e^{d-2^r})$ is a multiple of $e^0 \times e^d$. Now, the relation $\alpha_1 = \cdots = \alpha_{2^r} = 0$ we dispose means that $\bar{\alpha}$ is a linear combination of $e^0 \times e^{d-2^r}$ and $e^i \times e^{d-2^r-i}$ with $i = 2^r + 1, \dots, d - 2^r$. It follows that the element $\gamma = S^{2^r}(\alpha)$ is a linear combination of $e^0 \times e^d$ and $e^j \times e^{d-j}$ with $j > 2^r$. In particular, $\gamma_{2^r} = 0$.

Applying Proposition 1.1, we get that $\gamma_d = 0$. We are going to get a contradiction showing by a direct computation that $\gamma_d = 1$. Note that γ_d satisfies the relation $S^{2^r}(e^{d-2^r} \times e^0) = \gamma_d \cdot (e^d \times e^0)$ (here we recall that $\alpha_{d-2^r} = 1$) and, therefore, the simplified relation $S^{2^r}(e_{2^r}) = \gamma_d e_0$. We recall that the element $e_{2^r} \in \mathrm{CH}_{2^r}(\bar{X})$ is the class of a 2^r-dimensional projective space $P \subset \bar{X}$.

Therefore, by Corollary 3.3, we have:

$$S(e_{2^r}) = e_{2^r} \cdot (1 + e^1)^{d-2^r+1} \,.$$

In particular, $S^{2^r}(e_{2^r}) = \binom{d-2^r+1}{2^r} \cdot e_0$ and we get that $\gamma_d = \binom{d-2^r+1}{2^r} \mod 2$. Since the integer $d - 2^r + 1 = \dim(\phi) - i_1(\phi)$ is divisible by 2^r and is not divisible by 2^{r+1}, the binomial coefficient $\binom{d-2^r+1}{2^r}$ is odd. □

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