POINCARÉ DUALITY FOR PROJECTIVE VARIETIES

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In 1895 in his first topological memoir "Analysis Situs" [P] Henri Poincaré established a foundation of Algebraic Topology. Besides defining conceptions of homology and cohomology, he formulated and proved that Betti numbers of a (compact, orientable) manifold are symmetric. This result, together with a bunch of its refinements and consequences, is now called the Poincaré Duality. In a fancier language the theorem of Poincaré may be rewritten as follows.

Let us consider a complex oriented cohomology theory represented by a ring-spectrum E. Then, for every complex projective algebraic variety X one defines a fundamental class $[X] \in E_{2n}(X)$ (here n stays for the complex dimension of X). The cap-product $\frown[X] : E^*(X) \to E_{2n-*}(X)$ provides an isomorphism of cohomology and homology groups of X and called the Poincaré Duality isomorphism.

From the modern point of view it looks pretty interesting to obtain an analogue of this result in the context of Algebraic Geometry. In the current preprint we consider the category of smooth varieties Sm and two functors on it (contravariant and covariant) A^* and A_* with values in $\mathbb{Z}/2$-graded abelian groups. We postulate certain properties of these functors and derive the Poincaré Duality. We consider only theories which take values in $\mathbb{Z}/2$-graded abelian groups, however all the arguments work for $\mathbb{Z}$-grading as well.

In the first section we define a notion of pseudo-representable theory which mimics the pair (E^*, E_*) of theories represented by a spectrum E. In order to state the main result we need a whole bunch of products (two cross-products, two slant products, and two inner products) substituting ones appearing from a spectrum multiplication $\mu\colon E \wedge E \to E$. For this need we introduce a technical notion of a "multiplicative pair" which encodes all necessary structures. We also need a good replacement of the notion of orientability. For this end we axiomatize transfer structures in our theory which associate to every projective morphism $f\colon Y \to X$ two maps $f_!\colon A^*(Y) \to A^*(X)$ and $f^!\colon A_*(X) \to A_*(Y)$ satisfying a natural list of properties and compatible in certain sense (see Axiom B.5). It may be shown that the existence of these structures is related to the "nice" theory of characteristic classes for our functors (see [PS], where the transfer structure called "integration").

Having all these data in hands, we define a fundamental class of a projective variety X as $[X] \overset{\text{def}}{=} \pi^!(1) \in A_0(X)$, where $\pi\colon X \to \mathrm{pt}$ is the structure morphism. Then, our main result claims that the map

$$\frown [X]\colon A^*(X) \overset{\cong}{\to} A_*(X)$$

is the grade-preserving isomorphism.

In fact, our result holds for a rather wide class of theories represented by oriented multiplicative T-spectra in the sense of Voevodsky [Vo]. This includes, for example, Motivic Cohomology and Algebraic Cobordism (we should indicate here that the transfer structure for homology theories represented by an oriented T-spectra is still under construction). As a further refinement we can expect, for example, that the duality given in [FV] might be seen as a case of Poincaré Duality.

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Notation. Throughout the paper we use greek letters to denote elements of cohomology groups and latin for homological ones;

Δ always denotes a diagonal morphism;

$\mathbb{P}(V) = Proj(S^*(V^\vee))$ is the space of lines in a finite k-vector space V;

Symbol $\mathbf{1}$ denotes trivial bundle;

$\mathbb{P}(\mathcal{E}) = Proj(S^*(\mathcal{E}^\vee))$ is the space of lines in a vector bundle $\mathcal{E}$;

$\mathcal{O}_\mathcal{E}(-1)$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$;

$\mathcal{O}_\mathcal{E}(1)$ is the line bundle on $\mathbb{P}(\mathcal{E})$ dual to $\mathcal{O}_\mathcal{E}(-1)$;

We reserve letter z for the zero-section $X \xrightarrow{z} E$ and

s for the section $X \xrightarrow{s} \mathbb{P}(\mathbf{1} \oplus \mathcal{E})$ induced by the zero-section of $\mathcal{E}$.

Both the sections z and s are called below *the zero sections*;

$\mathrm{pt} = \operatorname{Spec} k$;

For the convenience of perception we usually move indexes up and down oppositely to the predefined positions of $*$ or $!$.

1. Initial data

Consider a category Sm/k of smooth algebraic varieties over a field k. Let A^* and A_* be functors (cohomology and homology pretheories) $A^*\colon (Sm/k)^{\mathrm{op}} \to \mathbb{Z}/2\text{-}\mathcal{A}b$ and $A_*\colon Sm/k \to \mathbb{Z}/2\text{-}\mathcal{A}b$ taking their values in the category of $\mathbb{Z}/2$-graded abelian groups.

Definition 1.1. Let functors A^* and A_* (cotravariant and covariant, respectively), be endowed with a product structure consisting of two cross-products

$$\underline{\times}\colon A_p(X) \otimes A_q(Y) \to A_{p+q}(X \times Y)$$
$$\overline{\times}\colon A^p(X) \otimes A^q(Y) \to A^{p+q}(X \times Y)$$

and two slant-products

$$/\colon A^p(X \times Y) \otimes A_q(Y) \to A^{p-q}(X)$$
$$\backslash\colon A^p(X) \otimes A_q(X \times Y) \to A_{q-p}(Y).$$

Define two inner products

$$\smile : A^p(X) \otimes A^q(X) \to A^{p+q}(X)$$
$$\frown : A^p(X) \otimes A_q(X) \to A_{q-p}(X),$$

as $\alpha \smile \beta = \Delta^*(\alpha \overline{\times} \beta)$ and $\alpha \frown a = \alpha \backslash \Delta_*(a)$. We say that functors A^* and A_* make a **multiplicative pair** (A^*, A_*) if the mentioned products satisfy the following five axioms.

(**A.1**) *The cup-product makes the group $A^*(X)$ an associative skew-commutative $\mathbb{Z}/2$-graded unitary ring and this structure is functorial.*

(**A.2**) *The cap-product makes the group $A_*(X)$ a unital $A^*(X)$-module (we have $1 \frown a = a$ for every $a \in A_*(X)$) and this structure is functorial in the sense that $\alpha \frown f_*(a) = f_*(f^*(\alpha) \frown a)$.*

(**A.3**) Associativity relations. *For $\alpha \in A^*(X \times Y)$, $\beta \in A^*(Y)$, $\gamma \in A^*(X)$, $a \in A_*(Y)$, and $b \in A_*(X)$, we have:*
 (i) $\alpha/(\beta \frown a) = (\alpha \smile p_Y^*(\beta))/a$
 (ii) $\gamma \smile (\alpha/a) = (p_X^*(\gamma) \smile \alpha)/a$
 (iii) $(\alpha/a) \frown b = p_*^X((\alpha \frown (a \underline{\times} b))$,
where morphisms p_X and p_Y are corresponding projections.

(**A.4**) Functoriality for slant-product: *For morphisms $f\colon X \to X'$, $g\colon Y \to Y'$, and elements $\alpha \in A^*(X' \times Y')$ and $a \in A_*(Y)$, one has: $(f \times g)^*(\alpha)/a = f^*(\alpha/g_*(a))$.*

(**A.5**) *In the homology group of the final object* pt *we are given an element* $[\mathrm{pt}] \in A_0(\mathrm{pt})$ *such that for every $\alpha \in A^*(\mathrm{pt})$, one has: $\alpha/[\mathrm{pt}] = \alpha$.*

One can easily verify that the latter property implies the existance of the isomorphism $\omega\colon A^*(\mathrm{pt}) \xrightarrow{\cong} A_*(\mathrm{pt})$ given as $\omega(\alpha) = \alpha \frown [\mathrm{pt}]$. Throughout the paper we implicitly use this identification and usually denote $[\mathrm{pt}]$ by 1, taking into account that $\omega(1) = [\mathrm{pt}]$.

Remark 1.2. In that follows we mostly make a deal with $\frown, \smile, /$, and $\underline{\times}$-products. As well as in topology, it is easy to recover the rest of the structure from these four products (see [Ad, Sw]).

To make more clear the origins of the notion of multiplicative pair, consider a simple example.

Example 1.3. Let E be a ring-spectrum and E^* , E_* denote (co-)homology theories built by E. Then, these functors make a multiplicative pair $\mathcal{E} = (E^*, E_*)$ such that all the products are corresponding topological ones. The proof consists of direct checking all the properties and may be found in any book on Algeraic Topology (see, for example, [Ad, Sw]).

Definition 1.4. A **pseudo-representable theory** $\mathcal{A}$ is a multiplicative pair (A^*, A_*) of functors which are defined at the begining of this section and both A^* and A_* are additive and strongly homotopy invariant. Namely, we require:

Additivity: Let $i_r\colon X_r \hookrightarrow X_1 \amalg X_2$ be natural inclusions ($r = 1, 2$). Then the induced maps $A^*(X_1 \amalg X_2) \to A^*(X_1) \oplus A^*(X_2)$ and $A_*(X_1) \oplus A_*(X_2) \to A_*(X_1 \amalg X_2)$ are isomorphisms.

Strong Homotopy Invariance: For any variety X, a vector bundle $\mathcal{E}$ over X, and an $\mathcal{E}$-torsor $\mathfrak{E}$ with the projection $p\colon \mathfrak{E} \to X$, the natural maps $p^*\colon A^*(X) \to A^*(\mathfrak{E})$ and $p_*\colon A_*(\mathfrak{E}) \to A_*(X)$ are isomorphisms.

Remark 1.5. In the spirit of [Pa, Definition 1.1.1] this makes the functor A^* a cohomology pretheory.

Definition 1.6. We say that a pseudo-representable theory $\mathcal{A}$ is **oriented** if for every projective morphism $f\colon Y \to X$ there are given maps $f_!\colon A^*(Y) \to A^*(X)$ and $f^!\colon A_*(X) \to A_*(Y)$ such that a triple $(A^*, \smile, f \mapsto f_!)$ is an oriented cohomology pretheory (see [Pa, Definition 1.1.7]) and the family of maps $f^!$ satisfies the followimg list of properties:

(**B.1**) Functoriality. *We have:* $(f \circ g)^! = g^! \circ f^!$ *and* $\mathrm{id}_X^! = \mathrm{id}$.

(**B.2**) Base-Change property for transversal squares. *For any transversal square (see* [Pa, 1.1.2] *or* [PY, 1.1] *for definitions)*

$$\begin{array}{ccc} Y' & \xhookrightarrow{\bar{f}} & X' \\ {\scriptstyle \bar{g}}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xhookrightarrow{f} & X \end{array}$$

the diagram

$$\begin{array}{ccc} A_*(Y') & \xleftarrow{\bar{f}^!} & A_*(X') \\ {\scriptstyle \bar{g}_*}\big\downarrow & & \big\downarrow{\scriptstyle g_*} \\ A_*(Y) & \xleftarrow{f^!} & A_*(X) \end{array}$$

commutes. For an arbitrary morphism $f\colon Y \to X$ *we have a commutative diagram*

$$\begin{array}{ccc} A_*(Y \times \mathbb{P}^n) & \xrightarrow{(f \times \mathrm{id})_*} & A_*(X \times \mathbb{P}^n) \\ \big\uparrow{\scriptstyle p_Y^!} & & \big\uparrow{\scriptstyle p_X^!} \\ A_*(Y) & \xrightarrow[f_*]{} & A_*(X). \end{array}$$

(**B.3**) Gysin exact sequence (localization) *Let* $i\colon Y \hookrightarrow X$ *be a closed embedding and* $j\colon X - Y \hookrightarrow X$ *the corresponding open inclusion. Then, there is an exact sequence*

$$A_*(X - Y) \xrightarrow{j_*} A_*(X) \xrightarrow{i^!} A_*(Y).$$

Before formulating the next axiom we should introduce the notion of an Euler class. For a line bundle $\mathcal{L}$ over X we set $e(\mathcal{L}) \overset{\mathrm{def}}{=} z^* z_!(1)$, where $z : X \to \mathcal{L}$ is the zero-section (see [Pa, 1.1.4] for details).

(**B.4**) Projective Bundle Theorem (PBT). *For* $X \in Sm/k$ *and a rank* n *vector bundle* $\mathcal{E} \xrightarrow{p} X$ *over* X *set* $\zeta = e(\mathcal{O}_E(1)) \in A^*(\mathbb{P}(\mathcal{E}))$. *Then, the map*

$$\oplus_{i=0}^{n-1} \psi_i\colon A_*(\mathbb{P}(\mathcal{E})) \xrightarrow{\simeq} \bigoplus_{i=0}^{r-1} A_*(X).$$

where $\psi_i = p_* \circ (\zeta^{\smile i} \frown -)$, *is an isomorphism.*

Finally, we require both transfer structures to be consistent in the following way:

(**B.5**) *Let* $\mathcal{L}$ *be a line bundle over* X. *Then, for the zero-section* $s\colon X \to \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$ *and every* $a \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{L}))$, *we have:* $s_* s^!(a) = s_!(1) \frown a$.

Corollary 1.7. *Let u_1 and u_2 be natural embeddings of varieties Y_1 and Y_2 to $Y = Y_1 \amalg Y_2$. For a closed embedding $i\colon Y \hookrightarrow X$ set $i_r = i \circ u_r : Y_r \hookrightarrow X$. Then,*

$$i^! = u^1_* \circ i^!_1 + u^2_* \circ i^!_2 \colon A_*(X) \to A_*(Y).$$

We do not prove this Corollary in this preprint but rather *add it to the list of axioms* for oriented pseudo-representable theory. However, similarly to the cohomological case, it can be derived from the other axioms (the proof is parallel to [Pa, 1.7.1]).

Remark 1.8. By the definition the triple $(A^*, \smile, f \mapsto f_!)$ is an oriented cohomology pretheory (see [Pa, Definition 1.1.7]). Thus, for a projective morphism $f : Y \to X$ the map $f_! : A^*(Y) \to A^*(X)$ is a two-side $A^*(X)$-module homomorphism, i.e.

$$f_!(f^*(\alpha) \smile \beta) = \alpha \smile f_!(\beta) \tag{1.1}$$
$$f_!(\alpha \smile f^*(\beta)) = f_!(\alpha) \smile \beta.$$

Remark 1.9. The reader could see that axioms B.1–B.4. are in obvious sense dual to the axioms for oriented cohomology pretheory given in [Pa].

Remark 1.10. Axiom B.5 may be replaced by a more natural one which gives an equivalent condition. For the zero-section $z\colon X \to \mathcal{L}$ we require the relation $z^! \circ z_* = e(\mathcal{L}) \frown\colon A_*(X) \to A_*(X)$ to be held.

To complete this section we formulate two simple consequences of the listed axioms which we need below.

Lemma 1.11. *For $p\colon X \times Y \to Y$ and $\alpha \in A^*(Y)$, we have: $p^*(\alpha) = 1 \times \alpha$.*

Lemma 1.12. *Let $\mathcal{L}$ be a line bundle over X. Then, for the zero-section $s\colon X \to \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$ and $\alpha \in A^*(\mathcal{L})$ one has: $s_! s^*(\alpha) = s_!(1) \smile \alpha$.*

2. Poincaré Duality Theorem

Definition 2.1. Let $\mathcal{A}$ be an oriented pseudo-representable theory and $X \in Sm/k$ projective variety with structure morphism $\pi\colon X \to \mathrm{pt}$. Then, we call an element $\pi^!(1) \in A_0(X)$ the **fundamental class** of X and denote it by $[X]$.

Theorem 2.2 (**Poincaré Duality**)**.** *Let $\mathcal{A}$ be an oriented pseudo-representable theory. For every projective $X \in Sm/k$, denote by $\mathcal{D}^\bullet\colon A^*(X) \to A_*(X)$ the map $\mathcal{D}^\bullet(\alpha) = \alpha \frown [X]$ and by $\mathcal{D}_\bullet\colon A_*(X) \to A^*(X)$ the map $\mathcal{D}_\bullet(a) = \Delta_!(1)/a$, where $\Delta\colon X \to X \times X$ is the diagonal morphism. Then, the maps $\mathcal{D}^\bullet$ and $\mathcal{D}_\bullet$ are mutually inverse isomorphisms.*

One can extract the following nice consequence of the Poincaré Duality theorem, which enables us to interpret transfer maps in a way topologists like to.

Corollary 2.3. *For projective $X, Y \in Sm/k$ and a morphism $f\colon X \to Y$, one has:*

$$f_! = \mathcal{D}^Y_\bullet f_* \mathcal{D}^\bullet_X$$
$$f^! = \mathcal{D}^\bullet_X f^* \mathcal{D}^Y_\bullet,$$

where $\mathcal{D}_X$ and $\mathcal{D}_Y$ are introduced above duality operators for varieties X and Y, respectively.

The proof of Theorem 2.2 is based on two projection formulae for cap- and slant-products. In this section we formulate these assertions and derive the theorem from them. The rest of the paper is devoted to the proofs of the assertions.

Theorem 2.4. *For $X, Y \in Sm/k$, an arbitrary projective morphism $f : Y \to X$, and any elements $\alpha \in A^*(Y)$ and $a \in A_*(X)$, the relation*

$$f_*(\alpha \frown f^!(a)) = f_!(\alpha) \frown a \tag{2.1}$$

holds in the group $A_*(X)$.

We need several corollaries of this theorem.

Corollary 2.5. *Let $\tau : X \times X \to X \times X$ be the transposition morphism. Then, for any elements $\alpha \in A^*(X)$, $\beta \in A^*(X \times X)$, and $a \in A_*(X \times X)$, we have:*

$$\begin{aligned} &\textbf{a)} \quad \Delta_!(\alpha) \frown a = \Delta_!(\alpha) \frown \tau_*(a) \\ &\textbf{b)} \quad \Delta_!(\alpha) \smile \beta = \Delta_!(\alpha) \smile \tau^*(\beta) \end{aligned}$$

in $A_*(X \times X)$ ($A^*(X \times X)$, respectively).

Proof. First of all, we show that $\Delta^! \circ \tau_* = \Delta^!$. In fact, this follows from transversality of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \tau} \\ X & \xrightarrow{\Delta} & X \times X. \end{array} \tag{2.2}$$

(Since the map τ is flat we may apply the transversality criterion given in [Fu].) The base change property for transfers in A_* tells us that $\Delta^! \circ \tau_* = \mathrm{id}_* \circ \Delta^!$. Since $\mathrm{id}_* = \mathrm{id}$, one gets the desired result.

To complete the proof of **a)**, it remains to observe that by Theorem 2.4 one has the chain of relations

$$\Delta_!(\alpha) \frown a = \Delta_*(\alpha \frown \Delta^!(a)) = \Delta_*(\alpha \frown \Delta^!(\tau_*(a))) = \Delta_!(\alpha) \frown \tau_*(a).$$

The proof of case **b)** goes similarly, using cohomological projection formula (1.1):

$$\Delta_!(\alpha) \smile \beta = \Delta_!(\alpha \smile \Delta^*(\beta)) = \Delta_!(\alpha \smile \Delta^*(\tau^*(\beta))) = \Delta_!(\alpha) \smile \tau^*(\beta).$$

□

Corollary 2.6. *For every projective $X \in Sm/k$ and its structure morphism $\pi \colon X \to \mathrm{pt}$, one has the following relation in $A(\mathrm{pt})$:*

$$\pi_*([X]) = \pi_!(1). \tag{2.3}$$

Proof. Apply Theorem 2.4 to the morphism π in the case: $\alpha = 1 \in A^*(X)$ and $a = 1 \in A_*(\mathrm{pt}) = A^*(\mathrm{pt})$. □

Corollary 2.7. *Let $\mathcal{V}$ be a vector bundle over $Y \in Sm/k$, a morphism $p \colon \mathbb{P}(\mathbf{1} \oplus \mathcal{V}) \to Y$ be the projection and let $s : Y \to \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$ be the zero-section. Then, for every elements $\alpha \in A^*(Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{V}))$, one has:*

$$\begin{aligned} &a) \qquad s^!(a) = p_*(s_!(1) \frown a) \quad \text{and} \\ &b) \qquad s_!(\alpha) = p^*(\alpha) \smile s_!(1). \end{aligned}$$

Proof. For a), we have:

$$p_*(s_!(1) \frown a) = p_* s_*(1 \frown s^!(a)) = s^!(a), \tag{2.4}$$

since $p_* s_* = \mathrm{id}$. As well, for b):

$$s_!(\alpha) = s_!(s^* p^*(\alpha) \smile 1) = p^*(\alpha) \smile s_!(1). \tag{2.5}$$

□

Let us state here the second important theorem concerning the interaction of the slant-product and transfers. First, we need to introduce a piece of notation which will be used pretty often throughout the paper.

Notation 2.8. Dealing with varieties X, Y, T, and a morphism $Y \xrightarrow{f} X$ we denote the morphism $F = (\mathrm{id} \times f)\colon T \times Y \to T \times Y$ by a corresponding capital letter.

Theorem 2.9. *Let $f : Y \to X$ be a projective morphism of smooth varieties. Let, also $T \in Sm/k$. Then, for every elements $\alpha \in A^*(T \times Y)$ and $a \in A_*(X)$ one has the following equality in $A^*(T)$:*

$$\alpha/f^!(a) = F_!(\alpha)/a.$$

Corollary 2.10. *Let $Y, T \in Sm/k$ with projective Y and let $q\colon Y \to \mathrm{pt}$ be the structure map of Y. Then, for every $\alpha \in A^*(T \times Y)$, one has the equality:*

$$\alpha/[Y] = Q_!(\alpha).$$

Proof. By Theorem 2.9 one has in $A^*(T)$

$$\alpha/[Y] = \alpha/q^!(1) = Q_!(\alpha)/1 = Q_!(\alpha), \tag{2.6}$$

which proves the Corollary. □

Corollary 2.11. *Let X be a smooth projective variety. Then, in $A^*(X)$, we have:*

$$\Delta_!(1)/[X] = 1. \tag{2.7}$$

Proof. Denoting by $p : X \times X \to X$ the projection morphism, we have a chain of equalities:

$$\Delta_!(1)/[X] = p_!(\Delta_!(1)) = (p \circ \Delta)_!(1) = \mathrm{id}_!(1) = 1, \tag{2.8}$$

which derives this Corollary from Corollary 2.10. □

Now we are ready to prove the main result.

Proof of Theorem 2.2. This is an easy consequence of Corollaries 2.11 and 2.5.

Let $p_1, p_2\colon X \times X \to X$ denote corresponding projections. Observe, that for every $\gamma \in A^*(X \times X)$ one has the relation $\Delta_!(1) \smile \gamma = \gamma \smile \Delta_!(1)$. In fact,

$$\Delta_!(1) \smile \gamma = \Delta_!(1 \smile \Delta^*(\gamma)) = \Delta_!(\Delta^*(\gamma) \smile 1) = \gamma \smile \Delta_!(1). \tag{2.9}$$

Applying Corollary 2.5.b, we have:

$$\begin{aligned} \Delta_!(1)/(\alpha \frown [X]) &= (\Delta_!(1) \smile p_2^*(\alpha))/[X] = (\Delta_!(1) \smile p_1^*(\alpha))/[X] \\ &= (p_1^*(\alpha) \smile \Delta_!(1))/[X] = \alpha \smile (\Delta_!(1)/[X]) \end{aligned} \tag{2.10}$$

By 2.11, we have: $\Delta_!(1)/[X] = 1$. Thus, $\Delta_!(1)/(\alpha \frown [X]) = \alpha$.

On the other hand, using 2.5.a, one has:

$$\begin{aligned} (\Delta_!(1)/a) \frown [X] &= p_*(\Delta_!(1) \frown (a \underline{\times} [X])) = p_*(\Delta_!(1) \frown ([X] \underline{\times} a)) \\ &= (\Delta_!(1)/[X]) \frown a. \end{aligned} \tag{2.11}$$

Since $\Delta_!(1)/[X] = 1$, one gets $(\Delta_!(1)/a) \frown [X] = a$. □

Thus, to prove Theorem 2.2 it is sufficient to prove Theorems 2.4 and 2.9. We start proving Theorem 2.4.

For this end, it is convinient to introduce a class $\mathfrak{V}$ of projective morphisms $f: Y \to X$ for which the relation

$$f_*(\alpha \frown f^!(a)) = f_!(\alpha) \frown a \tag{2.12}$$

holds in $A_*(X)$ for every elements $\alpha \in A^*(Y)$ and $a \in A_*(X)$.

We prove Theorem 2.4 showing consequently that the following classes of morphisms are contained in the class $\mathfrak{V}$.

- Zero-section morphisms of line bundles: $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$;
- Closed embeddings $i\colon D \hookrightarrow X$ of smooth divisors;
- Zero-sections of finite sum of line bundles:
$$s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_n));$$
- Zero-sections of arbitrary vector bundles: $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$;
- Closed embeddings $i\colon Y \hookrightarrow X$;
- Projections $p\colon X \times \mathbb{P}^n \to X$;
- Arbitrary projective morphisms.

3. Auxillary Facts

In this section we collect several useful lemmas, which are utilized repeatedly in proofs of Theorems 2.4 and 2.9 in consequent sections.

Lemma 3.1. *Suppose, we are given a transversal square*

$$\begin{array}{ccc} X' & \xleftarrow{\bar{f}} & Y' \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xleftarrow[f]{} & Y \end{array}$$

with projective morphism f. Let $\alpha \in A^*(Y)$ and $a \in A_*(X')$. Then, the following relations hold:

(1) $f_*(\alpha \frown f^!\bar{g}_*(a)) = \bar{g}_*\bar{f}_*(g^*(\alpha) \frown \bar{f}^!(a))$
(2) $f_!(\alpha) \frown \bar{g}_*(a) = \bar{g}_*(\bar{f}_!g^*(\alpha) \frown a)$
Moreover, for a variety $T \in Sm/k$ and an element $\beta \in A^*(T \times Y)$, we have:
(3) $\beta/f^!\bar{g}_*(a) = G^*(\beta)/\bar{f}^!(a)$
(4) $F_!(\beta)/\bar{g}_*(a) = \bar{F}_!G^*(\beta)/a.$

Proof. All these relations may be easily obtained using the base-change property and natural formulae from the first section. We illustrate it proving the first relation:

$$\begin{aligned} f_*(\alpha \frown f^!\bar{g}_*(a)) &= f_*(\alpha \frown g_*\bar{f}^!(a)) \\ &= f_*g_*(g^*(\alpha) \frown \bar{f}^!(a)) = \bar{g}_*\bar{f}_*(g^*(\alpha) \frown \bar{f}^!(a)). \end{aligned} \tag{3.1}$$

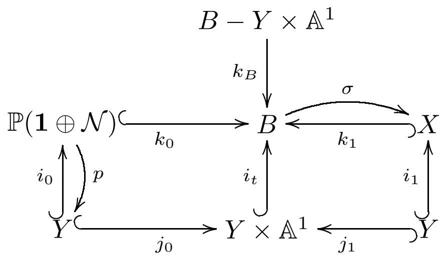
□

Remark 3.2. The reader could see that all the cases of the previous lemma may be considered as consequences of a general "lifting principle" formulation and proof of which is left to the reader.

The second fact that we need is a useful lemma, which is actually a "dualization" of "Useful Lemma 1.4.2" from [Pa].

To state this lemma, consider the following deformation diagram, in which B denotes the blowup of $X \times \mathbb{A}^1$ at $Y \times \{0\}$. This diagram has transversal squares.

(3.2)



The morphism σ here is the natural retraction $B = (X \times \mathbb{A}^1)\widetilde{}_{Y\times\{0\}} \xrightarrow{\sigma} X$.

Lemma 3.3 (Homological useful lemma). $\operatorname{Im} k_*^B + \operatorname{Im} k_*^0 = A_*(B)$

Proof. Let us take $b \in A_*(B)$. Since the map j_*^0 is an isomorphism and $i_0^! p^! = \mathrm{id}$, we can, using the transversal base-change property, lift b up to $\bar{b} = p^!(j_*^0)^{-1} i_t^!(b) \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathcal{N}))$, such that $i_t^!(k_*^0(\bar{b})) = i_t^!(b)$. Then, the Gysin exact sequence implies that $k_*^0(\bar{b}) - b \in \operatorname{Im} k_*^B$. □

This lemma plays an important role in the proofs of Theorems 2.4 and 2.9 because of the following proposition.

Proposition 3.4. *Let $i\colon Y \hookrightarrow X$ be a closed embedding of smooth varieties and $\mathcal{N}$ be the corresponding normal bundle. If the zero-section morphism $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1}\oplus\mathcal{N})$ belongs to $\mathfrak{V}$, then i itself belongs to $\mathfrak{V}$.*

Proof. Let us pay attention to Diagram 3.2. We take the morphism i as i_1 in this diagram and s as i_0. First of all, using the proposition assumption, we shall show that the morphism i_t in Diagram 3.2 belongs to the class $\mathfrak{V}$. Namely, we prove that for every elements $\alpha_t \in A^*(Y \times \mathbb{A}^1)$ and $a_t \in A_*(B)$, we have:

$$i_*^t(\alpha_t \frown i_t^!(a_t)) = i_!^t(\alpha_t) \frown a_t. \tag{3.3}$$

Using Lemma 3.3 we can rewrite a_t as a sum $k_*^B(a_B) + k_*^0(a_0)$, where $a_0 \in A_*(\mathbb{P}(\mathbf{1}\oplus \mathcal{N}))$ and $a_B \in A_*(B - Y \times \mathbb{A}^1)$. From the Gysin exact sequence, we have:

$$i_t^! k_*^B = 0 \quad \text{and} \tag{3.4}$$

$$k_B^* i_!^t = 0. \tag{3.5}$$

Therefore, $i_*^t(\alpha_t \frown i_t^! k_*^B(a_B)) = 0$ and $i_!^t(\alpha_t) \frown k_*^B(a_B) = 0$. In fact, to get the second relation, we use 3.5 as follows:

$$i_!^t(\alpha_t) \frown k_*^B(a_B) = k_*^B(k_B^* i_!^t(\alpha_t) \frown a_t) = 0. \tag{3.6}$$

This implies that

$$i_*^t(\alpha_t \frown i_t^!(a_t)) = i_*^t(\alpha_t \frown i_t^! k_*^0(a_0)). \tag{3.7}$$

Applying Lemma 3.1 to the left-hand-side square of Diagram 3.2 and denoting $j_0^*(\alpha_t)$ by α_0, one has:

$$i_*^t(\alpha_t \frown i_t^! k_*^0(a_0)) = k_*^0 i_*^0 \left(j_0^*(\alpha_t) \frown i_0^!(a_0)\right) = k_*^0 i_*^0 \left(\alpha_0 \frown i_0^!(a_0)\right). \tag{3.8}$$

Similarly, one has:

$$i_!^t(\alpha_t) \frown a_t = k_*^0\left(i_!^0(\alpha_0) \frown a_0\right). \tag{3.9}$$

By the proposition assumption, we have the relation $i_*^0(\alpha_0 \frown i_0^!(a_0)) = i_!^0(\alpha_0) \frown a_0$. Combining this with equalities (3.7), (3.8), and (3.9) finishes the proof of (3.3).

We now go further and move the desired relation one more step to the right in Diagram 3.2. Namely, we prove that for every elements $\alpha_1 \in A^*(Y)$ and $a_1 \in A_*(X)$, we have:

$$i_*^1(\alpha_1 \frown i_1^!(a_1)) = i_!^1(\alpha_1) \frown a_1. \tag{3.10}$$

For this end we observe that k_*^1 is a monomorphism, therefore it is sufficient to check that

$$k_*^1 i_*^1(\alpha_1 \frown i_1^!(a_1)) = k_*^1(i_!^1(\alpha_1) \frown a_1). \tag{3.11}$$

Using that the map j_1^* is an isomorphism we choose $\alpha_t \in A^*(Y \times \mathbb{A}^1)$ as $\alpha_t = (j_1^*)^{-1}(\alpha_1)$. We also set: $a_t = k_*^1(a_1) \in A_*(B)$. Applying Lemma 3.1 to the right-hand-side square of Diagram 3.2, we have:

$$k_*^1 i_*^1(\alpha_1 \frown i_1^!(a_1)) = k_*^1 i_*^1(j_1^*(\alpha_t) \frown i_1^!(a_1)) = i_*^t(\alpha_t \frown i_t^!(a_t)). \tag{3.12}$$

By the same arguments, we get:

$$k_*^1(i_!^1(\alpha_1) \frown a_1) = i_!^t(\alpha_t) \frown a_t. \tag{3.13}$$

Taking these two relations together with (3.3) finishes the proof of relation (3.10) and of the proposition as well. □

4. Proof of Theorem 2.4

We begin with the following very simple case of the Theorem.

Lemma 4.1. *Let $\mathcal{L}$ be a line bundle over a smooth variety Y. Then the zero-section $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$ belongs to $\mathfrak{V}$.*

Proof. Since the map s is a section of the projection map p, we can write $\alpha = s^*p^*(\alpha)$. Then, we have:

$$s_*(\alpha \frown s^!(a)) = s_*(s^*p^*(\alpha) \frown s^!(a)) = p^*(\alpha) \frown s_*s^!(a). \tag{4.1}$$

Using the normalization property, we can rewrite the latter expression as:

$$p^*(\alpha) \frown s_*s^!(a) = p^*(\alpha) \frown (s_!(1) \frown a) = (p^*(\alpha) \smile s_!(1)) \frown a = s_!(\alpha) \frown a. \tag{4.2}$$

□

Corollary 4.2. *Let $i\colon D \hookrightarrow X$ be a smooth divisor. Then, $i \in \mathfrak{V}$.*

Proof. Since, in this case, the corresponding normal bundle is a line bundle, the Corollary follows from Lemma 4.1 and Proposition 3.4. □

Proposition 4.3. *Let $\mathcal{W} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ be an n-dimensional vector bundle over a variety Y which splits in the sum of line bundles. Then, the zero-section morphism $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{W})$ belongs to the class $\mathfrak{V}$.*

Proof. The map s can be decomposed in the following obvious way:

$$Y \overset{i_1}{\hookrightarrow} \mathbb{P}(\mathbf{1} \oplus \mathcal{L}_1) \overset{i_2}{\hookrightarrow} \cdots \overset{i_n}{\hookrightarrow} \mathbb{P}(\mathbf{1} \oplus \mathcal{W}) \tag{4.3}$$

where the map i_j the zero-section of $\mathcal{L}_j$. Now we can rewrite the desired formula

$$\begin{aligned} s_*(\alpha \frown s^!(a)) &= i^n_* i^{n-1}_* \dots i^2_* i^1_*(\alpha \frown i^!_1 i^!_2 \dots i^!_{n-1} i^!_n(a)) \\ &= i^n_* i^{n-1}_* \dots i^2_*(i^1_!(\alpha) \frown i^!_2 \dots i^!_{n-1} i^!_n(a)) \\ &= \dots = i^n_! i^{n-1}_! \dots i^2_! i^1_!(\alpha) \frown a = s_!(\alpha) \frown a \end{aligned} \tag{4.4}$$

by iterative application of Corollary 4.2. □

In order to proceed with the case of an arbitrary vector-bundle, we need the homological analogue of the splitting principle. To state it, consider a vector bundle $\mathcal{E} \to Y$ of constant rank n over a smooth variety Y. Let $\mathcal{GL}_n$ be the corresponding principal GL_n-bundle over Y, let $T_n \subset GL_n$ be the diagonal tori, and let $Y' = \mathcal{GL}_n/T_n$ be the orbit variety with the projection morphism $p : Y' \to Y$. Finally, we denote by $\mathcal{E}' = \mathcal{E} \times_Y Y'$ the pull-back of the vector bundle $\mathcal{E}$.

Proposition 4.4. *The bundle $\mathcal{E}'$ splits in a direct sum of line bundles and the map $p_* \colon A_*(Y') \to A_*(Y)$ is a universal splitting epimorphism (which means for any base-change $Z \to Y$ the induced map $A_*(Z \times_Y Y') \to A_*(Z)$ is a splitting epimorphism).*

Proof. The projection $\mathcal{GL}_n \to Y'$ and the natural T_n-action on $\mathcal{GL}_n$ makes it a principal T_n-bundle over Y'. Moreover, if $\mathcal{GL}'_n = \mathcal{GL}_n \times_Y Y'$ is the bull-back of $\mathcal{GL}_n$, there is a natural isomorphism of principal GL_n-bundles

$$\mathcal{GL}_n \times_{T_n} GL_n \to \mathcal{GL}'_n \tag{4.5}$$

over Y'. The bundle $\mathcal{E}'$ over Y' corresponds exactly to the principal GL_n-bundle $\mathcal{GL}'_n$. Thus, the mentioned isomorphism of principal GL_n-bundles over Y' shows that the bundle $\mathcal{E}'$ splits in a direct sum of line bundles (say corresponding to the fundamental characters $\chi_1, \chi_2, \dots, \chi_n$ of the tori T_n). This proves the first assertion of the proposition.

To prove the second one, consider a Borel subgroup B_n in GL_n (say the subgroup of all upper triangle matrises) and let U_n be the maximal unipotent subgroup of B_n (the group of upper triangle matrises with 1's on the diagonal). Let $\mathcal{F} = \mathcal{GL}_n/B_n$ (this is just the flag bundle over Y assosiated to $\mathcal{E}$). The bundle $\mathcal{F}$ comes equipped with projections $q : \mathcal{F} \to Y$ and $r : Y' \to \mathcal{F}$, where the projection r is induced by the inclusion $T_n \subset B_n$. Using the natural U_n-action on $\mathcal{GL}_n$, it is easy to check that there is a tower of morphisms:

$$\mathcal{GL}_n = S_m \to S_{m-1} \to \cdots \to S_1 = \mathcal{F}, \tag{4.6}$$

which has on each level a principal G_a-bundle (so, every level is a torser over the trivial rank one vector bundle). By the strong homotopy invariance property, the induced map on homology $r_* : A_*(Y') \to A_*(\mathcal{F})$ is an isomorphism.

As it was already mentioned, $\mathcal{F}$ is a full flag bundle over Y associated to the bundle $\mathcal{E}$. Thus, there is a tower of morphisms

$$\mathcal{F} = Z_s \to Z_{s-1} \to \cdots \to Z_1 = Y \tag{4.7}$$

in which each level is a projective bundle associated to a vector bundle. Thus, by PBT, we have a split epimorphism in homology induced on each floor. Thus, the map $q_* \colon A_*(\mathcal{F}) \to A_*(Y)$ is a split epimorphism as well.

These proves that the map $p_* : A_*(Y') \to A_*(Y)$ is also an epimorphism.

Now let $Z \to Y$ be a morphism (with smooth variety Z) and $Z' = Z \times_Y Y'$. Consider a variety $\mathcal{F}_Z = Z \times_Y Y'$ and projections $p' : Z' \to Z$, $r' : Z' \to \mathcal{F}_Z$, and $q' : \mathcal{F}_Z \to Z$, which are base-changes of projections p, r, and q, respectively. Clearly, r' may be decomposed as a tower of principal G_a-bundles and thus it induces an isomorphism on the homology groups A_* by the strong homotopy invariance. The morphism q' is decomposable as a tower of projective bundles associated to certain vector bundles. By PBT, it induces a split epimorphism on the homology groups A_*. Thus the morphism p' induces a split epimorphism on the homology groups A_*. The proof of the Proposition is completed. □

Proposition 4.5. *Let $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1}\oplus\mathcal{V})$ be the zero-section of the finite-dimensional vector bundle $\mathcal{V}$. Then, $s \in \mathfrak{V}$.*

Proof. Let us take the variety Y' from Proposition 4.4 and the pull-back $\mathcal{V}'$ of the bundle $\mathcal{V}$ with respect to the morphism p. By 4.4, the bundle $\mathcal{V}'$ splits in a direct sum of line bundles. Consider the pull-back diagram

$$\begin{array}{ccc} \mathbb{P}(\mathbf{1}\oplus\mathcal{V}') & \xrightarrow{\bar{f}} & Y' \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{P}(\mathbf{1}\oplus\mathcal{V}) & \xrightarrow{f} & Y. \end{array} \tag{4.8}$$

The map

$$\bar{p}_*\colon A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V}')) \to A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V})) \tag{4.9}$$

is an epimorphism by Proposition 4.4. Let $s : Y \to \mathbb{P}(\mathbf{1}\oplus\mathcal{V})$ and $\bar{s}\colon Y' \to \mathbb{P}(\mathbf{1}\oplus\mathcal{V}')$ be sections of projections f and $\bar{f}$ induced by zero-sections of the corresponding vector bundles. Then, the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathbf{1}\oplus\mathcal{V}') & \xleftarrow{\bar{s}} & Y' \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{P}(\mathbf{1}\oplus\mathcal{V}) & \xleftarrow{s} & Y \end{array} \tag{4.10}$$

is transversal.

For every $\alpha \in A^*(Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V}))$, since the map $\bar{p}_*\colon A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V}')) \to A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V}))$ is an epimorphism, we can choose some $b \in A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V}'))$ such that $a = \bar{p}_*(b)$, and then apply Lemma 3.1:

$$s_*(\alpha \frown s^!(a)) = s_*(\alpha \frown s^!\bar{p}_*(b)) = \bar{p}_*\bar{s}_*(p^*(\alpha) \frown \bar{s}^!(b)). \tag{4.11}$$

On the other hand, by the same reason:

$$s_!(\alpha) \frown a = s_!(\alpha) \frown \bar{p}_*(b) = \bar{p}_*(\bar{s}_!p^*(\alpha) \frown b) \tag{4.12}$$

and two expressions on the right-hand-side coincide by Proposition 4.3. □

Proposition 4.6. *Let $i\colon Y \hookrightarrow X$ be a closed embedding. Then, $i \in \mathfrak{V}$.*

Proof. Applying Proposition 3.4 we reduce the question to the case of the zero-section morphism $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1}\oplus\mathcal{N})$ of the normal bundle $\mathcal{N} = \mathcal{N}_{X/Y}$. The morphism s belongs to $\mathfrak{V}$ by Proposition 4.5. □

In order to check that for every integer $n \geq 0$ the projection morphism $p : X \times \mathbb{P}^n \to X$ belongs to $\mathfrak{V}$ we need the following lemma.

Notation 4.7. We denote, from now on, for a projective morphism f the map $f_* f^!$ by $f^\diamond$ and $f_! f^*$ by $f_\diamond$

Fix now a variety $X \in Sm/k$ and take the n-dimensional projective space $\mathbb{P}^n_X$ over X. Consider the diagonal morphism $\Delta \colon \mathbb{P}^n_X \to \mathbb{P}^n_X \times \mathbb{P}^n_X$. (Up to the end of this section the scheme product is implicitly taken over X.) Due to the PBT, the element $\Delta_!(1)$ may be decomposed as

$$\Delta_!(1) = 1 \boxtimes \zeta^n + \zeta^n \boxtimes 1 + \sum_{i,j=1}^{n} a_{ij} \zeta^i \boxtimes \zeta^j, \tag{4.13}$$

where $a_{ij} \in A^*(X)$ and ζ is the canonical generator of $A^*(\mathbb{P}^n_X)$ as an $A^*(X)$-algebra.

Lemma 4.8. *For the projection morphism $p_n \colon \mathbb{P}^n_X \to X$, we have:*
a) $p_n^\diamond = -\sum_{j=1}^n a_{nj} p_{n-j}^\diamond$;
b) $p^n_\diamond = -\sum_{j=1}^n a_{nj} p_\diamond^{n-j}$.

Proof. Let us prove case *a*). For $n = 0$ we, trivially, have $p_0^\diamond = \mathrm{id}$.

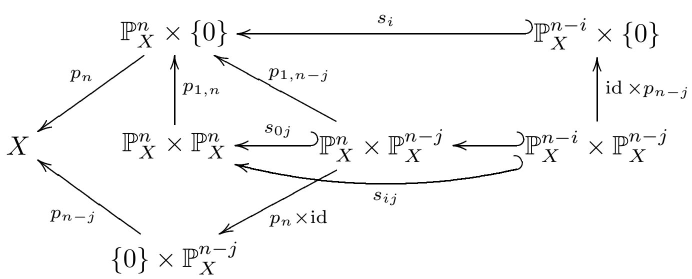
Starting from the mentioned decomposition for the diagonal morphism and taking into account the relation $s_{ij}^\diamond(x) = (\zeta^i \boxtimes \zeta^j) \frown x$, where $s_{ij} \colon \mathbb{P}^{n-i}_X \times \mathbb{P}^{n-j}_X \hookrightarrow \mathbb{P}^n_X \times \mathbb{P}^n_X$ is the standard embedding, we can rewrite the cap-product with $\Delta_!(1)$ operator in the form:

$$(\Delta_!(1) \frown) = s_{0n}^\diamond + s_{n0}^\diamond + \sum_{i,j=1}^{n} a_{ij} s_{ij}^\diamond. \tag{4.14}$$

This enables us to rewrite the map p^n_* as well:

$$p^n_* = p^n_* (p_{1,n}\Delta)^\diamond = p^n_* \sum_{i,j=0}^{n} a_{ij} (p_{1,n} s_{ij})^\diamond, \tag{4.15}$$

where the notation is taken from the diagram below.

(4.16)


Using the commutativity of this diagram and transversaltiy of the corresponding squares, we rewrite a typical summand in Formula 4.15 in the following way:
(4.17)
$$p^n_* p^{1,n}_* s^{ij}_* s^!_{ij} p^!_{1,n} = p^n_* p^{1,n}_* s^{ij}_* (\mathrm{id} \times p_{n-j})^! s^!_i = p^n_* p^{1,n}_* s^{0j}_* p^!_{1,n-j} s^\diamond_i = p^n_* p^\diamond_{1,n-j} (\zeta^i \frown).$$

Since $p^n_* p^\diamond_{1,n-j} = p^{n-j}_*(p_n \times \mathrm{id})_* p^!_{1,n-j} = p^\diamond_{n-j} p^n_*$, we can rewrite the expression (4.15) as follows:

$$p^n_* = p^\diamond_0 p^n_* + p^\diamond_n p^n_*(\zeta^n \frown) + \sum_{j=1}^{n} a_{nj} p^\diamond_{n-j} p^n_*(\zeta^n \frown) + \sum_{i,j=1}^{n-1} (\cdots) p^n_*(\zeta^i \frown). \tag{4.18}$$

Operators $p^n_*(\zeta^i \frown)$ coincide to ones ψ_i, which appear at the homology PBT. So that, we have:

$$p^n_* = \psi_0 = \psi_0 + \left(p^\diamond_n + \sum_{j=1}^{n} a_{nj} p^\diamond_{n-j} \right) \psi_n + \sum_{i,j=1}^{n-1} (\cdots) \psi_i. \tag{4.19}$$

Thus, one gets the relation:

$$\left(p^\diamond_n + \sum_{j=1}^{n} a_{nj} p^\diamond_{n-j} \right) \psi_n = - \sum_{i,j=1}^{n-1} (\cdots) \psi_i. \tag{4.20}$$

By PBT, for any $x \in A_*(X)$ we can choose an element $\varphi(x) \in A_*(\mathbb{P}^n_X)$ such that $\psi_i(\varphi(x)) = \begin{cases} 0, & i < n \\ x, & i = n. \end{cases}$

Applying operators on both sides of (4.20) to $\varphi(x)$, we get:

$$0 = p^\diamond_n + \sum_{j=1}^{n} a_{nj} p^\diamond_{n-j} \tag{4.21}$$

This finishes the proof of case a). The cohomological relation b) may be proved by dualization of these arguments or found in [Pa]. □

We need the following consequence of Lemma 4.8.

Proposition 4.9. *Let p_n denote, as before, the projection morphism $p_n \colon \mathbb{P}^n_X \to X$. Then, for every element $a \in A_*(X)$, one has:*

$$p^n_*(p^!_n(a)) = p^n_!(1) \frown a.$$

Proof. Recall that $p^\diamond_n(a) = p^n_*(p^!_n(a))$ and $p^n_!(1) = p^n_!(p^*_n(1)) = p^n_\diamond(1)$. Thus, we should verify the relation $p^\diamond_n(a) = p^n_\diamond(1) \frown a$. We proceed by induction on n. The case $n = 0$ obviously holds. Let the proposition hold for $n < N$. Then, for p_N we have by Lemma 4.8:

$$p^\diamond_N(a) = - \sum_{j=1}^{N} a_{Nj} p^\diamond_{N-j}(a) \tag{4.22}$$

and

$$p^N_\diamond(1) \frown a = - \sum_{j=1}^{N} a_{Nj} p^{N-j}_\diamond(1) \frown a \tag{4.23}$$

By the induction hypothesis the expressions on the right-hand-side coincide. The induction goes. □

Proposition 4.10. *For every integer $n \geq 0$ the projection map $p \colon \mathbb{P}^n_X \to X$ belongs to the class $\mathfrak{V}$.*

Proof. Given elements $\alpha \in A^*(\mathbb{P}^n_X)$ and $a \in A_*(X)$ we should verify that in $A_*(X)$

$$p_*(\alpha \frown p^!(a)) = p_!(\alpha) \frown a. \tag{4.24}$$

Clearly, both sides of (4.24) are $A^*(X)$-linear. By PBT, the group $A^*(\mathbb{P}^n_X)$ is generated as $A^*(X)$-module by elements ζ^j. Thus, it suffices to check the Proposition just for these elements. From [Pa], we have a relation $\zeta^j = i_!(1)$ in $A(\mathbb{P}^n_X)$, where $i\colon \mathbb{P}^{n-j}_X \hookrightarrow \mathbb{P}^n_X$ is the standard embedding map and the element $\zeta^j \in A^*(\mathbb{P}^n)$ is considered here as lying in $A^*(\mathbb{P}^n_X)$ via the pull-back operator for the projection $\mathbb{P}^n_X \to \mathbb{P}^n$. Denote by p_j the projection map $\mathbb{P}^{n-j}_X \to X$. Since $p \circ i = p_j$, we have by Proposition 4.6:

$$p_*(\zeta^j \frown p^!(a)) = p_*(i_!(1) \frown p^!(a)) = p_* i_*(1 \frown i^! p^!(a)) = p^j_* p^!_j(a). \tag{4.25}$$

Using Proposition 4.9, we get:

$$p^j_* p^!_j(a) = p^j_!(1) \frown a = p_! i_!(1) \frown a = p_!(\zeta^j) \frown a. \tag{4.26}$$

The desired relation $p_*(\zeta^j \frown p^!(a)) = p_!(\zeta^j) \frown a$ follows. □

Since any projective morphism f can be written as a composition of a closed embedding and a projection map ($f = p \circ i$), we have by Propositions 4.6 and 4.10:

$$\begin{aligned} f_*(\alpha \frown f^!(a)) &= p_* i_*(\alpha \frown i^! p^!(a)) = p_*(i_!(\alpha) \frown p^!(a)) \\ &= p_! i_!(\alpha) \frown a = f_!(\alpha) \frown a. \end{aligned} \tag{4.27}$$

This completes the proof of Theorem 2.4.

5. Proof of Theorem 2.9

The strategy of the proof of Theorem 2.9 is very similar to one used in the previous section. It is again convenient to introduce a class $\mathfrak{W}$ of projective morphisms $f : Y \to X$ for which the relation

$$F_!(\alpha)/a = \alpha/f^!(a) \tag{5.1}$$

holds. We show that the following classes of morphisms lie in $\mathfrak{W}$.

- Zero-sections of vector bundles: $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$;
- Closed embeddings $i\colon Y \hookrightarrow X$;
- Projections $p\colon X \times \mathbb{P}^n \to X$;
- All projective morphisms.

We start with the following lemma.

Lemma 5.1. *For any projective morphism $f\colon Y \to X$ and a variety T, one has the relation*

$$p^*_X f_!(1) = 1 \times f_!(1) = F_!(1)$$

in $A^*(T \times X)$. (We recall that here and below we use the notation, proclaimed in 2.8. So that, F in our case denotes the morphism $(\mathrm{id} \times f)\colon T \times X \to T \times Y$*.)*

Proof. Considering a transversal square

$$\begin{array}{ccc} T \times Y & \xrightarrow{F} & T \times X \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & X \end{array} \tag{5.2}$$

and using the base-change property, we have: $p^*_X(f_!(1)) = F_!(p^*_Y(1)) = F_!(1)$. □

Lemma 5.2. *Let $\mathcal{V}$ be a vector bundle over a smooth variety Y and let $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1}\oplus\mathcal{V})$ be the zero-section of the projection $p\colon \mathbb{P}(\mathbf{1}\oplus\mathcal{V}) \to Y$ Then, the morphism s belongs to the class $\mathfrak{W}$.*

Proof. Given a smooth variety T and elements $\alpha \in A^*(T\times Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{V}))$ we should verify that in $A^*(T)$

$$\alpha/s^!(a) = S_!(\alpha)/a. \tag{5.3}$$

Using Corollary 2.7, and Lemma 5.1 one gets a chain of relations:

$$\begin{aligned} \alpha/s^!(a) &= \alpha/p_*(s_!(1)\frown a) = P^*(\alpha)/(s_!(1)\frown a) \\ &= P^*(\alpha)\smile(1\times s_!(1))/a = P^*(\alpha)\smile S_!(1)/a = S_!(\alpha)/a \end{aligned} \tag{5.4}$$

which proves the required one. □

Now we proceed to the case of general closed embedding.

Proposition 5.3. *Any closed embedding morphism $i : Y \hookrightarrow X$ of smooth varieties belongs to the class $\mathfrak{W}$.*

Proof. Denote by $\mathbb{P}(\mathbf{1}\oplus\mathcal{N})$ the projectiovization corresponding to the normal bundle $\mathcal{N} = \mathcal{N}_{X/Y}$. It is endowed with a projection morphism $p : \mathbb{P}(\mathbf{1}\oplus\mathcal{N}) \to Y$ and the zero-section $s\colon Y \hookrightarrow \mathbb{P}(\mathbf{1}\oplus\mathcal{N})$.

As well as in the proof of Theorem 2.4 the proof settles on the deformation diagram, which actually obtained from (3.2) by multiplication on a variety $T \in Sm/k$. For convinience, we reproduce this diagram here.

$$\begin{array}{ccccc} & & T\times B - T\times Y\times\mathbb{A}^1 & & \\ & & \downarrow{\scriptstyle K_B} & & \\ T\times\mathbb{P}(\mathbf{1}\oplus\mathcal{N}) & \xrightarrow[K_0]{} & T\times B & \xleftarrow[K_1]{} & T\times X \\ \uparrow{\scriptstyle I_0} & & \uparrow{\scriptstyle I_t} & & \uparrow{\scriptstyle I_1} \\ T\times Y & \xrightarrow[J_0]{} & T\times Y\times\mathbb{A}^1 & \xleftarrow[J_1]{} & T\times Y \end{array} \tag{5.5}$$

First of all, we show that $I_t \in \mathfrak{W}$. Namely, we should prove that for any elements $\alpha_t \in A^*(T\times Y\times\mathbb{A}^1)$ and $a_t \in A_*(B)$ the relation

$$\alpha_t/i_t^!(a_t) = I_!^t(\alpha_t)/a_t. \tag{5.6}$$

holds in $A^*(T)$.

By Lemma 3.3, we can rewrite a_t as a sum $k_*^B(a_B)+k_*^0(a_0)$, where $a_0 \in A_*(\mathbb{P}(\mathbf{1}\oplus\mathcal{N}))$ and $a_B \in A_*(B - Y\times\mathbb{A}^1)$. Exactly in the same way as before (in section 3), we may use Gysin exact sequences (in homology and cohomology) to obtain the equalities:

$$i_t^! k_*^B = 0 \quad \text{and} \tag{5.7}$$

$$K_B^* I_!^t = 0. \tag{5.8}$$

Therefore, $\alpha_t/i_t^! k_*^B(a_B) = 0$ and $I_!^t(\alpha_t)/k_*^B(a_B) = K_B^* I_!^t(\alpha_t)/a_B = 0$. Now, applying Lemma 3.1, we get:

$$\alpha_t/i_t^!(a_t) = \alpha_t/i_t^! k_*^0(a_0) = \alpha_0/i_0^!(a_0), \tag{5.9}$$

where $\alpha_0 = J_0^*(\alpha_t)$.

Similarly, we obtain the relation:

$$I^t_!(\alpha_t)/a_t = I^0_!(\alpha_0)/a_0. \tag{5.10}$$

By Lemma 5.2, we have: $\alpha_0/i^!_0(a_0) = I^0_!(\alpha_0)/a_0$, which proves Formula 5.6.

We end up the proof by the same way as in 2.4. Since the map J^*_1 is an isomorphism, we can set $\alpha_t = (J^*_1)^{-1}(\alpha_1) \in A^*(T \times Y \times \mathbb{A}^1)$ and $a_t = k^1_*(a_1) \in A_*(B)$. Applying Lemma 3.1 again, we get:

$$\alpha_1/i^!_1(a_1) = J^*_1(\alpha_t)/i^!_1(a_1) = \alpha_t/i^!_t(a_t) \quad \text{and} \tag{5.11}$$

$$I^1_!(\alpha_1)/a_1 = I^1_! J^*_1(\alpha_t)/a_1 = I^t_!(\alpha_t)/a_t. \tag{5.12}$$

Combining these equalities with relation (5.6), we get: $\alpha_1/i^!_1(a_1) = I^1_!(\alpha_1)/a_1$, which proves the proposition. □

Consider now the case of projection.

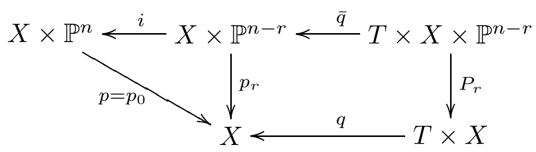
Proposition 5.4. *Let $X, T \in Sm/k$, let $p\colon X\times\mathbb{P}^n \to X$ be the projection morphism and $P = id\times p\colon T\times X\times\mathbb{P}^n \to T\times X$. Then, for every elements $\alpha \in A^*(T\times X\times\mathbb{P}^n)$ and $a \in A_*(X)$, one has a relation*

$$\alpha/p^!(a) = P_!(\alpha)/a$$

in $A^*(T)$.

Proof. Consider the following commutative diagram with transversal square:

(5.13)



Clearly, both sides of the required relation are $A^*(T)$-linear. So, we may assume that $\alpha = \zeta^r_{T\times X}$. (Here and below we denote by $\beta_Y \in A^*(Y\times\mathbb{P}^*)$ the pull-back of the element $\beta \in A^*(\mathbb{P}^*)$ with respect to the projection morphism $Y\times\mathbb{P}^* \to \mathbb{P}^*$.) From the relation $\zeta^r_X = i_!(1)$ in $A^*(X\times\mathbb{P}^n)$, we have: $\zeta^r_{T\times X} = I_!(1_{T\times X}) \in A^*(T\times X\times\mathbb{P}^n)$. This gives us the following chain of equalities:

$$\begin{aligned} \zeta^r_{T\times X}/p^!(a) &= I_!(1_{T\times X})/p^!(a) = 1_{T\times X}/i^!p^!(a) \\ &= 1/p^!_r(a) = P^*_r(1)/p^!_r(a) = 1/p^r_*p^!_r(a). \end{aligned} \tag{5.14}$$

By Proposition 4.9:

$$1/p^r_*p^!_r(a) = 1/(p^r_!(1_X) \frown a) = q^*p^r_!(1)/a. \tag{5.15}$$

Applying the base-change property to the square in the diagram above, we get:

$$q^*p^r_!(1_X) = P^r_!\,\bar{q}^*(1) = P^r_!(1_{T\times X}) = P_!(I_!(1)) = P_!(\zeta^r_{T\times X}), \tag{5.16}$$

which proves the relation $\zeta^r_{T\times X}/p^!(a) = P_!(\zeta^r_{T\times X})/a$ and the proposition as well. □

Finally, for an arbitrary projective morphism $f : Y \to X$ with a decomposition:

$$f\colon Y \overset{i}{\hookrightarrow} X\times\mathbb{P}^n \overset{p}{\to} X \tag{5.17}$$

one has:

$$\alpha/f^!(a) = \alpha/i^!p^!(a) = I_!(\alpha)/p^!(a) = P_!I_!(\alpha)/a = F_!(\alpha)/a, \tag{5.18}$$

which proves the relation $\alpha/f^!(a) = F_!(\alpha)/a$, and completes the proof of Theorem 2.9.

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