IZHBOLDIN'S RESULTS ON STABLY BIRATIONAL EQUIVALENCE OF QUADRICS

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Abstract. Our main goal is to give proofs of all results announced by Oleg Izhboldin in [13]. In particular, we establish Izhboldin's criterion for stable equivalence of 9-dimensional forms. Several other related results, some of them due to the author, are also included.

All the fields we work with are those of characteristic different from 2. In these notes we consider the following problem: for a given quadratic form ϕ defined over some field F, describe all the quadratic forms ψ/F which are stably birational equivalent to ϕ.

By saying "stably birational equivalent" we simply mean that the projective hypersurfaces $\phi = 0$ and $\psi = 0$ are stably birational equivalent varieties. In this case we also say "ϕ is *stably equivalent* to ψ"(for short) and write $\phi \overset{st}{\sim} \psi$.

Let us denote by $F(\phi)$ the function field of the projective quadric $\phi = 0$ (if the quadric has no function field, one set $F(\phi) = F$). Note that $\phi \overset{st}{\sim} \psi$ simply means that the quadratic forms $\phi_{F(\psi)}$ and $\psi_{F(\phi)}$ are isotropic (that is, the corresponding quadrics have rational points).

For an isotropic quadratic form ϕ, the answer to the question raised is easily seen to be as follows: $\phi \overset{st}{\sim} \psi$ if and only if the quadratic form ψ is also isotropic. Therefore, we may assume that ϕ is anisotropic.

One more class of quadratic forms for which the answer is easily obtained is given by the Pfister neighbors. Namely, for a Pfister neighbor ϕ one has: $\phi \overset{st}{\sim} \psi$ if and only if ψ is a neighbor of the same Pfister form as ϕ. Therefore, we may assume that ϕ is not a Pfister neighbor.

Let ϕ be an anisotropic quadratic form which is not a Pfister neighbor (in particular, $\dim \phi \geq 4$ since any quadratic form of dimension up to 3 is a Pfister neighbor) and assume that $\dim \phi \leq 6$. Then $\phi \overset{st}{\sim} \psi$ (with an arbitrary quadratic form ψ) if and only if ϕ is similar to ψ (in dimension 4 this is due to Wadsworth, [42]; 5 is done by Hoffmann, [4, main theorem]; 6 in the case of the trivial discriminant is served by Merkurjev's index reduction formula [33], see also [34, thm. 3]; the case of non-trivial discriminant is due to Laghribi, [32, th. 1.4(2)]).

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In this text we give a complete answer for the dimensions 7 and 9 (see §3 and §5). In dimension 8 the answer is almost complete (see §4). The only case where the criterion for $\phi \overset{st}{\sim} \psi$ with $\dim \phi = 8$ is not established is the case where the determinant of ϕ is non-trivial and the even Clifford algebra of ϕ (which is a central simple algebra of degree 8 over the quadratic extension of the base field given by the square root of the determinant of ϕ) is Brauer-equivalent to a biquaternion algebra not defined over the base field. In this exceptional case we only show that $\phi \overset{st}{\sim} \psi$ if and only if ϕ is motivic equivalent to ψ. This is not a final answer: it should be understood what the motivic equivalence means in this particular case.

The results on the 9-dimensional forms are due to Oleg Izhboldin and announced by himself (without proofs) in [13]. Here we also provide proofs for all other results announced in [13]. In particular, we prove the following two theorems (see Theorem 7.1.1 for the proof):

Theorem 0.0.1 (Izhboldin)**.** *Let ϕ be an anisotropic 10-dimensional quadratic form with* $\operatorname{disc} \phi = 1$ *and $i_S(\phi) = 2$. Let ψ be a quadratic form of dimension ≥ 9. Then $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ.*

Theorem 0.0.2 (Izhboldin)**.** *Let ϕ be an anisotropic 12-dimensional quadratic form from $I^3(F)$. Let ψ be a quadratic form of dimension ≥ 9. Then $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ.*

Also the theorem on the anisotropy of an arbitrary 10-dimensional form over the function of a non Pfister neighbor of dimension > 10 announced in [13] is proved here (see Theorem 7.2.1).

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Contents

1. Notation and results we are using

If the field of definition of a quadratic form is not explicitly given, we mean that this is a field F.

We use the following more or less standard notation concerning quadratic forms: $\det(\phi) \in F^*/F^{*2}$ is the determinant of the quadratic form ϕ, $\operatorname{disc}(\phi) = (-1)^{n(n-1)/2}\det(\phi)$ with $n = \dim\phi$ is its discriminant (or signed determinant); $i_W(\phi)$ is the Witt index of ϕ; $i_S(\phi)$ is the Schur index of ϕ, that is, the Schur index of the simple algebra $C_0(\phi)$ for $\phi \notin I^2(F)$ and the Schur index of the central simple algebra $C(\phi)$ for $\phi \in I^2(F)$. Here $I(F)$ is the ideal of the even-dimensional quadratic forms in the Witt ring $W(F)$. In the case where $\phi \in I^2(F)$, we also write $c(\phi)$ for the class of $C(\phi)$ in the Brauer group $\operatorname{Br}(F)$; this is the Clifford invariant of ϕ.

We write $\phi \sim \psi$ to indicate that two quadratic forms ϕ and ψ are similar, i.e., $\phi \simeq c\psi$ for some $c \in F^*$; $\phi \overset{st}{\sim} \psi$ stays for the stable equivalence (meaning that for any field extension E/F one has $i_W(\phi_E) \geq 1$ if and only if one has $i_W(\psi_E) \geq 1$); and $\phi \overset{m}{\sim} \psi$ denotes the motivic equivalence of ϕ and ψ meaning that for any field extension E/F and any integer n one has $i_W(\phi_E) \geq n$ if and only if one has $i_W(\psi_E) \geq n$.

Theorem 1.0.3 (Izhboldin, [12, cor. 2.9])**.** *Let ϕ and ψ be odd-dimensional quadratic forms over F. Then $\phi \overset{m}{\sim} \psi$ if and only if $\phi \sim \psi$.*

Theorem 1.0.4 (Hoffmann, [5, th. 1])**.** *Let ϕ and ψ be two anisotropic quadratic forms over F with $\dim\phi \leq \dim\psi$. If the form $\phi_{F(\psi)}$ is isotropic, then $\dim\phi$ and $\dim\psi$ are in the same interval $]2^{n-1}, 2^n]$ (for some n). In particular,* *the integer $n = n(\phi)$ such that $\dim\phi \in]2^{n-1}, 2^n]$ is a stably birational invariant of an anisotropic quadratic form ϕ.*

For an anisotropic ϕ, the *first Witt index* $i_1(\phi)$ is defined as $i_W(\phi_{F(\phi)})$.

Theorem 1.0.5 (Vishik, [22, th. 8.1])**.** *The integer $\dim\phi - i_1(\phi)$ is a stably birational invariant of an anisotropic form ϕ.*

1.1. **Pfister forms and neighbors.** A quadratic form isomorphic to a tensor product of several (say, n) binary forms representing 1 is called an (n-fold) Pfister form. Having a Pfister form π, we write π' for a pure subform of π, that is, for for a subform $\pi' \subset \pi$ (determined by π up to an isomorphism) such that $\pi = \langle 1\rangle \perp \pi'$. A quadratic form is called a Pfister neighbor, if it is similar to a subform of an n-fold Pfister form and has dimension bigger that 2^{n-1} (the half of the dimension of the Pfister form) for some n. Two quadratic forms ϕ and ψ with $\dim\phi = \dim\psi$ are called half-neighbors, if the orthogonal sum $a\phi \perp b\psi$ is a Pfister form for some $a, b \in F^*$.

1.2. **Similarity of 1-codimensional subforms.** We write $G(\phi) \subset F^*$ for the multiplicative group of the similarity factors of a quadratic form ϕ; $D(\phi) \subset F^*$ stays for the set of non-zero values of ϕ. The following observations are due to B. Kahn:

Lemma 1.2.1. *Let ϕ be an arbitrary quadratic form of even dimension. For every $a \in D(\phi)$, let ψ_a be a 1-codimensional subform of ϕ such that $\phi \simeq \langle a\rangle \perp \psi_a$. Then for every $a, b \in D(\phi)$, the forms ψ_a and ψ_b are similar if and only if $ab \in G(\phi)$.*

Proof. Comparing the determinants of the odd-dimensional quadratic forms ψ_a and ψ_b, we see that $\psi_a \sim \psi_b$ if and only if $b\psi_a \simeq a\psi_b$. By adding $\langle ab\rangle$ to both sides, the latter condition is transformed in $b\phi \simeq a\phi$, that is, to $ab \in G(\phi)$. □

Corollary 1.2.2. *Let ψ be a 1-codimensional subform of an even-dimensional anisotropic quadratic form $\phi = \langle a_0, a_1, \dots, a_n\rangle /F$. Let $\tilde F = F(x_0, x_1, \dots, x_n)/F$ be the purely transcendental field extension and let $\tilde\psi/\tilde F$ be a subform of $\phi_{\tilde F}$ complementary to the "generic value" $\tilde a = a_0x_0^2 + a_1x_1^2 + \dots + a_nx_n^2 \in \tilde F$ of ϕ (so that $\phi_{\tilde F} = \tilde\psi \perp \langle\tilde a\rangle$). Then $\psi_{\tilde F} \sim \tilde\psi$ if and only if ϕ is similar to a Pfister form.*

Proof. We may assume that $a_0 = 1$ and $\psi = \langle a_1, \dots, a_n\rangle$. Then

$$\psi_{\tilde F} \sim \tilde\psi \overset{\text{Lemma 1.2.1}}{\iff} \tilde a \in G(\phi_{\tilde F}) \overset{\text{[35, th. 4.4 of chap. 4]}}{\iff}$$
$$\phi_{\tilde F} \text{ is a Pfister form} \overset{\text{[5, prop. 7]}}{\iff} \phi \text{ is a Pfister form}$$

□

We will refer to the subform $\tilde\psi$ appearing in Corollary 1.2.2 as to the *generic 1-codimensional subform* of ϕ (although $\tilde\psi$ is a subform of $\phi_{\tilde F}$ and not of ϕ itself).

1.3. **Linkage of Pfister forms.** We need a result concerning the linkage of two n-fold Pfister forms. This result is an easy consequence of the results obtained in [2]. However, it is neither proved nor formulated in the article cited and we do not know any other reference for it. It deals with the *graded Witt ring* $GW(F)$ of a filed F which is the graded ring associated with the filtration of the ordinary Witt ring $W(F)$ by the powers of the fundamental ideal $I(F) \subset W(F)$. It will be applied in §5 to the case with $n = 3$ and $i = 2$.

Lemma 1.3.1 (cf. [37, th. 2.4.8])**.** *Let $a_1, \dots, a_n, b_1, \dots, b_n \in F^*$. We consider the elements α and β of the graded Witt ring $GW(F)$ given by the Pfister forms $\langle\langle a_1, \dots, a_n\rangle\rangle$ and $\langle\langle b_1, \dots, b_n\rangle\rangle$, and assume that they are non-zero (i.e., the Pfister forms are anisotropic). If there exist some $i < n$ and $c_1, \dots, c_i \in F^*$ such that the difference $\alpha - \beta$ is divisible by $\langle\langle c_1, \dots, c_i\rangle\rangle$ in $GW(F)$, then there exist some $d_1, \dots, d_i \in F^*$ such that $\langle\langle d_1, \dots, d_i\rangle\rangle$ divides both α and β in $GW(F)$.*

Proof. Let us make a proof using an induction on i. The case $i = 0$ is without contents.

If $\langle\langle c_1, \dots, c_i\rangle\rangle$ with some $i \geq 1$ divides the difference $\alpha - \beta$, then $\langle\langle c_1, \dots, c_{i-1}\rangle\rangle$ also divides it. By the induction hypothesis we can find some $\langle\langle d_1, \dots, d_{i-1}\rangle\rangle$ dividing both α and β. Therefore for some $a'_i, \dots, a'_n, b'_i, \dots, b'_n \in F^*$ we have isomorphisms of quadratic forms $\langle\langle a_1, \dots, a_n\rangle\rangle \simeq \langle\langle d_1, \dots, d_{i-1}, a'_i, \dots, a'_n\rangle\rangle$ and $\langle\langle b_1, \dots, b_n\rangle\rangle \simeq \langle\langle d_1, \dots, d_{i-1}, b'_i, \dots, b'_n\rangle\rangle$, whereby the difference $\alpha - \beta$ turns out to be represented by the quadratic form

$$\langle\langle d_1, \dots, d_{i-1}\rangle\rangle \otimes \left(\langle\langle a'_i, \dots, a'_n\rangle\rangle' \perp - \langle\langle b'_i, \dots, b'_n\rangle\rangle' \right)$$

of dimension $2^i(2^{n-i+1} - 1)$. We claim that this quadratic form is isotropic, and this gives what we need according to [2, prop. 4.4]. Indeed, assuming that this quadratic form is anisotropic, we can decompose it as $\langle\langle c_1, \dots, c_i\rangle\rangle \otimes \delta$ with some quadratic form δ. Counting dimension, we see that $\dim \delta = 2^{n-i+1} - 1$ is odd. This is a contradiction with the facts that $\langle\langle c_1, \dots, c_i\rangle\rangle \otimes \delta \in I^n(F)$, $n > i$, and $\langle\langle c_1, \dots, c_i\rangle\rangle$ is anisotropic. □

1.4. **Special forms, subforms, and pairs.** Here we recall (and slightly modify) some definitions given in [16, §8–9]. We will not work with the general notion of special pairs introduced in [16, def. 8.3]. We will only work with the degree 4 special pairs (see [16, examples 9.2 and 9.3]). Besides, it will be more convenient for us to call *special* also those pairs which are similar to the special pairs of [16, def. 8.3]. So, we give the definitions as follows:

Definition 1.4.1. A 12-dimensional quadratic form is called *special*, if it lies in $I^3(F)$. A 10-dimensional quadratic form is called *special*, if it has trivial discriminant and Schur index ≤ 2. A quadratic form is called *special*, if it is either a 12-dimensional or a 10-dimensional special form.

A 10-dimensional quadratic form is called a *special subform*, if it is divisible by a binary form. A 9-dimensional quadratic form is called a *special subform*, if it contains a 7-dimensional Pfister neighbor. A *special subform* is a quadratic form which is either a 10-dimensional or a 9-dimensional special subform.

A pair of quadratic forms ϕ_0, ϕ with $\phi_0 \subset \phi$ is called *special* if either ϕ is a 12-dimensional special form while ϕ_0 is a 10-dimensional special subform or ϕ is a 10-dimensional special form while ϕ_0 is a 9-dimensional special subform.

A special pair ϕ_0, ϕ is called *anisotropic*, if the form ϕ is anisotropic (in this case ϕ_0 is of course anisotropic as well).

Proposition 1.4.2 ([16, §§8–9]). *Special forms, subforms, and pairs have the following properties:*

1. *for any special subform ϕ_0, there exists a special form ϕ such that ϕ_0, ϕ is a special pair;*
2. *for any special form ϕ, there exists a special subform ϕ_0 such that ϕ_0, ϕ is a special pair;*
3. *for a given special pair ϕ_0, ϕ, the form ϕ is isotropic if and only if the form ϕ_0 is a Pfister neighbor;*
4. *for any anisotropic special pair ϕ_0, ϕ, the Pfister neighbor $(\phi_0)_{F(\phi)}$ is anisotropic.*

The items **3** and **4** give

Corollary 1.4.3 (cf. [16, prop. 8.13]). *Let ϕ_0, ϕ and ψ_0, ψ be two special pairs. If $\phi_0 \stackrel{st}{\sim} \psi_0$, then $\phi \stackrel{st}{\sim} \psi$.* □

1.5. **Anisotropic 9-dimensional forms of Schur index 2.** In this subsection, ϕ is an anisotropic 9-dimensional quadratic form with $i_S(\phi) = 2$.

Lemma 1.5.1. *There exist one and unique (up to an isomorphism) 10-dimensional special form μ containing ϕ. There exist one and unique (up to an isomorphism) 12-dimensional special form λ containing ϕ. Moreover,*

(i) *μ is isotropic if and only if ϕ contains an 8-dimensional subform divisible by a binary form;*
(ii) *λ is isotropic if and only if ϕ contains a 7-dimensional Pfister neighbor;*
(iii) *if μ and λ are both isotropic, then ϕ is a Pfister neighbor.*

Proof. The form μ is constructed as $\mu = \phi \perp \langle -\operatorname{disc}(\phi)\rangle$. The uniqueness of μ is evident.

The form λ is constructed as $\lambda = \phi \perp \operatorname{disc}(\phi)\beta'$, where β is a 2-fold Pfister form with $c(\beta) = c(\phi)$. If λ' is one more 12-dimensional special form containing ϕ, then the difference $\lambda - \lambda' \in W(F)$ is represented by a form of dimension 6. Since this difference lies in $I^3(F)$, it should be 0 by the Arason-Pfister-Hauptsatz.

Clearly, the form μ is isotropic if and only if ϕ represents its determinant, that is, if and only if ϕ contains an 8-dimensional subform ϕ' of trivial determinant. Since $i_S(\phi') = i_S(\phi) = 2$, the form ϕ' is divisible by some binary form ([29, example 9.12]).

The form λ is isotropic if and only if $\lambda = \pi$ for some form π similar to a 3-fold Pfister form. The latter condition holds if and only if ϕ and π contain a common 7-dimensional subform.

Note that the isotropy of λ implies that ϕ is a 9-dimensional special subform and ϕ, μ is a special pair. So, μ is isotropic if and only if ϕ is a Pfister neighbor in this case (Proposition 1.4.2). $\square$

2. Correspondences on odd-dimensional quadrics

In this section we give some formal rules concerning the game with the correspondences on odd-dimensional quadrics.

2.1. **Types of correspondences.** Let ϕ be a completely split quadratic form of an odd dimension and write $n = 2r + 1$ for the dimension of the projective quadric X_ϕ given by ϕ. We recall (see, e.g., [20, §2.1]), that there exists a filtration

$$X = X^{(0)} \supset X^{(1)} \supset \cdots \supset X^{(n)} \supset X^{(n+1)} = \emptyset$$

of the variety $X = X_\phi$ by closed subsets $X^{(i)}$ such that every successive difference $X^{(i)} \smallsetminus X^{(i+1)}$ is an affine space (so that X is cellular) and $\operatorname{codim}_X X^{(i)} = i$ for all $i = 0, 1, \ldots, n$. It follows (see [3]) that for every $i = 0, 1, \ldots, n$, the group $\mathrm{CH}^i(X)$ is infinite cyclic and is generated by the class of $X^{(i)}$. Note that for the class of a hyperplane section $h \in \mathrm{CH}^1(X)$ one has $[X^{(i)}] = h^i$ for $i < \dim X/2$ and $2 \cdot [X^{(i)}] = h^i$ for $i > \dim X/2$. In particular, the generators $[X^{(i)}]$ are canonical.

Since the product of two cellular varieties is also cellular, the group $\mathrm{CH}^*(X \times X)$ is also easily computed. Namely, this is the free abelian group on $[X^{(i)} \times X^{(j)}]$ for $i, j = 0, 1, \ldots, n$. In particular, $\mathrm{CH}^n(X \times X)$ is generated by $[X^{(i)} \times X^{(n-i)}]$, $i = 0, 1, \ldots, n$.

For any correspondence $\alpha \in \mathrm{CH}^n(X \times X)$, we define its *pretype* (cf. [23, §9]) as the sequence of the integer coefficients in the representation of α as the linear combination of the generators.

Moreover, refusing to assume that ϕ is split, we may still define the pretype of an $\alpha \in \mathrm{CH}^n(X_\phi \times X_\phi)$ as the pretype of $\alpha_{\bar{F}}$, where $\bar{F}$ is an algebraic closure of F. Note that the entries of the pretype of α can be also calculated as the half of the degrees of the 0-cycles $h^{n-i} \cdot \alpha \cdot h^i \in \mathrm{CH}_0(X_\phi \times X_\phi)$. This is an invariant definition of the pretype. In particular, the pretype of α does not depend on the choice of $\bar{F}$ (what can be also easily seen in the direct way).

Finally, we define the *type* as the pretype modulo 2.

2.2. **Formal notion of type.** We start with some quite formal (however convenient) definitions.

A *type* is an arbitrary sequence of elements of $\mathbb{Z}/2\mathbb{Z}$ of a finite length. For two types of the same length n, we define their *sum* and *product* as for the elements of $(\mathbb{Z}/2\mathbb{Z})^n$. We may also look at a type as at the diagram of a subset of the set $\{1, 2, \ldots, n\}$ (1 is on the i-th position iff the element i is in the subset). Using this interpretation of types, we may define the *union* and the *intersection* in the evident way (the intersection coincides with the product). We may also speak of the inclusion of types. In particular, we have the notion of a subtype of a given type (all these is defined for types of the same length).

The *reduction* (or 1-reduction) of a type of length ≥ 2 is the type obtained by erasing the two border entries. The *n-reduction* of a type is the result of n reductions successively applied to the type.

The *diagonal type* is the type with all the entries being 1. The *zero type* is the type with all the entries being 0.

We have two different notions of *weight* of a type: the sum of its entries (this is an element of $\mathbb{Z}/2$) and the number of 1-entries (this is an integer). To distinguish between them, we call the second number *cardinality*. So, the weight is the same as the cardinality modulo 2.

2.3. **Possible and minimal types.** Let ϕ be an odd-dimensional quadratic form. A type is called *possible* (for ϕ), if this is the type (in the sense of §2.1) of some correspondence on the quadric X_ϕ. Note that the possible types are of the length $\dim \phi - 1$. A possible non-zero type is called *minimal* (for ϕ), if no its proper subtype is possible.

We have the following rules (see [23, §9]): the diagonal and zero types are possible (the diagonal type is realized by the diagonal, [23, lemma 9.4]); moreover, sums, products, unions, and intersections of possible types are possible.

It follows that two different minimal types have no intersection. Moreover, a type is possible if and only if it is a union of minimal ones.

Therefore, in order to describe all possible types for a given quadratic form ϕ, it suffices to list the minimal types (see §2.7 as well as Propositions 3.0.9, 3.0.10, 3.0.12 or 5.0.22 for examples of such lists).

2.4. **Properties of possible types.** Here are some rules which help to detect the impossibility of certain types.

Assume that the quadratic form ϕ is anisotropic. Then the weight of every possible type is 0, [23, lemma 9.7].

And now we assume the contrary: ϕ is isotropic, say $\phi \simeq \psi \perp \mathbb{H}$ ($\mathbb{H}$ is the hyperbolic plane). Then the reduction of a type possible for ϕ is a type possible for ψ, [23, lemma 9.6].

These two rules (together with a trivial observation that a type possible for a ϕ is also possible for ϕ_E where E is an arbitrary field extension of the base field) have a useful consequence (cf. [22, th. 6.4]): if ϕ is an anisotropic form with the first Witt index n, then for any type possible for ϕ we have: the sum of the first n entries coincides with the sum of the last n entries.

Let us note that a types possible for ϕ_E is also possible for ϕ/F if the field extension E/F is unirational (this is easily seen by the homotopy invariance of the Chow group).

2.5. **Possible types and the Witt index.** Here is a way to determine the Witt index of a quadratic form ϕ looking at its possible types: for any integer $n \leq (\dim \phi)/2$, one has $i_W(\phi) \geq n$ if and only if the type with the only one 1 entry staying on the n-th position is possible. Note that the "only if" part is trivial while the "if" part follows from 2.4.

2.6. **The Rost type.** The *Rost type* of a given length is the type with 1 on the both border places and with 0 on all inner places. By definition, the Rost type is possible for a given odd-dimensional quadratic form ϕ if and only if there exists a correspondence $\rho \in \mathrm{CH}^n(X_\phi \times X_\phi)$ such that over an algebraic closure of the base field one has $\rho = a[X \times \mathbf{pt}] + b[\mathbf{pt} \times X]$ with some odd integers a, b, where $n = \dim X_\phi = \dim \phi - 2$ and where $\mathbf{pt}$ is a rational point. We will use this reformulation as definition for the expression "Rost type is possible" in the case of an even-dimensional quadratic form ϕ even though we do not have a definition of types possible for an even-dimensional quadratic form yet (cf. subsection 2.10).

As shown in [23, prop. 5.2], the Rost type is possible for any Pfister neighbor of dimension 2^n+1 (for any $n \geq 1$). The converse statement for the anisotropic forms is an extremely useful conjecture (cf. [23, conj. 1.6]) proved by A. Vishik in all dimensions $\neq 2^n + 1$: if $\dim \phi \neq 2^n + 1$ for all n, then the Rost type is not possible for ϕ (see [37] or [18, th. 6.1]). Vishik's proof uses existence and certain properties of the operations in the motivic cohomology obtained by Voevodsky and involved in his proof of the Milnor conjecture. In the original [40], the operations were constructed (or claimed to be constructed) only in characteristic 0 (this was enough for the Milnor conjecture because the Milnor conjecture in positive characteristics is a formal consequence of the Milnor conjecture in the characteristic 0, [40, lemma 5.2]). This is the reason why Vishik's result is announced only in characteristic 0 in [18]. The new version [41] of [40] is more characteristic independent. So, Vishik's result extends to any characteristic (cf. [38, th. 4.18]).

We also note that the conjecture on the Rost types is proved by simple and characteristic independent methods which do not use any unpublished result, in the following particular cases:

$i_S(\phi)$ is maximal ([23, cor. 6.6], cf. Lemma 3.0.8); note that this covers the cases of dimension 4 (because $i_S(\phi)$ of a 4-dimensional anisotropic form is always maximal) and 5 (because an anisotropic quadratic form ϕ with $\dim \phi = 5$ is not a Pfister neighbor if and only if $i_S(\phi)$ is maximal);

$\dim \phi = 7, 8$ and ϕ does not contain an Albert subform (see [23, prop. 9.10] for dimension 7; the same method works for dimension 8);

$\dim \phi = 9$, ϕ is arbitrary (this is the main result of [23]).

Finally, a simple and characteristic independent proof of the conjecture in all dimensions $\neq 2^n + 1$, using only the Steenrod operations on the Chow groups (constructed in an elementary way in [1]) is recently given in [26].

2.7. **Minimal types for 5-dimensional forms.** To give an example, we find the minimal types for a 5-dimensional anisotropic quadratic form ϕ (cf. [38, prop. 5.10]). Note that $i_S(\phi) = 2$ if and only if ϕ is a Pfister neighbor; otherwise

$i_S(\phi) = 4$. Also note that $i_1(\phi)$ is always 1. Therefore, the diagonal type (1111) is minimal for ϕ which is not a Pfister neighbor. For a Pfister neighbor ϕ, the minimal types are given by the Rost type (1001) and its complement (0110).

2.8. **Possible types for pairs of quadratic forms.** Let (ϕ, ψ) be a pair of quadratic forms (the order is important) having a same odd dimension n. A type is called possible for the pair (ϕ, ψ) if it is the type of a correspondence lying in the Chow group $\mathrm{CH}^n(X_\phi \times X_\psi)$. Here are some rules.

The product of a type possible for (ϕ, ψ) by a type possible for (ψ, τ) is a type possible for ϕ, τ. In particular, a product of a type possible for (ϕ, ψ) by a type possible for ψ (that is, possible for (ψ, ψ)) is still a type possible for (ϕ, ψ).

Therefore (see §2.5), one may compare the Witt indices of two quadratic forms ϕ and ψ (with $\dim \phi = \dim \psi$ being odd) over extensions E/F as follows: let n be an integer such that a type with 1 on the n-th place (the other entries can be arbitrary) is possible for (ϕ, ψ) as well as for (ψ, ϕ), let E/F be any field extension of the base field F; then $i_W(\phi_E) \geq n$ if and only if $i_W(\psi_E) \geq n$.

In particular, we get one part of Vishik's criterion of motivic equivalence of quadratic forms (cf. [21, criterion 0.1]): $\phi \overset{m}{\sim} \psi$ if the diagonal type is possible for the pair (ϕ, ψ).

2.9. **Rational morphisms and possible types.** Given some different ϕ and ψ, how can one construct at least one non-zero type possible for (ϕ, ψ)? In this article we use essentially only one method which works only if the form $\psi_{F(\phi)}$ is isotropic: we take the correspondence given by the closure of the graph of a rational morphism $X_\phi \to X_\psi$. Its type is non-zero because its first entry is 1.

Let us give an application. We assume that the diagonal type is minimal for an odd-dimensional ϕ and we show that $\phi \overset{st}{\sim} \psi$ (for some ψ with $\dim \psi = \dim \phi$) means $\phi \sim \psi$ in this case as follows: taking the product of the possible types for (ϕ, ψ) and (ψ, ϕ) given by the rational morphisms $X_\phi \to X_\psi$ and $X_\psi \to X_\phi$, we get a possible type for ϕ, starting with 1; therefore this is the diagonal type; therefore the types we have multiplied are diagonal as well; therefore the diagonal type is possible for (ϕ, ψ); therefore $\phi \overset{m}{\sim} \psi$ whereby $\phi \sim \psi$ by Theorem 1.0.3.

2.10. **Even-dimensional quadrics.** Even though this contradicts to the title of the current section, we briefly discuss the notion of a type possible for an even-dimensional quadratic form here. We need it in order to prove Proposition 4.0.14 on 8-dimensional quadratic forms (and only for this). So, let ϕ be an even-dimensional quadratic form and $X = X_\phi$. If ϕ is completely split (i.e., is hyperbolic), the variety X is also cellular (as it was the case with the odd-dimensional forms). So, $\mathrm{CH}^*(X)$ is a free abelian group, and one may choose the generators as follows: h^i for $\mathrm{CH}^i(X)$ with $i \leq \dim X/2$ and l_{n-i} for $\mathrm{CH}^i(X)$ with $i \geq \dim X/2$, where $h \in \mathrm{CH}^1(X)$ is the class of a hyperplane section while $l_i \in \mathrm{CH}^{n-i}(X)$ is the class of an i-dimensional linear subspace lying on X. Note that the "intermediate" group $\mathrm{CH}^r(X)$, where $r = \dim X/2$, has rang two (the other groups have rang 1). Moreover, the generator l_r is not canonical (the other generators are canonical).

It follows that $\mathrm{CH}^*(X\times X)$ is the free abelian group on the pairwise products of the listed above elements. In particular, $\mathrm{CH}^n(X\times X)$ with $n=\dim X$ is freely generated by the elements $h^i\times l_i$ $(i=0,\dots,r)$, $l_i\times h^i$ $(i=r,\dots,0)$, $h^r\times h^r$, and $l_r\times l_r$. We define the type of some $\alpha\in\mathrm{CH}^n(X\times X)$ as the sequence of the coefficients modulo 2 in the representation of α as a linear combination of the generators (in the order given) where the last two coefficients are erased (in other words, we do not care for the coefficients of $h^r\times h^r$ and $l_r\times l_r$).

Now, if the even-dimensional quadratic form ϕ is arbitrary (i.e., not necessarily split), we define the type of $\alpha\in\mathrm{CH}^n(X\times X)$ as the type of $\alpha_{\bar F}$, where $\bar F$ is an algebraic closure of F. As easily seen, the type does not depend on the choice neither of $\bar F$ nor of l_r. To justify our decision to forget two last coefficients, let us notice that the generator $h^r\times h^r$ is always defined over F, while the coefficient of $l_r\times l_r$ is necessarily even in the case of non-hyperbolic ϕ. It is also important that the diagonal class is the sum of all the generators (with coefficients 1) but the last two ones.

Now it is clear that one may define the notion of a type possible for some even-dimensional ϕ in the exactly same way as it was done in §2.3 for odd-dimensional forms (note that the length of a possible type equals now $\dim\phi$, in particular, it is still even). Moreover, all properties of possible types given above remain true.

Since the Rost type is not possible for an even dimensional form, we get the following

Proposition 2.10.1. *Let ϕ be an anisotropic even-dimensional form. Assume that the splitting pattern of ϕ "has no jumps" (i.e., $i_W(\phi_E)$ takes all values between 0 and $\dim\phi/2$ when E varies). Then the diagonal type is minimal for ϕ. In particular, if $\phi\overset{st}{\sim}\psi$, where ψ is some other quadratic form of the same dimension as ϕ, then $\phi\overset{m}{\sim}\psi$.* □

Remark 2.10.2. Since $i_1(\phi)=1$ for ϕ as in Proposition 2.10.1, such a form ϕ can not be stably equivalent to a form of a dimension $<\dim\phi$ (Theorem 1.0.5). One can also show that ϕ can not be stably equivalent to a form of dimension $>\dim\phi$. We do not give a proof for this fact, because we apply Proposition 2.10.1 to the 8-dimensional forms where this fact can be explained by Theorem 1.0.4.

3. Forms of dimension 7

Let ϕ be an anisotropic 7-dimensional quadratic form. In this section we give a complete answer to the problem of determining quadratic forms ψ such that $\phi\overset{st}{\sim}\psi$.

To begin, let us consider the even Clifford algebra $C_0(\phi)$ of the form ϕ. Since this is a central simple algebra of degree 8, the possible values of $i_S(\phi)$

are among 1, 2, 4, and 8. The condition $i_S(\phi) = 1$ is equivalent to the condition that ϕ is a Pfister neighbor; this is a case we do not consider.

Assume that $i_S(\phi) = 2$ and consider the quadratic form $\tau = \phi \perp \langle -\operatorname{disc}(\phi)\rangle$ which is a (unique up to an isomorphism) 8-dimensional quadratic form of trivial discriminant containing ϕ (as a subform). Since the Clifford algebra $C(\tau)$ is Brauer-equivalent with $C_0(\phi)$, we have $i_S(\tau) = i_S(\phi) = 2$. It is now easy to show that τ is anisotropic and $i_1(\tau) = 2$ (see, e.g., [8, th. 4.1] for the second statement). Therefore $\phi \overset{st}{\sim} \tau$, and, taking in account [30], we get

Theorem 3.0.3. *Let ϕ be an anisotropic 7-dimensional quadratic form with $i_S(\phi) = 2$, defined over a field F; let ψ be another quadratic form over F. The relation $\phi \overset{st}{\sim} \psi$ can hold only if $\dim \psi$ is 7 or 8. Moreover,*

- *for $\dim \psi = 7$, $\phi \overset{st}{\sim} \psi$ if and only if $\phi \perp \langle -\operatorname{disc}\phi\rangle \sim \psi \perp \langle -\operatorname{disc}\psi\rangle$;*
- *for $\dim \psi = 8$, $\phi \overset{st}{\sim} \psi$ if and only if $\phi \perp \langle -\operatorname{disc}\phi\rangle \sim \psi$.*

□

Example 3.0.4. For any given anisotropic 7-dimensional form ϕ/F with $i_S(\phi) = 2$, one may find a purely transcendental field extension $\tilde{F}/F$ and some 7-dimensional $\psi/\tilde{F}$ such that $\phi_{\tilde{F}} \overset{st}{\sim} \psi$ but $\phi_{\tilde{F}} \not\sim \psi$. Indeed, we may take as $\psi_{\tilde{F}}$ the "generic 1-codimensional subform" (§1.2) of the 8-dimensional form $\phi \perp \langle -\operatorname{disc}(\phi)\rangle$. Since this 8-dimensional form is not a Pfister neighbor (because its Schur index is 2 and not 1), we have $\phi_{\tilde{F}} \not\sim \psi$ according to Corollary 1.2.2.

It remains to handle the forms ϕ with $i_S(\phi)$ being 4 or 8. The main tool here is the following

Proposition 3.0.5 ([23, cor. 9.11], cf. [38, prop. 5.11(iii)])**.** *The diagonal type is minimal (see §2.3) for any 7-dimensional anisotropic quadratic form ϕ with $i_S(\phi) \geq 4$.*

Remark 3.0.6. The formulation of [23, cor. 9.11] includes one additional hypothesis: ϕ does not contain an Albert subform (that is, the form $\phi \perp \langle -\operatorname{disc}(\phi)\rangle$ is anisotropic). However this hypothesis is included only in order to avoid the use of the general theorem on the Rost types in dimension 7 which was known only in characteristic 0 in that time (see §2.6). Moreover, the proofs of Propositions 3.0.9 and 3.0.10 (generalizing Proposition 3.0.5) we give here are essentially the same as the proof of Proposition 3.0.5 given in [23].

Corollary 3.0.7. *Let ϕ be a 7-dimensional anisotropic quadratic form such that $i_S(\phi) \geq 4$, ψ an arbitrary quadratic form. Then $\phi \overset{st}{\sim} \psi$ if and only if $\phi \sim \psi$.*

Proof. Let ψ be a quadratic form stably equivalent with ϕ, and let us look at the dimension of ψ. We can not have $\dim \psi \leq 6$: one may either refer to the results on stable equivalence of forms of dimension ≤ 6 or to Theorem 1.0.5 and the fact that $i_1(\phi) = 1$.

If $\dim \psi = 7$, it follows from §2.9 and Proposition 3.0.5 that $\psi \sim \phi$.

Finally, if $\dim \psi = 8$, then all 1-codimensional subforms of ψ are similar (to ϕ). Moreover, this is still true over any purely transcendental extension of F. It follows by Corollary 1.2.2 that ψ is similar to a Pfister form, a contradiction. □

Proposition 3.0.5 and Corollary 3.0.7 can be generalized to any odd dimension as follows. We start with a statement concerning every (odd and even) dimension:

Lemma 3.0.8 ([23]). *If ϕ is a quadratic form with maximal $i_S(\phi)$ (i.e., such that the even Clifford algebra $C_0(\phi)$ is a division algebra or, in the case $\phi \in I^2$, a product of two copies of a division algebra), then the Rost type is not possible for ϕ.*

Proof. If the Rost type is possible for ϕ, then by [23, cor. 6.6] the class of a rational point in $K(\bar{X})$ is in the subgroup $K(X) \subset K(\bar{X})$, where $\bar{X}$ is X over an algebraic closure of F, while $K(X)$ is the Grothendieck group (of classes of quasi-coherent X-modules) of X. By the computation of $K(X)$ given in [36], it follows that $i_S(\phi)$ is not maximal, a contradiction. □

Proposition 3.0.9. *Let ϕ be an anisotropic quadratic form of an odd dimension $2n+1$. If $i_S(\phi) = 2^n$ (i.e., $i_S(\phi)$ is maximal), then the diagonal type is minimal for ϕ.*

Proof. First of all let us notice that $i_S(\phi_{F(\phi)}) = 2^{n-1}$. Consequently $i_1(\phi) = 1$, and the Schur index of the form $\big((\phi)_{F(\phi)}\big)_{an}$ is maximal. Therefore we can give a proof using induction on $\dim \phi$ as follows.

Let t be a minimal type (for ϕ) with 1 on the first position. By §2.4 we know that t has 1 on the last position as well. According to Lemma 3.0.8, the reduction (see §2.2) of t is a non-zero type. Moreover, this is a type possible for $(\phi_{F(\phi)})_{an}$. Therefore, by the induction hypothesis, the reduction of t is the diagonal type. It follows that the type t itself is diagonal. □

Proposition 3.0.10. *Let ϕ be an anisotropic quadratic form of an odd dimension $2n+1$ and assume that n is not a power of 2. If $i_S(\phi) = 2^{n-1}$ (i.e., $i_S(\phi)$ is "almost maximal"), then the diagonal type is minimal for ϕ.*

Proof. According to the index reduction formula for odd-dimensional quadrics ([34]), we have $i_S(\phi_{F(\phi)}) = i_S(\phi) = 2^{n-1}$. It follows that $i_1(\phi) = 1$ and that the odd-dimensional quadratic form $(\phi_{F(\phi)})_{an}$ has the maximal Schur index (so that we may apply Proposition 3.0.9 to it).

Let t be a minimal type (for ϕ) with 1 on the first position. We have to show that t is the diagonal type. Since t has 1 on the last position as well, it suffices to show that the reduction of t is diagonal. Since the reduction of t is a type possible for $(\phi_{F(\phi)})_{an}$ it suffices to show that the reduction of t is non-zero, that is, that t itself is not the Rost type. We finish the proof applying the theorem stating that the Rost type is not possible for a quadratic form of dimension different from a power of 2 plus 1, see §2.6. □

Theorem 3.0.11. *Let ϕ be as in Proposition 3.0.9 or as in Proposition 3.0.10. We assume additionally that* $\dim\phi \geq 5$. *Then ϕ is stably equivalent only with the forms similar to ϕ.*

Proof. We almost copy the proof of Corollary 3.0.7.

Let ψ be a quadratic form stably equivalent with ϕ, and let us look at the dimension of ψ. We can not have $\dim\psi < \dim\phi$ because of Theorem 1.0.5 and the fact that $i_1(\phi) = 1$.

If $\dim\psi = \dim\phi$, it follows by §2.9, Propositions 3.0.9, and 3.0.10 that $\psi \sim \phi$.

Finally, if $\dim\psi > \dim\phi$, then ψ is stably equivalent to any subform $\psi_0 \subset \psi$ of dimension $\dim\phi + 1$. Therefore it suffices to consider the case where $\dim\psi = \dim\phi + 1$. In this case all 1-codimensional subforms of ψ are similar (to ϕ). Moreover, this is still true over any purely transcendental extension of F. It follows by Corollary 1.2.2 that ψ is similar to a Pfister form. Therefore ϕ is a Pfister neighbor. However the Schur index $i_S(\phi)$ of a Pfister neighbor of dimension ≥ 5 is never maximal and it can be "almost maximal" only if $\dim\phi$ is a power of 2 plus 1. □

To complete the picture in dimension 7, we find the minimal types for 7-dimensional forms of Schur index 2:

Proposition 3.0.12 (cf. [38, prop. 5.11(ii)])**.** *Let ϕ be an anisotropic 7-dimensional quadratic form with $i_S(\phi) = 2$. Then the minimal types for ϕ are* (101101) *and its complement* (010010).

Proof. Let $t = (t_1t_2t_3t_4t_5t_6)$ be the minimal type with $t_1 = 1$. Since $i_1(\phi) = 1$ (see e.g. [8, th. 4.1]), $t_6 = 1$ as well (§2.4). Since the Rost type is not possible for ϕ (see §2.6; note that ϕ can not contain an Albert form because of $i_S(\phi) = 2$, therefore the Rost type is impossible by a simple reason, see §2.6), the reduction $t_2t_3t_4t_5$ of t is a non-zero type. Moreover, this reduction is a type which is possible for the 5-dimensional quadratic form $(\phi_{F(\phi)})_{an}$. Since $i_S(\phi_{F(\phi)})$ is still 2 ([34]), $t_2t_3t_4t_5$ is either 1111, or 1001, or 0110 (§2.7). So, there are three possibilities for t we have to consider:

(1) $t = (111111)$
(2) $t = (110011)$
(3) $t = (101101)$

In the first case we would be able to prove the following "theorem": for any purely transcendental field extension $\tilde{F}/F$ and for any 7-dimensional quadratic form $\psi/\tilde{F}$ such that $\psi \overset{st}{\sim} \phi_{\tilde{F}}$, one has $\psi \sim \phi_{\tilde{F}}$. This contradicts to Example 3.0.4. Therefore the diagonal type is not minimal for ϕ.

In the second case we would be able to prove the following "theorem": for any purely transcendental field extension $\tilde{F}/F$ and for any 7-dimensional quadratic form $\psi/\tilde{F}$ such that $\psi \overset{st}{\sim} \phi_{\tilde{F}}$, one has $i_W(\psi_E) \geq 2$ for some $E/\tilde{F}$ if and only if $i_S(\phi_{\tilde{F}}) \geq 2$. However for $\tilde{F}$ and $\psi/\tilde{F}$ as in Example 3.0.4, we additionally have

$$i_S(\psi_E) = 3 \Leftrightarrow i_S(\phi \perp \langle -\operatorname{disc}(\phi)\rangle)_E = 4 \Leftrightarrow i_S(\phi_E) = 3 \ .$$

It follows that $\phi_{\tilde{F}} \overset{m}{\sim} \psi$, whereby $\phi_{\tilde{F}} \sim \psi$ (Theorem 1.0.3), a contradiction. Therefore, the second case is not possible as well.

It follows that the only possible case is the third one, i.e., (101101) is a minimal type. Since its complement is evidently minimal as well (having the cardinality 2), we are done. □

The rest of the announcements of [13] concerning the 7-dimensional forms given in [13, th. 3.1] is covered by the following proposition. Note that we use [27] in the proof which is a tool that Izhboldin did not dispose.

Proposition 3.0.13. *Let ϕ be an anisotropic quadratic form of dimension 7 such that $i_S(\phi) \geq 4$. Let ψ be a form such that $\phi_{F(\psi)}$ is isotropic. Then*

1. *if ψ is not a 3-fold Pfister neighbor, then $\dim\psi \leq 7$;*
2. *if $\dim\psi = 7$ and ψ is not a 3-fold Pfister neighbor, then $\psi \sim \phi$;*
3. *if $\dim\psi = 7$ and $i_S(\phi) = 8$, then $\psi \sim \phi$.*

Proof. **1.** First of all, $\dim\psi \leq 8$ by Theorem 1.0.4. Furthermore, if $\dim\psi = 8$ then, since ψ is not a Pfister neighbor, we have $i_1(\psi) \leq 2$. It follows by [27] that $\psi \overset{st}{\sim} \phi$, a contradiction with Corollary 3.0.7.
2. Since $\dim\psi = 7$ and ψ is not a Pfister neighbor, one has $i_1(\psi) = 1$. Therefore $\psi \overset{st}{\sim} \phi$ by [27]. Applying Corollary 3.0.7, we get that $\psi \sim \phi$.
3. Since the form $\phi_{F(\psi)}$ is isotropic, one has $i_S(\phi_{F(\psi)}) < 8 = i_S(\phi)$. By the index reduction formula [34] it follows that $i_S(\psi) = 8$; in particular, ψ can not be a Pfister neighbor and we can apply **2**. □

4. Forms of dimension 8

We do not have a complete answer for the 8-dimensional forms, but the answer we give is almost complete. First we recall what is known.

Let ϕ be an anisotropic 8-dimensional quadratic form. We assume first that $\operatorname{disc}(\phi) = 1$ and we consider the Schur index of ϕ. Since $i_S(\phi) = 1$ if and only if ϕ is a Pfister neighbor (that is, a form similar to a 3-fold Pfister form), we start with the case $i_S(\phi) = 2$. In this case we have: $\phi \overset{st}{\sim} \psi$ for some ψ with $\dim\psi \geq 8$ if and only if $\phi \sim \psi$, [30].

For $i_S(\phi) = 4, 8$ one has: $\phi \overset{st}{\sim} \psi$ if and only if ϕ and ψ are half-neighbors: the case $i_S(\phi) = 4$ is done in [30] while the case $i_S(\phi) = 8$ is done in [31]. Note that ϕ and ψ can be non-similar in each of these two cases, [7, §4].

Now we assume that $\operatorname{disc}(\phi) \neq 1$ and $i_S(\phi) = 1$. Let $d \in F^* \smallsetminus F^{*2}$ be a representative of $\operatorname{disc}(\phi)$. As shown in [6], ϕ is similar to $\pi' \perp \langle d\rangle$ for some 3-fold Pfister form π. Clearly, the form $\pi_{F(\sqrt{d})} \simeq \phi_{F(\sqrt{d})}$ is anisotropic. By [10, lemma 3.5] one has: $\phi \overset{st}{\sim} \psi$ if and only if $\operatorname{disc}\psi = \operatorname{disc}\phi$, $i_S(\psi) = 1$, and the difference $\phi \perp -\psi$ is divisible by $\langle\langle d\rangle\rangle$ (that is, $\phi_{F(\sqrt{d})} \simeq \psi_{F(\sqrt{d})}$).

It follows that the open cases are the cases where $\det\phi \neq 1$ and (in the same time) $i_S(\phi) \geq 2$. In this case, the splitting pattern of ϕ is $\{0,1,2,3,4\}$ ([8, th. 4.1]), i.e., the splitting pattern of ϕ "has no jumps". Therefore we may apply Proposition 2.10.1 which gives us the following

Proposition 4.0.14. *Let ϕ be an anisotropic 8-dimensional quadratic forms of non-trivial discriminant and of Schur index ≥ 2. Then $\phi \overset{st}{\sim} \psi$ for some ψ if and only if $\phi \overset{m}{\sim} \psi$.*

Proof. If $\dim\psi = 8$, then the statement announced is a particular case of Proposition 2.10.1. If $\dim\psi \leq 7$, then the relation $\phi \overset{st}{\sim} \psi$ is not possible by Theorem 1.0.5 (because $i_1(\phi) = 1$; of course one may also refere to the results of previous sections on the stable equivalence of the quadratic forms of dimensions ≤ 7). Finally, $\dim\psi > 8$ is not possible by Theorem 1.0.4. □

Since the condition $\phi \overset{m}{\sim} \psi$ for two 8-dimensional forms ϕ and ψ "almost always" imply that the forms are half-neighbors ([15, th. 11.1]), we get the following

Theorem 4.0.15. *Let ϕ be an anisotropic 8-dimensional quadratic forms of non-trivial discriminant d and of Schur index ≥ 2. In the case where $i_S(\phi) = 4$ we assume additionally that the biquaternion division $F(\sqrt{d})$-algebra, which is Brauer equivalent to $C_0(\phi)$, is defined over F. Then $\phi \overset{st}{\sim} \psi$ for some ψ if and only if ϕ and ψ are half-neighbors.* □

Remark 4.0.16. In the case excluded (i.e., in the case where $\det\phi \neq 1$, $i_S(\phi) = 4$, and the underlying division algebra of $C_0(\phi)$ is not defined over F), we can only prove that $\phi \overset{st}{\sim} \psi \Leftrightarrow \phi \overset{m}{\sim} \psi$. We do not consider this as a final result. A further investigation should be undertaken in order to understand what the condition $\phi \overset{m}{\sim} \psi$ means in this case. Note that $\det\phi = \det\psi$ and $C_0(\phi) \simeq C_0(\psi)$ if $\phi \overset{m}{\sim} \psi$ ([21, lemma 2.6 and rem. 2.7]).

The rest of the announcements of [13] concerning the 8-dimensional forms which are given in [13, th. 3.3] is covered by the following proposition which is an immediate consequence of [27] (note that this is a tool that Izhboldin did not dispose).

Proposition 4.0.17. *Let ϕ be an anisotropic quadratic form of dimension 8. Let ψ be a form of dimension 8 such that the form $\phi_{F(\psi)}$ is isotropic. Suppose also that $i_1(\psi) = 1$ (i.e., $\psi \notin I^2$ or $i_S(\psi) \geq 4$). Then the form $\psi_{F(\phi)}$ is isotropic (and hence $\psi \overset{st}{\sim} \phi$).* □

5. Forms of dimension 9

In this section ϕ is a 9-dimensional quadratic form over F. We describe all quadratic forms ψ/F such that $\phi \overset{st}{\sim} \psi$.

We are going to use the following subdivision of anisotropic 9-dimensional forms ϕ:

kind 1: the forms ϕ which contain a 7-dimensional Pfister neighbor;
kind 2: the forms ϕ containing an 8-dimensional form divisible by a binary form;
kind 3: the rest.

Remark 5.0.18. A form of kind 1 is a 9-dimensional special subform (in the sense of §1.4) while a form of kind 2 is contained in certain 10-dimensional special subform (and is stably equivalent with it).

A form which is simultaneously of kind 1 and of kind 2 (this happens) is a Pfister neighbor (see Proposition 1.4.2(3) or Lemma 1.5.1 (iii)).

Theorem 5.0.19 (Izhboldin, cf. [13, th. 4.6])**.** *Let ϕ_1 and ϕ_2 be anisotropic 9-dimensional quadratic forms each of which is not a Pfister neighbor. The relation $\phi_1 \overset{st}{\sim} \phi_2$ can hold only if ϕ_1 and ϕ_2 are of the same kind. Moreover,*

3. *For ϕ_1 and ϕ_2 of kind 3, $\phi_1 \overset{st}{\sim} \phi_2$ if and only if $\phi_1 \sim \phi_2$.*
1. *For ϕ_1 and ϕ_2 of kind 1, $\phi_1 \overset{st}{\sim} \phi_2$ if and only if $\phi_i \sim \pi_i' \perp \langle u, v\rangle$ for $i = 1, 2$ with some 3-fold Pfister forms π_i and some $u, v \in F^*$ such that the Pfister form $\langle\langle u, v\rangle\rangle$ divides the difference $\pi_1 - \pi_2$ in $W(F)$.*
2. *For ϕ_1 and ϕ_2 of kind 2, let τ_i, $i = 1, 2$, be some 10-dimensional special subform containing ϕ_i. Then $\phi_1 \overset{st}{\sim} \phi_2$ if and only if some 9-dimensional subform of τ_1 is similar to some 9-dimensional subform of τ_2.*

Corollary 5.0.20 (Izhboldin)**.** *Let ϕ be an anisotropic 9-dimensional quadratic form which is not a Pfister neighbor. Let ψ be a quadratic form of dimension $\neq 9$. Then $\phi \overset{st}{\sim} \psi$ is possible only for ϕ of kind 2 and for ψ being a 10-dimensional special subform. Moreover, if τ is a special 10-dimensional subform containing ϕ while ψ is a 10-dimensional special subform as well, then $\phi \overset{st}{\sim} \psi$ if and only if some 9-dimensional subform of τ is similar to some 9-dimensional subform of ψ.*

Proof. The condition $\phi \overset{st}{\sim} \psi$ implies that $9 \leq \dim\psi \leq 16$ (Theorem 1.0.4) and that $i_1(\psi) = \dim\psi - 8$ (Theorem 1.0.5), i.e., the form ψ has the maximal splitting (meaning that the first Witt index has the maximal possible value among the quadratic forms of the same dimension as ψ). In particular, if $\dim\psi \geq 11$, then ψ is a Pfister neighbor, because there are no forms with maximal splitting but Pfister neighbors in dimensions from 11 up to 16, [19] (for a more elementary proof of this statement see [11]). Since ϕ is not a Pfister neighbor, the relation $\phi \overset{st}{\sim} \psi$ therefore implies that $\dim\psi = 10$.

If a 10-dimensional quadratic form ψ has the maximal splitting and is not a Pfister neighbor, then ψ is divisible by a binary form, [16, conj. 0.10]. In this case ψ is also stably equivalent to any 9-dimensional subform $\psi' \subset \psi$. Having $\phi \overset{st}{\sim} \psi'$ and applying Theorem 5.0.19 we get the required result. □

Remark 5.0.21. Let ϕ be an anisotropic 9-dimensional quadratic form. Let ψ be a quadratic form of a dimension ≥ 9. According to [14, th. 0.2], the form $\phi_{F(\psi)}$ is isotropic if and only if $\phi \overset{st}{\sim} \psi$. Therefore Theorem 5.0.19 with Corollary 5.0.20 give a criterion of isotropy of $\phi_{F(\psi)}$.

Proof of Theorem 5.0.19. The proof of the theorem takes the rest of the section. We refer to [13] for the proof that the conditions given in the theorem guaranty that $\phi \overset{st}{\sim} \psi$ (only the case where ϕ and ψ are of kind 1 requires some work; the rest is clear).

The proof that the conditions are necessary starts with the following

Proposition 5.0.22 (Izhboldin, cf. [38, prop. 5.8])**.** *Let ϕ be an anisotropic 9-dimensional quadratic form, and assume that ϕ is not a Pfister neighbor. Here are the minimal types for ϕ depending on the kind (for the kind 3 see Proposition 5.0.24):*

kind 1: (11011011) *and its complement;*
kind 2: (10100101) *and its complement.*

Proof. Let t by the minimal type with $t_1 = 1$. Since $i_1(\phi) = 1$, $t_8 = 1$ as well. Since ϕ is not a Pfister neighbor, the reduction t' of t is a non-zero type (§2.6). Moreover, t' is a type possible for the 7-dimensional form $\phi' = (\phi_{F(\phi)})_{an}$. Since $i_S(\phi') = i_S(\phi) = 2$ ([34]), we may apply Proposition 3.0.12 to ϕ' and conclude that t' is ether (101101), or (010010), or (111111). According to this, t is one of the following three types: (11011011), (10100101), or (11111111).

Let us assume that ϕ is of the first kind. By the reason of Corollary 6.2.2, the diagonal type can not be minimal for such ϕ. Assume that the second possibility for t takes place. Then we get the following "theorem": for any unirational field extension L/F and any 9-dimensional ψ/L with $\phi_L \overset{st}{\sim} \psi$ one has $i_W(\phi_E) \geq 3$ for some field extension E/L if and only if $i_W(\psi_E) \geq 3$. This contradicts however to Lemma 6.2.1. Therefore $t = (11011011)$ for ϕ of kind 1.

Now we assume that ϕ is of the second kind. By the reason of Proposition 6.1.1, the diagonal type can not be minimal for such ϕ. Assume that the first possibility for t takes place. Then we get the following "theorem": for purely transcendental field extension $\tilde{F}/F$, any 9-dimensional $\psi/\tilde{F}$ with $\phi_{\tilde{F}} \overset{st}{\sim} \psi$ and for $n = 2, 4$, one has $i_W(\phi_E) \geq n$ for some field extension $E/\tilde{F}$ if and only if $i_W(\psi_E) \geq n$. However for $\tilde{F}$ and ψ as in Proposition 6.1.1 we evidently have as well

$$i_W(\phi_E) \geq 3 \Leftrightarrow i_W(\tau_E) \geq 3 \Leftrightarrow i_W(\tau_E) \geq 4 \Leftrightarrow i_W(\psi_E) \geq 3\ .$$

It follows that $\phi_{\tilde{F}} \overset{m}{\sim} \psi$, whereby $\phi_{\tilde{F}} \sim \psi$, a contradiction. Therefore $t = (10100101)$ for ϕ of kind 2. □

Corollary 5.0.23. *A 9-dimensional and a 10-dimensional anisotropic special subforms are never stably equivalent.* □

Proposition 5.0.24. *Let ϕ be a 9-dimensional anisotropic form of kind 3, not a Pfister neighbor. Then the diagonal type is minimal for ϕ.*

Proof. Since ϕ is not a Pfister neighbor, we have $i_S(\phi) \geq 2$. If $i_S(\phi) \geq 4$, then the diagonal type is minimal for ϕ by [23, cor. 9.14]. So, we assume that $i_S(\phi) = 2$ in the rest of the proof.

Let t by the minimal type with $t_1 = 1$. As in the proof of Proposition 5.0.22, we show that t is either (11011011), or (10100101), or (11111111).

Let μ and λ be respectively the 10-dimensional and the 12-dimensional special forms containing ϕ (see §1.5). Over the function field $F(\lambda)$, the form $\mu_{F(\lambda)}$ is anisotropic ([16, th. 10.6]). Besides $\phi_{F(\lambda)}$ is a special subform of the special form $\mu_{F(\lambda)}$ (Lemma 1.5.1). It follows that the form $\phi_{F(\lambda)}$ is an anisotropic 9-dimensional form of kind 1 and is not a Pfister neighbor. We conclude that the type (10100101) is not possible for ϕ.

On the other hand, over the function field $F(\mu)$, the form $\lambda_{F(\mu)}$ is anisotropic (Proposition 7.1.3). Let τ be any 10-dimensional subform of λ containing ϕ. Besides $\tau_{F(\mu)}$ is a special subform of the special form $\lambda_{F(\mu)}$ (Lemma 1.5.1). ϕ is of kind 2 and still not a Pfister neighbor

So, we conclude that the type (11011011) is also not possible. The only remaining possibility is $t = (11111111)$. □

Corollary 5.0.25. *Let ϕ and ψ be anisotropic 9-dimensional quadratic forms, not Pfister neighbors. If $\phi \overset{st}{\sim} \psi$, then ϕ and ψ are of the same kind. Moreover, if the kind is 3, then $\phi \overset{st}{\sim} \psi$ is possible only if $\phi \sim \psi$.* □

5.1. **Stable equivalence for forms of kind 1.** For a 9-dimensional form ϕ of kind 1, we write μ_ϕ for the 10-dimensional special form $\phi \perp \langle -\operatorname{disc}(\phi)\rangle$ (so that ϕ, μ_ϕ is a special pair).

Let ϕ and ψ be 9-dimensional quadratic forms of kind 1 each of which is not a Pfister neighbor. We first prove

Proposition 5.1.1. *If $\phi \overset{st}{\sim} \psi$, then $\mu_\phi \sim \mu_\psi$.*

To prove this, we need

Lemma 5.1.2. *Let n be 2 or 4. If $\phi \overset{st}{\sim} \psi$, then for any field extension E/F one has:*

$$i_W(\phi_E) \geq n \quad \Leftrightarrow \quad i_W(\psi_E) \geq n \ .$$

Proof. Follows from the fact that the type 11011011 is minimal for ϕ (Proposition 5.0.22) as explained in §2.8. □

Proof of Proposition 5.1.1. Assuming that $\phi \overset{st}{\sim} \psi$, let us check that $\mu_\phi \overset{m}{\sim} \mu_\psi$, i.e., $i_W(\mu_\phi)_E \geq n \Leftrightarrow i_W(\mu_\psi)_E \geq n$ for any E/F and any $n \in \mathbb{Z}$. Since the possible values of $i_W(\mu_\phi)_E$ and $i_W(\mu_\psi)$ are 1, 3, and 5 (see, e.g., [8, th. 5.1]), it is enough to check the equivalence desired only for $n = 1, 3, 5$. The case $n = 1$ is served since $\phi \overset{st}{\sim} \psi \Rightarrow \mu_\phi \overset{st}{\sim} \mu_\psi$ by Corollary 1.4.3.

For $n = 3, 5$, one has

$$\begin{aligned} i_W(\mu_\phi)_E \geq n \quad \Rightarrow \quad i_W(\phi_E) \geq n-1 \quad & \overset{\text{Lemma 5.1.2}}{\Longrightarrow} \\ i_W(\psi_E) \geq n-1 \quad \Rightarrow \quad i_W(\mu_\psi)_E \geq n-1 \quad & \Rightarrow \quad i_W(\mu_\psi)_E \geq n \ . \end{aligned}$$

By symmetry, the converse holds as well.

We have shown that $\mu_\phi \overset{m}{\sim} \mu_\psi$. It follows that $\mu_\phi \sim \mu_\psi$ according to

Lemma 5.1.3. *Let π_1, π_2 be some 3-fold Pfister forms, and let τ_1, τ_2 be some 2-fold Pfister forms such that the 10-dimensional special forms $\mu_1 = \pi_1' \perp -\tau_1'$ and $\mu_2 = \pi_2' \perp -\tau_2'$ are anisotropic. The statements (1)–(5) are equivalent:*

(1) $\mu_1 \overset{m}{\sim} \mu_2$;

(2) (i) $\mu_1 \overset{st}{\sim} \mu_2$,
  (ii) $c(\mu_1) = c(\mu_2) \in \mathrm{Br}(F)$, *that is,* $\mu_1 \equiv \mu_2 \mod I^3(F)$ *in* $W(F)$;
  (iii) $(\mu_1)_{F(C)} \equiv (\mu_2)_{F(C)} \mod I^4(F)$ *in* $W(F(C))$, *where* C/F *is a Severi-Brauer variety corresponding to the element of (2-ii);*

(3) *the elements* τ_1 *and* τ_2 *of* $W(F)$ *coincide and divide the difference* $\pi_1 - \pi_2$;

(4) *for some* $u, v, a_1, a_2, b, c, k \in F^*$ *there are isomorphisms*
  (i) $\tau_1 \simeq \langle\langle u, v\rangle\rangle \simeq \tau_2$,
  (ii) $\pi_1 \simeq \langle\langle a_1, b, c\rangle\rangle$, $\pi_2 \simeq \langle\langle a_2, b, c\rangle\rangle$,
  (iii) $\langle\langle a_1 a_2, b, c\rangle\rangle \simeq \langle\langle k, u, v\rangle\rangle$;

(5) $\mu_1 \sim \mu_2$.

Remark 5.1.4. A stronger as Lemma 5.1.3 statement on the 10-dimensional special forms will be given in Proposition 7.1.3: $\mu_1 \sim \mu_2$ already if $\mu_1 \overset{st}{\sim} \mu_2$.

Proof of Lemma 5.1.3. We prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1).

(1) ⇒ (2). The property (2-i) constitutes a part of the definition of the property (1); (2-ii) follows from (1) by [21, rem. 2.7]. As to (2-iii), in the Witt ring of $F(C)$ we have $(\mu_1)_{F(C)} = (\pi_1)_{F(C)}$ and $(\mu_2)_{F(C)} = (\pi_2)_{F(C)}$. Therefore the Pfister forms $(\pi_1)_{F(C)}$ and $(\pi_2)_{F(C)}$ are stably equivalent, whereby $(\pi_1)_{F(C)} = (\pi_2)_{F(C)} \in W(F(C))$.

(2) ⇒ (3). Since $c(\mu_i) = c(\tau_i)$ for $i = 1, 2$, (2-ii) gives $c(\tau_1) = c(\tau_2)$ whereby $\tau_1 = \tau_2$ (because τ_i are 2-fold Pfister forms). Let τ be a quadratic form isomorphic to τ_1 and τ_2. Since $F(C) \simeq_F F(\tau')$ for C as in (2-iii), $(\pi_1)_{F(\tau)} \equiv (\pi_2)_{F(\tau)} \mod I^4(F)$ in $W(F(\tau))$. It follows that $(\pi_1)_{F(\tau)} = (\pi_2)_{F(\tau)} \in W(F(\tau))$ and therefore the difference $\pi_1 - \pi_2$ is divisible by τ in $W(F)$ ([28, lemma 4.4]).

(3) ⇒ (4). Since τ_1 and τ_2 are isomorphic 2-fold Pfister forms, we may find $u, v \in F^*$ satisfying (4-i).

Since the Witt class of $\langle\langle u, v\rangle\rangle$ divides the difference $\pi_1 - \pi_2$, the 3-fold Pfister forms π_1 and π_2 are *2-linked* (or, simply, *linked*), that is, divisible by a common 2-fold Pfister forms (Lemma 1.3.1). So, we may find a_1, a_2, b, c satisfying the condition (4-ii). Now, the difference $\pi_1 - \pi_2$ is represented by a quadratic form similar to the 3-fold Pfister form $\langle\langle a_1 a_2, b, c\rangle\rangle$. Since this 3-fold Pfister form is divisible by $\langle\langle u, v\rangle\rangle$, we may find $k \in F^*$ satisfying (4-iii).

(4) $\Rightarrow$ (5).[1] We write τ for $\langle\langle u, v\rangle\rangle$. Let us consider the difference $\gamma = \phi_1 - k\phi_2 \in W(F)$ with k from (4-iii). If $\gamma = 0$ then $\phi_1 \simeq k\phi_2$ and we are done. So, we assume that $\gamma \neq 0$. We have:

$$\gamma = (\pi_1 - \tau) - k(\pi_2 - \tau) = (\pi_1 - k\pi_2) - \langle\langle k\rangle\rangle\,\tau \equiv$$
$$(\pi_1 - \pi_2) - \langle\langle k\rangle\rangle\,\tau \equiv 0 \mod I^4(F)\ .$$

So, $\gamma \in I^4(F)$. Since the element $\gamma_{F(\pi_1)}$ can be evidently represented by a quadratic form of dimension < 16, the Arason-Pfister-Hauptsatz tells that $\gamma_{F(\pi_1)} = 0$, whereby π_1 divides γ in $W(F)$ ([28, lemma 4.4]). In particular, $\gamma \equiv \langle\langle s\rangle\rangle\,\pi_1 \mod I^5(F)$ for some $s \in F^*$. Having

$$\langle\langle s\rangle\rangle\,\pi_1 \equiv \gamma = (\pi_1 - \tau) - k\phi_2 \mod I^5(F)\ ,$$

we get

$$0 \equiv (s\pi_1 - \tau) - k\phi_2 \mod I^5(F)\ .$$

By the Arason-Pfister-Hauptsatz, this congruence turns out to be an equality, i.e., $(s\pi_1 - \tau) = k\phi_2$. In particular, the quadratic form $s\pi_1 \perp -\tau$ is isotropic. It follows (Elman-Lam, see [16, th. 8.1(1)]) that the anisotropic part of the form $s\pi_1 \perp -\tau$ is similar to $(\pi_1 \perp -\tau)_{an} = \phi_1$. Therefore $\phi_1 \sim \phi_2$.

(5) $\Rightarrow$ (1). This implication is trivial. □

We have checked the implication $\phi \overset{st}{\sim} \psi \Rightarrow \mu_\phi \sim \mu_\psi$. The proof of Proposition 5.1.1 is therefore finished. □

Lemma 5.1.5. *Let ϕ_1 and ϕ_2 be 9-dimensional quadratic forms of kind 1 each of which containing the pure subform of some (common) 3-fold Pfister form π. If $\phi_1 \overset{st}{\sim} \phi_2$, then $\phi_1 \sim \phi_2$.*

Proof. Using the hypothesis, we write ϕ_i (for $i = 1, 2$) as $\phi_i \simeq \pi' \perp \beta_i$, where β_1 and β_2 are some binary forms. Since the forms

$$\beta_1 \perp \langle -\det(\beta_1)\rangle \quad \text{and} \quad \beta_2 \perp \langle -\det(\beta_2)\rangle$$

are isomorphic (Proposition 5.1.1 with Lemma 5.1.3(3)), we can find some $u, v_1, v_2 \in F^*$ such that $\beta_i \simeq \langle u, v_i\rangle$.

Let μ be a 10-dimensional form isomorphic to $\phi_i \perp \langle -\operatorname{disc}(\phi_i)\rangle$ and let τ be a 2-fold Pfister form isomorphic to $\langle\langle u, v_i\rangle\rangle$. Since the form $\mu \simeq \pi' \perp -\tau'$ becomes isotropic over the function field $F(\mu)$, the forms π' and μ' over $F(\mu)$ have a common value d. Therefore, $\langle\langle d\rangle\rangle$ is a common divisor of π and μ over $F(\mu)$. Let $k \in F(\mu)^*$ be such that $\tau \simeq \langle\langle d, k\rangle\rangle$ over $F(\mu)$. Then $(\phi_i)_{F(\mu)}$ is a neighbor of the 4-fold Pfister form $\pi\,\langle\langle -uv_ik\rangle\rangle$. Since $(\phi_1)_{F(\mu)} \overset{st}{\sim} (\phi_2)_{F(\mu)}$, the Pfister forms $\pi\,\langle\langle -uv_1k\rangle\rangle$ and $\pi\,\langle\langle -uv_2k\rangle\rangle$ are isomorphic, i.e., $\pi\,\langle\langle v_1v_2\rangle\rangle = 0 \in W(F(\mu))$. Since μ is not a Pfister neighbor, it follows that $\pi\,\langle\langle v_1v_2\rangle\rangle = 0$ already in $W(F)$,

[1] A proof of this implication was found in the hand-written private notes of Oleg Izhboldin; we reproduce it here almost word by word.

that is, $v_1v_2 \in G(\pi)$. We note additionally that the relation $\langle\langle u, v_1\rangle\rangle = \langle\langle u, v_2\rangle\rangle$ implies that $v_1v_2 \in G(\langle\langle u\rangle\rangle)$. Now we get:

$$v_1v_2\phi_1 = v_1v_2(\pi - \langle\langle u\rangle\rangle + \langle v_1\rangle) = \pi - \langle\langle u\rangle\rangle + \langle v_2\rangle = \phi_2 \in W(F) ,$$

thereafter, ϕ_1 is similar to ϕ_2. □

Corollary 5.1.6. *Let ϕ_1 and ϕ_2 be 9-dimensional quadratic forms of kind 1 and assume that $\phi_1 \overset{st}{\sim} \phi_2$. Then there exists some linked 3-fold Pfister forms π_1 and π_2 and a binary form $\langle u, v\rangle$ such that $\phi_1 \sim \pi_1' \perp \langle u, v\rangle$, $\phi_2 \sim \pi_2' \perp \langle u, v\rangle$, and the difference $\pi_1 - \pi_2 \in W(F)$ is divisible by the 2-fold Pfister form $\langle\langle u, v\rangle\rangle$.*

Proof. By the definition of the first kind, up to similarity, we can write ϕ_1 and ϕ_2 as $\phi_i = \pi_i' \perp \langle u_i, v_i\rangle$ with some 3-fold Pfister forms π and some $u_i, v_i \in F^*$. We assume that $\phi_1 \overset{st}{\sim} \phi_2$. Then the difference $\pi_1 - \pi_2$ is divisible by $\langle\langle u_1, v_1\rangle\rangle$ according to Proposition 5.1.1 and Lemma 5.1.3. Let us consider the quadratic form $\phi_3 = \pi_2' \perp \langle u_1, v_1\rangle$. By [13, example 4.4] we have $\phi_1 \overset{st}{\sim} \phi_3$. It follows that $\phi_2 \overset{st}{\sim} \phi_3$. Applying Lemma 5.1.5 to the forms ϕ_2 and ϕ_3, we get that $\phi_2 \sim \phi_3$. Therefore, we may take $u = u_1$ and $v = v_1$. □

We have finished the proof of Theorem 5.0.19 for the 9-dimensional quadratic forms of kind 1.

5.2. **Stable equivalence for forms of kind 2.** The only thing to check here is the following

Proposition 5.2.1. *Let τ_1 and τ_2 be anisotropic 10-dimensional quadratic special subforms (see §1.4). We assume that neither τ_1 nor τ_2 are Pfister neighbors. Then $\tau_1 \overset{st}{\sim} \tau_2$ if and only if some 9-dimensional subform of τ_1 is similar with some 9-dimensional subform of τ_2.*

Proof. The "if" part of the statement is evident. We are going to prove the "only if" part.

For $i = 1, 2$, let ρ_i be a 12-dimensional special form containing τ_i. Let us choose some 11-dimensional form δ_i such that $\tau_i \subset \delta_i \subset \rho_i$. It is enough to show that $\delta_1 \sim \delta_2$ and we are going to do this.

According to [12], it suffices to check that $\delta_1 \overset{m}{\sim} \delta_2$, that is,

$$(*) \qquad i_W(\delta_1)_E \geq n \quad \Leftrightarrow i_W(\delta_2)_E \geq n$$

for any E/F and any integer n. Since the possible positive values of $i_W(\delta_i)_E$ are 1, 2, and 5 (see, e.g., [8, th. 5.4(ii)]), the relation $(*)$ has to be checked only for $n = 1, 2, 5$.

First of all, to handle the case of $n = 1$, let us check that $\delta_1 \overset{st}{\sim} \delta_2$. The condition $\tau_1 \overset{st}{\sim} \tau_2$ implies $\rho_1 \overset{st}{\sim} \rho_2$ by Corollary 1.4.3. Besides, since $i_1(\rho_i) = 2$, we have $\delta_i \overset{st}{\sim} \rho_i$ whereby the forms δ_1 and δ_2 are stably equivalent, indeed.

For $n = 2$ we have:

$$i_W(\delta_1)_E \geq 2 \;\Rightarrow\; i_W(\tau_1)_E \geq 1 \overset{\tau_1 \overset{st}{\sim} \tau_2}{\Longrightarrow}$$
$$i_W(\tau_2)_E \geq 1 \overset{i_1(\tau_2)=2}{\Longrightarrow} i_W(\tau_2)_E \geq 2 \;\Rightarrow\; i_W(\delta_2)_E \geq 2\,.$$

By the symmetry, $i_W(\delta_2)_E \geq 2 \Rightarrow\; i_W(\delta_1)_E \geq 2$ as well.

Finally, to handle the case $n = 5$, let us choose some 9-dimensional subforms $\phi_1 \subset \tau_1$ and $\phi_2 \subset \tau_2$. Since the quadratic forms ϕ_1 and ϕ_2 are of the 2nd kind and stably equivalent, it follows from Proposition 5.0.22 that $i_W(\phi_1)_E \geq 3$ if and only if $i_W(\phi_2)_E \geq 3$. Now we have:

$$i_W(\delta_1)_E = 5 \;\Rightarrow\; i_W(\phi_1)_E \geq 3 \;\Rightarrow$$
$$i_W(\phi_2)_E \geq 3 \;\Rightarrow\; i_W(\delta_2)_E \geq 3 \;\Rightarrow\; i_W(\delta_2)_E = 5$$

and $i_W(\delta_2)_E = 5 \Rightarrow\; i_W(\delta_1)_E = 5$ by the symmetry,. □

The proof of Theorem 5.0.19 is finished. □

The following corollary will be used in the proof of Theorem 0.0.2.

Corollary 5.2.2. *Let τ_1, ρ_1 and τ_2, ρ_2 be anisotropic special pairs with $\dim \tau_1 = \dim \tau_2 = 10$ (and $\dim \rho_1 = \dim \rho_2 = 12$). Let δ_1 and δ_2 be some 11-dimensional "intermediate" forms: $\tau_1 \subset \delta_1 \subset \rho_1$ and $\tau_2 \subset \delta_2 \subset \rho_2$. If $\tau_1 \overset{st}{\sim} \tau_2$, then $\delta_1 \sim \delta_2$ and $\rho_1 \sim \rho_2$.*

Proof. The relation $\delta_1 \sim \delta_2$ is checked in the proof of Proposition 5.2.1. It implies the relation $\rho_1 \sim \rho_2$ because $\rho_i \simeq \delta_i \perp \langle -\operatorname{disc}(\delta_i)\rangle$. □

6. Examples of non-similar stably equivalent 9-dimensional forms

The examples constructed in this section are good not only on its own: they also work in the proof of Proposition 5.0.22.

6.1. **Forms of kind 2.** For any given anisotropic 9-dimensional quadratic form ϕ of kind 2, we get another 9-dimensional form ψ (over a purely transcendental extension of the base field) such that $\psi \not\sim \phi$ while $\psi \overset{st}{\sim} \psi$ as follows:

Proposition 6.1.1. *Let ϕ be a 9-dimensional anisotropic form of kind 2. Let τ be a 10-dimensional special subform containing ϕ. Then there exists a purely transcendental field extension $\tilde{F}/F$ and a a 9-dimensional subform $\psi \subset \tau_{\tilde{F}}$ such that $\phi_{\tilde{F}} \overset{st}{\sim} \psi$ while $\phi_{\tilde{F}} \not\sim \psi$.*

Proof. Since $\phi \overset{st}{\sim} \tau$, the form τ is anisotropic. Since the dimension of τ is not a power of 2, τ is not a Pfister form. To finish, we apply Corollary 1.2.2 and use the fact that any two 1-codimensional subform of τ (or of $\tau_{\tilde{F}}$) are stably equivalent. □

6.2. **Forms of kind 1.** Let ϕ/F be an arbitrary 9-dimensional anisotropic quadratic form of the first kind, say, $\phi \simeq \langle\langle a,b,c\rangle\rangle' \perp \langle u,v\rangle$ with some $a,b,c,u,v \in F^*$. We assume that the 10-dimensional special form $\langle\langle a,b,c\rangle\rangle' \perp - \langle\langle u,v\rangle\rangle'$ is anisotropic (i.e., that ϕ is not a Pfister neighbor). Let us construct a new quadratic form over certain field extension of F as follows.

We consider a degree 2 purely transcendental extension $F(t,z)/F$ and the quadratic form $\psi = \langle\langle t,b,c\rangle\rangle' \perp \langle u,v\rangle$ over $F(t,z)$. Let $L/F(t,z)$ be the top of the generic splitting tower of the quadratic $F(t,z)$-form $\langle\langle at,b,c\rangle\rangle \perp - \langle\langle z,u,v\rangle\rangle$. We state that the data obtained this way have the following properties:

Lemma 6.2.1. (1) *the field extension L/F is unirational;*
(2) *the forms ϕ_L and ψ_L are stably equivalent;*
(3) *the forms ϕ_L and ψ_L are not similar;*
(4) *there exists a field extension E/L such that $i_W(\psi_E) \geq 3$ while $i_W(\phi_E) \leq 2$.*

Proof. 1. Over the field $F(\sqrt{at},\sqrt{z})$, the Pfister forms $\langle\langle at,b,c\rangle\rangle$ and $\langle\langle z,u,v\rangle\rangle$ are split. Therefore the filed extension $L(\sqrt{at},\sqrt{z})/F(\sqrt{at},\sqrt{z})$ is purely transcendental. Since the extension $F(\sqrt{at},\sqrt{z})/F$ is also purely transcendental, it follows that the extension $L(\sqrt{at},\sqrt{z})/F$ is purely transcendental and thereafter L/F is unirational.
2. According to the definition of L, the form $\langle\langle at,b,c\rangle\rangle_L$ is divisible by $\langle\langle u,v\rangle\rangle_L$. So, $\phi_L \overset{st}{\sim} \psi_L$ by [13, example 4.4].
3. Follows from 4.
4. We take $E = L(\langle\langle t,b,c\rangle\rangle)$. Since the form $\langle\langle t,b,c\rangle\rangle$ splits over E, the Witt index of $(\langle\langle t,b,c\rangle\rangle')_E$ is 3. Therefore $i_W(\psi_E) \geq 3$.

To see that $i_W(\phi_E) \leq 2$, it suffices to check that the form $\langle\langle a,b,c\rangle\rangle_E$ is anisotropic. We will check that this form is still anisotropic over a bigger extension, namely, over the field $E(\sqrt{t})$. For this we decompose the field extension $E(\sqrt{t})/F$ in a tower as follows:

$$F \subset F(\sqrt{t},z) \subset K \subset L' \cdot_F K \subset L \cdot_F K$$

where $K = F(\sqrt{t},z)(\langle\langle t,b,c\rangle\rangle)$ and where the field L', sitting between $F(t,z)$ and L, is the almost biggest field in the generic splitting tower of $\langle\langle at,b,c\rangle\rangle \perp - \langle\langle z,u,v\rangle\rangle$. Recall that L is the top of this tower and therefore $L = L'(\pi)$ where π/L' is a Pfister form similar with $\big((\langle\langle at,b,c\rangle\rangle \perp - \langle\langle z,u,v\rangle\rangle)_{L'}\big)_{an}$.

Since the extension K/F is purely transcendental (note that $\langle\langle t,b,c\rangle\rangle_{F(\sqrt{t},z)}$ is hyperbolic), the form $\langle\langle a,b,c\rangle\rangle_K$ is anisotropic. Since the extension $(L'\cdot K)/K$ is a tower of the function fields of some quadratic forms of dimension > 8, the form $\langle\langle a,b,c\rangle\rangle_{L'\cdot K}$ is still anisotropic (Theorem 1.0.4). In this situation the hyperbolicity of this form over $L \cdot K$ would mean that $\langle\langle a,b,c\rangle\rangle_{L'\cdot K} = \pi \in W(L'\cdot K)$. Since $\pi = \langle\langle at,b,c\rangle\rangle - \langle\langle z,u,v\rangle\rangle = \langle\langle a,b,c\rangle\rangle - \langle\langle z,u,v\rangle\rangle$, this would give hyperbolicity of $\langle\langle z,u,v\rangle\rangle_{L'\cdot K}$. However, the latter form is anisotropic by the reasons similar to those we have given already: the field extension $K/F(z)$ is purely transcendental (note that $\langle\langle z,u,v\rangle\rangle$ is defined over $F(z)$ and is of course anisotropic over $F(z)$ because $\langle\langle u, v\rangle\rangle$ is anisotropic over F) while the field extension $L' \cdot K/K$ is a tower of the function fields of some forms of dimensions > 8. $\square$

In particular, we get

Corollary 6.2.2. *Let ϕ/F be an anisotropic 9-dimensional quadratic form of the first kind. Then there exists a unirational field extension L/F and a 9-dimensional quadratic form ψ/L which is in the same time stably equivalent and non-similar to ϕ_L.* $\square$

7. Other related results

7.1. **Isotropy of special forms.**

Theorem 7.1.1 (Izhboldin)**.** *Let ϕ be an anisotropic special quadratic form and let ψ be a quadratic form of dimension ≥ 9. Then $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ.*

The proof will be given after certain preliminary observations.

Lemma 7.1.2. *If ϕ_0 is an anisotropic special subform while ψ is a special form, then the form $(\phi_0)_{F(\psi)}$ is anisotropic.*

Proof. We assume that the form $(\phi_0)_{F(\psi)}$ is isotropic (in particular, the form ψ is anisotropic). We have $\dim \phi_0 = 9$ or 10. Let $\phi_1 \subset \phi_0$ be a 9-dimensional subform of ϕ_0 (in the case $\dim \phi_0 = 9$ we set $\phi_1 = \phi_0$). We have $\phi_0 \overset{st}{\sim} \phi_1$ and therefore the form $(\phi_1)_{F(\psi)}$ is isotropic. Consequently $\phi_1 \overset{st}{\sim} \psi$ by Theorem [14, th. 0.2]. It follows that the form ψ has the maximal splitting. However this is not possible because ψ is special (and therefore $i_1(\psi) = 1$ for a 10-dimensional ψ while $i_1(\psi) = 2$ for a 12-dimensional ψ). $\square$

Proposition 7.1.3. *Let ϕ and ψ be special anisotropic quadratic forms. If the form $\phi_{F(\psi)}$ is isotropic, then the forms ϕ and ψ are similar.*

Proof. We can choose some subforms $\phi_0 \subset \phi$ and $\psi_0 \subset \psi$ such that ϕ_0, ϕ and ψ_0, ψ are anisotropic special pairs. Let E/F be the extension constructed in [16, prop. 6.10]. We recall that this extension is obtained as the union of an infinite tower of fields where each step is either an odd extension or the function field of some 4-fold Pfister form. By [16, lemma 10.1(1)] the special pairs $(\phi_0)_E, \phi_E$ and $(\psi_0)_E, \psi_E$ are still anisotropic. Since the form $\phi_{E(\psi)}$ is isotropic, the form $(\phi_0)_{E(\psi)}$ is a 4-fold Pfister neighbor (Proposition 1.4.2 (3)). Moreover, in view of Lemma 7.1.2 this 4-fold Pfister neighbor is anisotropic. By the same reason or by Proposition 1.4.2 (4), the form $(\psi_0)_{E(\psi)}$ is also an anisotropic 4-fold Pfister neighbor. By [16, lemma 6.7] we have $(\psi_0)_{E(\psi)} \overset{st}{\sim} (\phi_0)_{E(\psi)}$. Hence $(\phi_0)_{E(\psi,\psi_0)}$ is isotropic. Since $\psi_0 \subset \psi$, the form $(\phi_0)_{E(\psi_0)}$ is already isotropic. By [16, prop. 8.13], it follows that $(\phi_0)_E \overset{st}{\sim} (\psi_0)_E$. By Corollary 5.0.23, it follows that $\dim \phi_0 = \dim \psi_0$ and $\dim \phi = \dim \psi$. In the case where $\dim \phi_0 = \dim \psi_0 = 10$, that is, $\dim \phi = \dim \psi = 12$, we get that $\phi_E \sim \psi_E$ applying Corollary 5.2.2. In particular, $\phi_E \equiv \psi_E \mod I^4(E)$. It follows by [16, prop. 6.10, $n = 4$] that $\phi \equiv \psi \mod I^4(F)$. Therefore $\phi \sim \psi$ by [9, cor.].

In the case where $\dim \phi_0 = \dim \psi_0 = 9$, that is, $\dim \phi = \dim \psi = 10$, we get that $\phi_E \sim \psi_E$ by Proposition 5.1.1. In particular, $\phi_E \overset{st}{\sim} \psi_E$, $c(\phi_E) = c(\psi_E)$, and $\phi_{E(C)} = \psi_{E(C)} \in W(E(C))$ for C as in (2-iii) of Lemma 5.1.3. These three relations can be descended to F: the first one implies $\phi \overset{st}{\sim} \psi$ according to [16, lemma 10.1(2)]; the second one implies $c(\phi) = c(\psi)$ by [16, prop. 6.10(v), $n = 3$], while the third one gives $\phi_{F(C)} = \psi_{F(C)} \in W(F(C))$ according to the construction of E/F and [16, cor. 4.5, $n = 4$] with [16, lemma 1.2, odd extensions]. We have got the condition (2) of Lemma 5.1.3. Hence $\phi \sim \psi$. □

Lemma 7.1.4. *Let ϕ_0, ϕ be an anisotropic special pair and let ψ be a quadratic form with* $\dim \psi \geq 9$. *Let E/F be the field extension constructed in* [16, prop. 6.10]. *If the form $(\phi_0)_{E(\psi)}$ is isotropic, then ψ is similar to a subform of ϕ.*

Proof. Note that the forms $(\phi_0)_E, \phi_E$ are anisotropic by [16, lemma 10.1(1)]. We have $\dim \phi_0 = 9$ or 10. We consider first the case with $\dim \phi_0 = 9$. The isotropy of $(\phi_0)_{E(\psi)}$ implies the condition $(\phi_0)_E \overset{st}{\sim} \psi_E$ ([14, th. 0.2]). Moreover, since $(\phi_0)_E$ is a 9-dimensional form of the 1-st kind, ψ_E is 9-dimensional of the 1st kind as well (Theorem 5.0.19) and the forms $\phi_E = (\phi_0 \perp \langle -\operatorname{disc}(\phi_0)\rangle)_E$ and $(\psi \perp \langle -\operatorname{disc}(\psi)\rangle)_E$ are similar (Proposition 5.1.1). It follows by [16, lemma 10.1(2)] that the special forms ϕ and $\psi \perp \langle -\operatorname{disc}(\psi)\rangle$ are stably equivalent. Therefore these two forms are similar (Proposition 7.1.3), and we see that ψ is similar to a subform of ϕ in this case.

It remains to consider the case where $\dim \phi_0 = 10$. Note that any 9-dimensional subform $\phi_1 \subset (\phi_0)_E$ is of the 2nd kind and stably equivalent to $(\phi_0)_E$. Therefore, by Theorem 5.0.19 and Corollary 5.0.20, ψ_E is contained in a 10-dimensional special subform. It follows that ψ considered over F is also contained in a 10-dimensional special subform τ (in the case $\dim \psi = 10$ we simply take $\tau = \psi_E$). Moreover, τ_E is stably equivalent with $(\phi_0)_E$ (Corollary 5.0.20). Applying Corollary 5.2.2, we get that $\phi_E \sim \rho_E$ where ρ is the 12-dimensional special F-form containing τ. It follows by [16, lemma 10.1(2)] that the special forms ϕ and ρ are stably equivalent. Therefore these two forms are similar (Proposition 7.1.3), and we see that ψ is similar to a subform of ϕ in this case as well. □

Lemma 7.1.5. *Let F be a field such that $H^4(F) = 0$ (the degree 4 Galois cohomology group of F with coefficients $\mathbb{Z}/2$ is 0). Let ϕ_0, ϕ be a degree 4 anisotropic special pair over F and let ψ/F be a quadratic form of dimension ≥ 9. If the form $\phi_{F(\psi)}$ is isotropic while the form $(\phi_0)_{F(\psi)}$ is anisotropic, then* $\operatorname{Tors} \operatorname{CH}^3(X_\psi) \neq 0$, *where* $\operatorname{Tors} \operatorname{CH}^3(X_\psi)$ *stays for the torsion subgroup of the Chow group* $\operatorname{CH}^3(X_\psi)$.

Proof. Since the form $\phi_{F(\psi)}$ is isotropic, $(\phi_0)_{F(\psi)}$ is a neighbor of a 4-fold Pfister form $\pi/F(\psi)$ ([16, th. 8.6(2)]). Since the form $(\phi_0)_{F(\psi)}$ is anisotropic, the

Pfister form π is anisotropic and so the cohomological invariant $e^4(\pi)$ gives a non-zero element of $H^4(F(\psi))$. Since π contains a 9-dimensional subform defined over F, the element $e^4(\pi)$ is unramified over F ([16, lemma 6.2]). We conclude that the unramified cohomology group $H^4_{ur}(F(\psi)/F)$ is non-zero. Since $H^4(F) = 0$, we even get, that the cokernel of the restriction homomorphism $H^4(F) \to H^4_{ur}(F(\psi)/F)$ is non-zero. Since this cokernel is isomorphic to $\operatorname{Tors}\operatorname{CH}^3(X_\psi)$ ([16, th. 0.6]), the proof is finished (note that the hypothesis of [16, th. 0.6] saying that ψ is not a 4-fold Pfister neighbor is satisfied because otherwise the form ψ would be isotropic and $\phi_{F(\psi)}$ would be not). □

Lemma 7.1.6. *Let ψ/F be a quadratic form of dimension ≥ 9 and let E/F be the extension constructed in* [16, prop. 6.10]*. If $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_E}) \neq 0$, then $\operatorname{Tors}\operatorname{CH}^3(X_\psi) \neq 0$.*

Proof. If $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_E}) \neq 0$, then the form ψ_E is a form of one of the types (9-a), (9-b), (10-a), (10-b), (10-c), (11-a), (12-a) of forms listed in [16, th. 0.5]. Consider these types case by case.

$\psi_E \in$ **(9-a).** In this case, $(\psi \perp \langle -\operatorname{disc}(\psi)\rangle)_E$ is an element of $I^3(E)$ (represented by an anisotropic 3-fold Pfister form) which does not lie in $I^4(F)$. Therefore, the 10-dimensional F-form $\psi \perp \langle -\operatorname{disc}(\psi)\rangle$ gives an element of $I^3(F) \smallsetminus I^4(F)$ ([16, prop. 6.10(v)]). It follows that this element is represented by an anisotropic 3-fold Pfister F-form, whereby $\psi \in$(9-a).

$\psi_E \in$ **(9-b).** This type is characterized as follows: $\psi \in$ (9-b) for a 9-dimensional ψ iff $i_S(\psi) = 2$ and the both 10- and 12-dimensional special forms containing ψ (see §1.5) are anisotropic. Since $i_S(\psi_E) = i_S(\psi)$ ([16, prop. 6.10(ii)]), and a special F-form is anisotropic iff it is anisotropic over E ([16, lemma 10.1(2)]), it follows that $\phi \in$ (9-b) if $\phi_E \in$ (9-b).

$\psi_E \in$ **(10-a).** This condition means that the class of the 10-dimensional form ψ_E in $W(E)$ is represented by an anisotropic 3-fold Pfister form. As explained in the part (9-a), this is equivalent to the fact that the element $\psi \in W(F)$ is represented by an anisotropic 3-fold Pfister form, i.e., to the fact that $\psi \in$ (10-a).

$\psi_E \in$ **(10-b)** means that ψ_E is a 10-dimensional anisotropic special form. As explained above, this implies that ψ over F is a 10-dimensional anisotropic special form.

$\psi_E \in$ **(10-c).** Here ψ is an anisotropic 10-dimensional form with $\operatorname{disc}(\psi) \neq 1$ and $i_S(\psi) = 1$, because the form ψ_E has these properties (to see that $\operatorname{disc}(\psi) \neq 1$ one may use the binary form $\langle\langle \operatorname{disc}(\psi)\rangle\rangle$) and [16, prop. 6.10(v), $n = 2$]). Therefore, there exists a 12-dimensional special form ρ containing ψ (see, e.g., [16, lemma 1.19(i)]). Note that such ρ is also unique: if ρ' is another one, then the difference $\rho - \rho' \in I^3(F)$ is represented by a form of dimension 4 and hence is 0 by the Arason-Pfister-Hauptsatz. Since the special form ρ is anisotropic over E, is is anisotropic over F as well. Finally, the condition that $\psi_{F(\sqrt{d})}$ is not hyperbolic for a representative $d \in F^*$ of the discriminant of ψ is given by [16, prop. 6.10(vi)].

$\psi_E \in$ **(11-a)** means that $\psi_E \perp \langle -\operatorname{disc}(\psi)\rangle$ is a 12-dimensional anisotropic special form. In this case the 12-dimensional F-form $\psi \perp \langle -\operatorname{disc}(\psi)\rangle$ is also anisotropic and special.

$\psi_E \in$ **(12-a).** Here ψ is a 12-dimensional anisotropic special form because ψ_E is so. □

Lemma 7.1.7. *Let ψ/F be a quadratic form of one of the seven types (9-a)–(12-a) listed in* [16, th. 0.5]*. Then at least on of the following conditions hold:*

(i) *ϕ is isotropic or contains a 4-fold Pfister neighbor;*
(ii) *there exists a special form ρ containing ϕ and such that the form $\phi_{F(\rho)}$ is isotropic or contains a 4-fold Pfister neighbor;*
(iii) *there exist two special forms ρ and ρ' of different dimensions which (both) contain ϕ and such that the form $\phi_{F(\rho,\rho')}$ is isotropic or contains a 4-fold Pfister neighbor.*

Remark 7.1.8. Since any isotropic 9-dimensional quadratic form is a 4-fold Pfister neighbor, one may simplify the formulation of Lemma 7.1.7 by saying "contains a 4-fold Pfister neighbor" instead of "isotropic or contains a 4-fold Pfister neighbor" in (i), in (ii), and in (iii).

Proof of Lemma 7.1.7. We consider all the seven types (9-a)–(12-a) case by case.

If $\phi \in$ (9-a), then ϕ is a 4-fold Pfister neighbor; condition (i) is satisfied.

If $\phi \in$ (10-a), then ϕ is isotropic; condition (i) is satisfied as well.

If $\phi \in$ (9-b), then, by Lemma 1.5.1, there exists a (unique) 12-dimensional special form ρ containing a subform similar to ϕ and there exists a (unique) 10-dimensional special form ρ' containing ϕ. Moreover, both ρ and ρ' are anisotropic. Over the function field $F(\rho,\rho')$ the form ϕ becomes a 4-fold Pfister neighbor (Lemma 1.5.1).

If $\phi \in$ (10-b), then ϕ is a 10-dimensional special form.

If $\phi \in$ (12-a), then ϕ is a 12-dimensional special form.

If $\phi \in$ (11-a), then ϕ becomes isotropic over the function field of quadratic form $\phi \perp \langle -\operatorname{disc}(\phi)\rangle$ which is a 12-dimensional special form.

Finally, if $\phi \in$ (10-c), then ϕ is contained in some 12-dimensional special form ρ (mentioned in the definition of this type). Let us write $\rho = \phi + \beta \in W(F)$ with some binary quadratic form β. Since $\rho_{F(\rho)} = \pi$ in the Witt ring of the function field $F(\rho)$, where $\pi/F(\rho)$ is some 3-fold Pfister form, we have $\phi_{F(\rho)} = \pi - \beta_{F(\rho)}$. It follows that the form $\phi_{F(\rho)}$ contains a 3-fold Pfister form as a subform. Consequently, $\phi_{F(\rho)}$ contains a 9-dimensional 4-fold Pfister neighbor (one may take any 9-dimensional subform containing π). □

Proof of Theorem 7.1.1. Let us choose a special subform $\phi_0 \subset \phi$. So, we have an anisotropic special pair ϕ_0, ϕ. We assume that $\phi_{F(\psi)}$ is isotropic, where ψ is some quadratic form over F of a dimension ≥ 9. We write E/F for the field extension constructed in [16, prop. 6.10].

If the form $(\phi_0)_{E(\psi)}$ is isotropic, then ψ is similar to a subform of ϕ (Lemma 7.1.4) and the proof is finished. Otherwise, we have $\operatorname{Tors}\operatorname{CH}^3(X_{\psi_E}) \neq 0$ (Lemma 7.1.5, note that the special pair ϕ_0, ϕ remains anisotropic over E according to [16, lemma 10.1(1)]). Therefore one has $\operatorname{Tors}\operatorname{CH}^3(X_\psi) \neq 0$ already over F (Lemma 7.1.6). It follows that ψ is a quadratic form of one of the seven types listed in [16, th. 0.5], and we may apply Lemma 7.1.7.

Assume that condition (i) of Lemma 7.1.7 is fulfilled, i.e., ψ contains a 4-fold Pfister neighbor $\psi_0 \subset \psi$ (see Remark 7.1.8). Then the form ϕ becomes isotropic over the function field $F(\psi_0)$ which is a contradiction (cf. [16, lemma 10.1(1)].

Assume that condition (ii) of Lemma 7.1.7 is fulfilled, i.e., ψ is a subform of a special form ρ and the form $\psi_{F(\rho)}$ contains a 4-fold Pfister neighbor. Then the form ϕ becomes isotropic over the function field $F(\rho)$. Therefore $\phi \sim \rho$ (Proposition 7.1.3), whereby ψ is similar to a subform of ϕ.

Finally, assuming that condition (iii) of Lemma 7.1.7 is fulfilled, we get that ψ is contained in two special forms ρ and ρ' of different dimensions while the form $\psi_{F(\rho,\rho')}$ contains a 4-fold Pfister neighbor. Then the form ϕ becomes isotropic over the function field $F(\rho, \rho')$. Since the dimensions of ρ and ρ' are different, one of these two forms, say ρ, has the same dimension as the special form ϕ. If the form $\phi_{F(\rho)}$ would be anisotropic, the form $F(\rho, \rho')$ would be anisotropic as well, because $\rho'_{F(\rho)} \not\sim \phi_{F(\rho)}$ (the dimensions are different). Therefore $\phi_{F(\rho)}$ is isotropic, whereby $\phi \sim \rho$ (Proposition 7.1.3). Consequently ψ is similar to a subform of ϕ in this case too. □

7.2. **Anisotropy of 10-dimensional forms.** The following theorem will be proved with a help of [27]. The original proof is not known.

Theorem 7.2.1 (Izhboldin, cf. [13, th. 5.3]). *Let ϕ be an anisotropic 10-dimensional quadratic form. Let ψ be a quadratic form of dimension* > 10 *and assume that ψ is not a Pfister neighbor. Then the form $\phi_{F(\psi)}$ is anisotropic.*

Proof. It suffices to consider the case with $\dim\psi = 11$. In this case we have $i_1(\psi) \leq 3$ by Theorem 1.0.4. Since ψ is not a Pfister neighbor, $i_1(\psi) \neq 3$ ([19] or [11]). Besides, $i_1(\psi) \neq 2$ by [17, cor. 5.13] (see also: [24, th. 1.1], [39], or [25]). It follows that $i_1(\psi) = 1$; consequently, $\phi_{F(\psi)}$ is anisotropic by [27]. □

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