Hasse principle for Classical groups over function fields of curves over number fields

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Abstract

In ([CT]), Colliot-Thélène conjectures the following:
Let F be a function field in one variable over a number field, with field of constants k and G be a semisimple simply connected linear algebraic group defined over F. Then the map $H^1(F, G) \to \prod_{v \in \Omega_k} H^1(F_v, G)$ has trivial kernel, Ω_k denoting the set of places of k.

The conjecture is true if G is of type ${}^1A^*$, i.e., isomorphic to $SL_1(A)$ for a central simple algebra A over F of square free index, as pointed out by Colliot-Thélène, being an immediate consequence of the theorems of Merkurjev-Suslin ([S1]) and Kato ([K]). Gille ([G]) proves the conjecture if G is defined over k and $F = k(t)$, the rational function field in one variable over k. We prove that the conjecture is true for groups G defined over k of the types ${}^2A^*$, B_n, C_n, D_n (D_4 nontrialitarian), G_2 or F_4; a group is said to be of type ${}^2A^*$, if it is isomorphic to $SU(B, \tau)$ for a central simple algebra B of square free index over a quadratic extension k' of k with a unitary $k'|k$ involution τ.

1 Introduction

Let k be a number field and G a semisimple, simply connected linear algebraic group defined over k. Then the Hasse principle holds for principal homogeneous spaces for G over k, i.e., the natural map $H^1(k, G) \to \prod_{v \in V_k} H^1(k_v, G)$ is injective, V_k denoting the set of real places of k and for $v \in V_k$, k_v is the completion of k with respect to v, (cf. [PR]).

Let X be a smooth, geometrically integral curve over a number field. Let $k(X)$ be the function field of X, with field of constants k. Let Ω_k denote the set of places of k and for $v \in \Omega_k$, let $k_v(X)$ denote the function field of the curve X_{k_v}. Let G be a linear algebraic group defined over $k(X)$. Let $Ш^1(k(X), G)$ be the kernel of the map of pointed sets

$$H^1(k(X), G) \to \prod_{v \in \Omega_k} H^1(k_v(X), G).$$

The following conjecture was made by Colliot-Thélène ([CT]) in the 2-dimensional context.

Conjecture: If G is a semisimple, simply connected linear algebraic group defined over $k(X)$, then $Ш^1(k(X), G)$ is trivial.

In the case when G is defined over k and X is $\mathbb{P}^1$, Gille [G] has shown that $Ш^1(k(X), G)$ is trivial. The fact that $Ш^1(k(X), G)$ is trivial, if G is of type ${}^1A_n$, isomorphic to $SL_1(A)$ where A is a central simple algebra with square free index, follows immediately from the theorems of Merkurjev-Suslin (cf. 2.1) and Kato (cf. 2.3) and is known to experts for a long time. In this article we study $Ш^1(k(X), G)$, for G defined over the number field k. We show that this set is trivial if G is of type B_n, C_n and D_n (D_4 non-trialitarian). We also prove that if G is of type ${}^2A^*$, i.e., isomorphic to $SU(B, \tau)$ where B is a central simple algebra over a quadratic extension k' of k of square free index with a $k'|k$ involution τ, then $Ш^1(k(X), G)$ is trivial. We show from the structure theorems of Cayley algebras and exceptional Jordan algebras due to Springer, that if G is of type G_2 or F_4, then $Ш^1(k(X), G)$ is again trivial. The main ingredients in the proofs of the theorems stated above are higher dimensional class field theory results due to Kato (cf. [K]) and Jannsen (cf. [J]), results of Arason, Elman and Jacob concerning Witt groups of function fields in one and two variables over number fields (cf. [AEJ2], [AEJ3]), results of Merkurjev-Suslin on reduced norm criterion in terms of cohomology (cf. [S1], §24), theorems of Merkurjev on norm principle for algebraic groups (cf. [M2]) and results of Suresh on the structure of mod 2 Galois cohomology in degree 3 (cf. [Su]). The original conjecture is open for G defined over $k(X)$; *it is open even when G is defined over k.*

2 Some known results

We record in this section several results which we shall use in this paper. The first theorem is a result of Merkurjev and Suslin. It gives a criterion for an element in a central division algebra over a field E, to be a reduced norm, in terms of the Galois cohomology group $H^3(E, \mathbb{Q}/\mathbb{Z}(2))$.

Theorem 2.1 *(Suslin, [S1], §24, Theorem.24.4). Let E be a field of characteristic $p \geq 0$. Let D be a central division algebra of square free index n over E, with n coprime to p. Then $\lambda \in E^*$ is a reduced norm from D if and only if $(\lambda) \cup (D) = 0$ in $H^3(E, \mu_n^{\otimes 2})$.*

The next theorem is a norm principle due to Merkurjev for Spin groups. Let A be a central simple algebra of degree $2n \geq 4$ over a field E of characteristic different from 2 and σ be an orthogonal involution on A. Let h be a hermitian form over (A, σ). We have an exact sequence of algebraic groups (cf. §4 and §5 for details),

$$1 \to \mu_2 \to Spin(h) \to SU(h) \to 1$$

which induces the cohomology exact sequence,

$$SU(h)(E) \xrightarrow{\delta} E^*/E^{*2} \to H^1(E, Spin(h)) \to H^1(E, SU(h))$$

The map δ is the spinor norm map and we abbreviate $Sn(h_E)$ = image of δ in E^*/E^{*2}. The norm principle of Merkurjev states:

Theorem 2.2 *(Merkurjev, [M2], 6.2) With notation as above, the image of the spinor norm homomorphism $Sn(h_E)$ is equal to the subgroup of E^*/E^{*2} generated by the images of the norm groups $N_{L|E}(L^*)$ over all finite extensions $L|E$ such that the algebra A_L is split and the hermitian form h_L is isotropic.*

We next state a theorem due to Kato. Let X be a proper smooth geometrically integral curve defined over a number field k. Let F be the function field of X and F_v the function field of X_{k_v}.

Theorem 2.3 *(Kato, [K]) With notation as above and for any positive integer n, the canonical map*

$$H^3(F, \mathbb{Z}/n(2)) \to \prod_{v\in\Omega_k} H^3(F_v, \mathbb{Z}/n(2))$$

is injective.

The following theorem due to Jannsen is an analogue of Kato's theorem for surfaces.

Theorem 2.4 *(Jannsen, ([J]) Let E be a function field in two variables over a number field k, then the restriction map*

$$H^4(E, \mathbb{Q}/\mathbb{Z}(3)) \to \bigoplus_{v\in\Omega_k} H^4(E.k_v, \mathbb{Q}/\mathbb{Z}(3))$$

is injective.

Theorem 2.4 is true if we replace $\mathbb{Q}/\mathbb{Z}(3)$ by $\mathbb{Z}/2\mathbb{Z}$. This follows from the above result of Jannsen and due to the surjectivity of the map $K_3^M(E) \to H^3(E, \mathbb{Z}/2\mathbb{Z})$, where $K_3^M(E)$ is the Milnor K group, which is a consequence of theorems of Merkurjev-Suslin ([MS]) and Rost.

For a field E we denote the mod 2 Galois cohomology ring $H^*(E, \mathbb{Z}/2\mathbb{Z})$ by $H^*(E)$. Let $GW(E) = \bigoplus_{n=0}^{\infty} I^n(E)/I^{n+1}(E)$ be the graded Witt ring of E. We identify $H^1(E)$ with E^*/E^{*2} and for $a \in E^*$, we denote by (a) the corresponding element in $H^1(E)$. Arason (cf. [A], Satz 4.8) has shown that the assignment $< 1, -a_1 > \otimes \cdots \otimes < 1, -a_n > \mapsto (a_1) \cup \cdots \cup (a_n)$, for $a_1, \cdots, a_n \in E^*$ is a well defined map e_E^n from the set of n-fold Pfister forms to $H^n(E)$. The group $I^n(E)$ is generated by n-fold Pfister forms. The Milnor conjecture says that for every positive integer n, the maps e_E^n on the set of n-fold Pfister forms extend to homomorphisms from $I^n(E) \mapsto H^n(E)$, which are again denoted by e_E^n and the induced maps $\bar{e}_E^n : I^n(E)/I^{n+1}(E) \to H^n(E)$ are isomorphisms. Arason, Elman and Jacob have proved Milnor conjecture for function fields in two variables over a number field, (cf. [AEJ1], proposition 5.9 and [AEJ3], theorem 1.5). The deep theorems of Merkurjev-Suslin and Rost (cf. [MS]) and Jacob-Rost (cf. [JR]) are used in the proof. In particular, they prove the following:

Theorem 2.5 *Let E be a field of transcendence degree at most 2 over a number field. Then the map $\bar{e}^*_E$ induces an isomorphism of the graded Witt ring $GW(E)$ with the mod 2 Galois cohomology ring $H^*(E)$.*

Finally, we shall state a theorem of Suresh which will be used in this paper.

Theorem 2.6 *With the same notations as in (2.3), for any element ξ in $H^3(F)$ and a ternary form $< a, b, c >$ over F, there exists $f \in F^*$ such that*

1. *f is a value of $< a, b, c >$*
2. *For every finite non-dyadic place v of k, $\xi_{F_v(\sqrt{f})} = 0$.*
3. *For every dyadic place v of k, such that $-abc$ is a square in F_v, $\xi_{F_v(\sqrt{f})} = 0$.*

For a proof, see [Su].

3 The cases of inner type A_n and C_n

Let D be a central division algebra of index n over a field E with n coprime to the characteristic of E. We have an invariant (cf. [Se2]), for elements of $H^1(E, SL_{n,D})$ with values in $H^3(E, \mu_n^{\otimes 2})$, defined as follows. The set $H^1(E, SL_{n,D})$ is in bijection with $E^*/Nrd(D^*)$. Given $\lambda \in E^*$, the invariant associated with its class $(\lambda) \in E^*/Nrd(D^*)$ in $H^3(E, \mu_n^{\otimes 2})$ is the element $(\lambda) \cup (D)$.

Throughout this section, k denotes a number field and F the function field of a smooth geometrically integral curve X over k. Let Ω_k denote the set of places of k and for $v \in \Omega_k$, let $F_v = k_v(X)$ be the function field of the curve X_{k_v}. Let D be a central division algebra of square free index n over F. Then the map $H^1(F, SL_{n,D}) \to H^3(F, \mu_n^{\otimes 2})$ defined by this invariant is injective (cf. 2.1). By a theorem of Kato, the map $H^3(F, \mu_n^{\otimes 2}) \to \prod_{v \in \Omega_k} H^3(F_v, \mu_n^{\otimes 2})$ is injective (cf. 2.3). Hence the map $H^1(F, SL_{n,D}) \to \prod_{v \in \Omega_k} H^1(F_v, SL_{n,D})$ is injective. Thus, we have,

Proposition 3.1 *Let k be a number field and X a smooth geometrically integral curve over k. Let $F = k(X)$ be the function field of X. Let $G = SL_n(D)$, with D a central division algebra over F with square free index. Then, $\text{Ш}^1(F, G)$ is trivial.*

For non zero elements a, b in a field E, with char $E \neq 2$, we denote by $(a,\, b)_E$, the quaternion algebra over E, generated by the elements i, j, with $i^2 = a$, $j^2 = b$ and $ij = -ji$.

We now consider linear algebraic groups of type C_n. Let D be a quaternion division algebra over F and τ_0 the standard involution on D. Let h be a hermitian form over (D, τ_0) and $G = Sp(h)$, the symplectic group of h. Then G is a simply connected group of type C_n. The set $H^1(F, Sp(h))$ is in bijection with the set of isomorphism classes of hermitian forms over (D, τ_0) of the same rank as h. Given a hermitian form h' over (D, τ_0), there is an associated quadratic form $q_{h'}$ over F defined by $q_{h'}(y) = h'(y, y)$, for y in the underlying space which supports h'. In fact, if h' is represented by the diagonal matrix $< \lambda_1, \cdots, \lambda_r >$, $q_{h'}$ is represented by the matrix $< \lambda_1, \cdots, \lambda_r > \otimes n_D$, where n_D denotes the norm form on the quaternion algebra D. By a theorem of Jacobson (cf. [S], pg. 352), two hermitian forms h and h' are isomorphic over (D, τ_0) if and only if q_h and $q_{h'}$ are isomorphic as quadratic forms.

Let h_1 and h_2 be hermitian forms of the same rank as h, representing elements ξ_1 and ξ_2 in $H^1(F, Sp(h))$. Then $q_{h_1} \perp (-q_{h_2})$ is an element of $I^3(F)$. If $(\xi_1)_v = (\xi_2)_v$ in $H^1(F_v, Sp(h))$, for every $v \in \Omega_k$, then $h_1 \perp (-h_2)$ is hyperbolic over F_v, for all $v \in \Omega_k$. This implies that the class of $q_{h_1} \perp (-q_{h_2})$ is equal to zero in $I^3(F_v)$, for all $v \in \Omega_k$. By ([AEJ2], theorem 4), $q_{h_1} \perp (-q_{h_2})$ is hyperbolic over F; i.e., $h_1 \cong h_2$ and $\xi_1 = \xi_2$ in $H^1(F, Sp(h))$. Thus the map $H^1(F, Sp(h)) \to \prod_{v \in \Omega_k} H^1(F_v, Sp(h))$ is injective. In particular, we have

Proposition 3.2 *Let k be a number field and F be the function field of a smooth geometrically integral curve X over k. Let G be a simply connected group of type C_n defined over k. Then $Ш^1(F, G)$ is trivial.*

Proof. We just need to remark that the only division algebras with involutions of first kind over number fields are quaternion algebras (cf. [S], 10.2.3). □

4 The case of quadratic and hermitian forms

We continue with the same notation as in §2. The aim of this section is to prove the following two theorems.

Theorem 4.1 *Let q be a quadratic form of rank greater than or equal to 3, over a number field k. Then $Ш^1(F, Spin(q))$ is trivial.*

Let $K = k(\sqrt{d})$ be a quadratic field extension of k. Let $FK = F(\sqrt{d})$ and let τ denote the non-trivial automorphism of FK over F.

Theorem 4.2 *Let h be a hermitian form over (FK, τ), of rank at least 2. Then $Ш^1(F, SU(h))$ is trivial.*

We begin with the following

Proposition 4.3 *Let q be a quadratic form of rank greater than or equal to 3, over a number field k. The map*

$$\frac{F^*/F^{*2}}{Sn(q_F)} \to \prod_{v\in\Omega_k} \frac{F_v^*/F_v^{*2}}{Sn(q_{F_v})}$$

is injective.

Proof. **case.1.** rank$(q) = 3$: For any $\lambda \in F^*$, since $Sn(\lambda q) = Sn(q)$, after scaling we may assume that $q =< 1, a, b >$, for some $a, b \in k^*$. Let $D = (-a, -b)_F$. Then $Sn(q_F) = Nrd(D^*)$ modulo squares. If $\alpha \in F^*$ is a local spinor norm then α is a reduced norm from D locally and by (3.1), α is a reduced norm from D and hence a spinor norm from q_F.

case.2. rank$(q) = 4$: Suppose disc$(q) = 1$. After scaling we assume that $q =< 1, a, b, ab >$. Then $Sn(q_F) =$Nrd$((-a, -b)_F^*)$ modulo squares and the proof follows as in case 1.

Suppose disc$(q) = d$. By scaling we may assume that $q =< 1, a, b, abd >$. We have $Sn(q_F) = Nrd((-a, -b)_{F(\sqrt{d})}) \cap F^*$ modulo squares (cf. [KMRT], 15.11). Let $\alpha \in F^*$ be such that $\alpha \in Sn(q_{F_v})$, for every $v \in \Omega_k$. Then α is a reduced norm from $(-a, -b)_{(F(\sqrt{d}))_w}$, for all $w \in \Omega_{k(\sqrt{d})}$. By (3.1), $\alpha \in$ Nrd$(-a, -b)_{F(\sqrt{d})} \cap F^* = Sn(q_F)$ modulo squares.

case.3. rank$(q) = 5$: Let $d =$ disc(q). Then the form $q \perp< -d >$ is a six dimensional form over the number field k, which is indefinite and hence is isotropic (cf. [S], 6.6.6). Thus, q represents d and after scaling, we may assume that $q \cong< d, 1, a, b, ab >$. Hence q is a Pfister neighbour for the Pfister form $q_1 =< 1, a > \otimes < 1, b > \otimes < 1, d >$. By the norm principle (cf. 2.2), spinor norms for q_F are products of norms from finite extensions of F where q_F is isotropic. As q_F is isotropic if and only $(q_1)_F$ is hyperbolic, spinor norms for q_F are products of norms from finite extensions of F where $(q_1)_F$ is hyperbolic. Let $\alpha \in F^*$ be a spinor norm locally for all $v \in \Omega_k$, for q_F. Then for every $v \in \Omega_k$, α is a similarity factor for $(q_1)_{F_v}$ (cf. [L], Ch. 7, 4.5). Hence the form $< 1, -\alpha > q_1$ in $I^4(F)$ is zero in $I^4(F_v)$, for every $v \in \Omega_k$. As $I^4(F) \to \prod_{v\in\Omega_k} I^4(F_v)$ is injective (cf. [AEJ2], theorem 4), we have $< 1, -\alpha > q_1$ is zero in $W(F)$, i.e., α is a similarity factor for q_1 over F. Hence α is represented by q_1 over F. As q_1 is a Pfister form, α is a spinor norm of q_1 over F. By the norm principle (cf. 2.2), $Sn(q_{1F}) = Sn(q_F)$ and hence α is a spinor norm of q over F.

case.4. rank$(q) \geq 6$: We complete the proof by induction on rank(q). Let $q = q_1 \perp q_2$, with rank$(q_1) = 5$. Let disc$(q_1) = d$. After scaling q, we assume that $q_1 \cong< d, 1, a, b, ab >$, as in case.3. Let $\alpha \in F^*$ be a spinor norm locally for q_F. Let $l(Y) = F(\sqrt{-\alpha})$, with l denoting the field of constants in $F(\sqrt{-\alpha})$ and Y a curve over l.

Let $q' =< d, 1, a, b >\perp q_2$. Since $\mathrm{rank}(q') \geq 5$, q' is isotropic over l_w and hence over $l_w(Y)$, for every finite place w of l. Let w be a real place, where q' is definite. Since q' represents 1, the elements a, b and hence ab are all positive at l_w and hence over k_v, where v is the restriction of w to k. Since α is a spinor norm of q over F_v, α is a sum of squares in F_v and hence in $l_w(Y)$. Since $-\alpha$ is a square in $l_w(Y)$, it follows that -1 is a sum of squares in $l_w(Y)$, i.e., $l_w(Y)$ has no ordering. This implies that $cd(l_w(Y)) \leq 1$, (cf. [Se1]). Thus q' is isotropic over $l_w(Y)$. In particular, for each $w \in \Omega_l$, every element of $l_w(Y)^*$ is a spinor norm for $(q')_{l_w(Y)}$. By induction hypothesis, $Sn(q') = l(Y)^*/l(Y)^{*2}$. By the norm principle (cf. 2.2), α being a norm from $l(Y)$, is a spinor norm for q' and hence for q. $\square$

Remark 4.4 *In the case of quadratic forms of rank* 3 *or* 4*, the proposition* 4.3 *holds more generally for forms over the function field* F*, i.e., if* q *is a quadratic form over* F *of rank* 3 *or* 4*, then the map*

$$\frac{F^*/F^{*2}}{Sn(q_F)} \to \prod_{v\in\Omega_k} \frac{F_v^*/F_v^{*2}}{Sn(q_{F_v})}$$

is injective. The proof given in the proposition works as well in these cases.

Proof of theorem 4.1**.** We have an exact sequence of algebraic groups:

$$1 \longrightarrow \mu_2 \longrightarrow Spin(q) \xrightarrow{\eta} SO(q) \longrightarrow 1$$

which gives rise to an exact sequence of pointed sets:

$$SO(q)(F) \xrightarrow{\delta^0} F^*/F^{*2} \longrightarrow H^1(F, Spin(q)) \xrightarrow{\eta} H^1(F, SO(q)) \xrightarrow{\delta^1} H^2(F, \mu_2).$$

The map δ^0 is induced by the spinor norm. The set $H^1(F, SO(q))$ classifies isomorphism classes of quadratic forms, with the same rank and discriminant as q. For a class $[q'] \in H^1(F, SO(q))$, $\delta^1([q']) = c(q' \perp (-q))$, where c is the Clifford invariant of $(q' \perp (-q))$. Thus the image $H^1(F, Spin(q)) \to H^1(F, SO(q))$, consists of classes of quadratic forms q' with the same rank, discriminant and Clifford invariant as q; in particular, $q' \perp (-q) \in I^3(F)$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
1 \longrightarrow & \frac{F^*/F^{*2}}{Sn(q_F)} & \xrightarrow{\quad\delta^0\quad} & H^1(F, Spin(q)) & \xrightarrow{\quad\eta\quad} & H^1(F, SO(q)) \\
& \downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & \prod_{v\in\Omega_k} \frac{F_v^*/F_v^{*2}}{Sn(q_{F_v})} & \xrightarrow{\delta^0} & \prod_{v\in\Omega_k} H^1(F_v, Spin(q_{F_v})) & \xrightarrow{\eta} & \prod_{v\in\Omega_k} H^1(F_v, SO(q_{F_v}))
\end{array}$$

Let $\xi \in H^1(F, Spin(q))$ be such that $\xi_v = 1$, for all $v \in \Omega_k$. The element $\eta(\xi)$ corresponds to the class of a quadratic form q' over F with $q' \perp (-q) \in$

$I^3(F)$. By the commutativity of the above diagram, $(q' \perp (-q))_{F_v}$ is zero in $I^3(F_v)$, for all $v \in \Omega_k$. By ([AEJ2], theorem 4), we have an injection $I^3(F) \to \prod_{v\in\Omega_k} I^3(F_v)$. Thus $q' \perp (-q)$ is equal to zero in $I^3(F)$. By Witt's cancellation theorem, $q' \cong q$ and ξ lies in the kernel of η. Hence there exists $\alpha \in F^*$, such that $\delta^0([\alpha]) = \xi$. From the commutativity of the above diagram, it follows that α is locally a spinor norm, for all $v \in \Omega_k$. The theorem now follows from the proposition 4.3. $\square$

Recall that if E is a field of characteristic different from 2 and L is a quadratic extension of E, with σ denoting the non trivial automorphism of L over E, $W(L|E,\sigma)$ denotes the Witt group of σ-hermitian forms. We have a homomorphism of groups $W(L|E,\sigma) \to W(E)$, given by associating to any $h \in W(L|E,\sigma)$, the quadratic form q_h defined as $q_h(x,x) = h(x,x)$, for any x in the space supporting h. This gives rise to the following exact sequence:

$$1 \to W(L|E,\sigma) \to W(E) \to W(L)$$

where the map $W(E) \to W(L)$ is given by scalar extension from E to L. In fact if $L = E(\sqrt{d})$, for some $d \in E^*$, then the image of $W(L|E,\sigma)$ in $W(E)$ is the subgroup $W(E). < 1, -d >$, (cf. [S], 10.1.3).

Proof of theorem 4.2**.** We have the following exact sequence of algebraic groups

$$1 \to SU(h) \to U(h) \to R^1_{FK|F}(G_m) \to 1$$

where for any extension L of F,

$$R^1_{FK|F}(G_m)(L) = (LK)^{*1} = \{x \in (LK)^* \mid N_{LK|L}(x) = 1\}.$$

As $Nrd : U(h)(F) \to (FK)^{*1}$ is surjective, the above sequence gives rise to the following exact sequence of pointed sets,

$$1 \to H^1(F, SU(h)) \xrightarrow{\eta} H^1(F, U(h)).$$

The set $H^1(F, U(h))$ classifies isomorphism classes of hermitian forms, with the same rank as h. An element of $H^1(F, SU(h))$ maps under η to the class of a hermitian form with the same rank and discriminant as h. We have the following commutative diagram,

$$\begin{array}{ccccc}
1 & \longrightarrow & H^1(F, SU(h)) & \xrightarrow{\quad\eta\quad} & H^1(F, U(h)) \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v\in\Omega_k} H^1(F_v, SU(h)) & \xrightarrow{\eta} & \prod_{v\in\Omega_k} H^1(F_v, U(h))
\end{array}$$

Let $\xi \in H^1(F, SU(h))$ be locally trivial in $H^1(F_v, SU(h))$, for every $v \in \Omega_k$. The element $\eta(\xi)$ corresponds to the class of a hermitian form h' over (FK, τ) with rank and discriminant of h' same as those of h. Moreover, $(h \perp (-h'))_{F_v}$ is the hyperbolic form locally, for every $v \in \Omega_k$. The hermitian forms h and h' correspond to quadratic forms q_h and $q_{h'}$ over F respectively such that the rank, discriminant and Clifford invariants of $q_{h'}$ are the same as those of q_h. Hence the form $q_h \perp (-q_{h'}) \in I^3(F)$. Further, the form $q_h \perp (-q_{h'})$ is locally zero in $I^3(F_v)$, for every $v \in \Omega_k$. By ([AEJ2], theorem 4), $q_h \perp (-q_{h'})$ is zero in $I^3(F)$. Hence $h \cong h'$ over (FK, τ) and $\eta(\xi)$ is trivial. Since kernel(η) is trivial, ξ is trivial. $\square$

5 A classification theorem for hermitian forms over division algebras with an orthogonal involution

Let E be a field of characteristic different from 2 and L a quadratic field extension of E with σ denoting the nontrivial automorphism of L over E. Let $U_{2n}(L,\sigma)$ denote the unitary group of the hyperbolic form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ over (L,σ). If h is a hermitian form over (L,σ) of rank $2n$, it defines an element $\xi_h \in H^1(E, U_{2n}(L,\sigma))$. The set $H^1(E, SU_{2n}(L,\sigma))$ injects into $H^1(E, U_{2n}(L,\sigma))$, the image consisting of hermitian forms over (L,σ) of rank $2n$ and trivial discriminant. Hence if h has trivial discriminant, ξ_h defines an element in $H^1(E, SU_{2n}(L,\sigma))$. The Rost invariant of ξ_h is the Arason invariant of the quadratic form q_h associated to h (see §4 and [BP2], §3); i.e., the Rost invariant of an even rank hermitian form over (L,σ), with trivial discriminant is the same as the Arason invariant of the associated quadratic form in $I^3(E)$.

We next recall (cf. [BP2], §3) the Rost invariant associated to a hermitian form over a central division algebra D over any field E, with an orthogonal involution τ. Let h be a hermitian form over (D,τ). We denote by R_h the Rost invariant on $H^1(E, Spin(h))$ which takes values in $H^3(E, \mathbb{Q}/\mathbb{Z}(2))$. Its values on the subset $\frac{E^*/E^{*2}}{Sn(h_E)} \subset H^1(E, Spin(h))$ are given by $[\lambda] \mapsto (\lambda) \cup (D)$, (cf. [KMRT], §31.B, pp. 437). If h is a hermitian form of rank $2n$, trivial discriminant and trivial Clifford invariant, the class of h defines an element in $H^1(E, U_{2n}(D,\tau))$, where $U_{2n}(D,\tau)$ is the unitary group of the hyperbolic form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, which admits a lift $\xi \in H^1(E, Spin_{2n}(D,\tau))$ under the composite map :

$$H^1(E, Spin_{2n}(D,\tau)) \to H^1(E, SU_{2n}(D,\tau)) \to H^1(E, U_{2n}(D,\tau))$$

The Rost invariant of h, denoted as $R(h)$ is defined to be $R(h) = [R(\xi)] \in H^3(E, \mathbb{Q}/\mathbb{Z}(2))/H^1(E,\mu_2) \cup (D)$, (cf. [BP2], §3). If $D = E$ this invariant coincides with the Arason invariant. We recall the following lemma, (cf. [BP2], 3.6).

Lemma 5.1 *Let (D,τ) be a central division algebra with an orthogonal involution over a field E. Let h be a hermitian form over (D,τ). Let $\xi \in$*

$H^1(E, Spin(h))$ and h' the hermitian form over (D,τ), associated to the image of ξ in $H^1(E, U(h))$. Then $[R_h(\xi)] = R(h' \perp (-h))$ in $H^3(E, \mathbb{Q}/\mathbb{Z}(2))/H^1(E, \mu_2) \cup (D)$.

Let k be a number field. We denote by V_k, the set of real places of k.

Lemma 5.2 *Let k be a number field and M a function field in two variables over k. Then the map $H^n(M) \to \prod_{v \in V_k} H^n(M.k_v)$ is injective, for $n \geq 5$.*

Proof. Let $n \geq 5$. Let $\xi \in H^n(M)$ be trivial in $H^n(M.k_v)$, for every $v \in V_k$. As every real closure of M contains a real closure of k, by ([AEJ1], 2.2), ξ is a (-1)-torsion element in $H^n(M)$. We have the following exact sequence,

$$\begin{array}{ccc} H^n(M(\sqrt{-1})) & \xrightarrow{cores} & H^n(M) \\ & & \big\downarrow{\scriptstyle (-1)\cup} \\ H^{n+1}(M(\sqrt{-1})) & \xleftarrow[res]{} & H^{n+1}(M) \end{array}$$

As k is a number field, $vcd(k) \leq 2$ and hence $vcd(M) \leq 4$ and $H^r(M(\sqrt{-1})) = 0$, for $r \geq 5$. In view of the above exact sequence, as $n \geq 5$, we have $(-1) \cup : H^n(M) \to H^{n+1}(M)$ is an isomorphism. As ξ is (-1)-torsion in $H^n(M)$, ξ is zero in $H^n(M)$. $\square$

We record the following lemma, which is a consequence of a theorem of Jannsen (cf. 2.4) and a theorem of Arason-Elman-Jacob (cf. [AEJ1], 2.2).

Lemma 5.3 *Let k be a number field and M a function field in two variables over k. Then the map $I^4(M) \to \prod_{v \in V_k} I^4(M.k_v)$ is injective.*

Proof. Let $q \in I^4(M)$ with $q_{M.k_v} = 0$ locally for all $v \in \Omega_k$. Since e^n_M is well defined (cf. [AEJ1], 1.2), we have the following commutative diagram for each n:

$$\begin{array}{ccc} I^n(M) & \longrightarrow & \prod_{v \in \Omega_k} I^n(M.k_v) \\ {\scriptstyle e^n_M}\big\downarrow & & {\scriptstyle e^n_M}\big\downarrow \\ H^n(M) & \longrightarrow & \prod_{v \in \Omega_k} H^n(M.k_v) \end{array}$$

In view of this commutative diagram, the remark following (2.4) and since $\bar{e}^4_M$ is an isomorphism (2.5), it follows that $q \in I^5(M)$. Since q is locally zero, using the above commutative diagram for $n = 5$, we see that $e^5_M(q)$ is locally trivial in $H^5(M.k_v)$, for every $v \in \Omega_k$. By the preceding lemma (5.2), we have $e^5_M(q)$ is zero in $H^5(M)$. Hence $q \in I^6(M)$. Repeating this argument, we get that $q \in \bigcap_{n \geq 5} I^n(M)$ and hence is zero, by Arason-Pfister's theorem (cf. [S], 4.5.6). $\square$

Theorem 5.4 *Let k be a number field and let $F = k(X)$ be the function field of a smooth, geometrically integral curve X over k. Let D be a quaternion division algebra over F, with an orthogonal involution σ. Let h_1 and h_2 be two hermitian forms over (D, σ) with the same rank and discriminant. Suppose further that $c(h_1 \perp (-h_2)) = 0$ and $R(h_1 \perp (-h_2)) = 0$. Suppose h_1 and h_2 are equivalent over F_v for all $v \in \Omega_k$, then $h_1 \cong h_2$.*

Proof. Let L be a quadratic extension of F contained in D such that σ restricted to L is identity. Let $\mu \in D^*$ be such that $\sigma(\mu) = -\mu$ and $Int(\mu)$ restricted to L is the non-trivial automorphism τ_0 of L over F (cf. [BP2], §3.2). The involution $\tau = Int(\mu) \circ \sigma$ on D, being symplectic is the canonical involution on D. Let $L = l(Y)$, where l is the field of constants in L. For $v \in \Omega_k$, let $F_v = k_v(X)$ be the function field of the curve X_{k_v} and $L_v = L \otimes_F F_v$. We have the following commutative diagram with exact rows, (cf. [BP2], 3.2).

$$\begin{array}{ccccccc} W(D,\tau) & \xrightarrow{\pi_1} & W(L|F,\tau_0) & \xrightarrow{\tilde{\rho}} & W(D,\sigma) & \xrightarrow{\tilde{\pi}_2} & W(L) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod_{v\in\Omega_k} W(D_{F_v},\tau) & \xrightarrow{\pi_1} & \prod_{v\in\Omega_l} W(L_v|F_v,\tau_0) & \xrightarrow{\tilde{\rho}} & \prod_{v\in\Omega_v} W(D_{F_v},\sigma) & \xrightarrow{\tilde{\pi}_2} & \prod_{v\in\Omega_l} W(L_v) \end{array}$$

Let $h = h_1 \perp (-h_2)$. Then h has even rank, trivial discriminant, trivial Clifford invariant and trivial Rost invariant. Further h is zero in $W(D_{F_v}, \sigma)$, for every $v \in \Omega_k$. The element $\widetilde{\pi_2}(h) \in W(L)$ has even rank, trivial discriminant and trivial Clifford invariant and hence belongs to $I^3(L)$. Further, $\widetilde{\pi_2}(h)$ is zero in $W(L_w)$, for every $w \in \Omega_l$. By ([AEJ2], theorem 4), $\widetilde{\pi_2}(h)$ is zero in $I^3(L)$. Thus there exists $h_0 \in W(L|F, \tau_0)$ such that $\tilde{\rho}(h_0) = h$. The rank of h_0 is even. We show that the lift $h_0 \in W(L|F, \tau_0)$ may be modified so as to have trivial discriminant. Let $\alpha = disc(h_0) \in F^*/N_{L|F}(L^*)$. We have $c(\tilde{\rho}(h_0)) = (L) \cup (\alpha) \in Br(F)/(D)$, (cf. [BP1], 3.2.3). Since $c(\tilde{\rho}(h_0)) = c(h) = 0$, we have $(L) \cup (\alpha) = 0$ or $(L) \cup (\alpha) = (D) \in Br(F)$. If $(L) \cup (\alpha) = 0$, then $disc(h_0) = 1$. Suppose $(L) \cup (\alpha) = (D)$. Let $L = F(\sqrt{a})$ so that $D = (a, \alpha)_F$. The image of the form $< 1 > \in W(D, \tau)$ under the map π_1 in $W(L|F, \tau_0)$, is simply $(< 1, -\alpha >)$, which has discriminant α in $F^*/N_{L|F}(L^*)$. Modifying h_0 by $\pi_1(< 1 >)$, we may assume that $disc(h_0) = 1$.

We now show that the lift h_0 of h may be modified to have trivial Rost invariant. Let $rank(h_0) = 2n$. Let $SU(\mu^{-1}\sqrt{a}H_{2n})$ be the special unitary group with respect to the hermitian form $\mu^{-1}\sqrt{a}H_{2n}$ over (D, σ). The inclusion $SU_{2n}(L|F, \tau_0) \to SU(\mu^{-1}\sqrt{a}H_{2n})$ gives rise to an injection $SU_{2n}(L|F, \tau_0) \to SU_{2n}(D, \sigma)$ (by a choice of an isomorphism $\mu^{-1}\sqrt{a}H_{2n} \cong H_{2n}$ (cf. [BP2], pg. 671). This lifts to a homomorphism $\rho_0 : SU_{2n}(L|F, \tau_0) \to Spin_{2n}(D, \sigma)$. We have the following commuting diagram:

$$\begin{array}{ccc} SU_{2n}(L|F,\tau_0) & \xrightarrow{\rho_0} & Spin_{2n}(D,\sigma) \\ & {}_{\tilde{\tilde{\rho}}}\searrow \quad \swarrow & \\ & U_{2n}(D,\sigma) & \end{array}$$

which yields a corresponding diagram:

$$\begin{tikzcd}
H^1(F, SU_{2n}(L|F, \tau_0)) \arrow[rr, "\rho_0"] \arrow[dr, "\tilde{\tilde{\rho}}"'] & & H^1(F, Spin_{2n}(D, \sigma)) \arrow[dl] \\
& H^1(F, U_{2n}(D, \sigma)) &
\end{tikzcd}$$

The map $\tilde{\rho}$ at the level of Witt groups is induced by the map $\tilde{\tilde{\rho}}$, (for varying n). Indeed for the hermitian form h_0, $R(h_0) = R_{Spin_{2n}(D,\tau)}(\rho_0(h_0))$, (cf. [BP2], 3.20). Since $R(\tilde{\tilde{\rho}}(h_0)) = R(h) = 0$, there exists $\lambda \in F^*$, such that $R(h_0) = (\lambda) \cup (D)$. The element, $\pi_1(< 1, -\lambda >)$ has the associated quadratic form $< 1, -\lambda > \otimes n_D$, n_D denoting the norm form of D over F and has Rost invariant $(\lambda) \cup (D)$. Modifying h_0 by $\pi_1(< 1, -\lambda >)$, we may assume that $R(h_0) = 0$. Thus, the quadratic form associated to h_0, q_{h_0}, defines an element in $I^4(F)$.

The image of π_1 consists of hermitian forms f whose associated quadratic forms q_f, are multiples of n_D. Since $h = \tilde{\rho}(h_0)$ is locally trivial over F_v, for every $v \in \Omega_k$, h_{0F_v} is in the image of π_1 and hence q_{h_0} is a multiple of n_D over F_v, for every $v \in \Omega_k$.

Let C be the conic defined by $aX_1^2 + bX_2^2 - 1$ over F. Then $F(C)$ is a 2 dimensional field over k and n_D is zero over $F(C)$ (cf. [S], 5.2, (iv)). Hence the class of q_{h_0} in $I^4(F_v(C))$ is zero, for all $v \in \Omega_k$. The map $I^4(F(C)) \to \prod_{v \in \Omega_k} I^4(F_v(C))$ being injective (cf. 5.3), q_{h_0} is zero in $I^4(F(C))$ and hence is a multiple of of n_D (cf. [S], 5.4, (iv)). It follows that h_0 is in the image of π_1 and hence $\rho(h_0) = h = 0$ in $W(D, \sigma)$. $\square$

6 Hasse principle for groups of type D_n (D_4 non-trialitarian)

Let (D, σ) be a central simple algebra over a field E with an orthogonal involution. Let $L|E$ be an extension which splits D and let $\phi : (D, \sigma) \otimes_E L \cong (M_n(L), \tau_{q_0})$ be a splitting with $\sigma \otimes 1$ transported to the adjoint involution on $M_n(L)$ corresponding to a quadratic form q_0 over L. The form q_0 is determined upto a scalar. Let h be a hermitian form over $(D, \sigma) \otimes_E L$. Then by Morita theory with respect to ϕ, h is equivalent to a quadratic form q over L. The similarity class of q is uniquely determined by h and is independent of the choice of ϕ and q_0. The form h is isotropic if and only if q is isotropic. In particular, $Sn(h_L) = Sn(q_L)$.

Lemma 6.1 *Let (D, σ) be a quaternion algebra with an orthogonal involution over a local field k. Let h be a hermitian form of rank 3 over (D, σ) and σ_h the involution on $M_3(D)$, adjoint with respect to h. Suppose $disc(\sigma_h) \notin k^{*2}$. Then h is isotropic.*

Proof. Let τ be the canonical symplectic involution on D. Let $\sigma = Int\,u \circ \tau$, for some $u \in D^*$, such that $\tau(u) = -u$. The hermitian form h corresponds under scaling by u, to a skew hermitian form h_1 with respect to τ (cf. [BP1], §1.3). The involution τ_{h_1} on $M_3(D)$ adjoint with respect to h_1, corresponds with σ_h. Then $det(h_1) = disc(\tau_{h_1}) = disc(\sigma_h)$ (cf. [KMRT], 7.2). By the hypothesis on h, $disc(\sigma_h) \notin k^{*2}$. Hence $det(h_1)$ is not in k^{*2} and by ([S], 10.3.6), h_1 and hence h is isotropic. $\square$

Theorem 6.2 *Let (D, σ) be a quaternion division algebra over a number field k with an orthogonal involution σ and let h be a hermitian form over (D, σ) of rank at least 2. Let $F = k(X)$ be the function field of a smooth geometrically integral curve X over k. For each $v \in \Omega_k$, let F_v be the function field of the curve X_{k_v}. Then the map*

$$\frac{F^*/F^{*2}}{Sn(h_F)} \to \prod_{v \in \Omega_k} \frac{F_v^*/F_v^{*2}}{Sn(h_{F_v})}$$

is injective.

Proof. Suppose $rank(h) = 2$. Let $\delta = disc(h) \in k^*/k^{*2}$. The Clifford algebra $C = C(M_2(D), \tau_h))$, is a quaternion algebra over $k(\sqrt{\delta})$ and $Sn(h_F) = Nrd(C_{F(\sqrt{\delta})}) \cap F^*$ modulo squares, (cf. [KMRT], 15.11). Let $\lambda \in F^*$ be a local spinor norm for h_F. Then λ is a reduced norm from $C \otimes_F F_v$, for every place v of k and by (3.1), C being a quaternion algebra, λ is a reduced norm from $C_{F(\sqrt{\delta})}$ and belongs to $Nrd(C_{F(\sqrt{\delta})}) \cap F^* = Sn(h_F)$ modulo squares.

Let $rank(h) = n \geq 3$. Let $\lambda \in F^*$ be a local spinor norm for h_F. Then λ is a reduced norm from D_F, (cf. 2.2 and 3.1). Let L be a quadratic extension of F such that D_L is split and $\lambda = N_{L|F}(\mu)$, for some $\mu \in L^*$. The element λ is also a norm from $F(\sqrt{-\lambda})$. By ([W], Lemma 2.13), there exists $\theta \in L(\sqrt{-\lambda})$ such that $N_{L(\sqrt{-\lambda})|F}(\theta) = \nu^2\lambda$, for some $\nu \in F^*$. By (2.2), it suffices to show that every element of $L(\sqrt{-\lambda})^*$ modulo squares is contained in $Sn(h_{L(\sqrt{-\lambda})})$. We note that for every ordering v of k where D_{k_v} is split and h_{k_v} is definite, $\lambda \in F_v^*$ being a spinor norm of h_{F_v} is a sum of squares so that $L(\sqrt{-\lambda})\,.\,k_v$ has no orderings. In particular, if l is the field of constants of $L(\sqrt{-\lambda})$ and $L(\sqrt{-\lambda}) = l(Y)$, Y a curve over l, for any ordering w of l extending v, $l_w(Y)$ has no ordering. We rename $l = k$ and $Y = X$ and assume that $D \otimes_k k(X)$ is split and for every ordering v of k where D_{k_v} is split and h_{k_v} is definite, $k_v(X)$ has no orderings; in particular, $cd(k_v(X)) \leq 1$. We then show that every $\lambda \in k(X)^*$ is a spinor norm for $h_{k(X)}$. This is done by induction on $rank(h)$.

Suppose $rank(h) = 3$. Let S_1 be the set of real places of k such that D_{k_v} is split and h_{k_v} is indefinite. Let S_2 be the set of dyadic places of k such that D_{k_v} is split and $disc(\sigma_h) \notin k_v^{*2}$. Let S_3 be the set of dyadic places of k such that D_{k_v} is not split and $disc(\sigma_h) \notin k_v^{*2}$. For $v \in S_1 \cup S_2$, h_{k_v} corresponds under Morita equivalence to a quadratic form of rank 6 over k_v, which is isotropic. We choose a rank 1 subform $< X_{3v} >$ of h_{k_v}, such that under Morita equivalence, $< X_{3v} >$ corresponds to the quadratic form $< 1, -1 >$ over k_v. For $v \in S_1 \cup S_2$, let $< X_{1v}, X_{2v} >$ denote the orthogonal complement of $< X_{3v} >$ in h_{k_v}. For $v \in S_3$, since D_{k_v} is not split and $disc(\sigma_h) \notin k_v^{*2}$, h_{k_v} is isotropic in view of 6.1. We choose a rank 1 subform $< X_{1v} >$ of h_{k_v} such that $< X_{1v} >^{\perp} \cong < X_{2v}, X_{3v} >$ is hyperbolic. Using weak approximation, one can find a rank 1 subform $< X_1 >$ of h over k, such that for each $v \in S_1 \cup S_2 \cup S_3$, $< X_1 >_{k_v} \cong < X_{1v} >$. One can choose a subform $< X_2 >$ in $< X_1 >^{\perp}$ such that $< X_2 >_{k_v} \cong < X_{2v} >$, for each $v \in S_1 \cup S_2 \cup S_3$. Let $< X_1, X_2 >^{\perp} \cong < X_3 >$. Clearly, $< X_3 >_{k_v} \cong < X_{3v} >$, for $v \in S_1 \cup S_2 \cup S_3$. Thus $h \cong < X_1, X_2, X_3 >$. Since D is split over F, we choose an isomorphism $\phi : (D_F, \sigma) \to (M_2(F), \tau_{q_0})$, q_0 being a rank 2 quadratic form over F. The isomorphism ϕ yields a Morita correspondence between hermitian forms over D_F and quadratic forms over F. Let $< X_1 >_F$ correspond to $< a', b' >$ over F, $< X_2 >_F$ correspond to $< c', d' >$ over F and $< X_3 >_F$ correspond to $< e', f' >$ over F. Thus h_F corresponds to the rank 6 quadratic form $q =< a', b', c', d', e', f' >$. Since the spinor norm group is insensitive to scaling, we replace q by the form $(a'b'c') \,.\, q =< b'c', c'a', a'b', d'a'b'c', e'a'b'c', f'a'b'c' >$. Renaming, we set $q =< -a, -b, ab, c, d, -cd\delta >$, $\delta = disc(q) = disc(\sigma_h) \in k^*/k^{*2}$. We note that the form $< d, -cd\delta >= a'b'c' \; < e', f' >$. We choose $g \in F^*$ such that g is a value of the quadratic form $< a\,\delta, b\,\delta, -ab\,\delta >$ and such that for $\xi = (\lambda) \cup (c\,\delta) \cup (d\,\delta) \in H^3(F)$, $\xi_{F_v(\sqrt{g})} = 0$, for every finite nondyadic $v \in \Omega_k$ and for every dyadic $v \in \Omega_k$ where $\delta \in k_v^{*2}$, (cf. 2.6). Set $\alpha = g\,\delta \in F^*$. Then α is a value of the quadratic form $< a, b, -ab >$ over F. The form $< -a, -b, ab >$ being isotropic over $F(\sqrt{\alpha})$, we have, $q \cong \gamma < 1, -\alpha > \perp < -\alpha > \perp < c, d, -cd\delta >$, for some $\gamma \in F^*$. Let $q_1 =< -\alpha, c, d, -cd\delta >$. Then $disc(q_1) = g \in F^*/F^{*2}$. We claim that λ is a spinor norm for q_1 locally, for every $v \in \Omega_k$. Over $F(\sqrt{g})$, $q_1 \cong < -\delta, c, d, -cd\,\delta >$ and the Clifford algebra $C(q_1) \cong (c\,\delta, d\,\delta)_{F(\sqrt{g})}$. For a finite $v \in \Omega_k$ such that v is nondyadic or v is dyadic and $\delta \in k_v^{*2}$, over $F_v(\sqrt{g})$, $(\lambda) \cup C(q_1) = \xi_{F_v(\sqrt{g})} = 0$. As $C(q_1)$ is a quaternion algebra over $F_v(\sqrt{g})$, λ is a reduced norm from $C(q_1)$ and hence $[\lambda] \in Sn((q_1)_{F_v})$, (cf. [KMRT], 15.11). For $v \in S_1 \cup S_2$, by choice, the form $< d, -cd\delta >= a'b'c' \; < e', f' > \cong a'b'c' \; < X_{3v} > \cong < 1, -1 >$ over F_v. Hence q_1 being isotropic over F_v, $\lambda \in Sn((q_1)_{F_v})$. For $v \in S_3$, over F_v, $a'b'c' < ab, c, d, -cd\delta >$ corresponds under Morita equivalence to $< X_{2v}, X_{3v} >$. The form $< X_{2v}, X_{3v} >$ being hyperbolic, $< ab, c, d, -cd\delta >$ is hyperbolic and hence $< c, d, -cd\,\delta >$ is isotropic over F_v. In particular, q_1 is isotropic and $\lambda \in Sn((q_1)_{F_v}$. For a real $v \in \Omega_k$ such that D_{k_v} is split and h_{k_v} is equivalent to a definite quadratic form, $cd(k_v(X)) \leq 1$ and $(q_1)_{k_v}$ being 4 dimensional is isotropic. Hence $\lambda \in Sn((q_1)_{F_v})$. Let $v \in \Omega_k$ be a real place such that D_{k_v} is not split. We claim that $(q_1)_{F_v}$ is isotropic. Since every form of rank greater than 1 over D_{k_v} is isotropic, we have $< X_{3v} > \cong < -X_{3v} >$. As $< X_{3v} >$ corresponds to the quadratic form $< e', f' >$ over F_v, we have $2 < e', f' >= 0$. Since $< d, -cd\delta > \cong a'b'c' < e', f' >$, we have $< d, -cd\delta >$ is torsion in $W(F_v)$. To show that $(q_1)_{F_v}$ is isotropic, it is enough to show that q_1 is isotropic over $F_v(\sqrt{g})$. Over $F_v(\sqrt{g})$, $q_1 \cong < -\delta, c, d, -cd\delta > \cong d\,(< 1, -c\,\delta > \otimes < 1, cd >)$.

As $< 1, -c\delta >$ is torsion, we have $< 1, -c\delta > \otimes < 1, cd >$ is torsion over $F_v(\sqrt{g})$. As $vcd(F_v(\sqrt{g})) \leq 1$, $I^2(F_v(\sqrt{g}))$ is torsion free. Hence q_1 is isotropic over $F_v(\sqrt{g})$ and hence over F_v. Thus λ is a spinor norm for q_1 over F_v, for every place v of k and hence by (4.4), λ is a spinor norm for q_1 and hence for h.

Suppose $rank(h) = n \geq 4$. Let S_1 be the set of real places of k where D_{k_v} is split and h_{k_v} is isotropic. Let S_2 be the set of finite places of k where D_{k_v} is not split. Let $v \in S_2$. The form h_{k_v} being n dimensional, $n \geq 4$, is isotropic over D_{k_v}. Let $< \alpha_v >$ be a 1 dimensional subform of h_{k_v} such that $< \alpha_v >^{\perp}$ is isotropic. Let $v \in S_1$. Since h_{k_v} is isotropic, choose a 1 dimensional subform, $< \alpha_v >$ of h_{k_v}, such that $< \alpha_v >^{\perp}$ is isotropic. By weak approximation, one may choose a 1 dimensional subform $< \alpha >$ of h such that $< \alpha >_{F_v} \cong < \alpha_v >$, for $v \in S_1 \cup S_2$. Let $h_1 = < \alpha >^{\perp}$. We claim that $(h_1)_{F_v}$ is isotropic over F_v, for every place $v \in \Omega_k$. This is by choice for $v \in S_1 \cup S_2$; in fact, $(h_1)_{k_v}$ itself is isotropic. If $v \notin S_1 \cup S_2$, v real and D_{k_v} is split, then h_{k_v} is definite, $cd(F_v) \leq 1$ and $(h_1)_{F_v}$ being equivalent to a quadratic form of rank ≥ 3, is isotropic. If $v \notin S_1 \cup S_2$, v real and D_{k_v} is not split, $(h_1)_{F_v}$ being of rank ≥ 2 is isotropic. If $v \notin S_1 \cup S_2$, v finite, D_{k_v} being split, $(h_1)_{F_v}$ corresponds to a quadratic form of rank at least 6 and hence is isotropic. Thus $(h_1)_{F_v}$ is isotropic and since D_{F_v} is split, $Sn((h_1)_{F_v}) = F_v^*$ modulo squares, for every $v \in \Omega_k$. By induction, $Sn((h_1)_F) = F^*/F^{*2}$. This completes the proof of the theorem. □

Corollary 6.3 *With the same notation as in (6.2), let B be a central simple algebra of degree 4 over k. If $\lambda \in F^*$ is such that λ^2 is a reduced norm from B_{F_v}, for all $v \in \Omega_k$, then λ^2 is a reduced norm from B_F.*

Proof. With notation as in [KMRT], there is an equivalence of categories ${}^1A_3 \cong {}^1D_3$, (cf. [KMRT], 15.32). Under this equivalence, let the degree 4 algebra $(B \times B^{op})$ over $(k \times k)$, with the switch involution, correspond to the degree 6 algebra A over k with an orthogonal involution σ, i.e., $C(A, \sigma) \cong (B \times B^{op})$. We note that $(A, \sigma) \cong (M_3(H), \tau_h)$, H a quaternion algebra over k and h a rank 3 skew hermitian form over (H, τ), τ denoting the standard involution of H. Further, $Spin(A, \sigma) = Spin(h)$. We denote the extension of these algebras with involution to F by $(B_F \times B_F^{op})$ and (A_F, σ) respectively. Then,

$$Sn(h_F) = \{\rho \in F^* \mid \rho^2 \in Nrd_{B_F}(B_F^*)\}, modulo\ squares,$$

(cf. [KMRT], 15.34). Hence, the element λ as in the statement of the corollary, is locally a spinor norm for (A_{F_v}, σ), for every $v \in \Omega_k$. By the above theorem (6.2), λ is a spinor norm for (A_F, σ). By the description for the spinor norms of (A_F, σ) given above, λ^2 is a reduced norm for B_F. This completes the proof of the corollary. □

Remark 6.4 *One does not know, even in the setting of the corollary, whether local reduced norms are reduced norms from B_F.*

Theorem 6.5 *With the same notation as in (6.2), let G be a semisimple simply connected linear algebraic group defined over k, of type D_n (non-trialitarian). Then the map*

$$H^1(F,G) \to \prod_{v\in\Omega_k} H^1(F_v,G)$$

has trivial kernel.

Proof. We may assume without loss of generality that G is absolutely almost simple. Hence G is isomorphic to $Spin(h)$, where h is a hermitian form over (D,σ), for some central division algebra D with an orthogonal involution σ over k. Since D is 2 torsion, D is either a quaternion division algebra over k or $D = k$. If $D = k$, then h is a quadratic form over k with $rank(h) \geq 3$ and the theorem is proved in (4.1). So we may assume that D is a division algebra over k. Let $rank(h) = n$. We have an exact sequence of linear algebraic groups,

$$1 \to \mu_2 \to Spin(h) \to SU(h) \to 1$$

which in turn gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
SU(h)(F) & \longrightarrow & F^*/F^{*2} & \longrightarrow & H^1(F,Spin(h)) & \longrightarrow & H^1(F,SU(h)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{v\in\Omega_k} SU(h)(F_v) & \longrightarrow & \prod_{v\in\Omega_k} F_v^*/F_v^{*2} & \longrightarrow & \prod_{v\in\Omega_k} H^1(F_v,Spin(h)) & \longrightarrow & \prod_{v\in\Omega_k} H^1(F_v,SU(h))
\end{array}$$

Let $\xi \in H^1(F,Spin(h))$ be locally trivial in $H^1(F_v,Spin(h))$, for all $v \in \Omega_k$. Then under the composite map,

$$H^1(F,Spin(h)) \to H^1(F,SU(h)) \to H^1(F,U(h))$$

the image of ξ in $H^1(F,U(h))$, defines a hermitian form h' which has the same rank and discriminant as h and further $c(h' \perp (-h)) = 0$. Let $Spin_{2n}(D,\sigma)$ and $U_{2n}(D,\sigma)$ denote respectively the spin and unitary groups of the hyperbolic form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $\xi' \in H^1(F,Spin_{2n}(D,\sigma))$ be a lift of $h' \perp (-h)$ in $H^1(F,U_{2n}(D,\sigma))$. Then $R(\xi') = R_h(\xi)$, where $R_h : H^1(F,Spin(h)) \to H^3(F,\mathbb{Q}/\mathbb{Z}(2))$ is the Rost invariant map (cf. 5.1). Since ξ is locally trivial, $R_h(\xi) \in H^3(F,\mathbb{Q}/\mathbb{Z}(2))$ is locally trivial. Since D is a quaternion algebra, $R_h(\xi)$ in fact belongs to $H^3(F,\mathbb{Z}/4\mathbb{Z})$ and the map $H^3(F,\mathbb{Z}/4\mathbb{Z}) \to \prod_{v\in\Omega_k} H^3(F_v,\mathbb{Z}/4\mathbb{Z})$ is injective (cf. 2.3). Hence $R_h(\xi)$ is trivial in $H^3(F,\mathbb{Z}/4\mathbb{Z})$. Hence by the classification theorem (cf. 5.4), $h \cong h'$ and the image of ξ in $H^1(F,U(h))$ is trivial. Let η be the image of ξ in $H^1(F,SU(h))$. Since the nontrivial element in $H^1(F,SU(h))$ which maps to the trivial element in $H^1(F,U(h))$ is not in the image of $H^1(F,Spin(h))$ (cf. [BP2], 7.11), it follows that η is trivial and hence in view of the exact sequence above, ξ comes from an element $\tilde{\xi} \in \frac{F^*/F^{*2}}{Im(Sn(h_F))}$. By the commutative diagram above, $\tilde{\xi}$ is locally trivial and by (6.2), $\tilde{\xi}$ and hence ξ is trivial. □

7 Rost invariant for special unitary groups

Let E be a field of characteristic different from 2 and L a quadratic field extension of E. Let (D, τ) be a quaternion division algebra over L with a unitary $L|E$ involution. Let $D_0 \subset D$ be a quaternion division algebra over E such that $D = D_0.L$ and τ restricted to D_0 is the canonical symplectic involution on D_0. For a hermitian form h over (D, τ), we denote the unitary and the special unitary group with respect to h by $U(h)$ and $SU(h)$ respectively. We have the following exact sequence of algebraic groups,

$$1 \to SU(h) \to U(h) \to R^1_{L|E}(G_m) \to 1$$

which gives rise to the following exact sequence in Galois cohomology,

$$U(h)(E) \stackrel{Nrd}{\to} L^{*1} \stackrel{\delta}{\to} H^1(E, SU(h)) \to H^1(E, U(h)) \qquad (\star)$$

The next proposition computes the Rost invariant on the image of δ. The proposition is also a consequence of ([MPT], theorem 1.9) (see Appendix).

Proposition 7.1 *With the notation as above, for $\mu \in L^{*1}$, $R(\delta(\mu)) = N_{L|E}(\nu) \cup (D_0) \in H^3(E, \mathbb{Q}/\mathbb{Z}(2))$, where $\nu \in L^*$ is such that $\mu = \nu^{-1}\,\tau(\nu)$.*

Proof. The element $N_{L|E}(\nu) \cup (D_0)$ is well defined with respect to μ, since for any $\lambda \in E^*$, $N_{L|E}(\nu) \cup (D_0) = N_{L|E}(\lambda\nu) \cup (D_0)$ in $H^3(E, \mathbb{Q}/\mathbb{Z}(2))$. Let X_μ be the torsor corresponding to $\delta(\mu)$. Let $E(X_\mu)$ denote the function field of X_μ. Rost has shown (cf. [G1], §2.3, theorem 1) that the kernel $\mathcal{K}_\mu$ of the map

$$H^3(E, \mathbb{Q}/\mathbb{Z}(2)) \stackrel{res}{\to} H^3(E(X_\mu), \mathbb{Q}/\mathbb{Z}(2)),$$

is a finite cyclic group generated by $R(\delta(\mu))$. We claim that $R(\delta(\mu))$ has order at most 2. We choose a quadratic extension field M of E such that D_{0M} is split. Set $ML = M \otimes_E L$. Then D_{ML} is split and $Nrd : U(h)(M) \to (ML)^{*1}$ is surjective. Hence $res(R(\delta(\mu)))$ is trivial in $H^3(M, \mathbb{Q}/\mathbb{Z}(2))$ and $cores(res(R(\delta(\mu)))) = 2.\, R(\delta(\mu)) = 0$.

As the torsor X_μ has a rational point over the field $E(X_\mu)$, $\delta(\mu)$ is trivial in $H^1(E(X_\mu), SU(h))$. Hence $\mu \in Nrd(U(h)(E(X_\mu)))$ and by (cf. [KMRT], pg. 202), $\mu = \theta^{-1}\,\tau(\theta)$, for some $\theta \in Nrd(D_{E(X_\mu)})$. Thus, $N_{L|E}(\nu) \cup (D_{0E(X_\mu)}) = N_{L\otimes_E E(X_\mu)\,|E(X_\mu)}(\theta) \cup (D_{0E(X_\mu)})$ in $H^3(E(X_\mu), \mathbb{Q}/\mathbb{Z}(2))$. Since $\theta \in Nrd(D_{E(X_\mu)})$, by the norm principle (2.2), $N_{L\otimes_E E(X_\mu)\,|E(X_\mu)}(\theta) \in Nrd(D_{0E(X_\mu)})$. Hence $N_{L|E}(\nu) \cup (D_{0E(X_\mu)}) = 0$ in $H^3(E(X_\mu), \mathbb{Q}/\mathbb{Z}(2))$ and $N_{L|E}(\nu) \cup (D_0) \in \mathcal{K}_\mu$. Since $\mathcal{K}_\mu$ is generated by $R(\delta(\mu))$, $N_{L|E}(\nu) \cup (D_0) = R(\delta(\mu))$ or $N_{L|E}(\nu) \cup (D_0) = 0$. Suppose $N_{L|E}(\nu) \cup (D_0) = 0$. Then there exists a quadratic extension P of E, such that D_0 is split over P and $N_{L|E}(\nu) = N_{P|E}(\alpha)$, for some $\alpha \in P^*$. Set $PL = P \otimes_E L$. By (cf. [W], lemma 2.13), there exist $\beta \in (PL)^*$ and $\delta \in E^*$, such that $N_{PL\,|L}(\beta) = \nu.\,\delta$. As D is split over PL, by the norm principle, (2.2), $\nu.\,\delta \in Nrd(D)$. As $\mu = (\nu.\,\delta)^{-1}\tau(\nu.\,\delta)$, by (cf. [KMRT], pg. 202),

$\mu \in Nrd(U(h)(E))$, $\delta(\mu)$ is trivial and $R(\delta(\mu)) = 0$. Hence if $\mu \in L^{*1}$ is not in $Nrd(U(h)(E))$, then $N_{L|E}(\nu) \cup (D_0)$ is not zero and hence coincides with $R(\delta(\mu))$. Thus in either case, $N_{L|E}(\nu) \cup (D_0) = R(\delta(\nu))$. □

Let $U_{2n}(D_0, \tau_0)$ denote the unitary group of the hyperbolic form $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ over (D_0, τ_0). We denote the unitary group and the special unitary group with respect to the hyperbolic form $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ by $U_{2n}(D, \tau)$ and $SU_{2n}(D, \tau)$ respectively. We have a natural inclusion $U_{2n}(D_0, \tau_0) \hookrightarrow U_{2n}(D, \tau)$. Since τ_0 is symplectic, the reduced norm of an element in $U_{2n}(D_0, \tau_0)$ has reduced norm 1 and we have the following diagram

$$\begin{tikzcd}
U_{2n}(D_0,\tau_0) \arrow[rr, "\rho_0"] \arrow[dr, "\tilde{\tilde{\rho}}"'] & & SU_{2n}(D,\tau) \arrow[dl] \\
& U_{2n}(D,\tau) &
\end{tikzcd}$$

which induces the following commutative diagram

$$\begin{tikzcd}
H^1(E, U_{2n}(D_0,\tau_0)) \arrow[rr, "\rho_0"] \arrow[dr, "\tilde{\tilde{\rho}}"'] & & H^1(E, SU_{2n}(D,\tau)) \arrow[dl] \\
& H^1(E, U_{2n}(D,\tau)) &
\end{tikzcd}$$

Proposition 7.2 *With the notation as above, if* $[h] \in H^1(E, U(D_0, \tau_0))$ *then* $R(h) = R(\rho_0(h))$.

Proof. By (cf. [KMRT], pg. 436), there exists an integer n_{ρ_0} such that $n_{\rho_0} R(h) = R(\rho_0(h))$. We show that $n_{\rho_0} = 1$. Let $X = R_{L|E}(X_D)$ where X_D is the Brauer Severi variety of D over L. Let $M = E(X)(X_1, \cdots, X_{2n})$. Then D_{0M} is not split, since $Br(E) \to Br(E(X))$ is injective, (cf. [MT], corollary 2.12) and $D_{0ML} = D_M$ is split. Let $L = E(\sqrt{d})$. Then $D_{0M} = (a, d)_M$, for some $a \in M^*$. Let i, $j \in D_{0M}$ be such that $i^2 = a$, $j^2 = d$, $ij = -ji$. We have the splitting $\phi : D_{0M} \otimes_M ML \cong M_2(ML)$, defined by,

$$\phi(i \otimes 1) = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad \phi(j \otimes 1) = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}.$$

An explicit computation shows that $\phi \circ \tau_{ML} \circ \phi^{-1} = Int(q_1) \circ T$, where

$$T \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \tau(x) & \tau(z) \\ \tau(y) & \tau(w) \end{pmatrix}$$

and $q_1 =< 1, -a >$. Under Morita equivalence, through ϕ, every τ-hermitian form over (D_{ML}, τ) corresponds to a $ML|M$ hermitian form. The (D_{ML}, τ)

hermitian form $h =< X_1, \cdots, X_{2n} >$ corresponds to an $ML|M$ hermitian form represented by $< X_1, \cdots, X_{2n} > \otimes < 1, -a >$, whose Rost invariant is $((-1)^n X_1. \cdots X_{2n}) \cup (a) \cup (d) = Pf(h) \cup (D_{0M}) \neq 0$, where $Pf(h)$ is the Pfaffian norm of h (cf. [KMRT], pg. 19). Since $R(h) = Pf(h) \cup (D_0)$ (cf. [KMRT], pg. 440), it follows that $n_{\rho_0} = 1$. □

8 Classification theorems for hermitian forms over quaternion division algebras with a unitary involution

Let $K = k(\sqrt{d})$ be a quadratic field extension of a field k of characteristic different from 2 and (D, τ) be a quaternion algebra over K with a $K|k$ involution τ. Let $D_0 \subset D$ be a quaternion k algebra such that τ restricted to D_0 is τ_0, the canonical involution of D_0 and $D = D_0 K$. We have $D = D_0 \oplus D_0\sqrt{d}$. For any hermitian form h over (D, τ), let

$$h(x, y) = h_1(x, y) + h_2(x, y)\sqrt{d}, \quad h_i(x, y) \in D_0, \text{ for } i = 1, 2.$$

Since $\tau(h(y, x)) = h(x, y)$ and $\tau(\sqrt{d}) = -\sqrt{d}$, it follows that $\tau_0(h_1(y, x)) = h_1(x, y)$ and $\tau_0(h_2(y, x)) = -h_2(x, y)$. Thus h_1 is a hermitian form over (D_0, τ_0) and h_2 is a skew-hermitian form over (D_0, τ_0). Let $p_1(h) = h_1$ and $p_2(h) = h_2$. Clearly $p_i(h \perp h') = p_i(h) \perp p_i(h')$ for $i = 1, 2$. Suppose that h is hyperbolic. Let W be a totally isotropic subspace of h, then W is also a totally isotropic subspace for $p_i(h)$, for $i = 1, 2$. Thus we have homomorphisms

$$p_1 : W(D, \tau) \to W(D_0, \tau_0)$$

and

$$p_2 : W(D, \tau) \to W^{-1}(D_0, \tau_0).$$

Let $\tilde{\rho} : W(D_0, \tau_0) \to W(D, \tau)$ be the homomorphism defined as follows: Let f be a hermitian form over D_0 and V_0 its underlying D_0 vector space. Let $V = V_0 \otimes_k K$ and write $V = V_0 \oplus V_0\sqrt{d}$. Define

$$\tilde{\rho}(f)(x_1 \oplus y_1\sqrt{d}, x_2 \oplus y_2\sqrt{d}) = f(x_1, x_2) + f(x_1, y_2)\sqrt{d} - f(y_1, x_2)\sqrt{d} - f(y_1, y_2)d.$$

It is easy to check that $\tilde{\rho}$ is a well defined homomorphism. We also have homomorphisms $\pi_i : W(K) \to W(k)$, for $i = 1, 2$, defined as follows. For any quadratic form q over K, write $q(x, y) = q_1(x, y) + q_2(x, y)\sqrt{d}$, where, $q_i(x, y) \in k$, for $i = 1, 2$. Then q_1 and q_2 are quadratic forms over k and $\pi_i(q) = q_i$, for $i = 1, 2$. Let $\tilde{\pi_1}$ be the composition $W(K) \xrightarrow{\pi_1} W(k) \to W(D_0, \tau_0)$, where the map $W(k) \to W(D_0, \tau_0)$ is induced by base change.

Proposition 8.1 *(Suresh) The following sequence:*

$$W(K) \stackrel{\tilde{\pi}_1}{\to} W(D_0, \tau_0) \stackrel{\tilde{\rho}}{\to} W(D, \tau) \stackrel{p_2}{\to} W^{-1}(D_0, \tau_0) \qquad (\star\star)$$

is exact.

Proof. Let f be a hermitian form over D_0 and V_0 its underlying D_0-vector space. Then the underlying vector space for $p_2\tilde{\rho}(f)$ is $V_0 \otimes_k K = V_0 \oplus V_0\sqrt{d}$ and $p_2\tilde{\rho}(f)(x_1 \oplus y_1\sqrt{d}, x_2 \oplus y_2\sqrt{d}) = f(x_1, y_2) - f(y_1, x_2)$. Thus the space $W = \{x \oplus 0 \mid x \in V_0\}$ is a totally isotropic subspace for $p_2\tilde{\rho}(f)$ and $W^{\perp} = W$. Therefore $p_2\tilde{\rho}(f) = 0$. Let h be an anisotropic hermitian form over D such that $p_2(h) = 0$. In particular, there exists a vector $x \neq 0$ such that $p_2(h)(x, x) = h_2(x, x) = 0$. This implies that $h(x, x) = h_1(x, x) = \alpha \in k$. Since h is anisotropic $\alpha \neq 0$. Therefore we can write $h =< \alpha >\perp h'$. It is easy to see that $\tilde{\rho}(< \alpha >) =< \alpha >$ and induction on the rank of h, yields the exactness at $W(D, \tau)$. We next show that $\tilde{\rho}\,\tilde{\pi}_1 = 0$. For $\theta = a + b\sqrt{d} \in K^*$, with $a, b \in k^*$, $\tilde{\pi}_1(< \theta >) \in W(D_0, \tau_0)$ is represented by the matrix $\begin{pmatrix} a & bd \\ bd & ad \end{pmatrix}$, which is equivalent to the diagonal form $< a, adN_{K|k}(\theta) >$. The form $\tilde{\rho}\,\tilde{\pi}_1(< \theta >) \in W(D, \tau)$, is also represented by the form $< a, adN_{K|k}(\theta) >$. Since $< 1, dN_{K|k}(\theta) >$ is equivalent to $< 1, -1 >$ over (D, τ), $\tilde{\rho}\,\tilde{\pi}_1(< \theta >) = 0$. Thus $\tilde{\rho}\,\tilde{\pi}_1 = 0$. Suppose (V_0, h) is an anisotropic hermitian form over (D_0, τ_0) such that $\tilde{\rho}(h) = 0$. Then there exists a vector $x_1 + y_1\sqrt{d} \neq 0 \in V_0 \oplus V_0\sqrt{d}$ such that $\tilde{\rho}(h)(x_1 + y_1\sqrt{d}, x_1 + y_1\sqrt{d}) = 0$. Then $h(x_1, x_1) = h(y_1, y_1)d$ and $h(x_1, y_1) = h(y_1, x_1)$. Set $a = h(y_1, y_1)$ and $bd = h(x_1, y_1)$. Then $\tilde{\pi}_1(< a + b\sqrt{d} >)$ is represented by the matrix $\begin{pmatrix} a & bd \\ bd & ad \end{pmatrix}$, which is the matrix representing h restricted to the subspace of V_0 spanned by (x_1, y_1). The proof of the proposition now follows by induction on the rank of h. □

Let $K = k(\sqrt{d})$ be a quadratic field extension of a field k of characteristic different from 2 and let D be a central division algebra over K with an involution τ of second kind over $K|k$. Let $SU_{2n}(D, \tau)$ be the special unitary group with respect to the hyperbolic form $H_{2n} = \begin{pmatrix} o & I_n \\ I_n & 0 \end{pmatrix}$. Let h be a hermitian form over (D, τ) of even rank $2n$ and trivial discriminant. Then there exists $\xi \in H^1(k, SU_{2n}(D, \tau))$, such that the image of ξ in $H^1(k, U_{2n}(D, \tau))$ is the class of h. We say that the Rost invariant $R(h)$ of h is zero, if there exists a $\xi \in H^1(k, SU_{2n}(D, \tau))$ lifting the class of h and such that $R(\xi) = 0$, where $R(\xi)$ is the Rost invariant associated to ξ.

Lemma 8.2 *Let K be a field such that $vcd(K) = n$. For any field extension E of K, with $[E : K] \leq 2$ assume that the maps $\overline{e_r} : I^r(E)/I^{r+1}(E) \to H^r(E)$ are well defined isomorphisms for all $r \geq 0$. Then the map $I^{n+1}(K) \to C(\mathcal{X}_K, 2^{n+1}\mathbb{Z})$ is surjective, $\mathcal{X}_K$ denoting the space of orderings of K.*

Proof. Let $\phi \in C(\mathcal{X}_K, 2^{n+1}\mathbb{Z})$. By ([S], 3.6.1), there exists a quadratic form $q \in W(K)$, such that $sgn(q) = 2^m\,\phi$, for some $m \geq 0$. Multiplying q by

$< 1, 1 >^{\otimes s}$, if necessary, we may assume that $q \in I^{n+1}(K)$. Suppose $m > 0$. We have the following commutative diagram:

$$\begin{array}{ccc} I^{n+1}(K) & \overset{sgn}{\to} & C(\mathcal{X}_K, 2^{n+1}\mathbb{Z}) \\ \downarrow e_{n+1} & & \downarrow mod 2^{n+2} \\ H^{n+1}(K) & \overset{h_{n+1}}{\to} & C(\mathcal{X}_K, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

where h_{n+1} is as defined in (cf. [AEJ1], remark following theorem 2.3). Since $m > 0$, the signature of q modulo 2^{n+2} is zero. We have an exact sequence in Galois cohomology,

$$H^r(K(\sqrt{-1})) \overset{cores}{\to} H^r(K) \overset{\cup(-1)}{\to} H^{r+1}(K) \to H^{r+1}(K(\sqrt{-1})).$$

Since $vcd(K) \leq n$, $H^r(K(\sqrt{-1})) = 0$, for $r \geq n+1$, so that $\cup(-1)$ is an isomorphism. Thus $H^{n+1}(K)$ is (-1)-torsion free. By ([AEJ1], 2.2 and 2.3), h_{n+1} is injective. Since $h_{n+1}(e_{n+1}(q)) = 0$, $e_{n+1}(q) = 0$. Since $\overline{e_{n+1}}$ is an isomorphism, $q \in I^{n+2}(K)$. Since the map $I^{n+1}(K) \overset{\otimes<1,1>}{\to} I^{n+2}(K)$ is surjective (cf.[AEJ1], pg. 22, remark following 1.16), there exists $q_1 \in I^{n+1}(K)$, such that $[< 1, 1 > \otimes q_1] = [q]$. We have $sgn(q_1) = 2^{m-1}\phi$. Repeating the process, we arrive at $q \in I^{n+1}(K)$ with $sgn(q) = \phi$. $\square$

We have the following classification theorem for hermitian forms.

Theorem 8.3 *Let $K = k(\sqrt{d})$ be a quadratic extension of a number field k. Let $k(X)$ be the function field of a smooth geometrically integral curve X over k and $K(X) = K \otimes_k k(X)$. Let (D, τ) be a quaternion division algebra over $K(X)$, with a $K(X)|k(X)$ unitary involution τ. Let h_1 and h_2 be hermitian forms over (D, τ) which have the same rank, discriminant and such that $R(h_1 \perp (-h_2)) = 0$. Suppose further that h_1 and h_2 are equivalent over $k_v(X)$, for every $v \in \Omega_k$. Then $h_1 \cong h_2$.*

Proof. Let $h = h_1 \perp (-h_2)$. Let $D_0 = (a, b)_{k(X)} \subset D$ be a quaternion algebra over $k(X)$, such that $D = D_0 .K(X)$ and τ restricted to D_0 is τ_0, τ_0 denoting the canonical involution on D_0. Let C be the conic, $aX_1^2 + bX_2^2 - 1 = 0$. The algebra $D \otimes_{k(X)} k(X)(C)$ is split and the hermitian form h over $D_{k(X)(C)}$ corresponds by Morita equivalence to a hermitian form over $K(X)(C)|k(X)(C)$, which in turn corresponds to a quadratic form $q(h)$ over $k(X)(C)$, of even rank, trivial discriminant and trivial Clifford and Rost invariants. Hence $[q(h)] \in I^4(k(X)(C))$. Further, $[q(h)]$ is zero in $W(k_v(X)(C))$, for every $v \in \Omega_k$. By (5.3), $I^4(k(X)(C)) \to \prod_{v\in\Omega_k} I^4(k_v(X)(C))$ is injective. Hence h is zero in $W(D_{k(X)(C)}, \tau)$. We have the following commutative diagram:

$$\begin{array}{ccc} W(D, \tau) & \overset{p_2}{\to} & W^{-1}(D_0, \tau_0) \\ \downarrow & & \downarrow \\ W(D_{k(X)(C)}, \tau) & \overset{p_2}{\to} & W^{-1}(D_{0_{k(X)(C)}}, \tau_0) \end{array}$$

with the second vertical map injective by (cf. [PSS]), so that $p_2(h)$ is zero in $W^{-1}(D_0, \tau_0)$. Hence by 8.1, there exists $h' \in W(D_0, \tau_0)$, such that $\tilde{\rho}(h') = h$.

We show that h' can be chosen to have trivial Pfaffian norm (cf. [KMRT], pg. 19). Since $R(h) = 0$, there exists a lift $\xi \in H^1(k(X), SU_{2n}(D,\tau))$ of h such that $R(\xi) = 0$. Since $\rho_0(h')$ is also a lift of h in $H^1(k(X), SU_{2n}(D,\tau))$, by (cf. [KMRT], pg. 387, last paragraph), there exists $\mu \in K(X)^{*1}$ such that $\rho_0(h')_{\tilde{\xi}} = \delta(\mu)$, where $\tilde{\xi}$ is a cocycle representing the cohomology class ξ and δ is the connecting map in ($\star$) for the groups $(SU_{2n}(h))_{\tilde{\xi}}$ and $(U_{2n}(h))_{\tilde{\xi}}$. By (cf. [G1], §2.3, lemma 7), $R(\rho_0(h')_{\tilde{\xi}}) = R(\rho_0(h')) + R(\xi)$. As $R(\xi) = 0$ we have, $R(\delta(\mu)) = R(\rho_0(h'))$. By (7.2), $R(\rho_0(h') = Pf(h') \cup (D_0)$. Let $\mu = \nu^{-1}\tau(\nu)$, for some $\nu \in K(X)^*$. Then by (7.1), $R(\delta(\mu)) = N_{K(X)\,|k(X)}(\nu) \cup (D_0) = Pf(h') \cup (D_0)$. Hence $Pf(h') = N_{K(X)\,|k(X)}(\nu).\,Nrd(x)$, for some $x \in D_0$. If $h' \cong < \lambda_1, \cdots, \lambda_{2n} >$, then replacing h' by the equivalent form $< \lambda_1\, x\tau(x), \cdots, \lambda_{2n} >$, we assume that $Pf(h') = N_{K(X)\,|k(X)}(\nu)$. Now replacing h' by the form $h' \perp < 1, -N_{K(X)\,|k(X)}(\nu) >$, we assume that $Pf(h')$ is trivial, noting that $\tilde{\rho}(< 1, -N_{K(X)\,|k(X)}(\nu) >) = 0$ in $W(D,\tau)$.

We have, $W(D_0,\tau_0) \cong W(k(X)).n_{D_0}$, under the map $f \mapsto q_f$, where $q_f(x,x) = f(x,x)$ and n_{D_0} denotes the norm form of D_0, (cf. §3). If $f \cong < \lambda_1, \cdots, \lambda_n > \in W(D_0,\tau_0)$ then $q_f = < \lambda_1, \cdots, \lambda_n > \otimes n_{D_0}$. We set $Q_f = < \lambda_1, \cdots, \lambda_n >$ as an element of $W(k(X))$. We note that for $f \in W(D_0,\tau_0)$, $Pf(f) = disc(Q_f)$.

As $Pf(h') = 1$, we have $Q_{h'} \in I^2(k(X))$. We claim that h' is in the image of $\tilde{\pi}_1$.

Consider the exact sequence ($\star\star$) locally, for a real place v of k such that $K_v = K \otimes k_v$ is a proper quadratic extension of k_v. Since $\tilde{\rho}((h')_{k_v(X)}) = 0$, there exists $f_v \in W(K_v(X))$ such that $[(h')_{k_v(X)}] = [\tilde{\pi}_1(f_v)]$. Hence $[q_{h'}] = [(Q_{h'} \otimes n_{D_0})_{k_v(X)}] = [\pi_1(f_v) \otimes n_{D_0}]$. Since $cd(K_v(X)) \leq 1$, $Br(K_v(X)) = 0$, so that $D_{0K_v(X)}$ is split. Hence $\pi_1(f_v) \otimes n_{D_0} = \pi_1(f_v \otimes n_{D_{0K_v(X)}}) = 0$. In particular, $(h')_{k_v(X)} = 0$. Consider a real place v of k, such that $K_v = K \otimes k_v$ is isomorphic to $K_{w_1} \times K_{w_2}$, where w_1 and w_2 are two orderings of K, extending the ordering v of k. Then the map $I^2(K_v(X)) \stackrel{\pi_1}{\to} I^2(k_v(X))$ is surjective, so that there exists $f_v \in I^2(K_v(X))$, such that $\pi_1(f_v) = (Q_{h'})_{k_v(X)}$. Let $f_v = (f_{w_1}, f_{w_2})$. We define a continuous function ϕ on $\mathcal{X}_{K(X)}$, as follows. The space $\mathcal{X}_{K(X)}$ is the union of open and closed sets $\mathcal{X}_{K_w(X)}$, w varying over the real orderings of K. For an ordering w of K lying over an ordering v of k, we set $\phi_w = sgn_w(f_v \otimes (n_{D_0})_{K_v(X)})$. Since $f_v \in I^2(K_v(X))$, $\phi_w \in C(\mathcal{X}_{K_w(X)}, 16\mathbb{Z})$, for every $w \in \mathcal{X}_{K_w(X)}$. By (8.2), there exists a quadratic form $q_2 \in I^4(K(X))$, such that $sgn_w(q_2) = \phi_w$. We claim that q_2 is a multiple of n_{D_0}. Consider the following commutative diagram:

$$\begin{array}{ccc} I^4(K(X)) & \stackrel{i_C}{\to} & I^4(K(X)(C)) \\ \downarrow & & \downarrow \\ \prod_{w\in\mathcal{X}_K} I^4(K_w(X)) & \to & \prod_{w\in\mathcal{X}_K} I^4(K_w(X)(C)) \end{array}$$

If w is a finite place of K, $I^4(K_w(X))$ is zero, so that, $(i_C(q_2))_w$ is zero. Let w be a real place of K. Since $sgn_w(q_2) = sgn_w(f_w \otimes n_{D_0})$, q_2 is Witt equivalent to $f_w \otimes n_{D_0}$, since the signature is the only invariant for quadratic forms in $I^4(K_w(X))$. Hence q_2 is split over $K_w(X)(C)$ and the element $i_C(q_2) \in I^4(K(X)(C))$ is locally zero, for every $w \in \mathcal{X}_K$. By (5.3), $i_C(q_2) = 0$. Hence $q_2 = q_3 \otimes n_{D_0}$, for some $q_3 \in W(K(X))$. Clearly, q_3 is even dimensional. Since $q_2 = q_3 \otimes n_{D_0} \in I^4(K(X))$ and $(q_3 \perp < 1, -disc(q_3) >) \otimes n_{D_0} \in I^4(K(X))$, $< 1, -disc(q_3) > \otimes n_{D_0} \in I^4(K(X))$ and being of rank 8 is zero. Replacing q_3 by $q_3 \perp < 1, -(disc(q_3)) >$ if necessary, we assume that $q_3 \in I^2(K(X))$. We have,

$$\begin{array}{rcl} sgn_v(\tilde{\pi}_1(q_3)) & = & sgn_v(\pi_1(q_3) \otimes n_{D_0}) \\ & = & sgn_v(\pi_1(q_3 \otimes n_{D_0})) \\ & = & sgn_v(\pi_1(q_2)) \\ & = & sgn_v(\pi_1(f_v \otimes n_{D_0})) \\ & = & sgn_v((Q_{h'})_{k_v(X)} \otimes n_{D_0}). \end{array}$$

Hence the form $q_{h'} \perp (-q_{\tilde{\pi}_1(q_3)}) \in I^4(K(X))$ is torsion. Since $I^4(K(X))$ is torsion free (cf. [AEJ2], cor.3), $q_{h'} \perp (-q_{\tilde{\pi}_1(q_3)})$ is equivalent to zero. Hence $h' = \tilde{\pi}_1(q_3)$ and $\tilde{\rho}(h') = h$ is zero in $W(D, \tau)$. $\Box$

9 A classification theorem for hermitian forms over division algebras of odd degree with a unitary involution

Let k be a number field and X a smooth geometrically integral curve over k. Let $k(X)$ be the function field of X and for $v \in \Omega_k$, let $k_v(X)$ denote the function field of the curve X_{k_v}. Let K be a quadratic field extension of k and $K(X) = K \otimes_k k(X)$ and for $v \in \Omega_k$, let $K_v(X) = K \otimes_k k_v(X)$. Let (D, τ) denote a central division algebra of odd degree over $K(X)$ with a $K(X)|k(X)$ unitary involution τ. We prove the following classification theorem:

Theorem 9.1 *Let the notation be as in the previous paragraph. Let h_1 and h_2 in $W(D, \tau)$ be hermitian forms of the same rank and discriminant and such that $h_1 \cong h_2$, locally over $k_v(X)$, for every $v \in \Omega_k$. Then $h_1 \cong h_2$ over $k(X)$.*

Proof. Let $h = h_1 \perp (-h_2)$. Then h has even rank, trivial discriminant and is locally zero in $W(D_{K_v(X)}, \tau)$. Let L be an odd degree field extension of $k(X)$ such that $D_{L \otimes_{k(X)} K(X)}$ is split, (cf. [BP1], 3.3.1). Let $L = l(Y)$, where l is the field of constants of L. By Morita equivalence h corresponds to a hermitian form over $L \otimes_{k(X)} K(X) | L$ and hence to a quadratic form $q(h)$ over L. Moreover, $q(h)$ has even rank, trivial discriminant, trivial Clifford invariant and is locally zero in $W(l_w(Y))$, for every $w \in \Omega_l$. Hence $q(h) \in I^3(l(Y))$ and is locally zero in $I^3(l_w(Y))$, for every $w \in \Omega_l$. By ([AEJ2], theorem 4), $q(h)$ is zero in $W(l(Y))$. As L is an odd degree extension of $k(X)$, by ([BL], theorem 2.1), h is zero in $W(D, \tau)$. Hence $h_1 \cong h_2$. $\Box$

10 Hasse principle for some groups of type ^2A_n

We begin with a result on the Hasse principle for special unitary groups of hermitian forms over quaternion algebras with unitary involutions.

Theorem 10.1 *Let (D, τ) be a quaternion division algebra over a number field K, with a $K|k$ unitary involution τ. Let X be a smooth geometrically integral curve over k. Let $k(X)$ be the function field of X and for each $v \in \Omega_k$, let $k_v(X)$ be the function field of the curve X_{k_v}. Let $K(X) = K \otimes_k k(X)$ and for $v \in \Omega_k$, let $K_v(X) = K \otimes_k k_v(X)$. Let h be a hermitian form over (D, τ). Let $SU(h)$ denote the special unitary group of h. Then the natural map $H^1(k(X), SU(h)) \to \prod_{v \in \Omega_k} H^1(k_v(X), SU(h))$ has trivial kernel.*

Proof. Let $\xi \in H^1(k(X), SU(h))$ be such that ξ is locally trivial in $H^1(k_v(X), SU(h))$, for every $v \in \Omega_k$. Under the map $H^1(k(X), SU(h)) \to H^1(k(X), U(h))$, let ξ map to the hermitian form h'. Then the hermitian form $h' \perp (-h)$ has even rank, trivial discriminant and is locally trivial. We claim that the Rost invariant, $R(h' \perp (-h))$ is trivial. We first note that as ξ is locally trivial, $R(\xi)$ is locally trivial in $H^3(k_v(X), \mathbb{Q}/\mathbb{Z}(2))$ for every $v \in \Omega_k$. Hence $R(\xi)$ is zero in $H^3(k(X), \mathbb{Q}/\mathbb{Z}(2))$, by (2.3). We now consider the map $SU(h) \to SU(h \perp (-h))$, given by, $f \mapsto (f, 1)$. This gives rise to a map from $H^1(F, SU(h)) \xrightarrow{i} H^1(F, SU(h \perp (-h)))$, and the image of ξ under this map corresponds to the hermitian form $h' \perp -h$ in $H^1(k(X), U(h \perp -h))$. By (cf. [KMRT], pg. 436), there exists an integer n_i, such that $n_i R(\xi) = R(i(\xi))$. By going over to a suitable field extension of k, where D is split and the Rost invariant is computed, we see that $n_i = 1$. Hence $R(i(\xi)) = 0$ and in particular, $R(h' \perp (-h)) = 0$. Since $h' \perp (-h)$ is a hermitian form of even rank, trivial discriminant, trivial Rost invariant and is locally trivial, by (8.3), we have $h' \cong h$ in $W(D, \tau)$. We have the following exact sequence of algebraic groups,

$$1 \to SU(h) \to U(h) \to R^1_{K(X)|k(X)}(G_m) \to 1$$

The above sequence gives rise to the following cohomology exact sequence,

$$U(h)(k(X)) \xrightarrow{Nrd} K^{*1} \to H^1(k(X), SU(h)) \to H^1(k(X), U(h)).$$

Since ξ maps to the trivial element in $H^1(k(X), U(h))$, there exists $\nu \in K(X)^{*1}$ such that under the connecting map $K(X)^{*1} \to H^1(k(X), SU(h))$, the image of ν is ξ. Since ξ is locally trivial, we have $\nu \in Nrd(U(h)(k_v(X)))$ for every $v \in \Omega_k$. We show that the natural map

$$K(X)^{*1}/\,Nrd(U(h)(k(X))) \to \prod_{v \in \Omega_k} K_v(X)^{*1}/\,Nrd(U(h)(k_v(X)))$$

is an injection. By (cf. [KMRT], pg. 202), we have,

$$\begin{aligned} Nrd(U(h)(k(X))) &= \{z\,\tau(z)^{-1} \mid z \in Nrd(D)\} \\ &= Nrd(U_2(D, \tau)(k(X))), \end{aligned}$$

where $U_2(D,\tau)$ is the unitary group of the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, in dimension 2. We have the following commutative diagram,

$$\begin{array}{ccccc} 1 & \longrightarrow & K(X)^{*1}/\,Nrd(U(h)(k(X))) & \longrightarrow & H^1(k(X), SU_2(D,\tau)) \\ & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod_{v\in\Omega_k} K_v(X)^{*1}/\,Nrd(U(h)(k_v(X))) & \longrightarrow & \prod_{v\in\Omega_k} H^1(k_v(X), SU_2(D,\tau)) \end{array} \qquad (\star\star\star)$$

Thus, to complete the proof of the theorem, we show that the natural map

$$H^1(k(X), SU_2(D,\tau)) \to \prod_{v\in\Omega_k} H^1(k_v(X), SU_2(D,\tau))$$

has trivial kernel.

Let $D = D_0\,.K$ with the restriction of τ to D_0 being the canonical involution on D_0. By (cf. [KMRT], 15.35 and 15.36), we have $SU_2(D,\tau) = Spin(q)$, where $q =< 1, -d >\perp n_{D_0}$, where $K = k(\sqrt{d})$ and n_{D_0} denotes the norm form on the quaternion algebra D_0. Hence there is a bijection

$$i : H^1(k(X), SU_2(D,\tau)) \overset{\cong}{\to} H^1(k(X), Spin(q))$$

and by (cf. 4.1), $H^1(k(X), Spin(q)) \to \prod_{v\in\Omega_k} H^1(k_v(X), Spin(q))$ has trivial kernel and hence $H^1(k(X), SU_2(D,\tau)) \to \prod_{v\in\Omega_k} H^1(k_v(X), SU_2(D,\tau))$ has trivial kernel. In particular, in diagram $(\star\star\star)$, the left vertical map is injective. This completes the proof of the theorem. □

The following proposition will be used in the proof of (10.4).

Proposition 10.2 *Let L be a quadratic field extension of a field E of characteristic not 2. Let (A,σ) be a central division algebra over L of even degree, with a $L|E$ unitary involution. Let h be a hermitian form over (A,σ). Then for any field extension M of E, we have,*

$$N_{M\otimes_E L\,|L}(Nrd(U(h)(M))) \subset Nrd(U(h)(E)).$$

Proof. Set $ML = M \otimes_E L$. Let $\phi_{L|E}$ and $\phi_{ML\,|M}$ denote the non trivial automorphisms of L over E and ML over M respectively. By (cf. [KMRT], pg. 202), $Nrd(U(h)(M)) = \{z\,\phi_{ML\,|M}(z)^{-1} \mid z \in Nrd(D_{ML})\}$. Let $x \in N_{ML\,|L}(Nrd(U(h)(M)))$. Then $x = N_{ML\,|L}(y\,\phi_{ML\,|M}(y)^{-1})$, for some $y \in Nrd(D_{ML})$. We note that $N_{ML\,|L}(\phi_{ML\,|M}(y)) = \phi_{L|E}(N_{ML\,|L}(y))$. As $N_{ML\,|L}(Nrd(D_{ML})) \subset Nrd(D)$, setting $t = N_{ML\,|L}(y)$, we have $t \in Nrd(D)$ and $x = t\,\phi_{L|E}(t^{-1})$, proving the proposition. □

Let (D,τ) be a division algebra with square free index over a number field K, with a $K|k$ unitary involution τ. Let X be a smooth geometrically integral curve over k. Let $k(X)$ be the function field of X and for each $v \in \Omega_k$, let $k_v(X)$ be the function field of the curve X_{k_v}. Let $K(X) = K \otimes_k k(X)$ and for $v \in \Omega_k$, let $K_v(X) = K \otimes_k k_v(X)$. In the next part of this section we prove the Hasse principle for groups of the form $SU(h)$, where h is a hermitian form over (D, τ). We begin with the following proposition.

Proposition 10.3 *With notation as above, suppose further that (D, τ) has odd degree over K. Let h be a hermitian form over (D, τ). Let $K(X)^{*1} = \{x \in K(X)^* \mid N_{K(X)|k(X)}(x) = 1\}$. Then the natural map*

$$K(X)^{*1}/\, Nrd(U(h)(k(X))) \to \prod_{v \in \Omega_k} K_v(X)^{*1}/\, Nrd(U(h)(k_v(X)))$$

is injective.

Proof. Let $\lambda \in K(X)^{*1}$ be locally in $Nrd(U(h)(k_v(X)))$, for every $v \in \Omega_k$. As degree D is odd, by a result of Suresh, (cf. [KMRT], pg. 202), $Nrd(U(h)(k(X))) = Nrd(D^*_{k(X)}) \cap K(X)^{*1}$. As D has square free index and λ is locally a reduced norm from $D_{k_v(X)}$, for every $v \in \Omega_k$, by (3.1), λ is a reduced norm for $D_{k(X)}$. Hence $\lambda \in Nrd(D^*_{k(X)}) \cap K(X)^{*1} = Nrd(U(h)(k(X)))$. $\square$

Theorem 10.4 *Let (D, τ) be a division algebra with square free index over a number field K, with a $K|k$ unitary involution τ. Let X be a smooth geometrically integral curve over k. Let $k(X)$ be the function field of X. Let h be a hermitian form over (D, τ). Let $SU(h)$ denote the special unitary group of h. Then the natural map $H^1(k(X), SU(h)) \to \prod_{v \in \Omega_k} H^1(k_v(X), SU(h))$ has trivial kernel.*

Proof. Let $\xi \in H^1(k(X), SU(h))$ be such that ξ is locally trivial in $H^1(k_v(X), SU(h))$, for every $v \in \Omega_k$. Under the map $H^1(k(X), SU(h)) \to H^1(k(X), U(h))$, let ξ map to the hermitian form h'. Then the hermitian form $h' \perp (-h)$ has even rank, trivial discriminant and is locally trivial. As ξ is locally trivial, the Rost invariant of ξ, $R(\xi)$ is locally trivial in $H^3(k_v(X), \mathbb{Q}/\mathbb{Z}(2))$ for every $v \in \Omega_k$. Hence $R(\xi)$ is zero in $H^3(k(X), \mathbb{Q}/\mathbb{Z}(2))$, by (2.3). Consider the map $SU(h) \to SU(h \perp (-h))$, given by, $f \mapsto (f, 1)$, which gives rise to a map from $H^1(F, SU(h)) \xrightarrow{i} H^1(F, SU(h \perp (-h)))$. The image of ξ under this map corresponds to the hermitian form $h' \perp -h$ in $H^1(k(X), U(h \perp -h))$. As in the proof of 10.1, one shows that $R(i(\xi)) = 0$. In particular, $R(h' \perp (-h)) = 0$. Hence $h' \perp (-h)$ is a hermitian form of even rank, trivial discriminant, trivial Rost invariant and is locally trivial. We claim that $h \cong h'$ over $k(X)$.

Suppose the degree of D is odd. Then by the classification theorem (9.1), $h \cong h'$.

Suppose the degree of D is even. Let $D \cong H \otimes_K D'$, where H is a quaternion division algebra over K and D' is an odd degree division algebra over K. Let L be an odd degree extension of k such that $(D \otimes_k L, \tau) \cong (M_r(H \otimes_k L), \sigma_f)$, where σ is a unitary $L \otimes_k K\,|L$ involution on $H \otimes_k L$ and σ_f, the adjoint involution on $M_r(H \otimes_k L)$ with respect to the hermitian form f over $(H \otimes_k L, \sigma)$, (cf. [BP1], 3.3.1). Let $l(Y) = L \otimes_k k(X)$, where l is the field of constants in $l(Y)$. Over $l(Y)$, by Morita theory, $h' \perp (-h)$ corresponds to a hermitian form h_1 over $(H_{l(Y)}, \sigma)$ of even rank, trivial discriminant, trivial Rost invariant and such that h_1 is locally zero in $W(H_{l_w(Y)}, \sigma)$, for every $w \in \Omega_l$. By (8.3), h_1 is zero in $W(H_{l(Y)}, \sigma)$ and hence $h' \perp (-h)$ is zero in $W(D_{l(Y)}, \tau)$. Since $[l(Y) : k(X)] = [L : k]$ is odd, by ([BL], theorem 2.1), $h' \perp (-h)$ is zero in $W(D_{k(X)}, \tau)$ and hence $h \cong h'$ and ξ maps to the trivial element in $H^1(k(X), U(h))$.

We have the following exact sequence of algebraic groups,

$$1 \to SU(h) \to U(h) \to R^1_{K(X)|k(X)}(G_m) \to 1$$

The above sequence gives rise to the following cohomology exact sequence,

$$U(h)(k(X)) \xrightarrow{Nrd} K^{*1} \to H^1(k(X), SU(h)) \to H^1(k(X), U(h)).$$

Since ξ maps to the trivial element in $H^1(k(X), U(h))$, there exists $\nu \in K(X)^{*1}$ such that under the natural map $K(X)^{*1} \to H^1(k(X), SU(h))$, the image of ν is ξ. Since ξ is locally trivial, we have $\nu \in Nrd(U(h)(k_v(X)))$ for every $v \in \Omega_k$. We show that the natural map from

$$K(X)^{*1} / \, Nrd(U(h)(k(X))) \to \prod_{v \in \Omega_k} K_v(X)^{*1} / \, Nrd(U(h)(k_v(X)))$$

is injective. If the degree of D is odd, then this follows from proposition 10.3. Hence we assume that the degree of D is even. Let $\lambda \in K(X)^{*1}$ be locally in $Nrd(U(h)(k_v(X)))$, for every $v \in \Omega_k$. Let H, D', L, $l(Y)$ and σ be as in the previous paragraph. As $H^1(l(Y), SU(h)) \to \prod_{w \in \Omega_l} H^1(l_w(Y), SU(h))$ has trivial kernel, (10.1), λ considered as an element of $l(Y)^*$ is in $Nrd(U(h)(l(Y)))$. By proposition (10.2), we have $N_{l(Y) \otimes_{k(X)} K(X) \, | K(X)}(U(h)(l(Y))) \subset Nrd(U(h)(k(X)))$. As the dimension of L over k is odd, $\lambda^{2r+1} \in Nrd(U(h)(k(X)))$, for some positive integer r. We show that $\lambda^2 \in Nrd(U(h)(k(X)))$. We choose a quadratic field extension N of k such that $H_{N \otimes_k K}$ is split. Then $(D_{N \otimes_k K}, \tau) \cong (M_2(D'_{N \otimes_k K}), \tau')$, for some $N \otimes_k K | N$ unitary involution τ'. The division algebra D' has odd degree and arguing as in the case of odd degree algebras, we have, $\lambda \in Nrd(U(h)(N \otimes_k k(X)))$. Hence $\lambda^2 \in Nrd(U(h)(k(X)))$. Thus, $\lambda \in Nrd(U(h)(k(X)))$ and the proof of the theorem is complete. □

11 The groups G_2 and F_4

For any field E, characteristic $E \neq 2$, if G is a semisimple simply connected absolutely almost simple linear algebraic group defined over E of type G_2, G is isomorphic to $Aut(C)$ where C is a Cayley algebra defined over E. The pointed set $H^1(E, G)$ classifies isomorphism classes of Cayley algebras over E. Given two Cayley algebras C and C', they are isomorphic if and only if their norm forms n_C and $n_{C'}$ are isomorphic. The norm form of a Cayley algebra is a 3-fold Pfister form over E.

Let k be a number field and X be a smooth geometrically integral curve defined over k. Let $F = k(X)$ be its function field and for every $v \in \Omega_k$ let $F_v = k_v(X)$ be the function field of X_{k_v}. Let G be as above of type G_2 over the field F. Then $G \cong Aut(C)$ for some Cayley algebra C over F. Let ξ be an element in $H^1(F, G)$ which is trivial in $H^1(F_v, G)$, for every $v \in \Omega_k$. The element ξ corresponds to a Cayley algebra $C(\xi)$ over F. By hypothesis, $n_C \cong n_{C(\xi)}$ over F_v for every $v \in \Omega_k$. Since the map $I^3(F) \to \prod_{v\in\Omega_k} I^3(F_v)$ is injective, (cf. [AEJ2], theorem 4), $n_C \cong n_{C(\xi)}$ over F so that $C \cong C(\xi)$ i.e., ξ is trivial.

For any field E of characteristic not 2 or 3, if G is a semisimple simply connected absolutely almost simple linear algebraic group defined over E, of type F_4, G is isomorphic to $Aut(J)$, J being a 27 dimensional central simple Jordan algebra over E. The set $H^1(E, G)$ classifies isomorphism classes of exceptional central simple Jordan algebras over E. Given such a Jordan algebra J over E, there are three invariants, $f_3(J) \in H^3(E)$, $f_5(J) \in H^5(E)$ and $g_3(J) \in H^3(E, \mathbb{Z}/3\mathbb{Z})$, (cf. [Se2], §9). The algebra J is reduced if and only if $g_3(J) = 0$. If J is reduced, the two invariants $f_3(J)$ and $f_5(J)$ completely determine the isomorphism class of J, thanks to the classification theorems of Springer (cf. [Sp], theorem 1).

Let k be an algebraic number field and $k(X)$ as above. Let J be a 27 dimensional exceptional central simple Jordan algebra over k and $G = Aut(J)$. Since $H^1(k(\sqrt{-1}), F_4) = (1)$, (cf. [Se2], §9.4), J is split over $k(\sqrt{-1})$. Hence $g_3(J) = 0$ and J is reduced. Let $\xi \in H^1(F, G)$ be trivial locally at all places of k. Let ξ correspond to an exceptional Jordan algebra J' over F. Since $J' \cong J \otimes F_v$ locally for all v in Ω_k, $g_3(J') = g_3(J \otimes F_v)$, for all $v \in \Omega_k$. Since $H^3(F, \mathbb{Z}/3\mathbb{Z}) \to \prod_{v\Omega_k} H^3(F_v, \mathbb{Z}/3\mathbb{Z})$ is injective (cf. 2.3), $g_3(J') = g_3(J \otimes F) = 0$. Hence J' is reduced. Similarly, as $f_3(J') = f_3(J \otimes F_v)$, for every $v \in \Omega_k$, we have $f_3(J') = f_3(J \otimes F)$. Since $f_5(J') = f_5(J \otimes F_v)$, for every $v \in \Omega_k$, we have $f_5(J') - f_5(J \otimes F)$ is in the kernel of the natural map $H^5(F) \to \prod_{w\in\mathcal{X}_F} H^5(F_w)$, $\mathcal{X}_F$ denoting all the orderings of F and hence is torsion. As $vcd(F) = 3$, $H^5(F)$ is torsion free. Hence $f_5(J') = f_5(J \otimes F)$, so that by Springer's theorem, $J' \cong J \otimes F$ and ξ is trivial.

12 The Hasse principle

The aim of this section is to prove the Hasse principle stated in the introduction. We say that a semisimple simply connected absolutely simple group over a field E is of type A^* if it is isomorphic to $SL_1(A)$ for a central simple algebra A over E of square free index or if it is isomorphic to $SU(B, \tau)$ for a central simple algebra B over a quadratic extension L of E of square free index with an $L|E$ involution τ.

Theorem 12.1 *Let k be a number field and X a smooth geometrically integral curve defined over k. Let $k(X)$ denote the function field of X and for every $v \in \Omega_k$, let $k_v(X)$ denote the function field of the curve X_{k_v}. Let G be a semisimple simply connected linear algebraic group defined over k, which is the product of the Weil restrictions of absolutely simple groups of types A^*, B_n, C_n, D_n (D_4 non-trialitarian), G_2, and F_4. Then the map*

$$H^1(k(X), G) \to \prod_{v \in \Omega_k} H^1(k_v(X), G)$$

has trivial kernel.

Proof. Recall that for a finite field extension L of a field E, if $G = R_{L|E}(G')$ is the Weil restriction of a linear algebraic group G' defined over L, then $H^1(E, G) = H^1(L, G')$. The theorem is now a consequence of (3.1, 3.2, 4.1, 4.2, 6.5, 10.1, 10.4 and §11). □

Appendix

Rost invariant for the special unitary groups

Let E be a field of characteristic different from 2 and $L = E(\sqrt{d})$ be a quadratic field extension of E. Let (D, τ) be a central division algebra over L with a unitary $L|E$ involution. For a hermitian form h over (D, τ), we denote the unitary and the special unitary groups with respect to h by $U(h)$ and $SU(h)$ respectively. We have the following exact sequence of algebraic groups,

$$1 \to SU(h) \to U(h) \to R^1_{L|E}(G_m) \to 1$$

which gives rise to the following exact sequence in Galois cohomology,

$$U(h)(E) \xrightarrow{Nrd} L^{*1} \xrightarrow{\delta} H^1(E, SU(h)) \to H^1(E, U(h)).$$

The next theorem computes the Rost invariant on the image of δ.

Theorem *With the notation as above, for $\mu \in L^{*1}$,*

$$R(\delta(\mu)) = Cores_{L|E}((\nu) \cup (D)) \in H^3(E, \mathbb{Q}/\mathbb{Z}(2)),$$

where $\nu \in L^*$ is such that $\mu = \nu\, \tau(\nu)^{-1}$.

Proof. We first show that $Cores_{L|E}((\nu) \cup (D))$ is well defined. Indeed, for $\lambda \in E^*$, we have

$$\begin{aligned} Cores_{L|E}((\nu\,\lambda) \cup (D)) &= Cores_{L|E}((\nu) \cup (D)) + Cores_{L|E}((\lambda) \cup (D)) \\ &= Cores_{L|E}((\nu) \cup (D)) + (\lambda) \cup Cores_{L|E}(D) \\ &= Cores_{L|E}((\nu) \cup (D)), \end{aligned}$$

since $Cores_{L|E}(D) = 0$. Set $\xi = Cores_{L|E}((\nu) \cup (D))$. If $\delta(\mu) = 1$, i.e., $\mu \in Nrd(U(h)(E))^*$ then ν can be chosen to be in $Nrd(D)^*$ (cf. [KMRT], pg.

202). Hence $(\nu) \cup (D) = 0$ and $\xi = 0$. Further, $R(\delta(\mu)) = 0$. Hence, in this case, $R(\delta(\mu)) = \xi = 0$. We now assume that $\delta(\mu) \neq 1$. By ([KMRT], pg.438), we have, $R(\delta(\mu))_L = (\mu) \cup (D) = (\nu) \cup (D) + (\tau(\nu)) \cup (D^{-1}) = \xi_L$. Hence corestricting to E, we get, $2.\, R(\delta(\mu)) = 2.\, \xi$.

case.1. Suppose degree (D) is odd. We choose a field extension M of E of degree n, with n odd, such that $D \otimes_E (M \otimes_E L)$ is split. Set $ML = M \otimes_E L$. Since D is split over ML, $\xi_M = 0$. Further, $U(h)(M) \overset{Nrd}{\to} (ML)^{*1}$ is surjective, so that $\delta(\mu)_M = 1$. Hence $R(\delta(\mu))_M = 0$. Since $Cores_{M|E} \circ res$ coincides with multiplication by n, we have $n.\, \xi = n.\, R(\delta(\mu)) = 0$. As $2.\, \xi = 2.\, R(\delta(\mu))$, we have $\xi = R(\delta(\mu))$.

case.2. Suppose degree $(D) = 2^n$, for some positive integer n. Let $\nu = a + b\sqrt{d}$, for some $a,\, b \in E$. As $\mu \notin Nrd(U(h)(E))$, we have, $b \neq 0$. Consider the rational function field $E(t)$. We extend the base field E to $E(t)$. Set $\mu_t = \frac{t+b\sqrt{d}}{t-b\sqrt{d}}$ and $\nu_t = t + b\sqrt{d}$. Let X_{μ_t} be the torsor corresponding to $\delta(\mu_t) \in H^1(E(t), SU(h))$. Let $E(t)(X_{\mu_t})$ denote the function field of X_{μ_t}. By a result of Rost (cf. [G1], §2.3, theorem 1), the kernel $\mathcal{K}_{\mu_t}$, of the map

$$H^3(E(t), \mathbb{Q}/\mathbb{Z}(2)) \overset{res}{\to} H^3(E(t)(X_{\mu_t}), \mathbb{Q}/\mathbb{Z}(2)),$$

is a finite cyclic group generated by $R(\delta(\mu_t))$. Since $\delta(\mu_t)$ is trivial over $E(t)(X_{\mu_t})$, $\mu_t \in Nrd(U(h)(E(t)))$. Hence there exists $\lambda \in E(t)(X_{\mu_t})^*$ such that $\lambda.\, \nu_t \in Nrd(D_{E(t)(X_{\mu_t})})$ (cf. [KMRT], pg. 202). Set $\xi_t = Cores_{L(t)|E(t)}((\nu_t) \cup (D))$. Then over $E(t)(X_{\mu_t})$, we have,

$$\xi_{t\,E(t)(X_{\mu_t})} = Cores_{L(t)(X_{\mu_t})|E(t)(X_{\mu_t})}((\lambda.\, \nu_t) \cup (D)) = 0.$$

Therefore $\xi_t \in \mathcal{K}_{\mu_t}$. Let s be the order of $R(\delta(\mu_t))$. Then there exists a positive integer $r \leq s$ such that $\xi_t = r.\, R(\delta(\mu_t))$. Since $\xi_{t\,L(t)} = R(\delta(\mu_t))_{L(t)}$, $2.\, \xi_t = 2.\, R(\delta(\mu_t))$ and hence $(2r-2)\, R(\delta(\mu_t)) = 0$. Hence $2r - 2 = sl$, for some positive integer l and $r = \frac{sl}{2} + 1$. If l is even, we have $\xi_t = R(\delta(\mu_t))$. Suppose l is an odd integer. Then $\xi_t = (\frac{s}{2} + 1) R(\delta(\mu_t))$. In this case, we show that $s = 2m$, where m denotes the exponent of D. Suppose $s \neq 2m$. We first note that $\frac{s}{2}.\, R(\delta(\mu_t))_{L(t)} = (\xi_t - R(\delta(\mu_t)))_{L(t)} = 0$. We have, $m.\, R(\delta(\mu_t))_{L(t)} = m.\, \xi_{L(t)} = m.\, ((\mu_t) \cup (D)) = (\mu_t) \cup (D^m) = 0$. Hence over $E(t)$, $2m.\, R(\delta(\mu_t)) = 0$. As s is the order of $R(\delta(\mu_t))$, s divides $2m$. As m is a power of 2, $\frac{s}{2}.\, R(\delta(\mu_t))_{L(t)} = 0$ and $s \neq 2m$, we have $\frac{m}{2}.\, R(\delta(\mu_t))_{L(t)} = 0$. Let $\partial_{(t-a)} : H^3(L(t), \mathbb{Q}/\mathbb{Z}(2)) \to H^2(L, \mathbb{Q}/\mathbb{Z}(1))$ denote the residue with respect to the prime $(t - a)$ in $L(t)$ (cf. [G1], §1.3). We have, $\partial_{(t-a)}((\mu_t) \cup (D)) = (D)$. Since $R(\delta(\mu_t))_{L(t)} = (\mu_t) \cup (D)$ and $\frac{m}{2}.\, R(\delta(\mu_t))_{L(t)} = 0$, we have $D^{\frac{m}{2}} = 0$ in $Br(L)$, which is a contradiction. Hence $s = 2m$. Since $m.\, \xi_t = Cores_{L(t)|E(t)}((\nu_t) \cup (D^m)) = 0$, we have

$$\begin{aligned} (m+1).\, \xi_t &= \xi_t \\ &= (\tfrac{s}{2} + 1).\, R(\delta(\mu_t)) \\ &= (m+1).\, R(\delta(\mu_t)). \end{aligned}$$

As $2.\, \xi_t = 2.\, R(\delta(\mu_t))$ and $m + 1$ is odd, we have $\xi_t = R(\delta(\mu_t))$.

Let $\mathcal{O}$ be the ring of integers of the completion $L((t-a))$ of $L(t)$ with respect to the discrete valuation corresponding to the prime $(t-a)$ on $L(t)$. Let $\mathcal{G}$ be a semi simple simply connected $\mathcal{O}$ group scheme with the special fibre isomorphic to $SU(h)$ over the residue field L at the prime $(t-a)$. We have the following commutative diagram (cf. [G1], theorem 2)

$$\begin{array}{ccc} H^1(L((t-a)), \mathcal{G}_{L((t-a))}) & \overset{R_{L((t-a))}}{\to} & H^3(L((t-a)), \mathbb{Q}/\mathbb{Z}(2)) \\ \uparrow & & \uparrow \\ H^1_{et}(\mathcal{O}, \mathcal{G}) & & \Big| \\ \wr\downarrow & & \Big| \\ H^1(L, SU(h)) & \overset{R_L}{\to} & H^3(L, \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

The torsor $\delta(\mu_t)$ over $L((t-a))$ comes from a torsor for $\mathcal{G}$ over $\mathcal{O}$, since μ_t is a unit in $\mathcal{O}$ and it specialises to $\delta(\mu)$ in $H^1(L, SU(h))$. In view of the above commutative diagram, $R(\delta(\mu))_{L((t-a))} = R(\delta(\mu_t)) = Cores_{L((t-a))|E((t-a))}((\nu_t) \cup (D))$. Since characteristic E is coprime to m, $\nu_t = b\sqrt{d} + t = b\sqrt{d} + a + (t-a) = (a + b\sqrt{d}).\alpha^m$, for some $\alpha \in L((t-a))$. Set $M = E((t-a))$ and $ML = L((t-a))$. Hence $Cores_{ML|M}((\nu_t) \cup (D)) = Cores_{ML|M}(((a + b\sqrt{d}).\alpha^m) \cup (D)) = Cores_{L|E}((a + b\sqrt{d}) \cup (D))_{ML} + Cores_{ML|M}((\alpha^m) \cup (D))$. Since $Cores_{ML|M}((\alpha^m) \cup (D)) = Cores_{ML|M}((\alpha) \cup (D^m)) = 0$, we have $R(\delta(\mu))_{ML} = Cores_{L|E}((a+b\sqrt{d}) \cup (D))_{ML}$. Since the map $H^3(L, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(ML, \mathbb{Q}/\mathbb{Z}(2))$ is injective, (cf. [G1], §1.3), we have $R(\delta(\mu)) = Cores_{L|E}((a + b\sqrt{d}) \cup (D))$.

case.3. Suppose degree $(D) = 2^l.m$, where m is odd. In this case, we choose an extension M of E of odd degree n such that $D_{M\otimes_E L}$ has degree some power of 2. Set $ML = M \otimes_E L$. By the previous case, $R(\delta(\mu))_M = Cores_{ML|M}((\nu) \cup (D_{ML})) = Cores_{L|E}((\nu) \cup (D))_M$. Since $Cores_{ML|M} \circ res$ coincides with multiplication by n, we have $n.R(\delta(\mu)) = n.Cores_{L|E}((\nu) \cup (D))$. As $2.R(\delta(\mu)) = 2.Cores_{L|E}((\nu) \cup (D))$, we have $R(\delta(\mu)) = Cores_{L|E}((\nu) \cup (D))$. $\square$

Remark The above result is also a consequence of a theorem of Merkurjev-Parimala-Tignol, (cf. [MPT], theorem 1.9), in view of the following commutative diagram

$$\begin{array}{ccccccc} U(h)(E) & \xrightarrow{Nrd} & L^{*1} & \xrightarrow{\delta} & H^1(E, SU(h)) & \longrightarrow & H^1(E, U(h)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ PGU(h)(E) & \xrightarrow{\delta} & H^1(E, \mu_{n[L]}) & \longrightarrow & H^1(E, SU(h)) & \longrightarrow & H^1(E, U(h)) \end{array}$$

where $PGU(h)$ is the projective unitary group with respect to h and $\mu_{n[L]} = kernel(R_{L|E}(\mu_n) \overset{N_{L|E}}{\to} \mu_n)$. The proof of Merkurjev-Parimala-Tignol, uses invariants of quasi-trivial tori.

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