FUNDAMENTAL HERMITE CONSTANTS OF LINEAR ALGEBRAIC GROUPS

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Abstract. Let G be a connected reductive algebraic group defined over a global field k and Q a maximal k-parabolic subgroup of G. The constant $\gamma(G,Q,k)$ attached to (G,Q) is defined as an analogue of Hermite's constant. This constant depends only on G, Q and k in contrast to the previous definition of generalized Hermite constants ([W1]). Some functorial properties of $\gamma(G,Q,k)$ are proved. In the case that k is a function field of one variable over a finite field, $\gamma(GL_n,Q,k)$ is computed.

Let k be an algebraic number field of finite degree over $\mathbb{Q}$ and let G be a connected reductive algebraic group defined over k. In [W1], we introduced a constant γ_π^G attached to an absolutely irreducible strongly k-rational representation $\pi\colon G \longrightarrow GL(V_\pi)$ of G. More precisely, if $G(\mathbb{A})$ denotes the adele group of G and $G(\mathbb{A})^1$ the unimodular part of $G(\mathbb{A})$, it is defined by

$$\gamma_\pi^G = \max_{g\in G(\mathbb{A})^1} \min_{\gamma\in G(k)} ||\pi(g\gamma)x_\pi||^{2/[k:\mathbb{Q}]}\,,$$

where x_π is a non-zero k-rational point of the highest weight line in the representation space V_π and $||\cdot||$ is a height function on the space $GL(V_\pi(\mathbb{A}))V_\pi(k)$. This constant is called a generalized Hermite constant by the reason that, in the case when $k=\mathbb{Q}$, $G=GL_n$ and $\pi=\pi_d$ is the d-th exterior representation of GL_n, $\gamma_{\pi_d}^{GL_n}$ is none other than the Hermite - Rankin constant ([R]):

$$\gamma_{n,d} = \max_{g\in GL_n(\mathbb{R})} \min_{\substack{x_1,\cdots,x_d\in\mathbb{Z}^n\\ x_1\wedge\cdots\wedge x_d\neq 0}} \frac{\det({}^t x_i {}^t g g x_j)_{1\le i,j\le d}}{|\det g|^{2d/n}}\,.$$

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When GL_n is defined over a general k, then $\gamma_{\pi_d}^{GL_n}$ coincides with the following generalization of $\gamma_{n,d}$ due to Thunder ([T2]):

$$\gamma_{n,d}(k) = \max_{g\in GL_n(\mathbb{A})} \min_{X\in \mathrm{Gr}_d(k^n)} \frac{H_g(X)^2}{|\det g|_{\mathbb{A}}^{2d/(n[k:\mathbb{Q}])}},$$

where $\mathrm{Gr}_d(k^n)$ is the Grassmannian variety of d-dimensional subspaces in k^n and H_g a twisted height on $\mathrm{Gr}_d(k^n)$. In a general G, γ_π^G has a geometrical representation similarly to $\gamma_{n,d}(k)$. In order to describe this, we change our primary object from a representation π to a parabolic subgroup of G. Thus, we first fix a k-parabolic subgroup Q of G, and then take a representation π such that the stabilizer Q_π of the highest weight line of π in G is equal to Q. The mapping $g \mapsto \pi(g^{-1})x_\pi$ gives rise to a k-rational embedding of the generalized flag variety $Q\backslash G$ into the projective space $\mathbb{P}V_\pi$. Taking a k-basis of $V_\pi(k)$, we get a height H_π on $\mathbb{P}V_\pi(k)$, and on $Q(k)\backslash G(k)$ by restriction. In this notation, γ_π^G is represented as

$$\gamma_\pi^G = \max_{g\in G(\mathbb{A})^1} \min_{x\in Q(k)\backslash G(k)} H_\pi(xg)^2.$$

In this paper, we investigate γ_π^G more closely when Q is a maximal k-parabolic subgroup of G. Especially, we shall show that π and H_π are not essentials of the constant γ_π^G, to be exact, there exists a constant $\gamma(G,Q,k)$ depending only on G, Q and k such that the equality $\gamma_\pi^G = \gamma(G,Q,k)^{c_\pi}$ holds for any π with $Q_\pi = Q$, where c_π is a positive constant depending only on π. This $\gamma(G,Q,k)$ is called the fundamental Hermite constant of (G,Q) over k. We emphasize that there is a similarity between the definition of $\gamma(G,Q,k)$ and a representation of the original Hermite's constant $\gamma_{n,1}$ as the maximum of some lattice constants. Remember that $\gamma_{n,1}$ is represented as

$$\gamma_{n,1}^{1/2} = \max_{\substack{g\in GL_n(\mathbb{R})\\ |\det g|=1}} \min\{T>0\colon B_T^n \cap g\mathbb{Z}^n \neq \{0\}\},$$

where B_T^n stands for the ball of radius T with center 0 in $\mathbb{R}^n$. Corresponding to $\mathbb{R}^n$, we consider the adelic homogeneous space $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ as a base space. The set X_Q of k-rational points of $Q\backslash G$ plays a role of the standard lattice $\mathbb{Z}^n$. In addition, there is a notion of "the ball" B_T of radius T in Y_Q, whose precise definition will be given in Section 2. Then $\gamma(G,Q,k)$ is defined by

$$\gamma(G,Q,k) = \max_{g\in G(\mathbb{A})^1} \min\{T>0\colon B_T \cap X_Q g \neq \emptyset\}.$$

Independency of $\gamma(G,Q,k)$ on π and H_π allows us to study some functorial properties of fundamental Hermite constants. For instance, the following theorems will be verified in Section 4.

Theorem. *If $\beta\colon G \longrightarrow G'$ is a surjective k-rational homomorphism of connected reductive groups defined over k such that its kernel is a central k-split torus in G, then $\gamma(G, Q, k) = \gamma(G', \beta(Q), k)$.*

Theorem. *If $R_{k/\ell}$ denotes the functor of restriction of scalars for a subfield $\ell \subset k$, then $\gamma(R_{k/\ell}(G), R_{k/\ell}(Q), \ell) = \gamma(G, Q, k)$.*

Theorem. *If both Q and R are standard maximal k-parabolic subgroups of G and M_R is a standard Levi subgroup of R, then one has an inequality of the form*

$$\gamma(G, Q, k) \leq \gamma(M_R, M_R \cap Q, k)^{\omega_1} \gamma(G, R, k)^{\omega_2} ,$$

where ω_1 and ω_2 are rational numbers explicitly determined from Q and R.

These theorems are including the duality theorem: $\gamma_{n,j}(k) = \gamma_{n,n-j}(k)$ for $1 \leq j \leq n-1$ and Rankin's inequality ([R], [T2]): $\gamma_{n,i}(k) \leq \gamma_{j,i}(k)\gamma_{n,j}(k)^{i/j}$ for $1 \leq i < j \leq n-1$ as a particular case.

Since no any serious problem arises from replacing k with a function field of one variable over a finite field, we shall develop a theory of fundamental Hermite constants for any global field. In the case of number fields, the main theorem of [W1] gives a lower bound of $\gamma(G, Q, k)$. An analogous result will be proved for the case of function fields in the last half of this paper. The case of $G = GL_n$ is especially studied in detail because this case gives an analogue of the classical Hermite – Rankin constants. When k is a function field, it is almost trivial from definition that $\gamma(G, Q, k)$ is a power of the cardinal number q of the constant field of k. Thus, the possible values of $\gamma(G, Q, k)$ are very restricted if both lower and upper bounds are given. This is a striking difference between the number fields and the function fields. For example, it will be proved that $\gamma(GL_n, Q, k) = 1$ for all maximal Q and all $n \geq 2$ provided that the genus of k is zero, i.e., k is a rational function field over a finite field.

The paper is organized as follows. In Section 1, we recall the Tamagawa measures of algebraic groups and homogenous spaces. In Sections 2 and 3, the constant $\gamma(G, Q, k)$ is defined, and then a relation between $\gamma(G, Q, k)$ and γ_π^G is explained. The functorial properties of $\gamma(G, Q, k)$ is proved in Section 4. In Section 5, we will give a lower bound of $\gamma(G, Q, k)$ when k is a function field, and compute $\gamma(GL_n, Q, k)$ in Section 6.

Notation. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the field of rational, real and complex numbers, respectively. The group of positive real numbers is denoted by $\mathbb{R}_+^\times$.

Let k be a global field, i.e., an algebraic number field of finite degree over $\mathbb{Q}$ or an algebraic function field of one variable over a finite field. In the latter case, we identify the constant field of k with the finite field $\mathbb{F}_q$ with q elements. Let $\mathfrak{V}$ be the set of all places of k. We write $\mathfrak{V}_\infty$ and $\mathfrak{V}_f$ for the sets of all infinite places and all finite places of k, respectively. For $v \in \mathfrak{V}$, k_v denotes the completion of k at v. If v is finite, $\mathfrak{O}_v$ denotes the ring of integers in k_v, $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{O}_v$, $\mathfrak{f}_v$ the residual field $\mathfrak{O}_v/\mathfrak{p}_v$ and q_v the order of $\mathfrak{f}_v$. We fix, once and for all, a Haar measure μ_v on k_v normalized so that $\mu_v(\mathfrak{O}_v) = 1$ if $v \in \mathfrak{V}_f$, $\mu_v([0,1]) = 1$ if v is a real place and $\mu_v(\{a \in k_v\colon a\overline{a} \leq 1\}) = 2\pi$ if v is an imaginary place. Then the absolute value $|\cdot|_v$ on k_v is defined as $|a|_v = \mu_v(aC)/\mu_v(C)$, where C is an arbitrary compact subset of k_v with nonzero measure.

Let $\mathbb{A}$ be the adele ring of k, $|\cdot|_{\mathbb{A}} = \prod_{v\in\mathfrak{V}} |\cdot|_v$ the idele norm on the idele group $\mathbb{A}^\times$ and $\mu_{\mathbb{A}} = \prod_{v\in\mathfrak{V}} \mu_v$ an invariant measure on $\mathbb{A}$. The measure $\mu_{\mathbb{A}}$ is characterized by

$$\mu_{\mathbb{A}}(\mathbb{A}/k) = \begin{cases} |D_k|^{1/2} & \text{(if } k \text{ is an algebraic number field of discriminant } D_k\text{).} \\ q^{g(k)-1} & \text{(if } k \text{ is a function field of genus } g(k)\text{).} \end{cases}$$

In general, if μ_A and μ_B denote Haar measures on a locally compact unimodular group A and its closed unimodular subgroup B, respectively, then $\mu_B\backslash\mu_A$ (resp. μ_A/μ_B) denotes a unique right (resp. left) A-invariant measure on the homogeneous space $B\backslash A$ (resp. A/B) matching with μ_A and μ_B.

1. Tamagawa measures.

Let G be a connected affine algebraic group defined over k. For any k-algebra A, $G(A)$ stands for the set of A-rational points of G. Let $\mathbf{X}^*(G)$ and $\mathbf{X}^*_k(G)$ be the free $\mathbb{Z}$-modules consisting of all rational characters and all k-rational characters of G, respectively. The absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on $\mathbf{X}^*(G)$. The representation of $\mathrm{Gal}(\overline{k}/k)$ in the space $\mathbf{X}^*(G)\otimes_{\mathbb{Z}}\mathbb{Q}$ is denoted by σ_G and the corresponding Artin L-function is denoted by $L(s,\sigma_G) = \prod_{v\in\mathfrak{V}_f} L_v(s,\sigma_G)$. We set $\sigma_k(G) = \lim_{s\to 1}(s-1)^n L(s,\sigma_G)$, where $n = \operatorname{rank}\mathbf{X}^*_k(G)$. Let ω^G be a nonzero right invariant gauge form on G defined over k. From ω^G and the fixed Haar measure μ_v on k_v, one can construct a right invariant Haar measure ω^G_v on $G(k_v)$. Then, the Tamagawa measure on $G(\mathbb{A})$ is well defined by

$$\omega^G_{\mathbb{A}} = \mu_{\mathbb{A}}(\mathbb{A}/k)^{-\dim G}\omega^G_\infty\omega^G_f\,,$$

where

$$\omega^G_\infty = \prod_{v\in\mathfrak{V}_\infty}\omega^G_v \ \text{ and } \ \omega^G_f = \sigma_k(G)^{-1}\prod_{v\in\mathfrak{V}_f} L_v(1,\sigma_G)\omega^G_v\,.$$

For each $g \in G(\mathbb{A})$, we define the homomorphism $\vartheta_G(g)\colon \mathbf{X}^*_k(G) \longrightarrow \mathbb{R}^\times_+$ by $\vartheta_G(g)(\chi) = |\chi(g)|_{\mathbb{A}}$ for $\chi \in \mathbf{X}^*_k(G)$. Then ϑ_G is a homomorphism from $G(\mathbb{A})$ into $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}^*_k(G),\mathbb{R}^\times_+)$.

We write $G(\mathbb{A})^1$ for the kernel of ϑ_G. The Tamagawa measure $\omega_{G(\mathbb{A})^1}$ on $G(\mathbb{A})^1$ is defined as follows:

• The case of $\mathrm{ch}(k) = 0$. If a $\mathbb{Z}$-basis $\chi_1, \cdots, \chi_n$ of $\mathbf{X}_k^*(G)$ is fixed, then $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(G), \mathbb{R}_+^\times)$ is identified with $(\mathbb{R}_+^\times)^n$ and ϑ_G gives rise to an isomorphism from $G(\mathbb{A})^1 \backslash G(\mathbb{A})$ onto $(\mathbb{R}_+^\times)^n$. Put the Lebesgue measure dt on $\mathbb{R}$ and the invariant measure dt/t on $\mathbb{R}_+^\times$. Then $\omega_{G(\mathbb{A})^1}$ is the measure on $G(\mathbb{A})^1$ such that the quotient measure $\omega_{G(\mathbb{A})^1} \backslash \omega_{\mathbb{A}}^G$ is the pullback of the measure $\prod_{i=1}^n dt_i/t_i$ on $(\mathbb{R}_+^\times)^n$ by ϑ_G. The measure $\omega_{G(\mathbb{A})^1}$ is independent of the choice of a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$.

• The case of $\mathrm{ch}(k) > 0$. The value group of the idele norm $|\cdot|_{\mathbb{A}}$ is the cyclic group $q^{\mathbb{Z}}$ generated by q (cf. [We2]). Thus the image $\mathrm{Im}\vartheta_G$ of ϑ_G is contained in $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(G), q^{\mathbb{Z}})$ and $G(\mathbb{A})^1$ is an open normal subgroup of $G(\mathbb{A})$. Since the index of $\mathrm{Im}\vartheta_G$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(G), q^{\mathbb{Z}})$ is finite ([Oe, I, Proposition 5.6]),

$$d_G^* = (\log q)^{\mathrm{rank}\mathbf{X}_k^*(G)} [\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(G), q^{\mathbb{Z}}) \colon \mathrm{Im}\vartheta_G] \tag{1.1}$$

is well defined. The measure $\omega_{G(\mathbb{A})^1}$ is defined to be the restriction of the measure $(d_G^*)^{-1}\omega_{\mathbb{A}}^G$ to $G(\mathbb{A})^1$.

In both cases, we put the counting measure $\omega_{G(k)}$ on $G(k)$. The volume of $G(k)\backslash G(\mathbb{A})^1$ with respect to the measure $\omega_G = \omega_{G(k)} \backslash \omega_{G(\mathbb{A})^1}$ is called the Tamagawa number of G and denoted by $\tau(G)$.

In the following, let G be a connected reductive group defined over k. We fix a maximally k-split torus S of G, a maximal k-torus S_1 of G containing S, a minimal k-parabolic subgroup P of G containing S and a Borel subgroup B of P containing S_1. Denote by Φ_k and Δ_k the relative root system of G with respect to S and the set of simple roots of Φ_k corresponding to P, respectively. Let M be the centralizer of S in G. Then P has a Levi decomposition $P = MU$, where U is the unipotent radical of P. For every standard k-parabolic subgroup R of G, R has a unique Levi subgroup M_R containing M. We denote by U_R the unipotent radical of R. Throughout this paper, we fix a maximal compact subgroup K of $G(\mathbb{A})$ satisfying the following property; For every standard k-parabolic subgroup R of G, $K \cap M_R(\mathbb{A})$ is a maximal compact subgroup of $M_R(\mathbb{A})$ and $M_R(\mathbb{A})$ possesses an Iwasawa decomposition $(M_R(\mathbb{A}) \cap U(\mathbb{A}))M(\mathbb{A})(K \cap M_R(\mathbb{A}))$. We set $K^{M_R} = K \cap M_R(\mathbb{A})$, $P^R = M_R \cap P$ and $U^R = M_R \cap U$.

Let R be a standard k-parabolic subgroup of G and Z_R be the greatest central k-split torus in M_R. The restriction map $\mathbf{X}_k^*(M_R) \longrightarrow \mathbf{X}_k^*(Z_R)$ is injective. Since $\mathbf{X}_k^*(M_R)$ has the same rank as $\mathbf{X}_k^*(Z_R)$, both indexes

$$d_R = [\mathbf{X}_k^*(Z_R) : \mathbf{X}_k^*(M_R)] \quad \text{and} \quad \widehat{d}_R = [\mathbf{X}_k^*(Z_R/Z_G) : \mathbf{X}_k^*(M_R/Z_G)]$$

are finite. We define another Haar measure $\nu_{M_R(\mathbb{A})}$ of $M_R(\mathbb{A})$ as follows. Let $\omega_{\mathbb{A}}^M$ and $\omega_{\mathbb{A}}^{U^R}$ be the Tamagawa measures of $M(\mathbb{A})$ and $U^R(\mathbb{A})$, respectively. The modular character $\delta_{P^R}^{-1}$ of $P^R(\mathbb{A})$ is a function on $M(\mathbb{A})$ which satisfies the integration formula

$$\int_{U^R(\mathbb{A})} f(mum^{-1})d\omega_{\mathbb{A}}^{U^R}(u) = \delta_{P^R}(m)^{-1} \int_{U^R(\mathbb{A})} f(u)d\omega_{\mathbb{A}}^{U^R}(u).$$

Let $\nu_{K^{M_R}}$ be the Haar measure on K^{M_R} normalized so that the total volume equals one. Then the mapping

$$f \mapsto \int_{U^R(\mathbb{A})\times M(\mathbb{A})\times K^{M_R}} f(nmh)\delta_{P^R}(m)^{-1}d\omega_{\mathbb{A}}^{U^R}(u)d\omega_{\mathbb{A}}^{M}(m)d\nu_{K^{M_R}}(h)\,, \quad (f \in C_0(M_R(\mathbb{A})))$$

defines an invariant measure on $M_R(\mathbb{A})$ and is denoted by $\nu_{M_R(\mathbb{A})}$. There exists a positive constant C_R such that

$$\omega_{\mathbb{A}}^{M_R} = C_R \nu_{M_R(\mathbb{A})}\,.$$

We have the following compatibility formula:

$$(1.2) \qquad \int_{G(\mathbb{A})} f(g)d\omega_{\mathbb{A}}^G(g) = \frac{C_G}{C_R}\int_{U_R(\mathbb{A})\times M_R(\mathbb{A})\times K} f(umh)\delta_R(m)^{-1}d\omega_{\mathbb{A}}^{U_R}d\omega_{\mathbb{A}}^{M_R}(m)d\nu_K(h)$$

for $f \in C_0(G(\mathbb{A}))$, where δ_R^{-1} is the modular character of $R(\mathbb{A})$.

On the homogeneous space $Y_R = R(\mathbb{A})^1\backslash G(\mathbb{A})^1$, we define the right $G(\mathbb{A})^1$-invariant measure ω_{Y_R} by $\omega_{R(\mathbb{A})^1}\backslash\omega_{G(\mathbb{A})^1}$. We note that both $G(\mathbb{A})^1$ and $R(\mathbb{A})^1$ are unimodular.

2. Definition of fundamental Hermite constants.

Throughout this paper, Q denotes a standard maximal k-parabolic subgroup of G. There is an only one simple root $\alpha \in \Delta_k$ such that the restriction of α to Z_Q is non-trivial. Let n_Q be the positive integer such that $n_Q^{-1}\alpha|_{Z_Q}$ is a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(Z_Q/Z_G)$. We write α_Q and $\widehat{\alpha}_Q$ for $n_Q^{-1}\alpha|_{Z_Q}$ and $\widehat{d}_Q n_Q^{-1}\alpha|_{Z_Q}$, respectively. Then $\widehat{\alpha}_Q$ is a $\mathbb{Z}$-basis of the submodule $\mathbf{X}_k^*(M_Q/Z_G)$ of $\mathbf{X}_k^*(Z_Q/Z_G)$. If we set $e_Q = n_Q \dim U_Q$ and $\widehat{e}_Q = n_Q \dim U_Q/\widehat{d}_Q$, then

$$\delta_Q(z) = |\alpha_Q(z)|_{\mathbb{A}}^{e_Q} \quad \text{and} \quad \delta_Q(m) = |\widehat{\alpha}_Q(m)|_{\mathbb{A}}^{\widehat{e}_Q}$$

holds for $z \in Z_Q(\mathbb{A})$ and $m \in M_Q(\mathbb{A})$.

Define a map $z_Q \colon G(\mathbb{A}) \longrightarrow Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A})$ by $z_Q(g) = Z_G(\mathbb{A})M_Q(\mathbb{A})^1 m$ if $g = umh$, $u \in U_Q(\mathbb{A}), m \in M_Q(\mathbb{A})$ and $h \in K$. This is well defined and a left $Z_G(\mathbb{A})Q(\mathbb{A})^1$-invariant. Since $Z_G(\mathbb{A})^1 = Z_G(\mathbb{A}) \cap G(\mathbb{A})^1 \subset M_Q(\mathbb{A})^1$, z_Q gives rise to a map from $Y_Q =$

$Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ to $M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A})\cap G(\mathbb{A})^1)$. Namely, we have the following commutative diagram:

$$\begin{array}{ccc} Y_Q & \xrightarrow{z_Q} & M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A})\cap G(\mathbb{A})^1) \\ \downarrow & & \downarrow \\ Z_G(\mathbb{A})Q(\mathbb{A})^1\backslash G(\mathbb{A}) & \xrightarrow{z_Q} & Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A}) \end{array}$$

In this diagram, the vertical arrows are injective, and in particular, these are bijective if $\mathrm{ch}(k)=0$. We further define a function $H_Q\colon G(\mathbb{A})\longrightarrow\mathbb{R}_+^\times$ by $H_Q(g)=|\widehat{\alpha}_Q(z_Q(g))|_{\mathbb{A}}^{-1}$ for $g\in G(\mathbb{A})$. This has the following property:

• The case of $\mathrm{ch}(k)=0$. Let Z_G^+ and Z_Q^+ be the subgroups of $Z_G(\mathbb{A})$ and $Z_Q(\mathbb{A})$, respectively, defined as in [W1]. Then H_Q gives a bijection from $Z_G^+\backslash Z_Q^+$ onto $\mathbb{R}_+^\times$. If $(H_Q|_{Z_G^+\backslash Z_Q^+})^{-1}$ denotes the inverse map of this bijection, then the map

$$i_Q\colon \mathbb{R}_+^\times\times K\longrightarrow Y_Q\colon (t,h)\mapsto Q(\mathbb{A})^1(H_Q|_{Z_G^+\backslash Z_Q^+})^{-1}(t)h$$

is surjective.

• The case of $\mathrm{ch}(k)>0$. The value group $|\widehat{\alpha}_Q(M_Q(\mathbb{A})\cap G(\mathbb{A})^1)|_{\mathbb{A}}$ is a subgroup of $q^{\mathbb{Z}}$. Let $q_0=q_0(Q)$ be the generator of $|\widehat{\alpha}_Q(M_Q(\mathbb{A})\cap G(\mathbb{A})^1)|_{\mathbb{A}}$ that is greater than one. Then H_Q gives a surjection from Y_Q onto the cyclic group $q_0^{\mathbb{Z}}$.

We set $X_Q=Q(k)\backslash G(k)$, which is regarded as a subset of Y_Q. Let $B_T=\{y\in Y_Q\colon H_Q(y)\le T\}$ for $T>0$. The volume of B_T is given by

$$\omega_{Y_Q}(B_T)=\begin{cases} \dfrac{C_G d_Q}{C_Q d_G e_Q}T^{\widehat{e}_Q} & (\mathrm{ch}(k)=0) \\ \dfrac{C_G d_Q^*}{C_Q d_G^*}\dfrac{q_0^{[\log_{q_0}T]\widehat{e}_Q}}{1-q_0^{-\widehat{e}_Q}} & (\mathrm{ch}(k)>0) \end{cases}$$

where $[\log_{q_0}T]$ is the largest integer which is not exceeding $\log_{q_0}T$ (cf. [W1, Lemma1] and Lemma 1 in §5).

Proposition 1. *For $T>0$ and any $g\in G(\mathbb{A})^1$, $B_T\cap X_Qg$ is a finite set. Hence, one can define the function*

$$\Gamma_Q(g)=\min\{T>0\colon B_T\cap X_Qg\neq\emptyset\}=\min_{y\in X_Qg}H_Q(y)$$

on $G(\mathbb{A})^1$. Then the maximum

$$\gamma(G,Q,k)=\max_{g\in G(\mathbb{A})^1}\Gamma_Q(g)$$

exists.

Proposition 1 will be proved in the next section.

Definition. The constant $\gamma(G,Q,k)$ is called the fundamental Hermite constant of (G,Q) over k.

We often write γ_Q for $\gamma(G,Q,k)$ if k and G are clear from the context. The constant γ_Q is characterized as the greatest positive number T_0 such that $B_T \cap X_Q g_T = \emptyset$ for any $T < T_0$ and some $g_T \in G(\mathbb{A})^1$. It is obvious by definition that $\gamma_Q \in q_0^{\mathbb{Z}}$ if $\mathrm{ch}(k) > 0$.

Remark. Let $\widetilde{Y}_Q = Z_G(\mathbb{A})Q(\mathbb{A})^1\backslash G(\mathbb{A})$. Then, for any $g \in G(\mathbb{A})$, $X_Q g$ is regarded as a subset of $\widetilde{Y}_Q$. In some cases, it is more convenient to consider the constant

$$\widetilde{\gamma}(G,Q,k) = \max_{g\in G(\mathbb{A})} \min_{y\in X_Q g} H_Q(y)\,.$$

In general, $\gamma(G,Q,k) \leq \widetilde{\gamma}(G,Q,k)$ holds. If $\mathrm{ch}(k) = 0$ or G is semisimple, then $\gamma(G,Q,k) = \widetilde{\gamma}(G,Q,k)$ because of $\widetilde{Y}_Q = Y_Q$.

Remark. If $\mathrm{ch}(k) = 0$, one can consider the more general Hermite constant defined by

$$\gamma(G,Q,D,k) = \max_{g\in G(\mathbb{A})^1} \min\{T > 0 \colon i_Q((0,T] \times D) \cap X_Q g \neq \emptyset\}$$

for an open and closed subset D of K.

3. A relation between γ_Q and a generalized Hermite constant.

We recall the definition of generalized Hermite constants ([W1, §2]). Let V_π be a finite dimensional $\overline{k}$-vector space defined over k and $\pi\colon G \longrightarrow GL(V_\pi)$ be an absolutely irreducible k-rational representation. The highest weight space in V_π with respect to B is denoted by x_π. Let Q_π be the stabilizer of x_π in G and λ_π the rational character of Q_π by which Q_π acts on x_π. In the following, we assume $Q = Q_\pi$ and π is strongly k-rational, i.e., x_π is defined over k. Then λ_π is a k-rational character of Q_π. It is known that such π always exists (cf. [Ti1], [W1]). We use a right action of G on V_π defined by $a \cdot g = \pi(g^{-1})a$ for $g \in G$ and $a \in V_\pi$. Then the mapping $g \mapsto x_\pi \cdot g$ gives rise to a k-rational embedding of $Q\backslash G$ into the projective space $\mathbb{P}V_\pi$. We fix a k-basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$ of the k-vector space $V_\pi(k)$ and define a local height H_v on $V_\pi(k_v)$ for each $v \in \mathfrak{V}$ as follows:

$$H_v(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = \begin{cases} (|a_1|_v^2 + \cdots + |a_n|_v^2)^{1/2} & \text{(if } v \text{ is real).} \\ |a_1|_v + \cdots + |a_n|_v & \text{(if } v \text{ is imaginary).} \\ \sup(|a_1|_v, \cdots, |a_n|_v) & \text{(if } v \in \mathfrak{V}_f\text{).} \end{cases}$$

The global height H_π on $V_\pi(k)$ is defined to be a product of all H_v, that is, $H_\pi(a) = \prod_{v\in\mathfrak{V}} H_v(a)$. By the product formula, H_π is invariant by scalar multiplications. Thus, H_π defines a height on $\mathbb{P}V_\pi(k)$, and on X_Q by restriction. The height H_π is extended to $GL(V_\pi(\mathbb{A}))\mathbb{P}V_\pi(k)$ by

$$H_\pi(\xi\overline{a}) = \prod_{v\in\mathfrak{V}} H_v(\xi_v a)$$

for $\xi = (\xi_v) \in GL(V_\pi(\mathbb{A}))$ and $\overline{a} = ka \in \mathbb{P}V_\pi(k)$, $a \in V_\pi(k) - \{0\}$. Put

$$\Phi_{\pi,\xi}(g) = H_\pi(\xi(x_\pi \cdot g))/H_\pi(\xi x_\pi), \qquad (g \in G(\mathbb{A})).$$

Since this satisfies

$$\Phi_{\pi,\xi}(gg') = |\lambda_\pi(g)^{-1}|_{\mathbb{A}}\Phi_{\pi,\xi}(g'), \qquad (g \in Q(\mathbb{A}),\ \ g' \in G(\mathbb{A})),$$

$\Phi_{\pi,\xi}$ defines a function on Y_Q. We can and do choose a $\xi \in GL(V_\pi(\mathbb{A}))$ so that $\Phi_{\pi,\xi}$ is right K-invariant. Then, in the case of $\mathrm{ch}(k) = 0$, the generalized Hermite constant attached to π is defined by

$$\gamma_\pi = \max_{g\in G(\mathbb{A})^1} \min_{x\in X_Q} \Phi_{\pi,\xi}(xg)^{2/[k:\mathbb{Q}]}\,. \tag{3.1}$$

Let us prove Proposition 1. We take positive rational numbers e_π and $\widehat{e}_\pi$ such that

$$|\lambda_\pi(z)|_{\mathbb{A}} = |\alpha_Q(z)|_{\mathbb{A}}^{e_\pi} \quad \text{and} \quad |\lambda_\pi(m)|_{\mathbb{A}} = |\widehat{\alpha}_Q(m)|_{\mathbb{A}}^{\widehat{e}_\pi}$$

for $z \in Z_Q(\mathbb{A}) \cap G(\mathbb{A})^1$ and $m \in M_Q(\mathbb{A}) \cap G(\mathbb{A})^1$. Then, by definition,

$$\Phi_{\pi,\xi}(y) = H_Q(y)^{\widehat{e}_\pi}\,, \qquad (y \in Y_Q).$$

Therefore, one has

$$B_T \cap X_Q = \{x \in X_Q\colon H_\pi(\xi x) \le H_\pi(\xi x_\pi)T^{\widehat{e}_\pi}\}.$$

Since $\sharp\{x \in \mathbb{P}V_\pi(k)\colon H_\pi(\xi x) \le c\}$ is finite for a fixed constant c (cf. [S]), $B_T \cap X_Q$ is a finite set. If $g \in G(\mathbb{A})^1$ is given, then there is a $T_g > 0$ depending on g such that $B_T g^{-1} \subset B_{T_g}$. This implies that $\sharp(B_T \cap X_Q g) = \sharp(B_T g^{-1} \cap X_Q)$ is also finite. Furthermore, we obtain

$$\Gamma_Q(g) = \min_{x\in X_Q} \Phi_{\pi,\xi}(xg)^{1/\widehat{e}_\pi}\,.$$

In [W1, Proposition 2], we proved in the case of $\mathrm{ch}(k) = 0$ that the function in $g \in G(\mathbb{A})^1$ defined by the right hand side attains its maximum. The same proof works well for the case of $\mathrm{ch}(k) > 0$ by using the reduction theory due to Harder ([H]). We mention its proof for the sake of completeness. If necessary, by replacing G with $G/(\mathrm{Ker}\pi)^0$, we may assume $\mathrm{Ker}\pi$ is finite. Let

$$S(\mathbb{A})_c = \{z \in S(\mathbb{A}) \colon |\beta(z)|_{\mathbb{A}}^{-1} \leq c \text{ for all } \beta \in \Delta_k\}$$

and

$$S(\mathbb{A})'_c = \{z \in S(\mathbb{A}) \colon c^{-1} \leq |\beta(z)|_{\mathbb{A}}^{-1} \leq c \text{ for all } \beta \in \Delta_k\}$$

for a sufficiently large constant $c > 1$. By reduction theory, there are compact subsets $\Omega_1 \in P(\mathbb{A})$ and $\Omega_2 \in G(\mathbb{A})$ such that $K \subset \Omega_2$ and $G(\mathbb{A}) = G(k)\Omega_1 S(\mathbb{A})_c \Omega_2$. Set $\mathfrak{S}(c) = \Omega_1 S(\mathbb{A})_c \Omega_2 \cap G(\mathbb{A})^1$ and $\mathfrak{S}(c)' = \Omega_1 S(\mathbb{A})'_c \Omega_2 \cap G(\mathbb{A})^1$. There is a constant c' such that

$$\min_{x \in X_Q} \Phi_{\pi,\xi}(x\omega_1 z\omega_2) \leq \Phi_{\pi,\xi}(\omega_1 z \omega_2) \leq c' |\lambda_\pi(z)|_{\mathbb{A}}^{-1}$$

holds for all $\omega_1 \in \Omega_1, z \in S(\mathbb{A})_c$ and $\omega_2 \in \Omega_2$. The highest weight λ_π can be written as a $\mathbb{Q}$-linear combination of simple roots modulo $\mathbf{X}_k^*(Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e.,

$$\lambda_\pi|_S \equiv \sum_{\beta \in \Delta_k} c_\beta \beta \mod \mathbf{X}_k^*(Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

A crucial fact is $c_\beta > 0$ for all $\beta \in \Delta_k$ (cf. [W1, Proof of Proposition 2]). From this and the above inequality, it follows

$$\sup_{g \in \mathfrak{S}(c)} \min_{x \in X_Q} \Phi_{\pi,\xi}(xg) = \sup_{g \in \mathfrak{S}(c)'} \min_{x \in X_Q} \Phi_{\pi,\xi}(xg)\,.$$

This implies that the function $g \mapsto \min_{x \in X_Q} \Phi_{\pi,\xi}(xg)$ attains its maximum since $\mathfrak{S}(c)'$ is relatively compact in $G(\mathbb{A})^1$ modulo $G(k)$. Therefore, the maximum

$$\gamma_Q = \max_{g \in G(\mathbb{A})^1} \min_{x \in X_Q} \Phi_{\pi,\xi}(xg)^{1/\widehat{e}_\pi} \tag{3.2}$$

exists. This completes the proof of Proposition 1.

Next theorem is obvious by (3.1), (3.2), $e_\pi = \widehat{d}_Q \widehat{e}_\pi$, $e_Q = \widehat{d}_Q \widehat{e}_Q$ and [W1, Theorem 1].

Theorem 1. *If* $\mathrm{ch}(k) = 0$*, then the Hermite constant attached to a strongly* k*-rational representation* π *is given by*

$$\gamma_\pi = \gamma_Q^{2\widehat{e}_\pi/[k:\mathbb{Q}]}\,.$$

One has an estimate of the form

$$\left(\frac{C_Q d_G e_Q \tau(G)}{C_G d_Q \tau(Q)}\right)^{1/\widehat{e}_Q} \leq \gamma_Q\,. \tag{3.3}$$

Example 1. Let V be an n dimensional vector space defined over an algebraic number field k and $\mathbf{e}_1, \cdots, \mathbf{e}_n$ a k-basis of $V(k)$. We identify the group of linear automorphisms of V with GL_n. For $1 \leq j \leq n-1$, Q_j denotes the stabilizer of the subspace spanned by $\mathbf{e}_1, \cdots, \mathbf{e}_j$ in GL_n and $\pi_j \colon GL_n \longrightarrow GL(\bigwedge^j V)$ the j-th exterior representation. A k-basis of $V_{\pi_j}(k) = \bigwedge^j V(k)$ is formed by the elements $\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_j}$ with $I = \{1 \leq i_1 < i_2 < \cdots < i_j \leq n\}$. The global height H_{π_j} is defined similarly as above with respect to the basis $\{\mathbf{e}_I\}_I$. By definition and $H_{\pi_j}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_j) = 1$, we have

$$\begin{aligned}\gamma_{n,j}(k) = \gamma_{\pi_j} &= \max_{g \in GL_n(\mathbb{A})^1} \min_{x \in Q_j(k)\backslash GL_n(k)} H_{\pi_j}(x \cdot g)^{2/[k:\mathbb{Q}]} \\ &= \max_{g \in GL_n(\mathbb{A})} \min_{\substack{x_1, \cdots, x_j \in V(k) \\ x_1 \wedge \cdots \wedge x_j \neq 0}} \frac{H_{\pi_j}(gx_1 \wedge \cdots \wedge gx_j)^{2/[k:\mathbb{Q}]}}{|\deg g|_{\mathbb{A}}^{2j/(n[k:\mathbb{Q}])}}.\end{aligned}$$

Let $\gcd(j, n-j)$ be the greatest common divisor of j and $n-j$. It is easy to see that

$$\widehat{d}_{Q_j} = \frac{j(n-j)}{\gcd(j,n-j)}, \qquad \widehat{e}_{Q_j} = \gcd(j,n-j), \qquad \widehat{e}_{\pi_j} = \frac{\gcd(j,n-j)}{n}. \tag{3.4}$$

Therefore,

$$\gamma(GL_n, Q_j, k) = \gamma_{n,j}(k)^{n[k:\mathbb{Q}]/(2\gcd(j,n-j))},$$

and in particular, $\gamma(GL_n, Q_1, \mathbb{Q})^{2/n}$ is none other than the classical Hermite's constant $\gamma_{n,1}$. By [T2] and [W1, Example 2], we have

$$\left(\frac{|D_k|^{j(n-j)/2} n}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod_{i=n-j+1}^{n} Z_k(i)}{\prod_{j=2}^{j} Z_k(j)} \right)^{1/\gcd(j,n-j)} \leq \gamma(GL_n, Q_j, k),$$

$$\gamma(GL_n, Q_j, k) \leq \left(\frac{2^{r_1+r_2}|D_k|^{1/2}}{\pi^{r/2}} \Gamma(1+\frac{n}{2})^{r_1/n} \Gamma(1+n)^{r_2/n} \right)^{jn/\gcd(j,n-j)},$$

where $\zeta_k(s)$ denotes the Dedekind zeta function of k, $\Gamma(s)$ the gamma function, $Z_k(s) = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s)$, r_1 and r_2 the numbers of real and imaginary places of k, respectively. When $j = 1$, the next inequality was proved in [O-W]:

$$\gamma(GL_n, Q_1, k) \leq |D_k|^{1/[k:\mathbb{Q}]} \frac{\gamma(GL_{n[k:\mathbb{Q}]}, Q_1, \mathbb{Q})}{[k:\mathbb{Q}]}.$$

4. Some properties of fundamental Hermite constants.

First, we consider the exact sequence

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{\beta} G' \longrightarrow 1$$

of connected reductive groups defined over a global field k. We assume the following two conditions for Z:

(4.1) Z is central in G.

(4.2) Z is isomorphic to a product of tori of the form $R_{k'/k}(GL_1)$, where each k'/k is a finite separable extension and $R_{k'/k}$ denotes the functor of restriction of scalars from k' to k.

By [B, Theorem 22.6], the assumption (4.1) implies that $P' = \beta(P)$, $S' = \beta(S)$ and $Q' = \beta(Q)$ give a minimal k-parabolic subgroup, a maximal k-split torus and a maximal standard k-parabolic subgroup of G', respectively, and furthermore, the homomorphism $(\beta|_S)^*\colon \mathbf{X}_k^*(S') \longrightarrow \mathbf{X}_k^*(S)$ induced from β maps bijectively the relative root system Φ'_k of (G', S') onto Φ_k. From the assumption (4.2), it follows that β gives rise to the isomorphisms $G(k)/Z(k) \cong G'(k)$, $G(\mathbb{A})/Z(\mathbb{A}) \cong G'(\mathbb{A})$ and $X_Q \cong X_{Q'}$ (cf. [Oe, III 2.2]). By the commutative diagram

$$\begin{array}{ccccc}
Z(\mathbb{A})^1 & \longrightarrow & G(\mathbb{A})^1 & \xrightarrow{\beta} & G'(\mathbb{A})^1 \\
\downarrow & & \downarrow & & \downarrow \\
Z(\mathbb{A}) & \longrightarrow & G(\mathbb{A}) & \xrightarrow{\beta} & G'(\mathbb{A}) \\
\vartheta_Z \downarrow & & \vartheta_G \downarrow & & \vartheta_{G'} \downarrow \\
\mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(Z), \mathbb{R}_+^\times) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(G), \mathbb{R}_+^\times) & \xrightarrow{(\beta^*)^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(G'), \mathbb{R}_+^\times)
\end{array}$$

we obtain the isomorphisms $G(\mathbb{A})^1/Z(\mathbb{A})^1 \cong G'(\mathbb{A})^1$, $Q(\mathbb{A})^1/Z(\mathbb{A})^1 \cong Q'(\mathbb{A})^1$ and $Y_Q \cong Y_{Q'}$. Since $Z \cap Z_G$ is the greatest k-split subtorus of Z, the character group $\mathbf{X}_k^*(Z/Z \cap Z_G)$ is trivial. Therefore, β induces an isomorphism $\mathbf{X}_k^*(M_{Q'}/Z_{G'}) \longrightarrow \mathbf{X}_k^*(M_Q/Z_G)$ and maps $\widehat{\alpha}_{Q'}$ to $\widehat{\alpha}_Q$. The next proposition is now obvious.

Theorem 2. *If the exact sequence*

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{\beta} G' \longrightarrow 1$$

of connected reductive groups defined over k satisfies the conditions (4.1) *and* (4.2), *then $\gamma(G, Q, k)$ equals $\gamma(G', \beta(Q), k)$.*

Example 2. If $\beta\colon GL_n \longrightarrow PGL_n$ denotes a natural quotient morphism, then $\gamma(GL_n, Q, k) = \gamma(PGL_n, \beta(Q), k)$.

Example 3. Let D be a division algebra of finite dimension m^2 over k and D° the opposition algebra of D. There are inner k-forms G and G' of GL_{mn} such that $G(k) = GL_n(D)$ and $G'(k) = GL_n(D^\circ)$. We put

$$w_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in GL_n(D^\circ).$$

Then the morphism $\beta\colon G \longrightarrow G'$ defined by $\beta(g) = w_0({}^t g^{-1})w_0^{-1}$ yields a k-isomorphism. If we take a maximal k-parabolic subgroup Q_j of G as

$$Q_j(k) = \left\{ \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} : a \in GL_j(D),\ \ b \in GL_{n-j}(D) \right\}$$

for $1 \le j \le n-1$, then $\beta(Q_j(k))$ equals

$$Q'_{n-j}(k) = \left\{ \begin{pmatrix} a' & * \\ 0 & b' \end{pmatrix} : a' \in GL_{n-j}(D^\circ),\ \ b' \in GL_j(D^\circ) \right\}.$$

Therefore,

$$\gamma(G, Q_j, k) = \gamma(G', Q'_{n-j}, k).$$

This relation was first proved in [W3]. Particularly, if $m = 1$, this is none other than the duality relation

$$\gamma(GL_n, Q_j, k) = \gamma(GL_n, Q_{n-j}, k).$$

Remark. When $\mathrm{ch}(k) = 0$, for a given connected reductive k-group G, there exists a group extension

$$1 \longrightarrow Z \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

defined over k such that Z satisfies (4.1) and (4.2), and in addition , the derived group of $\widetilde{G}$ is simply connected. Such an extension of G is called z-extension(cf. [K, §1]).

Second, we consider a restriction of scalars. Take a subfield ℓ of k such that k/ℓ is a finite separable extension and put $G' = R_{k/\ell}(G)$, $P' = R_{k/\ell}(P)$ and $Q' = R_{k/\ell}(Q)$. The adele ring of ℓ is denoted by $\mathbb{A}_\ell$. Since the functor $R_{k/\ell}$ yields a bijection from the set of k-parabolic subgroups of G to the set of ℓ-parabolic subgroups of G' ([B-Ti, Corollaire 6.19]), P' and Q' give a minimal ℓ-parabolic subgroup and a maximal standard ℓ-parabolic subgroup of G', respectively. Although the torus $R_{k/\ell}(S)$ does not necessarily split over ℓ, the greatest ℓ-split subtorus S' of $R_{k/\ell}(S)$ gives a maximal ℓ-split torus of G'. For an arbitrary connected k-subgroup R of G and $R' = R_{k/\ell}(R)$, we introduce a canonical homomorphism $\beta^*\colon \mathbf{X}_k^*(R) \longrightarrow \mathbf{X}_\ell^*(R')$. If A is an ℓ-algebra, there is a canonical identification $R'(A)$ with $R(A \otimes_\ell k)$. Then, for $\chi \in \mathbf{X}_k^*(R)$, $\beta^*(\chi)$ is defined to be

$$\beta^*(\chi)(a) = N_{A\otimes k/A}(\chi(a))\,, \qquad (a \in R'(A) = R(A \otimes_\ell k))$$

for any ℓ-algebra A, where $N_{A\otimes k/A}\colon (A\otimes_\ell k)^\times \longrightarrow A^\times$ denotes the norm. This β^* is bijective ([Oe, II Theorem 2.4]), and if $R = S$, then β^* maps Φ_k to the relative root system Φ'_ℓ of (G', S') ([B-Ti, 6.21]). From the commutative diagram

$$\begin{array}{ccc} R(\mathbb{A}) & =\!=\!= & R'(\mathbb{A}_\ell) \\ {\scriptstyle\vartheta_R}\big\downarrow & & {\scriptstyle\vartheta_{R'}}\big\downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(R), \mathbb{R}_+^\times) & \xrightarrow{(\beta^*)^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}_k^*(R'), \mathbb{R}_+^\times) \end{array}$$

it follows $R(\mathbb{A})^1 = R'(\mathbb{A}_\ell)^1$. Accordingly, $Q(\mathbb{A})^1\backslash G(\mathbb{A})^1 = Q'(\mathbb{A}_\ell)^1\backslash G'(\mathbb{A}_\ell)^1$. Since $Z_{G'}$ is the greatest ℓ-split torus in $R_{k/\ell}(Z_G)$, the natural quotient morphism $M_{Q'}/Z_{G'} \longrightarrow M_{Q'}/R_{k/\ell}(Z_G)$ induces an isomorphism $\mathbf{X}_\ell^*(M_{Q'}/R_{k/\ell}(Z_G)) \cong \mathbf{X}_\ell^*(M_{Q'}/Z_{G'})$. The composition of this and β^* yields an isomorphism between $\mathbf{X}_k^*(M_Q/Z_G)$ and $\mathbf{X}_\ell^*(M_{Q'}/Z_{G'})$. This maps $\widehat{\alpha}_Q$ to $\widehat{\alpha}_{Q'}$. Then, by definition of β^*,

$$|\widehat{\alpha}_{Q'}(m)|_{\mathbb{A}_\ell} = |N_{\mathbb{A}/\mathbb{A}_\ell}(\widehat{\alpha}_Q(m))|_{\mathbb{A}_\ell} = |\widehat{\alpha}_Q(m)|_{\mathbb{A}}$$

for $m \in M_{Q'}(\mathbb{A}_\ell) \cap G'(\mathbb{A}_\ell)^1 = M_Q(\mathbb{A}) \cap G(\mathbb{A})^1$, In other words, $H_{Q'}$ is equal to H_Q on $Y_{Q'} = Y_Q$. As a consequence, we proved the following

Theorem 3. *If k/ℓ is a finite separable extension, then $\gamma(R_{k/\ell}(G), R_{k/\ell}(Q), \ell)$ is equal to $\gamma(G, Q, k)$.*

Finally, we show a generalization of Rankin's inequality. Let R and Q be two different maximal standard k-parabolic subgroups of G. We set $Q^R = M_R \cap Q$, $M_Q^R = M_R \cap M_Q$, $U_Q^R = M_R \cap U_Q$ and $X_Q^R = Q^R(k)\backslash M_R(k)$. Then Q^R is a maximal standard parabolic subgroup of M_R with a Levi decomposition $U_Q^R M_Q^R$. We write $\widehat{\alpha}_Q^R$ for the $\mathbb{Z}$-basis $\widehat{\alpha}_{Q^R}$ of $\mathbf{X}_k^*(M_Q^R/Z_R)$, z_Q^R for the map $z_{Q^R}\colon M_R(\mathbb{A}) \longrightarrow Z_R(\mathbb{A})M_Q^R(\mathbb{A})^1\backslash M_Q^R(\mathbb{A})$ and H_Q^R for the function $H_{Q^R}\colon M_R(\mathbb{A}) \longrightarrow \mathbb{R}_+^\times$ defined by $m \mapsto |\widehat{\alpha}_Q^R(z_Q^R(m))|_{\mathbb{A}}^{-1}$. The fundamental Hermite constants of (M_R, Q^R) are given by

$$\gamma(M_R, Q^R, k) = \max_{m\in M_R(\mathbb{A})^1} \min_{y\in X_Q^R m} H_Q^R(y) \quad\text{and}\quad \widetilde{\gamma}(M_R, Q^R, k) = \max_{m\in M_R(\mathbb{A})} \min_{y\in X_Q^R m} H_Q^R(y).$$

The exact sequence

$$1 \longrightarrow Z_R/Z_G \longrightarrow M_Q^R/Z_G \longrightarrow M_Q^R/Z_R \longrightarrow 1$$

induces the exact sequence

$$1 \longrightarrow \mathbf{X}_k^*(M_Q^R/Z_R) \longrightarrow \mathbf{X}_k^*(M_Q^R/Z_G) \longrightarrow \mathbf{X}_k^*(Z_R/Z_G).$$

From $\widehat{\alpha}_R|_{Z_R} = \widehat{d}_R\alpha_R \neq 0$, it follows that the $\mathbb{Q}$-vector space $\mathbf{X}_k^*(M_Q^R/Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is spanned by $\widehat{\alpha}_Q^R$ and $\widehat{\alpha}_R|_{M_Q^R}$, and hence there are $\omega_1, \omega_2 \in \mathbb{Q}$ such that

$$\widehat{\alpha}_Q|_{M_Q^R} = \omega_1\widehat{\alpha}_Q^R + \omega_2\widehat{\alpha}_R|_{M_Q^R}\,. \tag{4.3}$$

Theorem 4. *Being notations as above, one has the inequality*

$$\gamma(G,Q,k) \leq \widetilde{\gamma}(M_R, Q^R, k)^{\omega_1}\gamma(G,R,k)^{\omega_2}.$$

Proof. Since X_Q^R is naturally regarded as a subset of X_Q, the inequality

$$\Gamma_Q(g) = \min_{x \in X_Q} H_Q(xg) \leq \min_{x \in X_Q^R} H_Q(xg)$$

holds for $g \in G(\mathbb{A})^1$. By the Iwasawa decomposition, we write $g = umh$, where $u \in U_R(\mathbb{A})$, $m \in M_R(\mathbb{A}) \cap G(\mathbb{A})^1$ and $h \in K$. Then, for $x \in M_R(k)$, $xux^{-1} \in U_R(\mathbb{A}) \subset Q(\mathbb{A})^1$, and

$$H_Q(xg) = H_Q((xux^{-1})xmh) = H_Q(xm) = |\widehat{\alpha}_Q(z_Q(xm))|_{\mathbb{A}}^{-1}\,.$$

If we write $xm = u_1m_1h_1$, $u_1 \in U_Q^R(\mathbb{A})$, $m_1 \in M_Q^R(\mathbb{A})$ and $h_1 \in K^{M_R}$ by the Iwasawa decomposition $M_R(\mathbb{A}) = U_Q^R(\mathbb{A})M_Q^R(\mathbb{A})K^{M_R}$, then

$$\begin{aligned} H_Q(xm) &= |\widehat{\alpha}_Q(m_1)|_{\mathbb{A}}^{-1} = |\widehat{\alpha}_Q^R(m_1)|_{\mathbb{A}}^{-\omega_1}|\widehat{\alpha}_R(m_1)|_{\mathbb{A}}^{-\omega_2} \\ &= |\widehat{\alpha}_Q^R(z_Q^R(xm))|_{\mathbb{A}}^{-\omega_1}|\widehat{\alpha}_R(xm)|_{\mathbb{A}}^{-\omega_2} = H_Q^R(xm)^{\omega_1}|\widehat{\alpha}_R(m)|_{\mathbb{A}}^{-\omega_2} \\ &= H_Q^R(xm)^{\omega_1}H_R(g)^{\omega_2}\,. \end{aligned}$$

Therefore,

$$\Gamma_Q(g) \leq \left(\min_{x \in X_Q^R} H_Q^R(xm)\right)^{\omega_1} H_R(g)^{\omega_2} \leq \widetilde{\gamma}(M_R, Q^R, k)^{\omega_1}H_R(g)^{\omega_2}\,.$$

As Γ_Q is left $G(k)$-invariant, the inequality

$$\Gamma_Q(g) \leq \widetilde{\gamma}(M_R, Q^R, k)^{\omega_1}H_R(xg)^{\omega_2}$$

holds for all $x \in G(k)$. Taking the minimum, we get

$$\Gamma_Q(g) \leq \widetilde{\gamma}(M_R, Q^R, k)^{\omega_1}\Gamma_R(g)^{\omega_2}\,.$$

The assertion follows from this. □

Notice that $\widetilde{\gamma}(M_R, Q^R, k) = \gamma(M_R, Q^R, k)$ in the case of number fields.

Corollary. *If* $\mathrm{ch}(k) = 0$, *then* $\gamma(G, Q, k) \leq \gamma(M_R, Q^R, k)^{\omega_1} \gamma(G, R, k)^{\omega_2}$.

Example 4. We use the same notations as in Example 1. For $i, j \in \mathbb{Z}$ with $1 \leq i < j \leq n-1$, both $R = Q_j$ and $Q = Q_i$ are maximal standard k-parabolic subgroups of GL_n. Then, $M_R = GL_j \times GL_{n-j}$, $M_Q = GL_i \times GL_{n-i}$ and $M_Q^R = GL_i \times GL_{j-i} \times GL_{n-j}$. We denote an element of M_Q^R by

$$\mathrm{diag}(a, b, c) = \begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}, \qquad (a \in GL_i,\ b \in GL_{j-i},\ \ c \in GL_{n-j}).$$

It is easy to see

$$\begin{aligned} \widehat{\alpha}_Q^R(\mathrm{diag}(a, b, c)) &= (\det a)^{(j-i)/\gcd(i,j-i)} (\det b)^{-i/\gcd(i,j-i)} \\ \widehat{\alpha}_R|_{M_Q^R}(\mathrm{diag}(a, b, c)) &= (\det a)^{(n-j)/\gcd(j,n-j)} (\det b)^{(n-j)/\gcd(j,n-j)} (\det c)^{-j/\gcd(j,n-j)} \\ \widehat{\alpha}_Q|_{M_Q^R}(\mathrm{diag}(a, b, c)) &= (\det a)^{(n-i)/\gcd(i,n-i)} (\det b)^{-i/\gcd(i,n-i)} (\det c)^{-i/\gcd(i,n-i)}. \end{aligned}$$

Thus,

$$\omega_1 = \frac{n}{j} \frac{\gcd(i, j-i)}{\gcd(i, n-i)}, \qquad \omega_2 = \frac{i}{j} \frac{\gcd(j, n-j)}{\gcd(i, n-i)}.$$

Theorem 4 deduces

$$\gamma(GL_n, Q_i, k) \leq \widetilde{\gamma}(M_{Q_j}, Q_i^{Q_j}, k)^{\frac{n}{j} \frac{\gcd(i,j-i)}{\gcd(i,n-i)}} \gamma(GL_n, Q_j, k)^{\frac{i}{j} \frac{\gcd(j,n-j)}{\gcd(i,n-i)}}.$$

If $\mathrm{ch}(k) = 0$, then, by Example 1, this reduces to Rankin's inequality

$$\gamma_{n,i}(k) \leq \gamma_{j,i}(k) \gamma_{n,j}(k)^{i/j}.$$

5. A lower bound of γ_Q.

We prove an analogous inequality to (3.3) when $\mathrm{ch}(k) > 0$. Thus we assume $\mathrm{ch}(k) > 0$ throughout this section.

Lemma 1. *If f is a right K-invariant measurable function on Y_Q,*

$$\int_{Y_Q} f(y) d\omega_{Y_Q}(y) = \frac{C_G d_Q^*}{C_Q d_G^*} \sum_{M_Q(\mathbb{A})^1 \xi \in M_Q(\mathbb{A})^1 \backslash (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)} \delta_Q(\xi)^{-1} f(\xi).$$

Proof. Let $\phi \in C_0(G(\mathbb{A})^1)$ be a right K-invariant function. By the definition of invariant measures, we have

$$\begin{aligned}\int_{G(\mathbb{A})^1} \phi(g) d\omega_{G(\mathbb{A})^1}(g) &= (d_G^*)^{-1} \int_{G(\mathbb{A})^1} \phi(g) d\omega_{\mathbb{A}}^G(g) \\ &= \frac{C_G}{C_Q d_G^*} \int_{U_Q(\mathbb{A}) \times (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)} \phi(um) \delta_Q(m)^{-1} d\omega_{\mathbb{A}}^{U_Q}(u) d\omega_{\mathbb{A}}^{M_Q}(m) \\ &= \frac{C_G d_Q^*}{C_Q d_G^*} \sum_{M_Q(\mathbb{A})^1 \xi \in M_Q(\mathbb{A})^1 \backslash (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)} \delta_Q(\xi)^{-1} f(\xi)\,,\end{aligned}$$

where

$$f(\xi) = \int_{U_Q(\mathbb{A}) \times M_Q(\mathbb{A})^1} \phi(um\xi) d\omega_{\mathbb{A}}^{U_Q}(u) d\omega_{M_Q(\mathbb{A})^1}(m) = \int_{Q(\mathbb{A})^1} \phi(g\xi) d\omega_{Q(\mathbb{A})^1}(g).$$

On the other hand,

$$\begin{aligned}\int_{G(\mathbb{A})^1} \phi(g) d\omega_{G(\mathbb{A})^1}(g) &= \int_{Y_Q} \int_{Q(\mathbb{A})^1} \phi(gy) d\omega_{Q(\mathbb{A})^1}(g) d\omega_{Y_Q}(y) \\ &= \int_{Y_Q} f(y) d\omega_{Y_Q}(y). \qquad \square\end{aligned}$$

Theorem 5. *If* $\mathrm{ch}(k) > 0$, *one has*

$$\left(\frac{C_Q d_G^* \tau(G)}{C_G d_Q^* \tau(Q)} (1 - q_0^{-\widehat{e}_Q}) \right)^{1/\widehat{e}_Q} < q_0^{j_0+1} \leq \gamma_Q\,,$$

where the integer j_0 *is given by*

$$j_0 = \max\{ j \in \mathbb{Z} \colon q_0^{j\widehat{e}_Q} \leq \frac{C_Q d_G^* \tau(G)}{C_G d_Q^* \tau(Q)} (1 - q_0^{-\widehat{e}_Q}) \}$$

and $q_0 = q_0(Q)$ *is the generator of the value group* $|\widehat{\alpha}_Q(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)|_{\mathbb{A}}$ *which is greater than one.*

Proof. For $j \in \mathbb{Z}$, we define the function $\psi_j \colon q_0^{\mathbb{Z}} \longrightarrow \mathbb{R}$ by

$$\psi_j(q_0^i) = \begin{cases} 1 & (i \leq j). \\ 0 & (i > j). \end{cases}$$

Then, by Lemma 1,

$$I_j = \int_{Y_Q} \psi_j(H_Q(y))d\omega_{Y_Q}(y)$$
$$= \frac{C_G d_Q^*}{C_Q d_G^*} \sum_{M_Q(\mathbb{A})^1 \xi \in M_Q(\mathbb{A})^1 \backslash (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)} \delta_Q(\xi)^{-1} \psi_j(H_Q(\xi)).$$

Since H_Q is bijective from $M_Q(\mathbb{A})^1 \backslash (M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)$ to $q_0^{\mathbb{Z}}$ and $\delta_Q(m)^{-1} = H_Q(m)^{\widehat{e}_Q}$ for $m \in M_Q(\mathbb{A})$, we have

$$I_j = \frac{C_G d_Q^*}{C_Q d_G^*} \sum_{i=-\infty}^{j} q_0^{i\widehat{e}_Q} = \frac{C_G d_Q^*}{C_Q d_G^*} \frac{q_0^{j\widehat{e}_Q}}{1 - q_0^{-\widehat{e}_Q}}.$$

If j satisfies $I_j < \tau(G)/\tau(Q)$, then

$$I_j = \frac{1}{\tau(Q)} \int_{G(k)\backslash G(\mathbb{A})^1} \sum_{x \in X_Q} \psi_j(H_Q(xg))d\omega_G(g) < \frac{\tau(G)}{\tau(Q)}.$$

Therefore, at least one $g_0 \in G(\mathbb{A})^1$,

$$\sum_{x \in X_Q} \psi_j(H_Q(xg_0)) < 1$$

holds, and hence $\psi_j(H_Q(xg_0)) = 0$ for all $x \in X_Q$. This implies

$$\min_{x \in X_Q} H_Q(xg_0) \geq q_0^{j+1},$$

and

$$\gamma_Q \geq q_0 \sup\{q_0^j \colon \frac{C_G d_Q^*}{C_Q d_G^*} \frac{q_0^{j\widehat{e}_Q}}{1 - q_0^{-\widehat{e}_Q}} < \frac{\tau(G)}{\tau(Q)}\} = q_0^{1+j_0}$$
$$> \left(\frac{C_Q d_G^* \tau(G)}{C_G d_Q^* \tau(Q)} (1 - q_0^{-\widehat{e}_Q}) \right)^{1/\widehat{e}_Q}. \qquad \square$$

Remark. In §6, Example 5, we will see an example of γ_Q satisfying

$$\left(\frac{C_Q d_G^* \tau(G)}{C_G d_Q^* \tau(Q)} (1 - q_0^{-\widehat{e}_Q}) \right)^{1/\widehat{e}_Q} < \gamma_Q < q_0 \left(\frac{C_Q d_G^* \tau(G)}{C_G d_Q^* \tau(Q)} (1 - q_0^{-\widehat{e}_Q}) \right)^{1/\widehat{e}_Q}.$$

If G splits over k, this lower bound is described more precisely. For $v \in \mathfrak{V}_f$, we choose each v component K_v of K as follows:

(5.1) K_v is a hyperspecial maximal compact subgroup $\mathcal{G}_v(\mathfrak{O}_v)$ of $G(k_v)$, and

(5.2) $K_v \cap M_Q(k_v)$ is a hyperspecial maximal compact subgroup $\mathcal{M}_{Q,v}(\mathfrak{O}_v)$ of $M_Q(k_v)$, where $\mathcal{G}_v$ and $\mathcal{M}_{Q,v}$ stand for the smooth affine group schemes defined over $\mathfrak{O}_v$ with generic fiber G and M_Q, respectively (cf. [Ti2]).

Then it is known by [Oe, I Proposition 2.5] that

$$\omega_{\mathbb{A}}^G(K) = \mu_{\mathbb{A}}(\mathbb{A}/k)^{-\dim G}\sigma_k(G)^{-1} \prod_{v\in\mathfrak{V}_f} L_v(1,\sigma_G) q_v^{-\dim G} |\mathcal{G}_v(\mathfrak{f}_v)|$$

$$\omega_{\mathbb{A}}^{M_Q}(K^{M_Q}) = \mu_{\mathbb{A}}(\mathbb{A}/k)^{-\dim M_Q}\sigma_k(M_Q)^{-1} \prod_{v\in\mathfrak{V}_f} L_v(1,\sigma_{M_Q}) q_v^{-\dim M_Q} |\mathcal{M}_{Q,v}(\mathfrak{f}_v)|$$

$$\omega_{\mathbb{A}}^{U_Q}(K\cap U_Q(\mathbb{A})) = \mu_{\mathbb{A}}(\mathbb{A}/k)^{-\dim U_Q}\,.$$

In the integral formula (1.2), if we put the characteristic function of K as f, then

$$\frac{C_G}{C_Q} = \frac{\omega_{\mathbb{A}}^G(K)}{\omega_{\mathbb{A}}^{U_Q}(K\cap U_Q(\mathbb{A}))\omega_{\mathbb{A}}^{M_Q}(K^{M_Q})}\,.$$

Since G splits over k, σ_G is the trivial representation of $\mathrm{Gal}(\overline{k}/k)$ of dimension $\mathrm{rank}\mathbf{X}^*(G) = \dim Z_G$. As Q is a maximal parabolic subgroup, we have

$$\frac{\sigma_k(G)}{\sigma_k(M_Q)} = \frac{(\mathrm{Res}_{s=1}\zeta_k(s))^{\dim Z_G}}{(\mathrm{Res}_{s=1}\zeta_k(s))^{\dim Z_Q}} = \frac{1}{\mathrm{Res}_{s=1}\zeta_k(s)} = \frac{q^{g(k)-1}(q-1)\log q}{h_k}\,,$$

where $\zeta_k(s)$ denotes the congruence zeta function of k and h_k the divisor class number of k. Summing up, we obtain

Theorem 6. *If* $\mathrm{ch}(k) > 0$ *and* G *splits over* k, *then*

$$\left(\frac{(1-q_0^{-\widehat{e}_Q})q^{(g(k)-1)\dim G/Q}}{\mathrm{Res}_{s=1}\zeta_k(s)}\frac{d_G^*\tau(G)}{d_Q^*\tau(Q)}\prod_{v\in\mathfrak{V}}(1-q_v^{-1})q_v^{\dim G/M_Q}\frac{|\mathcal{M}_{Q,v}(\mathfrak{f}_v)|}{|\mathcal{G}_v(\mathfrak{f}_v)|}\right)^{1/\widehat{e}_Q} < \gamma_Q\,.$$

6. Computations of $\gamma(GL_n, Q, k)$ when $\mathrm{ch}(k) > 0$.

In this section, we assume $\mathrm{ch}(k) > 0$. We concentrate our attention on $G = GL_n$ because this case gives an analogue of classical Hermite's constant. We use the same notations as in Example 1 of §3. Namely, V denotes an n dimensional vector space defined over k, $\mathbf{e}_1, \cdots, \mathbf{e}_n$ a k-basis of $V(k)$, Q_j the stabilizer of the subspace spanned by $\mathbf{e}_1, \cdots, \mathbf{e}_j$ in GL_n and $\pi_j\colon GL_n \longrightarrow GL(V_{\pi_j})$ the j-th exterior representation of GL_n for $1 \leq j \leq n-1$. We take K as $\prod_{v \in \mathfrak{V}} GL_n(\mathfrak{O}_v)$. The global height $H_j = H_{\pi_j}$ on $V_{\pi_j}(k)$ is defined to be

$$H_j(\sum_I a_I \mathbf{e}_I) = \prod_{v \in \mathfrak{V}} \sup_I(|a_I|_v)\,.$$

As an analogue of the number fields case, we can define the constant

$$\gamma_{n,j}(k) = \max_{g \in GL_n(\mathbb{A})} \min_{\substack{x_1, \cdots, x_j \in V(k) \\ x_1 \wedge \cdots \wedge x_j \neq 0}} \frac{H_j(gx_1 \wedge \cdots \wedge gx_j)}{|\det g|_{\mathbb{A}}^{j/n}}.$$

It is immediate to see that

$$\frac{H_j(g^{-1}\mathbf{e}_1 \wedge \cdots \wedge g^{-1}\mathbf{e}_j)}{|\det g^{-1}|_{\mathbb{A}}^{j/n}} = H_{Q_j}(g)^{\gcd(j,n-j)/n}$$

for $g \in GL_n(\mathbb{A})$, and hence

$$\gamma_{n,j}(k) = \widetilde{\gamma}(GL_n, Q_j, k)^{\gcd(j,n-j)/n}\,.$$

In general, $Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1$ is not equal to $GL_n(\mathbb{A})$ in contrast to the number fields case. It is obvious that $Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1$ is an index finite normal subgroup of $GL_n(\mathbb{A})$. Let $\Xi = \{\xi\}$ be a complete set of representatives for the cosets of $Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1 \backslash GL_n(\mathbb{A})$. If we put

$$\begin{aligned} \gamma_{n,j}(k)_\xi &= \max_{g \in Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1\xi} \min_{\substack{x_1, \cdots, x_j \in V(k) \\ x_1 \wedge \cdots \wedge x_j \neq 0}} \frac{H_j(gx_1 \wedge \cdots \wedge gx_j)}{|\det g|_{\mathbb{A}}^{j/n}} \\ &= \frac{1}{|\det \xi|_{\mathbb{A}}^{j/n}} \max_{g \in GL_n(\mathbb{A})^1\xi} \min_{\substack{x_1, \cdots, x_j \in V(k) \\ x_1 \wedge \cdots \wedge x_j \neq 0}} H_j(gx_1 \wedge \cdots \wedge gx_j) \end{aligned}$$

for $\xi \in \Xi$, then

$$\gamma_{n,j}(k) = \max_{\xi \in \Xi} \gamma_{n,j}(k)_\xi,$$

and in particular, for the unit element $\xi = 1$,

$$\gamma_{n,j}(k)_1 = \gamma(GL_n, Q_j, k)^{\gcd(j,n-j)/n}.$$

Since $1 \leq \gamma_{n,j}(k)_1$ by the definition of H_j, we obtain

$$1 \leq \gamma(GL_n, Q_j, k) \leq \gamma_{n,j}(k)^{n/\gcd(j,n-j)}\,. \tag{6.1}$$

Lemma 2. $\gamma_{n,j}(k) \leq q^{jg(k)}$.

Proof. By [T1, §5, Corollary 1], for a given $g \in GL_n(\mathbb{A})$, there are linearly independent vectors $x_1, \cdots, x_n$ of $V(k)$ with

$$H_1(gx_1) \cdots H_1(gx_n) \leq q^{ng(k)} |\det g|_{\mathbb{A}} .$$

We may assume $H_1(gx_1) \leq H_1(gx_2) \leq \cdots \leq H_1(gx_n)$. Then,

$$\begin{aligned} H_j(gx_1 \wedge \cdots \wedge gx_j) &\leq H_1(gx_1) \cdots H_1(gx_j) \\ &\leq (H_1(gx_1) \cdots H_1(gx_n))^{j/n} \\ &\leq q^{jg(k)} |\det g|_{\mathbb{A}}^{j/n} . \end{aligned}$$

This implies the assertion. We note that our definition of the global height H_j is slightly different from [T1]. □

Theorem 7. *We have the following estimate.*

$$\begin{aligned} &\left(\frac{q^{(g(k)-1)(j(n-j)+1)}(q-1)(1-q^{-n})}{h_k} \frac{\prod_{i=n-j+1}^{n} \zeta_k(i)}{\prod_{i=2}^{j} \zeta_k(i)} \right)^{1/\gcd(j,n-j)} \\ &\quad < \gamma(GL_n, Q_j, k) \leq \widetilde{\gamma}(GL_n, Q_j, k) \leq q^{njg(k)/\gcd(j,n-j)} = q_0(Q_j)^{jg(k)} . \end{aligned}$$

Proof. Recall that $q_0(Q_j)$ is the generator of the value group $|\widehat{\alpha}_{Q_j}(M_{Q_j}(\mathbb{A}) \cap GL_n(\mathbb{A})^1)|_{\mathbb{A}}$ which is greater than one. Since

$$M_{Q_j} = \{\mathrm{diag}(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in GL_j, \ \ b \in GL_{n-j}\} ,$$

any $\mathrm{diag}(a,b) \in M_{Q_j}(\mathbb{A}) \cap GL_n(\mathbb{A})^1$ satisfies

$$|\det a|_{\mathbb{A}} = |\det b|_{\mathbb{A}}^{-1}.$$

The $\mathbb{Z}$-basis $\widehat{\alpha}_{Q_j}$ of $\mathbf{X}^*(M_{Q_j}/Z_{GL_n})$ is given by

$$\widehat{\alpha}_{Q_j}(\mathrm{diag}(a,b)) = (\det a)^{(n-j)/\gcd(j,n-j)} (\det b)^{-j/\gcd(j,n-j)} .$$

Hence, $|\widehat{\alpha}_{Q_j}(\mathrm{diag}(a,b))|_{\mathbb{A}} = |\det a|^{n/\gcd(j,n-j)}$ holds for $\mathrm{diag}(a,b) \in M_{Q_j}(\mathbb{A}) \cap GL_n(\mathbb{A})^1$. This and $\{|\det a|_{\mathbb{A}} \colon a \in GL_j(\mathbb{A})\} = q^{\mathbb{Z}}$ conclude $q_0(Q_j) = q^{n/\gcd(j,n-j)}$. The upper estimate is obvious from Lemma 2 and (6.1). Since the order of the finite group $GL_n(\mathfrak{f}_v)$ is equal to $(q_v^n - 1)(q_v^n - q_v)\cdots(q_v^n - q_v^{n-1})$, one has

$$\prod_{v\in\mathfrak{V}}(1-q_v^{-1})q_v^{\dim GL_n/M_{Q_j}}\frac{|GL_j(\mathfrak{f}_v)\times GL_{n-j}(\mathfrak{f}_v)|}{|GL_n(\mathfrak{f}_v)|} = \frac{\displaystyle\prod_{i=n-j+1}^{n}\zeta_k(i)}{\displaystyle\prod_{i=2}^{j}\zeta_k(i)}\,.$$

It is known that $\tau(GL_n) = \tau(GL_j \times GL_{n-j}) = 1$ (cf. [Wel, Theorem 3.2.1] and [Oe, III Theorem 5.2]). From the surjectivity of ϑ_{GL_n}, it follows $d^*_{GL_n} = \log q$, $d^*_{Q_j} = d^*_{GL_j\times GL_{n-j}} = (\log q)^2$ and

$$\frac{1}{\mathrm{Res}_{s=1}\zeta_k(s)}\frac{d^*_{GL_n}\tau(GL_n)}{d^*_{Q_j}\tau(Q_j)} = \frac{q^{g(k)-1}(q-1)}{h_k}\,.$$

Then, the lower bound is a result of Theorem 6 and $\widehat{e}_{Q_j} = \gcd(j, n-j)$. □

Corollary 1. *If $g(k) = 0$, i.e., k is a rational function field over $\mathbb{F}_q$, then $\gamma(GL_n, Q_j, k) = \widetilde{\gamma}(GL_n, Q_j, k) = 1$ for all n and j.*

It is known that the zeta function $\zeta_k(s)$ is of the form

$$\zeta_k(s) = \frac{L_k(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $L_k(t)$ is a polynomial of degree $2g(k)$ with integer coefficients. If we write $L_k(t)$ as

$$L_k(t) = a_0 + a_1 t + \cdots + a_{2g(k)}t^{2g(k)}\,,$$

then a_i's have the following properties:

1) $a_0 = 1$, $a_{2g(k)} = q^{g(k)}$ and $a_{2g(k)-i} = q^{g(k)-i}a_i$ for $1 \le i \le g(k)$.

2) $a_1 = N(k) - (q+1)$, where $N(k) = \sharp\{v \in \mathfrak{V} \colon [\mathfrak{f}_v : \mathbb{F}_q] = 1\}$.

3) $L_k(1) = h_k$.

In this notation, Theorem 4 deduces the following inequality.

Corollary 2. *If $j = 1$, then*

$$\frac{q^{g(k)n}(q-1)L_k(q^{-n})}{h_k(q^n-q)} < \gamma(GL_n, Q_1, k) \leq \widetilde{\gamma}(GL_n, Q_1, k) \leq q^{g(k)n} = q_0(Q_1)^{g(k)} .$$

Example 5. If $g(k) = 0$, then $L_k(t) = 1$ and $h_k = 1$. So that we have

$$\frac{q-1}{q^n-q} < \gamma(GL_n, Q_1, k) = 1 < q^n \frac{q-1}{q^n-q} = q_0(Q_1)\frac{q-1}{q^n-q} .$$

Put

$$\epsilon_n(k) = \frac{q^n(q-1)L_k(q^{-n})}{h_k(q^n-q)} .$$

By Corollary 2, if $1 \leq \epsilon_n(k)$ holds for k, then both $\gamma(GL_n, Q_1, k)$ and $\widetilde{\gamma}(GL_n, Q_1, k)$ must be equal to $q^{g(k)n}$.

Example 6. If $g(k) = 1$, then

$$\epsilon_n(k) = \frac{(q-1)(q^{2n}+a_1q^n+q)}{(q+a_1+1)(q^{2n}-qq^n)} .$$

We have the inequality:

$$1 \leq \frac{q^{2n}+a_1q^n+q}{q^{2n}-qq^n}.$$

This is obvious by the Hasse – Weil bound $|a_1| \leq 2\sqrt{q}$. Hence, if $a_1 \leq -2$, i.e., $h_k \leq q-1$, then $\gamma(GL_n, Q_1, k) = \widetilde{\gamma}(GL_n, Q_1, k) = q^n$ for all $n \geq 2$.

Remark. In the case of number fields, the explicit values of $\gamma(GL_n, Q_1, k)$ are very little known. One knows only $\gamma(GL_n, Q_1, \mathbb{Q})$ for $2 \leq n \leq 8$ and $\gamma(GL_2, Q_1, k)$ for a few quadratic number fields k (cf. [BCIO], [O-W]).

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