Arithmetic of linear algebraic groups over two-dimensional geometric fields

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Dedicated to M. S. Raghunathan on his sixtieth birthday

§0. Introduction

Let K be the field of fractions of a two-dimensional, excellent, henselian, local domain with algebraically closed residue field of characteristic zero. In this introduction, we shall refer to such a field as a two-dimensional, strictly henselian field (indeed a henselian local ring with a separably closed residue field is called a strictly henselian local ring). An example of such a field is the field of fractions $\mathbf{C}((X,Y))$ of the formal power series ring $\mathbf{C}[[X,Y]]$.

In the paper [CTOP], Ojanguren and two of the authors investigated quadratic forms over such a field K, and established properties analogous to those familiar over number fields. An analogue of the local-global principle for the Brauer group (going back to work of M. Artin) was also established.

In the present paper, we show that most well-known properties of linear algebraic groups and of their homogeneous spaces over totally imaginary number fields have counterparts for such groups over a two-dimensional strictly henselian field.

To prove this, we recall (§1) that both types of fields share a number of properties :

(i) their cohomological dimension is two ;

(ii) index and exponent of central simple algebras coincide ;

(iii) for any semisimple simply connected group G over K, we have $H^1(K,G)=1$;

(iv) on fields of either type, there is a natural set of rank one discrete valuations (i.e. with values in $\mathbf{Z}$), with respect to which one can take completions and then investigate such natural questions as the local-global principle and weak approximation for homogeneous spaces of linear algebraic groups.

We recall how properties (i) and (ii) are enough to deduce (iii) for groups without E_8-factors.

Properties (i), (ii) and (iii) are satified by other fields of interest, such as the Laurent series field $l((t))$ over a field l (of characteristic zero) of cohomological dimension one (such fields occur as completions of two-dimensional strictly henselian fields).

For function fields in two variables over an algebraically closed field of characteristic zero, (i) holds and (ii) is known for algebras of 2-primary or 3-primary index(1).

In §2, we show that if a field K satifies properties (i) and (ii), then for the standard isogeny

$$1 \to \mu \to G^{\mathrm{sc}} \to G^{\mathrm{ad}} \to 1$$

between a simply connected algebraic group and its adjoint group, the associated boundary map $H^1(K,G^{\mathrm{ad}}) \to H^2(K,\mu)$ is onto. This is a generalization of a classical result over number fields. Using the results from §1, we simultaneously prove that if a semisimple group G over such a field K is not purely of type A, then it is isotropic. Again this is the generalization of a classical result over number fields.

In §3, for tori over two-dimensional strictly henselian fields, we establish the analogues of results of [CTS1] for tori over number fields : finiteness of R-equivalence on the set of rational points, computation of groups measuring the failure of weak approximation and of the Hasse

(1) This actually holds for any index, as recently proved by A. J. de Jong. For most problems studied in our paper, only the 2-primary and 3-primary parts of Brauer groups are relevant. One notable exception is Theorem 2.1, which by de Jong's theorem now applies to function fields in two variables over an algebraically closed field of characteristic zero.

principle for homogeneous spaces. Finiteness of R-equivalence holds for higher dimensional local henselian fields, and also for function fields.

In §4, we study the same problems for arbitrary connected linear algebraic groups. Here the restriction to fields of cohomological dimension two is crucial. We first discuss semisimple simply connected groups. Using a series of earlier works and the isotropy result of §2, we show that if the field K satisfies properties (i) and (ii) (and an additional property if factors of type E_8 are allowed) then for any semisimple simply connected connected group G over K, the group $G(K)/R$ is trivial. If K is a two-dimensional strictly henselian field, then weak approximation holds for such a group G. We then go on to study arbitrary linear algebraic groups over a two-dimensional strictly henselian field. The analysis of [Gi1] and [Gi2] can be carried out : we thus show that for such a group G, the group $G(K)/R$ is finite, and we give formulas for that group and for the group measuring the lack of weak approximation on G (with respect to a finite set of places). For G without E_8-factor, a number of properties, in particular finiteness of $G(K)/R$, also hold for K a two-dimensional function field.

In §5, we study the Hasse principle for homogeneous spaces. We prove that the Hasse principle holds for principal homogeneous spaces of a semisimple group G over a two-dimensional strictly henselian field, if G is adjoint or if the underlying K-variety of G is K-rational. The main result is a proof of the Hasse principle for projective homogeneous spaces of an arbitrary connected linear group over such a field. The proof uses the surjectivity statement established in §2, which enables us to adapt Borovoi's proof in the number field case to the present context.

The main results of the paper were announced in the note [CTGP].

R-equivalence

Let F be a covariant functor from commutative k-algebras to sets. Let O denote the semilocal ring of the polynomial algebra $k[t]$ at the points $t = 0$ and $t = 1$. Let us say that two elements $a, b \in F(k)$ are elementarily related if there exists $\xi \in F(O)$ such that $\xi(0) = a$ and $\xi(1) = b$. By definition, R-equivalence on $F(k)$ is the equivalence relation generated by the previous elementary relation. Thus two elements $a, b \in F(k)$ are R-equivalent if and only if there exists a finite set of elements $x_0, \ldots, x_{n+1} \in F(k)$, with $x_0 = a$ and $x_{n+1} = b$, such that x_i is related to x_{i+1} for $0 \leq i \leq n$. One lets $F(k)/R$ denote the quotient set for this equivalence relation. For any field K containing k, one defines a similar equivalence relation on $F(K)$ by using the semilocal ring of $K[t]$ at the points $t = 0$ and $t = 1$. There is a natural, functorial map $F(k)/R \to F(K)/R$.

If $F = F_X$ is the functor associated to a k-variety X, namely $F_X(A) = X(A)$, we get the R-equivalence on $X(k)$, as defined by Manin. Suppose $F = F_G$, where G/k is a linear algebraic group (see [CTS1], [Gi1]). Then the set of k-points R-equivalent to $1 \in G(k)$ is a normal subgroup; if k is infinite, then any k-point R-equivalent to $1 \in G(k)$ is elementarily related to 1 ([Gi1], Lemme II.1.1). The quotient $G(k)/R$ is thus equipped with a natural group structure. For G linear connected, it is an open question whether this group is commutative (commutativity is known for simply connected groups of classical type).

In these notes, as in [Gi2], we shall also be concerned with the functors F defined by $F(A) = H^1_{\text{ét}}(A, S)$, where S is a commutative k-group scheme.

Weak approximation

Let K be a field equipped with a set Ω of rank one discrete valuations (i.e. valuations the value group of which is $\mathbf{Z}$). For each $v \in \Omega$, let K_v denote the completion of K at v. For any K-variety X, and for any $v \in \Omega$, the topology on K_v induces a topology on $X(K_v)$. One says that *weak approximation* holds for X and a finite set $S \subset \Omega$ if the diagonal map $X(K) \to \prod_{v \in S} X(K_v)$ has dense image.

Acknowledgments We thank the Indo-French Centre for the Promotion of Advanced Research (IFCPAR) for continuous support of this project. Part of the paper grew out of lectures given at the Ecole de Printemps de l'Université d'Artois à Lens (June 2001). We thank the referees for their careful reading.

§1. **The fields under consideration and their basic properties (a review)**

Let k be an algebraically closed field of characteristic zero. We shall mainly be interested in fields of one of the following types :

(gl) *a function field K in two variables over k, i.e. the function field of a smooth, projective, connected surface over k*;

(ll) *the field of fractions K of a henselian, excellent, two-dimensional local domain A with residue field k*;

(sl) *the Laurent series field $l((t))$ over a field l of characteristic zero and cohomological dimension 1.*

For the definition of an excellent ring, see [EGA IV_2] 7.8.2 and 5.6.2 (and refer to [EGA IV_1] 16.1.4 and 14.3.2 for the definition of a "catenary" ring). See also [Ma], chap. 13.

Note that any finite field extension of a field of one of these types is of the same type.

For type (sl), we shall mainly be interested in the case where l is either the function field of a curve over k, or is itself a Laurent series field in one variable $k((u))$.

Let us now consider the following properties of a field K, which we assume to be of characteristic 0.

(a) The field K has the C_2 property : that is, for any $r \geq 1$ and any system of homogeneous forms $f_j(X_1, \ldots, X_n)$ $(j = 1, \ldots, r)$ with coefficients in K, if $n > \sum_j d_j^2$, where d_j is the degree of f_j, then there is a nontrivial common zero in K^n for these r forms. This property is preserved under finite field extensions ([Pf], Chap. 5, Thm. 1.3).

(b) The cohomological dimension $\mathrm{cd}(K)$ is at most 2. This property is preserved under algebraic field extensions ([Se2] §I.3.3, Prop. 14 p. 17).

(c) The reduced norm is surjective. That is, for any finite field extension L/K and any central simple algebra D/L, the reduced norm map $\mathrm{Nrd} : D^* \to L^*$ is surjective.

(d) For a given prime p, index and exponent of p-primary algebras coincide : That is, for any finite field extension L/K, and any central simple p-primary division algebra D/L, the index of D, i.e. the square root of its dimension, and the exponent of D, i.e. the order of its class in the Brauer group of L, coincide. (An algebra is called p-primary if its order in the Brauer group is a power of p; equivalently, it is equivalent to a central simple algebra of index a power of p.)

(e) For a given prime p, over any finite field extension L of K, the tensor product of two central simple algebras of index p is of index at most p.

(f) Over any finite field extension L/K, any quadratic form in at least 5 variables has a nontrivial zero.

(g) The cohomological dimension $\mathrm{cd}(K^{ab})$ of the maximal abelian extension of K is at most one (this property is stable under algebraic field extensions).

Here are the known relations between these various properties.

Theorem 1.1 *Let K be a field of characteristic zero. The following implications hold :*
(i) (a) implies (b)
(ii) (b) is equivalent to (c).
(iii) (a) implies (f).
(iv) (d) is equivalent to (e).

(v) (f) implies (d) for $p = 2$.

(vi) (a) implies (d) for $p = 2$ and $p = 3$.

(vii) the combination of (b) and (e) for $p = 2$ implies (f).

Proof Statement (ii) is a consequence of the Merkurjev-Suslin results ([Su], Thm. 24.8). Statement (i) is a direct consequence of (ii). Statement (iii) is trivial. For statement (iv), which also relies on results of Merkurjev and Suslin, we refer to the proof of Prop. 7 in [CTG] and the references given there. Statement (v) follows from a well-known criterion of Albert (see [CTG]) together with Statement (iv) for $p = 2$. For the proof of (vi), see [CTG], Prop. 7. For the proof of (vii), see [CTOP], proof of Theorem 3.4.

Recall Serre's conjecture II : for a perfect field K of cohomological dimension 2 and a simply connected group G/K, the set $H^1(K, G)$ is reduced to one element. The following theorem gathers results known in that direction.

Theorem 1.2 *Let K be a field of characteristic zero and of cohomological dimension at most 2. Let G/K be a simply connected group.*

(i) If G is absolutely almost simple of classical type, or is of type G_2 or F_4, then $H^1(K, G) = 1$.

(ii) Suppose that index and exponent coincide for 2-primary algebras over finite field extensions of K. Then $H^1(K, G) = 1$ for G of trialitarian type D_4 and for G of type E_7.

(iii) Suppose that index and exponent coincide for 3-primary algebras over finite field extensions of K. Then $H^1(K, G) = 1$ for G of type E_6.

(iv) Suppose that the cohomological dimension of K^{ab} is at most one. If G is of type E_8, then $H^1(K, G) = 1$.

(v) Suppose that index and exponent coincide for 2-primary and 3-primary algebras over finite field extensions of K. If G has no factor of type E_8, then $H^1(K, G) = 1$, and the same holds in general if moreover the cohomological dimension of K^{ab} is at most one.

Proof For G of type ${}^1A_n$, statement (i) is a consequence of the Merkurjev-Suslin theorems, as noticed by Suslin [Su]. For the other groups of classical type, the result is due to Bayer-Fluckiger and Parimala [BFP]. When exponent and index of algebras coincide, note that the proofs of [BFP] may be shortened. Statements (ii), (iii), (iv) are due to Gille. For trialitarian type D_4, this is [Gi2], Thm. 8 p. 313. For E_6 (either ${}^1E_6$ or ${}^2E_6$), this is [Gi2], Thm. 9. p. 314. For E_7, this is [Gi2], Thm. 10 p. 316. The argument for the E_8 case (briefly sketched on p. 322 of [Gi2]) is inspired by the number theory case (Chernousov). Let G_0 be the split group of type E_8. Over any field K, any element of $H^1(K, G_0)$ is in the image of $H^1(K, T)$ for the natural embedding $T \subset G_0$ of a suitable maximal torus T of G (Steinberg, see [Se2], §III.2.3, Corollaire, p. 140). For $T \subset G_0$ a maximal torus, the group $H^1(K, T)$ is killed by a power of 2.3.5 (Harder, Tits, cf. [Se3], Thm. 3 p. 234). Under the assumption that $\mathrm{cd}(K^{ab}) \leq 1$, this implies that any element of $H^1(K, G_0)$ belongs to the image of $H^1(L/K, G_0)$ for some finite abelian extension of degree $2^a.3^b.5^c$. Theorem 11, p. 317 of [Gi2], states that for G_0 split of type E_8 over a field K of characteristic zero and of cohomological dimension at most 2, and L/K cyclic of degree 2, 3 or 5, we have $H^1(L/K, G_0) = 1$. By induction, we conclude that $H^1(K, G_0) = 1$. Since the centre of G_0 is trivial and there are no outer automorphisms, we conclude that the only form of G_0 is the split form, which completes the proof of (iv).

The proof of (v) involves a well-known result, which will be used a number of times in this paper, and which we shall refer to as *the standard reduction to the almost simple case*. Here is the statement : If a semisimple group G/K is simply connected, resp. adjoint, then it is K-isomorphic to a product $\prod_i R_{K_i/K} G_i$, where the K_i/K are finite field extensions, $R_{K_i/K}$ denotes the Weil restriction of scalars, and each G_i is an absolutely almost simple K_i-group which is simply connected, resp. adjoint. In the simply connected case, the centre of G is the product of the $R_{K_i/K} C_i$, where C_i denotes the centre of G_i.

Statement (v) now follows from the other statements thanks to the Shapiro type formula $H^1(K, G) = \prod_i H^1(K_i, G_i)$.

Remark For the exceptional groups, an independent proof of Theorem 1.2 has been given by Chernousov [Ch].

We now specialize to the fields of "geometric type" of interest in this paper.

Theorem 1.3 (case (gl)) *Let K be a function field in two variables over an algebraically closed field k of characteristic zero. The field K is a C_2-field, it has* $\mathrm{cd}(K) = 2$. *Over K, index and exponent of a 2-primary or a 3-primary algebra coincide*[(2)]. *If G/K is a semisimple simply connected group without E_8-factor, then $H^1(K, G) = 1$.*

Proof That K is a C_2-field is a special case of the Tsen-Lang theorem (see [Pf], Chap. 5). That K has cohomological dimension 2 is a consequence of Theorem 1.1 (i) above, but it had been known from the very beginning of Galois cohomology ([Se2], II.4.2, Corollaire, p. 95). The rest of the theorem follows from Theorems 1.1 and 1.2.

Theorem 1.4 (case (ll)) *Let K be the field of fractions of a two-dimensional, excellent, henselian local domain A with residue field k algebraically closed of characteristic zero. The field K has* $\mathrm{cd}(K) = 2$. *All central simple algebras over K are cyclic, of exponent equal to the index. Quadratic forms in at least 5 variables have a nontrivial zero. The cohomological dimension* $\mathrm{cd}(K^{ab})$ *of the maximal abelian extension of K is one. For any semisimple simply connected group G/K, we have $H^1(K, G) = 1$.*

Proof That $\mathrm{cd}(K) = 2$ is a special case of result of M. Artin ([SGA 4] XIX Cor. 6.3). That index and exponent coincide is also a result of M. Artin ([A], Theorem 1.1). That algebras are cyclic (of index equal to the exponent) is a result of T. Ford and D. Saltman in the geometric case, see Thm 2.1 of [CTOP] for the general statement. That $\mathrm{cd}(K^{ab}) \leq 1$ is Thm. 2.2 of [CTOP]. The rest of the theorem follows from Theorems 1.1 and 1.2.

Theorem 1.5 (case (sl)) *Let K be the Laurent series field $l((t))$ over a field l of characteristic zero and cohomological dimension 1. Such a field is of cohomological dimension 2. Quadratic forms in at least 5 variables have a nontrivial zero. Index and exponent of algebras coincide. For any semisimple simply connected group G/K, we have $H^1(K, G) = 1$.*

Proof For the property $\mathrm{cd}(K) = 2$, see [Se2], II.4.3, Prop. 12 p. 95. The proof of the statement for quadratic forms is well-known ([Sc], §6.2, Corollary 2.6.(iv), p. 209). Any central simple algebra over $K = k((t))$ is similar to a cyclic algebra $(K/k, \sigma, t)$, where K/k is a finite cyclic field extension and σ is a generator of the Galois group of K/k. Such an algebra being clearly a division algebra, its index is equal to its exponent. That Serre's conjecture $H^1(K, G) = 1$ holds in this context is a theorem of Bruhat and Tits ([BT], Théorème 4.7).

Remark Fields of type (sl) were studied by J.-C. Douai in his thesis (Lille, 1976). Building upon the work of Bruhat and Tits, for such fields he established Theorem 2.1 (a), Prop. 5.3 and Prop. 5.4 of the present paper.

For fields of type (gl), there are various natural completions. More generally, let K be a function field of transcendence degree d over k (we shall mostly be interested in the case $d = 2$). For X/k a smooth projective model of K, let X^1 denote the set of codimension 1 points on X, and let Ω_X denote the set of discrete valuations v associated to points $x \in X^1$. The completion $K_x = K_v$ of K at such a discrete valuation ring is isomorphic to $l((t))$, where $l = \kappa(x)$ is the

(2) This actually holds for arbitrary algebras, as recently proved by A. J. de Jong.

residue field at $v = v_x$, which is a function field in $d-1$ variables. One then denotes by Ω the union over all models X of the Ω_X.

For any X, the kernel of the diagonal restriction map on Brauer groups

$$\mathrm{Br}(K) \to \oplus_{v\in\Omega_X}\mathrm{Br}(K_v)$$

is contained in the subgroup $\mathrm{Br}(X) \subset \mathrm{Br}(K)$, and it is equal to that subgroup in the (gl) case under consideration above, namely if $d = 2$. The same two statements hold for the kernel of the diagonal restriction map

$$\mathrm{Br}(K) \to \prod_{v\in\Omega}\mathrm{Br}(K_v).$$

This torsion group is an extension of a finite group by a group of the shape $(\mathbf{Q}/\mathbf{Z})^r$. For all this, see Grothendieck [Gr].

Similarly, for fields of type (ll), there are various natural completions. Let K be the field of fractions of a strictly henselian, two-dimensional, excellent, local domain A with residue field k algebraically closed of characteristic zero.

Let X be a regular, integral, two-dimensional scheme equipped with a proper birational morphism $X \to \mathrm{Spec}(A)$. Such schemes exist (for references, see [CTOP]). The function field of X is K. Let Ω_X denote the set of discrete valuations v associated to points $x \in X^1$. If x lies above a codimension 1 point on $\mathrm{Spec}(A)$, then K_v is isomorphic to $F_v((x))$, where the field F_v is the field of fractions of a discrete, henselian valuation ring with residue field k. If x lies above the closed point of $\mathrm{Spec}(A)$, then K_v is isomorphic to $F_v((x))$, and F_v is the function field of a smooth, projective curve over k. In both cases, Theorem 1.5 ensures that K_v satisfies properties (a) to (g) listed before Theorem 1.1.

Given $A \subset K$ as above, we let $\Omega = \Omega_A$ be the union of all Ω_X for X varying among the regular integral schemes X equipped with a proper birational morphism $X \to \mathrm{Spec}(A)$.

We refer to §1 of [CTOP] for the proof of the following theorem.

Theorem 1.6 *Let K be the field of fractions of a strictly henselian two-dimensional local domain A with residue field k. Let $X \to \mathrm{Spec}(A)$ be a regular proper model. With notation as above, the kernel of the diagonal restriction map on Brauer groups*

$$\mathrm{Br}(K) \to \oplus_{v\in\Omega_X}\mathrm{Br}(K_v)$$

is trivial. So is a fortiori the kernel of the diagonal restriction map on Brauer groups

$$\mathrm{Br}(K) \to \prod_{v\in\Omega}\mathrm{Br}(K_v).$$

§2. Surjectivity of the boundary map and isotropy

Part (a) of the following theorem was stated as Conjecture 5.4 in [CTOP]. It will play a key rôle in §5 (Hasse principle for projective homogeneous spaces over fields of type (ll)). Part (b) will play a key rôle in §4 (R-equivalence and weak approximation).

Theorem 2.1 *Let k be a field of characteristic zero. Assume :*
(i) the cohomological dimension $\mathrm{cd}(k)$ of k is at most 2 ;
(ii) for central simple algebras over any finite field extension of k, index coincides with exponent.

Let G be a semisimple, simply connected group over k. If G has some factor of type E_8, assume that the cohomological dimension of the maximal abelian extension k^{ab} of k is at most one. Let

$$1 \to \mu \to G \to G^{\mathrm{ad}} \to 1$$

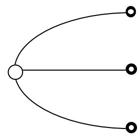
be the central isogeny associated to the centre μ of G. Then :

(a) The boundary map $\delta : H^1(k, G^{\mathrm{ad}}) \to H^2(k, \mu)$ is a bijection.

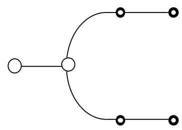
(b) If the group G is not purely of type A, then it is isotropic.

(c) Groups of type F_4, G_2, E_8 are split, and groups of type ${}^{3,6}D_4$, ${}^{1,2}E_6$, E_7 which are not split have the following Tits indices :

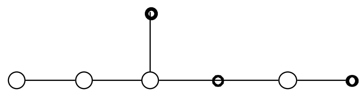
${}^{3,6}D_4$



${}^{1,2}E_6$



E_7



Proof The proof will occupy the whole section.

By the standard reduction recalled in the proof of Theorem 1.2, to prove the theorem one may assume that G is absolutely almost simple.

By Theorem 1.2, we have $H^1(k, G) = 0$ for any semisimple simply connected group G. This already implies that the connecting map $H^1(k, G^{\mathrm{ad}}) \to H^2(k, \mu)$ is one-to-one.

Let G_0, resp. G_0^{ad} be the unique, inner quasisplit form of G, resp. G^{ad}. We have the two exact sequences

$$1 \to \mu \to G_0 \to G_0^{\mathrm{ad}} \to 1$$

and

$$1 \to \mu \to G \to G^{\mathrm{ad}} \to 1.$$

Let $\gamma \in Z^1(k, G_0^{\mathrm{ad}})$ be a cocycle such that the twist $G_{0,\gamma}$ is equal to G. Let $\varepsilon \in H^2(k, \mu)$ be the image of the class of γ under the connecting map $\delta_0 : H^1(k, G_0^{\mathrm{ad}}) \to H^2(k, \mu)$. We have a bijection $\theta_\gamma : H^1(k, G^{\mathrm{ad}}) \to H^1(k, G_0^{\mathrm{ad}})$ sending the trivial class to the class of γ. From the commutative diagram (28.12) p. 388 of [KMRT], we draw the following two conclusions :

(1) Surjectivity of the map $\delta_0 : H^1(k, G_0^{\mathrm{ad}}) \to H^2(k, \mu)$ is equivalent to surjectivity of the map $\delta : H^1(k, G^{\mathrm{ad}}) \to H^2(k, \mu)$.

(2) The Tits class t_G of G (cf. [KMRT] p. 426) is the image under δ_0 of the class of γ.

By statement (1), to prove the theorem we may restrict to the case where the absolutely almost simple simply connected group G is quasisplit, $G = G_0$.

For several types, the surjectivity statement in (a) and the isotropy statement in (b) (in the cases where (b) holds) will be proved simultaneously. What we shall prove is that for any class $c \in H^2(k, \mu)$ there exists an explicit isotropic internal form G of G_0, whose class in $H^1(k, G_0^{\mathrm{ad}})$ has image c under the boundary map δ_0. That will clearly be enough to prove surjectivity and isotropy, because δ_0 is one-to-one.

Groups of type ^1A_n
Let $m = n+1$. The basic exact sequence here is

$$1 \to \mu_m \to SL_m \to PGL_m \to 1$$

and the boundary map δ_0 sends the isomorphy class of a central simple algebra A/k of degree m to its class in the m-torsion subgroup ${}_m\mathrm{Br}(k)$. Hypothesis (ii) of the theorem guarantees that δ is onto.

Groups of type B_n $(n \geq 2)$
Here $\mu = \mu_2$ and G_0 is the split group. By hypothesis (ii), $H^2(k, \mu_2) = {}_2\mathrm{Br}(k)$ consists of classes of quaternion algebras. Let $D/k = (a, b)$ be a quaternion algebra. We have to show that the class $[D]$ of D in $\mathrm{Br}(k)$ is equal to the Tits class t_G ([KMRT], p. 426) for some (automatically internal) form G of G_0.

By [KMRT] (31.9) p. 427, given q a quadratic form of dimension $2n+1$, the Tits class of $\mathrm{Spin}(q)$ is the class of the even Clifford algebra $C_0(q)$. Let q be the (isotropic) quadratic form $\langle a, b, -ab\rangle \perp (n-1)\mathbf{H}$, where $\mathbf{H} = \langle 1\rangle \perp \langle -1\rangle$ and $\langle a, b, -ab\rangle$ is the standard notation for the quadratic form $aX^2 + bY^2 - abZ^2$. Since $C_0(\langle a, b, -ab\rangle) = (a, b)$, then $[C_0(q)] = [(a, b)]$ (see [Sc], Chap. 9, §2). The associated group $\mathrm{Spin}(q)$ is isotropic, and its Tits class is $[C_0(q)]$ ([KMRT], p. 378), hence equal to $[D]$.

Groups of type C_n $(n \geq 2)$
Here again $\mu = \mu_2$, and the group G_0 is split (all forms are inner). By hypothesis, any class in ${}_2\mathrm{Br}(k)$ is represented by a quaternion algebra D. There is a unique symplectic involution γ on the quaternion algebra D/k ([KMRT], Prop. 2.21). Let $P \in M_n(k)$ be the symmetric matrix with associated quadratic form $2x_1x_2 + \sum_{i=3}^n x_i^2$. The tensor product of the involution γ on D and the involution $X \mapsto -P^tXP$ on M_n is an isotropic symplectic involution on $M_n(D)$, which we denote by $\widetilde{\gamma}$. The group $Sp(M_n(D), \widetilde{\gamma})$ is then an isotropic group of type C_n with Tits class $[D]$ ([KMRT] (31.10) p. 427).

Groups of type $^3D_4$ or $^6D_4$
Let G_0 have type $^3D_4$ or $^6D_4$, with associated cubic field extension L/k. The associated exact sequence of quasisplit groups

$$1 \to \mu \to G_0 \to G_0^{\mathrm{ad}} \to 1$$

is sequence (44.11) of [KMRT], p. 563. The centre μ of G_0 (which is denoted C_δ in [KMRT]) is the group $\mu = R^1_{L/k}(\mu_2)$ ([KMRT], p. 564). Since the degree of L over k is 3, the composite of the diagonal embedding $\mu_2 \to R_{L/k}(\mu_2)$ and of the norm map $R_{L/k}(\mu_2) \to \mu_2$ is identity. Thus the exact sequence

$$1 \to R^1_{L/k}(\mu_2) \to R_{L/k}(\mu_2) \to \mu_2 \to 1$$

is split, hence induces an isomorphism

$$H^2(k, \mu) = \ker[\mathrm{Cores} : {}_2\mathrm{Br}(L) \to {}_2\mathrm{Br}(k)].$$

By hypothesis (ii) of the Theorem, an element in $H^2(k, \mu)$ is represented by a quaternion algebra D over L whose corestriction in k is trivial in the Brauer group of k. The main Theorem of [Ga] states that there exists a unique (simply connected) trialitarian group G/k with Allen algebra $M_4(D)$. This is an internal form of G_0 and its Tits index of the type described in the Theorem. Let $\xi \in H^1(k, G_0^{\mathrm{ad}})$ represent G/k. The map δ_0 is the map denoted by Sn^1 in sequence (44.12) of [KMRT]. The image of ξ under δ_0 is called the Clifford invariant of G, it is the class of the Allen algebra, and is equal in that case to $[D] \in Br(L)$.

Statements (b) and (c) now follow from the general argument mentioned before the discussion of case ^1A_n.

Groups of type E_6

Let G_0 be the split simply connected group of type E_6. Its centre is the group μ_3. Fix a (split) Borel subgroup. Deleting the central root from the extended Dynkin diagram, we see that there is a natural k-homomorphism of split k-groups $(SL_2)^3 \to G_0^{\mathrm{ad}}$ with finite kernel. Let $H \subset G_0$ denote its image. One shows (see [Ti1], §6.4.4 and [Ti2], §1.2) that this homomorphism induces a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_3 & \longrightarrow & G_0 & \longrightarrow & G_0^{\mathrm{ad}} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & ((\mu_3)^3)^0 & \longrightarrow & (SL_3)^3 & \longrightarrow & H & \longrightarrow & 1.
\end{array}
$$

Here $((\mu_3)^3)^0$ is the kernel of the product map $(\mu_3)^3 \to \mu_3$ sending (λ, μ, ν) to $\lambda.\mu.\nu$, and the map $((\mu_3)^3)^0 \to \mu_3$ in the diagram is the map which sends an element $(\lambda, \mu, \nu) \in ((\mu_3)^3)^0$ to $\lambda/\mu = \mu/\nu = \nu/\lambda$. Note that this map is split. Applying Galois cohomology to the above diagram, one sees that the image of δ_0 in $H^2(k, \mu_3)$ contains the image of the composite map $H^1(k, H) \to H^2(k, ((\mu_3)^3)^0) \to H^2(k, \mu_3)$. Because of the splitting mentioned above, the last map is surjective. Now the image of the first map consists exactly of the triples $([D_1], [D_2], [D_3]) \in H^2(k, ((\mu_3)^3)^0) \subset {}_3\mathrm{Br}(k)$ with sum zero, each D_i being a central simple algebra over k of degree 3. By our assumption on the field k, any class in $H^2(k, ((\mu_3)^3)^0)$ is of this shape. This concludes the proof of surjectivity in the ${}^1E_6$ case. For the isotropy, and the possible Tits indices, see [Gi2], Thm. 9. p. 314.

There is a natural action of the group $\mathbf{Z}/2$ on the above diagram of morphisms of groups. On the upper row, it is given by the "opposition involution" associated to the nontrivial automorphism of the Dynkin diagram of E_6 ([Ti1], 1.5.1) which induces the inverse map $x \to x^{-1}$ on the centre μ_3 ([Ti1], 1.5.3 (c)). This involution respects the exended Dynkin diagram deprived of its central point. It thus gives rise to an automorphism of the group H which lifts to an automorphism of the group $(SL_3)^3$ which is given by $(x, y, z) \to (x, z, y)$. Let K/k be a quadratic field extension. One can twist the above diagram by means of the obvious nontrivial cocyle in $H^1(\mathrm{Gal}(K/k), \mathbf{Z}/2)$. This gives rise to the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R^1_{K/k}(\mu_3) & \longrightarrow & G'_0 & \longrightarrow & G_0'^{\mathrm{ad}} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & (\mu_3 \times R_{K/k}\mu_3)^0 & \longrightarrow & SL_3 \times R_{K/k}(SL_3) & \longrightarrow & H & \longrightarrow & 1,
\end{array}
$$

where G'_0 is the quasisplit form ${}^2E_6$ corresponding to the extension K/k and $(\mu_3 \times R_{K/k}\mu_3)^0$ denotes the kernel of the map $\mu_3 \times R_{K/k}\mu_3 \to \mu_3$ given by $(\lambda, \xi) \mapsto \lambda.N_{K/k}(\xi)$. The left vertical map sends (λ, ξ) to ξ/λ. This map is clearly split. To prove that δ_0 is onto, it thus suffices to show that the boundary map $H^1(k, H) \to H^2(k, (\mu_3 \times R_{K/k}\mu_3)^0)$ is onto. The target of this map consists of pairs $(a, b) \in {}_3\mathrm{Br}(k) \oplus {}_3\mathrm{Br}(K)$ such that $a + N_{K/K}(b) = 0$. The image of the map $H^1(k, H) \to H^2(k, (\mu_3 \times R_{K/k}\mu_3)^0)$ consists of such pairs (a, b) having the additional property that $a \in {}_3\mathrm{Br}(k)$ is represented by an algebra over k of degree 3 and $b \in {}_3\mathrm{Br}(K)$ is represented by an algebra over K of degree 3. Under hypothesis (ii) of the Theorem, this last property is always satisfied.

For statements (b) and (c), see [Gi2], Thm. 9 p. 314.

Groups of type E_7

Let G_0 be the simply connected split form of E_7. The centre of G_0 is μ_2. Consideration of the extended Dynkin diagram and of [Ti2], §1.2 shows (see [MPW], §8.II, p. 156) that there exists a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mu_2 & \to & G_0 & \to & G_0^{\mathrm{ad}} & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & \mu_4/\mu_2 & \to & SL_8/\mu_2 & \to & SL_8/\mu_4 & \to & 1. \end{array}$$

The diagonal map $SL_4 \to SL_4 \times SL_4 \subset SL_8$ induces the commutative diagram of exact sequences (where the last vertical arrow is an embedding)

$$\begin{array}{ccccccccc} 1 & \to & \mu_4/\mu_2 & \to & SL_8/\mu_2 & \to & SL_8/\mu_4 & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & \mu_4 & \to & SL_4 & \to & PGL_4 & \to & 1. \end{array}$$

Combining these two diagrams and taking the boundary map, we get the commutative diagram

$$\begin{array}{ccc} H^1(k, G_0^{\mathrm{ad}}) & \xrightarrow{\delta_0} & H^2(k, \mu_2) \\ \uparrow & & {\scriptstyle \times 2}\uparrow \\ H^1(k, PGL_4) & \to & H^2(k, \mu_4). \end{array}$$

Since the cohomological dimension of k is at most 2, the induced map $H^2(k, \mu_4) \to H^2(k, \mu_2)$ is onto. The boundary map $H^1(k, PGL_4) \to H^2(k, \mu_4) = {}_4\mathrm{Br}(k)$ has image the set of classes of algebras of degree 4. Under the assumption that index coincides with exponent, that boundary map is surjective. Hence so is the map $\delta_0 : H^1(k, G_0^{\mathrm{ad}}) \to H^2(k, \mu_2)$.

For the isotropy statement and the more precise description of the possible Tits indices, we refer to [Gi2], Thm. 10 p. 316.

Groups of type G_2, F_4, E_8

In this case $\mu = 1$ and forms are classified by $H^1(k, G_0)$, where G_0 is the simply connected split group. By Theorem 1.2, $H^1(k, G_0) = 1$, hence any group of this type is split and in particular isotropic.

Groups of type ${}^2A_n$

Let K/k denote a quadratic field extension; assume $\mathrm{char}(k) \neq 2$.

Lemma 2.2 *Let k be a field such that quadratic forms of dimension 5 over k are isotropic. Let B be a quaternion algebra over K equipped with a K/k-involution τ. Given any $\lambda \in k^*$ there exists $x \in B^*$ such that $\tau x = x$ and $\mathrm{Nrd} x = \lambda$.*

Proof Let τ_0 denote the nontrivial automorphism of K/k. There exists a k-quaternion subalgebra $B_0 \subset B$ such that $B = B_0.K$ and τ restricted to B_0 is the canonical involution

σ on B_0. (The algebra B_0 is the fixed ring of the product $\sigma\tau$, where σ denotes the canonical involution on the K-quaternion algebra B.) Let $K = k(\sqrt{a})$. A τ-symmetric element of B is of the form $\mu + y\sqrt{a}$ with $\mu \in k$, $y \in B_0$ satisfying $\sigma y = -y$. We then have $\mathrm{Nrd}(\mu + y\sqrt{a}) = \mu^2 - y^2 a = \mu^2 + aN'(y)$, where N' denotes the restriction of the reduced norm on B_0 to trace zero elements. By the hypothesis on k, any 4-dimensional quadratic form over k represents any nonzero element in k.

Lemma 2.3 *Let B/K be a central simple algebra with a K/k-involution τ. Let $x \in B^*$ be an element such that $\tau x = -x$ and $x^2 = \nu \in k^*$. Let $\varepsilon \in k^*$ be a norm from the quadratic extension $k(x)/k$. Then the matrix $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ in $M_2(B)$ is conjugate to a τ-hermitian symmetric matrix.*

Proof Let $\varepsilon = c^2 - \nu d^2, c, d \in k$. Then the product

$$\begin{pmatrix} 1 & c \\ 0 & xd \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & xd \end{pmatrix}^{-1} = \begin{pmatrix} c & -xd \\ xd & -c \end{pmatrix}$$

is hermitian symmetric with respect to τ.

Lemma 2.4 *Let B be a central division algebra over K, of degree $2^n, n \geq 1$. Let $\varepsilon \in k^*$ be the reduced norm of a symmetric element with respect to some K/k-involution on B. Then there exist an odd degree field extension F/k, a KF/F-involution τ on $B \otimes_k F$ and $x \in B \otimes_k F$ such that $\tau x = -x$, $x^2 \in F^*$ and ε is a norm from the quadratic extension $F(x)/F$.*

Proof Let τ_1 be a K/k-involution on B such that there exists $u \in B^*$ with $\tau_1 u = u$ and $\mathrm{Nrd}(u) = \varepsilon$. There exists a maximal subfield E of B/K such that u belongs to E and $N_{E/K}(u) = \varepsilon$. Since the degree $[k(u) : k]$ is a power of 2, after replacing k by an odd degree extension, we may assume that $k(u)$ contains a quadratic extension $k(x)/k$ with $x^2 \in k$. Let η be the automorphism of $K(x)/k$ which restricts to the nontrivial automorphism of K/k and to the nontrivial automorphism of $k(x)/k$. Then η extends to a K/k-involution τ on B such that $\tau(x) = -x$ ([Kn1], Theorem on p. 37). Let $w = N_{E/K(x)}(u)$. Since $K(x)$ is stable under τ_1, we have $\tau_1 w = w$. Since $k(x)$ is the field of τ_1-invariant elements in $K(x)$, this implies $w \in k(x)$ and $N_{k(x)/k}(w) = \varepsilon$.

To any central simple algebra B/K equipped with a K/k-involution τ there is an associated central simple algebra over k, called the discriminant algebra, denoted $D(B, \tau)$ ([KMRT], Definition 10.28 p. 128).

Lemma 2.5 *Let B/K be a central division algebra with a K/k-involution τ. Let $H_{2n}, n \geq 1$ be the standard hyperbolic form and $\tau_{H_{2n}}$ the involution on $M_{2n}(B)$ adjoint to H_{2n}. Then the discriminant algebra of $(M_{2n}(B), \tau_{H_{2n}})$ is a matrix algebra over k.*

Proof Let X/K be the Severi-Brauer variety attached to B and $Y = R_{K/k}(X)$ be its Weil restriction of scalars to k. The restriction map from $\mathrm{Br}(k)$ to $\mathrm{Br}(k(Y))$ is injective ([MT]). To prove the lemma, we may replace k by $k(Y)$ and thus assume that B/K is split. In this case the result follows from [KMRT] (10.35) p. 131.

Lemma 2.6 *Let B/K be a central division algebra with a K/k-involution τ. Let $\tilde{\tau}$ be the involution on $M_2(B)$ given by $x \mapsto (\tau x)^t$. If the cohomological dimension of k is at most 2, then the discriminant algebra of $(M_2(B), \tilde{\tau})$ is a matrix algebra over k.*

Proof The involution $\tilde{\tau}$ is adjoint with respect to the hermitian form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ over (B, τ) ([KMRT] (4.1) and (4.2) pp. 42, 43). Since this hermitian form has the same dimension and discriminant as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathrm{cd}(k) \leq 2$, the two forms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are isometric

([BFP]). Thus the adjoint involutions with respect to these two forms are isomorphic. The result now follows from Lemma 2.5.

We record the following well known result (cf. [CTG], Prop. 7).

Proposition 2.7 *Let k be a field,* $\mathrm{char}(k) \neq 2$, *such that every 5-dimensional quadratic form over every finite extension of k is isotropic. Then over any finite extension of k, for an algebra of exponent a power of 2, the exponent is equal to the index.*

Theorem 2.8 *Let B/K be a central simple algebra with a K/k-involution τ. Suppose :*
(a) $\mathrm{cd}(k) \leq 2$ *;*
(b) every 5-dimensional quadratic form over every finite extension of k is isotropic.
Then every element of k^*, up to a square, is the reduced norm of a τ-symmetric element.
Proof Let $\lambda \in k^*$.

Suppose the theorem is true for division algebras. Write $B = M_n(D)$, with D/K a central division algebra equipped with a K/k-involution τ_0 ([KMRT], (3.1)). There exists a τ_0-hermitian symmetric matrix $H \in B^*$ such that τ is the adjoint involution $\tau_{0,H}$ with respect to H. Since H is a hermitian symmetric matrix, $\mathrm{Nrd}_B(H) \in K^*$ actually belongs to k^*. By the assumption, there exists $u \in D^*$ such that $\tau_0(u) = u$ and $\lambda.\mathrm{Nrd}_B H = \mathrm{Nrd}_D(u).\mu^2 \in k^*$ with $\mu \in k^*$. Let U be the diagonal matrix $< 1, \cdots, 1, u >$ and let $X = U.H^{-1}$. Then $\tau(X) = X$ and $\mathrm{Nrd}_B(X) = \mathrm{Nrd}_D(u).\mathrm{Nrd}_B(H^{-1}) = \lambda.\mu^{-2}$. We are reduced to the case where B is a central division algebra.

Suppose the theorem is true for division algebras with degree a power of 2. Write $B = B_1 \otimes_K B_2$ with $\deg_K(B_1)$ a power of 2 and $\deg_K(B_2) = m$ with m odd. By the characterisation of K/k-involutions (KMRT], (3.1)), $\mathrm{Cores}_{K/k}(B) = 0$. The algebras B_1 and B_2 being of coprime order in the Brauer group, $\mathrm{Cores}_{K/k}(B_i) \sim 0$ for $i = 1, 2$. By the quoted characterisation, there exists a K/k-involution τ_i on B_i for $i = 1, 2$. By [KMRT] (2.7) there exists $u \in B^*$ with $\tau u = u$ and $\tau = Int(u) \circ (\tau_1 \otimes \tau_2)$. By the assumption, there exists $v \in B_1^*$ such that $\tau_1 v = v$ and $\mathrm{Nrd}_{B_1}(v) = \mathrm{Nrd}_B(u).\nu^2$ for some $\nu \in k^*$. Let $\tau_1' = Int(v) \circ \tau_1$. Then $\tau = Int(u) \circ Int(v^{-1} \otimes 1) \circ (\tau_1' \otimes \tau_2)$. By the assumption, there exists $w \in B_1$ with $\tau_1' w = w$ and $\mathrm{Nrd}_{B_1}(w) = \lambda\delta^2$ for some $\delta \in k^*$. The element $(w \otimes 1)(u.(v^{-1} \otimes 1))^{-1}$ is τ-invariant. We have

$$\mathrm{Nrd}_B((w \otimes 1)(u.(v^{-1} \otimes 1))^{-1}) = \mathrm{Nrd}_{B_1}(w)^m.\mathrm{Nrd}_{B_1}(v)^m.\mathrm{Nrd}_B(u)^{-1}$$

which is λ up to squares.

We are reduced to the case where B/K is a central division algebra of degree 2^n. In this case we prove more : *every $\lambda \in k^*$ is the norm of a τ-symmetric element.*

Note that, by Proposition 2.7, the exponent of B in $\mathrm{Br}(K)$ is 2^n. The proof is by induction on n. For $n = 1$, the theorem follows from Lemma 2.2. Suppose $n > 1$. Let $D(B, \tau)$ be the discriminant algebra of (B, τ). Its class is of order 2 in the Brauer group of k. Since every 5-dimensional quadratic form is isotropic, the tensor product of two quaternion algebras is the class of a quaternion algebra. Combining this with Merkurjev's theorem we conclude that the class of $D(B, \tau)$ in $\mathrm{Br}(k)$ is represented by a quaternion algebra A/k. The class of A_K in the Brauer group of K coincides with that of $B^{\otimes 2^{n-1}}$ ([KMRT], Prop. 10.30 p. 129). By Proposition 2.7, the index of the latter is equal to its exponent, which is 2. Thus A_K and therefore A are division algebras.

Write $K = k(\sqrt{a})$ with $a \in k^*$. Let N_A' denote the restriction of the reduced norm of A to trace zero elements of A. By hypothesis (b), the 5-dimensional quadratic form $N_A' \perp a < 1, -\lambda >$ is isotropic over k. There exists $\varepsilon = x^2 - \lambda y^2 \in k^*$, with $x, y \in k$, such that $-a\varepsilon$ is a value of N_A'. Since A is a quaternion division algebra, $a\varepsilon$ is not a square and $k(\sqrt{a\varepsilon})$ splits A. Since ε is a norm from $k(\sqrt{\lambda})$, λ is a norm from $k(\sqrt{\varepsilon})$. Let $\eta \in k(\sqrt{\varepsilon})$ be such that $N_{k(\sqrt{\varepsilon})/k}(\eta) = \lambda$. In the Brauer group of $K(\sqrt{\varepsilon}) = k(\sqrt{a}, \sqrt{\varepsilon})$, we have

$$B^{2^{n-1}} \otimes_K K(\sqrt{\varepsilon}) \sim A \otimes_k K \otimes_K K(\sqrt{\varepsilon}) \sim A \otimes_k k(\sqrt{a\varepsilon}) \otimes_{k(\sqrt{a\varepsilon})} K(\sqrt{\varepsilon}) \sim 0.$$

Hence the exponent of $B \otimes_K K(\sqrt{\varepsilon})$ is at most 2^{n-1}. By Proposition 2.7 and hypothesis (b), the index of $B \otimes_K K(\sqrt{\varepsilon})$ is also at most 2^{n-1}; it is in fact equal to 2^{n-1}, because the index can be divided at most by 2 under a quadratic extension. Hence $K(\sqrt{\varepsilon})$ embeds into B over K. The nontrivial automorphism of $K(\sqrt{\varepsilon})/k(\sqrt{\varepsilon})$ extends to a K/k-involution τ_1 on B with $\tau_1(\sqrt{\varepsilon}) = \sqrt{\varepsilon}$ (Kneser, cf. [Sch], Chap. 8.10, Thm. 10.1 p. 311). Let C be the commutant of $K(\sqrt{\varepsilon})$ in B. Then τ_1 restricts to a $K(\sqrt{\varepsilon})/k(\sqrt{\varepsilon})$ involution on C. The index of $C/K(\sqrt{\varepsilon})$ is 2^{n-1}. By the induction hypothesis applied to $C/k(\sqrt{\varepsilon})$ and τ_1, there exists $x \in C$ with $\tau_1 x = x$ and $\mathrm{Nrd}_C(x) = \eta$. We have

$$\mathrm{Nrd}_B(x) = N_{K(\sqrt{\varepsilon})/K}(\mathrm{Nrd}_C(x)) = N_{k(\sqrt{\varepsilon})/k}(\eta) = \lambda.$$

There exists $u \in B^*$ such that $\tau_1 = Int(u) \circ \tau$ with $\tau_1 u = u$. Let $t = \mathrm{Nrd}_B(u)$. Then $\tau(xu) = xu$ and $\mathrm{Nrd}_B(xu) = \lambda t$.

Our next step is to get rid of t by replacing $k(\sqrt{\varepsilon})$ by a conjugate field $yk(\sqrt{\varepsilon})y^{-1}$ for a suitable $y \in B^*$.

For $y \in B^*$ to be chosen later, let $\tau_1' = Int(y) \circ \tau_1 \circ Int(y)^{-1}$ and $u' = yu\tau(y)$. Then $\tau_1' = Int(u') \circ \tau$. We look for y such that there exists $x' \in yCy^{-1}$ which is τ_1'-symmetric and $\mathrm{Nrd}_B(x'u') = \lambda$. This will complete the proof of the theorem because $x'u'$ is τ-symmetric.

Suppose we have y and x' as above. Then

$$N_{yk(\sqrt{\varepsilon})y^{-1}/k}(\mathrm{Nrd}_{yCy^{-1}}(x')).N_{K/k}(\mathrm{Nrd}_B(y)).t = \lambda.$$

As quadratic forms, we have $N_{yk(\sqrt{\varepsilon})y^{-1}/k} \simeq < 1, -\varepsilon >$ and $N_{K/k} \simeq < 1, -a >$.

The existence of y and x' guarantees the existence of a nontrivial zero of the quadratic form

$$q = t < 1, -\varepsilon > \perp -\lambda < 1, -a >$$

over k.

Conversely, isotropy of q would give $\mu \in yk(\sqrt{\varepsilon})^* y^{-1}$ and $\nu \in K^*$ such that

$$t.N_{yk(\sqrt{\varepsilon})^* y^{-1}/k}(\mu).N_{K/k}(\nu) = \lambda.$$

Since $\mathrm{cd}(k) \leq 2$, there exists $y \in B^*$ such that $\mathrm{Nrd}_B(y) = \nu$. Let $\tau_1' = Int(y) \circ \tau_1 \circ Int(y)^{-1}$ and $u' = yu\tau(y)$. By the induction hypothesis, there exists $x' \in yCy^{-1}$ such that $\tau_1'(x') = x'$ and $\mathrm{Nrd}_{yCy^{-1}}(x') = \mu$. Then $\tau(x'u') = x'u'$ and $\mathrm{Nrd}_B(x'u') = \lambda$.

Thus we need only to show that q is isotropic over k. It suffices to show that q is isotropic over the discriminant extension $k(\sqrt{a\varepsilon})$.

By base change we replace k by $k(\sqrt{a\varepsilon})$, K by $K(\sqrt{a\varepsilon}) = k(\sqrt{a}, \sqrt{\varepsilon})$.

Over these new fields k and K, we have $B = M_2(B_1)$ for B_1/K a central division algebra of degree 2^{n-1} which admits a K/k-involution. By the induction hypothesis, given any K/k-involution on B_1, ε is the reduced norm of an element which is symmetric with respect to this involution.

To prove isotropy of q over k we may go over to an odd-degree extension of k. Using Lemma 2.4 we replace k by an odd-degree extension and assume that there exist a K/k-involution τ_0 on B_1 and $x \in B_1^*$ such that $\tau_0(x) = -x$, $x^2 = d \in k^*$ and $\varepsilon = \xi_1^2 - d\xi_2^2$, with $\xi_1, \xi_2 \in k$. The involution τ on B is adjoint with respect to a τ_0-hermitian form $W \in M_2(B_1)$, i.e. $(\tau_0 W)^t = W$. Let $\tilde{\tau}_0$ be the involution on B adjoint to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

By [KMRT] (10.36) p. 131, in the Brauer group of k, we have

$$D(B,\tau) \sim ((\mathrm{Nrd}_B W) \cup (a)) \otimes_k D(B,\tilde{\tau}_0).$$

By Lemma 4, $D(B,\tilde{\tau}_0) \sim 0$. We note that $D(B,\tau) \sim A$ is split over $k(\sqrt{a\varepsilon}) = k$ by our new choice of k. Hence $\mathrm{Nrd}_B W \in N_{K/k}K^*$. Let $\mathrm{Nrd}_B W = N_{K/k}(\theta)$, $\theta \in K^*$.

Since q is isotropic if ε is a square in k, we may assume that ε is not a square. The element $\sqrt{\varepsilon} \in B$ is conjugate to $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ in B (Skolem-Noether). By Lemma 2.3, $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ is conjugate to a τ_0-hermitian symmetric matrix in $B = M_2(B_1)$. Hence there exists $X \in GL_2(B_1)$ such that

$$\tau_0(X\sqrt{\varepsilon}X^{-1})^t = X\sqrt{\varepsilon}X^{-1}$$

i.e.

$$W^{-1}\tau(X\sqrt{\varepsilon}X^{-1})W = X\sqrt{\varepsilon}X^{-1}.$$

We have $\tau(\sqrt{\varepsilon}) = u^{-1}\tau_1(\sqrt{\varepsilon})u = u^{-1}\sqrt{\varepsilon}u$, so that

$$W^{-1}(\tau(X))^{-1}u^{-1}\sqrt{\varepsilon}u\tau(X)W = X\sqrt{\varepsilon}X^{-1}.$$

Thus the element $b = u\tau(X)WX$ commutes with $k(\sqrt{\varepsilon})$ hence belongs to C. From $\tau(X).WX = u^{-1}b$ we get

$$N_{K/k}(\mathrm{Nrd}_B(X)).\mathrm{Nrd}_B(W) = t^{-1}.N_{K(\sqrt{\varepsilon})/K}((\mathrm{Nrd}_C(b)) \in K^*.$$

Since W is τ-symmetric, and $\tau_1 = Int(u) \circ \tau$, the element $b = u\tau(X)WX$ is τ_1-symmetric. Thus $\mathrm{Nrd}_C(b)$ lies in $k(\sqrt{\varepsilon})$ and $N_{K(\sqrt{\varepsilon})/K}((\mathrm{Nrd}_C(b)) = N_{k(\sqrt{\varepsilon})/k}((\mathrm{Nrd}_C(b))$. We now have

$$N_{K/k}(\mathrm{Nrd}_B(X)).N_{K/k}(\theta) = t^{-1}.N_{k(\sqrt{\varepsilon})/k}(\mathrm{Nrd}_C(b)) \in k^*.$$

Thus q is isotropic over k.

Let $n = 2m$ be an even integer and let k be a field, with char(k) prime to n. Let K be a separable field extension of k, let $\Gamma = \mathrm{Gal}(K/k) = \{1, \sigma\}$ and let μ be the kernel of the norm map $R_{K/k}(\mu_n) \to \mu_n$; this is the group denoted $\mu_{n[K]}$ in the book [KMRT].

Lemma 2.9 *For μ as above, there is an exact sequence of k-groups of multiplicative type*

$$1 \to \mu \to R_{K/k}\mathbf{G}_m \times \mathbf{G}_m \to R_{K/k}\mathbf{G}_m \times \mathbf{G}_m \to 1,$$

where the map $R_{K/k}\mathbf{G}_m \times \mathbf{G}_m \to R_{K/k}\mathbf{G}_m \times \mathbf{G}_m$ is given by

$$(\alpha, b) \mapsto (b.\alpha^m, N_{K/k}(\alpha)).$$

Proof Let $\hat{\mu}$ be the character group of μ. The dual of the sequence of k-groups

$$1 \to \mu \to R_{K/k}(\mu_n) \to \mu_n \to 1$$

is the sequence

$$0 \to \mathbf{Z}/n \to \mathbf{Z}/n[\Gamma] \to \hat{\mu} \to 0$$

where the map $\mathbf{Z}/n \to \mathbf{Z}/n[\Gamma]$ is given by $1 \mapsto 1 + \sigma$. As an abelian group, $\hat{\mu}$ is isomorphic to $\mathbf{Z}/n$.

The map

$$M_1 = \mathbf{Z}[\Gamma] \oplus \mathbf{Z} \to M_2 = \mathbf{Z}[\Gamma] \oplus \mathbf{Z}$$

given by $(a + b\sigma, c) \mapsto ((ma + c) + (mb + c)\sigma, a + b)$ is dual to the map $R_{K/k}\mathbf{G}_m \times \mathbf{G}_m \to R_{K/k}\mathbf{G}_m \times \mathbf{G}_m$ given in the proposition.

Consider the additive map $\mathbf{Z}[\Gamma] \oplus \mathbf{Z} \to \mathbf{Z}/n[\Gamma]$ given by $(a+b\sigma, c) \mapsto (a+b\sigma+cm)$ modulo n. Composition with the projection $\mathbf{Z}/n[\Gamma] \to \hat{\mu}$ produces a Γ-invariant, surjective homomorphism $M_2 \to \hat{\mu}$. One checks that the composite map $M_1 \to M_2 \to \hat{\mu}$ is zero and that the determinant of $M_1 \to M_2$ has absolute value n. This gives the exact sequence of Γ-modules of finite type

$$0 \to \mathbf{Z}[\Gamma] \oplus \mathbf{Z} \to \mathbf{Z}[\Gamma] \oplus \mathbf{Z} \to \hat{\mu} \to 0.$$

Dualizing gives the lemma.

Proposition 2.10 *Let K/k and μ be as above. Then the group $H^2(k, \mu)$ is isomorphic to the subgroup of* $\mathrm{Br}(K) \oplus \mathrm{Br}(k)$ *consisting of pairs (x, y) satisfying* $\mathrm{Res}_{k/K}(y) + mx = 0 \in \mathrm{Br}(K)$ *and* $\mathrm{Cores}_{K/k}(x) = 0$.

Proof This follows from the Galois cohomology of the exact sequence given in Lemma 2.9, noting that $H^1\big(k, R_{K/k}(\mathbf{G}_m) \times \mathbf{G}_m\big) = 0$ by Hilbert's Theorem 90.

Remark Let $n > 1$ be an odd integer. One easily checks that the sequence

$$1 \to \mu \to R_{K/k}(\mu_n) \to \mu_n \to 1$$

is split. From this one identifies $H^2(k, \mu)$ with the kernel of the corestriction map ${}_n\mathrm{Br}(K) \to {}_n\mathrm{Br}(k)$. One can use the same argument to compute $H^1(k, \mu)$ (compare [KMRT] (30.13) p. 418).

Theorem 2.11 *Let $n > 1$ be an integer, let k be a field of characteristic prime to n, let G be a semisimple, simply connected, absolutely simple group of k of type ${}^2A_{n-1}$. Let $G \to G^{\mathrm{ad}}$ be the isogeny whose kernel is the center μ of G. Assume that k satisfies the two conditions :*

(a) $\mathrm{cd}(k) \leq 2$ *;*

(b) over any finite extension of k, the index of a central simple algebra coincides with its exponent.

Then the induced map $H^1(k, G^{\mathrm{ad}}) \to H^2(k, \mu)$ is surjective.

Remark Condition (b) implies that over any finite extension of k, 5-dimensional quadratic forms are isotropic.

Proof of Theorem 2.11 It is enough to prove the theorem when $G = G_0$ is quasisplit.

The group G_0 determines a quadratic extension K/k. To any $\xi \in H^1(k, G_0^{\mathrm{ad}})$ there is an associated central simple algebra B/K of degree n, equipped with a K/k-involution τ. The group $G_{0,\xi}$ is isomorphic to $SU(B, \tau)$.

Suppose that n is odd. Let x be an element of $H^2(k, \mu)$, identified with the kernel of the corestriction map ${}_n\mathrm{Br}(K) \to {}_n\mathrm{Br}(k)$. By assumption (b), there exists a central simple algebra B/K of degree n which in view of $\mathrm{Cores}_{K/k}(B) \sim 0$ admits a K/k-involution τ. Let $G = SU(B, \tau)^{\mathrm{ad}}$. Then $\delta_0(G) = t_G$ which is equal to the class of $B \in H^2(k, \mu)$ ([KMRT] (31.8) p. 427).

Suppose now that $n = 2m$. Under the identification of Proposition 2.10, the element $\delta_0(\xi) = t_G \in H^2(k, \mu)$ is the pair $([B], D(B, \tau))$ ([KMRT] (31.8)).

By Proposition 2.10, any element of $H^2(k, \mu)$ is represented by a pair $(x, y) \in \mathrm{Br}(K) \oplus \mathrm{Br}(k)$ such that $\mathrm{Res}_{k/K}(y) + mx = 0$ in $\mathrm{Br}(K)$ and $\mathrm{Cores}_{K/k}(x) = 0$. Observe that $2y = 0$ and $2mx = 0$. By assumption (b), there exists a central simple algebra B/K of degree n which represents x. In view of $\mathrm{Cores}_{K/k}(B) \sim 0$, it admits a K/k-involution τ. The discriminant algebra $D(B, \tau)$ is of exponent 2 hence is represented by a quaternion algebra A/k. From $D(B, \tau) \otimes_k K \sim B^{\otimes m}$ we deduce that $y - [A] \in \mathrm{Br}(k)$ vanishes in $\mathrm{Br}(K)$. Let $K = k(\sqrt{a})$. There exists a $\lambda \in k^*$ such that $y = [A] + (a, \lambda) \in \mathrm{Br}(k)$. By Theorem 2.8, there exists $u \in B^*$ which is τ-symmetric and such that $\mathrm{Nrd}_B(u) = \lambda.\mu^2$ for some $\mu \in k^*$. Let $\tau' = Int(u) \circ \tau$. Then $D(B, \tau') \sim (\mathrm{Nrd}_B(u), a) \otimes D(B, \tau)$

by [KMRT] 10.36 p. 131, hence $D(B,\tau') \sim (\mathrm{Nrd}_B(u), a) \otimes A$ which coincides with the class of y in $\mathrm{Br}(k)$. We have now written (x,y) as $\delta_0(G)$, where is G is the group adjoint to $SU(B,\tau')$.

Nontrialitarian groups of type D_m $(m \geq 2)$

Let G_0/k be a quasisplit, semisimple, simply connected of type D_m, $m \geq 2$. The group G_0/k is isomorphic to $\mathrm{Spin}(A_0,\sigma_0)$ where $A_0 = M_{2m}(k)$ is the split algebra and σ_0 is an orthogonal involution corresponding to a quasisplit quadratic form $q_0 = (m-1) < 1,-1 > \perp < 1,-d >$. Let K be the centre of the Clifford algebra $C(A_0,\sigma_0)$. This is a quadratic étale algebra over k isomorphic to $k[x]/(x^2-d)$.

Let μ be the centre of G_0. If m is odd, $\mu = R^1_{K/k}(\mu_4)$; if m is even, then $\mu = R_{K/k}(\mu_2)$ ([KMRT], p. 371).

Let

$$1 \to \mu \to G_0 \to G_0^{\mathrm{ad}} \to 1$$

be the standard isogeny.

For any $\xi \in H^1(k, G_0^{\mathrm{ad}})$, there is an associated central simple algebra A/k of degree $2m$ and an orthogonal involution σ on A such that the twisted group $G = G_{0,\xi}$ is isomorphic to $\mathrm{Spin}(A,\sigma)$. Let $C(A,\sigma)$ denote the Clifford algebra of (A,σ) ([KMRT], p. 91). The cocycle ξ gives rise to an isomorphism η of its centre with K. Conversely, a triple (A,σ,η), where A is a central simple algebra over k of degree $2m$ with an orthogonal involution σ and an isomorphism η of the centre of $C(A,\sigma)$ with K, determines a class $\xi \in H^1(k, G_0^{\mathrm{ad}})$ (cf. [KMRT] p. 409).

Suppose m is odd. By Proposition 2.10, $H^2(k,\mu)$ is isomorphic to the group of pairs $(x,y) \in \mathrm{Br}(K) \oplus \mathrm{Br}(k)$ such that $\mathrm{Res}_{k/K}(y) + 2x = 0 \in \mathrm{Br}(K)$ and $\mathrm{Cores}_{K/k}(x) = 0$. The element $\delta_0(\xi) = t_G$ is the pair $(\eta_*[C(A,\sigma)], [A])$, where η_* is the induced map from the Brauer group of the centre of $C(A,\sigma)$ to the Brauer group of K ([KMRT] (31.13) p. 428).

Suppose m is even. Then $H^2(k,\mu) = {}_2\mathrm{Br}(K)$ and $\delta_0(\xi) = t_G$ is the class of $\eta_*([C(A,\sigma)])$ in ${}_2\mathrm{Br}(K)$.

Proposition 2.12 *Let $m > 1$ be an integer, k a field of characteristic prime to 2, and G a semisimple, quasisplit, simply connected group over k, of nontrialitarian type D_m. Let $G \to G^{\mathrm{ad}}$ be the isogeny whose kernel is the center μ of G. Assume that k satisfies the two conditions :*

(a) $\mathrm{cd}(k) \leq 2$ *;*

(b) over any finite extension of k, every 5-dimensional quadratic form is isotropic.

Then the induced map $H^1(k, G^{\mathrm{ad}}) \to H^2(k,\mu)$ is surjective. Moreover, if $m \geq 4$, then G is isotropic.

Proof

Suppose $m = 2$. Because $D_2 = A_1^2$ (for a rational description, see [KMRT] (15.B)), this case follows by the standard reduction from the case of A_1 (over a field extension), handled earlier on. More explicitly, let $x \in {}_2\mathrm{Br}(K)$. By hypothesis (b) we may represent x as the class of a degree 2 central separable algebra B over K. The construction in [KMRT] (15.B) p. 210 yields a degree 4 algebra A over k and an orthogonal involution σ on A such that $C(A,\sigma)$ is k-isomorphic to B. A suitable choice of an isomorphism of the centre of $C(A,\sigma)$ with K defines a class $\xi \in H^1(k, G^{\mathrm{ad}})$ with $\delta_0(\xi) = B$.

Suppose $m = 3$. We have $D_3 = A_3$ (for a rational description, see [KMRT] (15.D) p. 220). This case has already been handled.

Suppose $m \geq 4$ and for $m = 4$ exclude the trialitarian case.

Suppose m is odd. Let $2m = 6 + 4l$ for $l \geq 1$. Let $(x,y) \in \mathrm{Br}(K) \oplus \mathrm{Br}(k)$ be an element representing a class in $H^2(k,\mu)$. By the case $m = 3$, there exists a degree 6 algebra D/k with an orthogonal involution σ such that the Tits class of $\mathrm{Spin}(A,\sigma)$ is (x,y). There exists a quaternion algebra A/k such that $D = M_3(A)$. We fix an orthogonal involution τ on A. Then σ is the adjoint involution τ_h to a τ-hermitian form h on A^3. Let h_l be the hyperbolic hermitian form on A^{2l}. Let σ' be the orthogonal involution on $A' = M_{3+2l}(A)$ which is adjoint to $h \perp h_l$. A suitable choice of an isomorphism of the centre of $C(A', \sigma')$ with K defines a class $\xi' \in H^1(k, G_0^{\mathrm{ad}})$ with $\delta_0(\xi') = (x, y)$ ([DLT], Proposition 3). By construction, the group $\mathrm{Spin}(A', \sigma') = SU(A, h \perp h_l)$ is isotropic.

Suppose m is even. Let $2m = 4 + 4l$ for $l \geq 1$. Let $x \in {}_2\mathrm{Br}(K)$ be an element representing a class in $H^2(k, \mu)$. By the case $m = 2$, there exists a degree 4 algebra D/k with an orthogonal involution σ such that the Tits class of $\mathrm{Spin}(A, \sigma)$ is x. There exists a quaternion algebra A/k such that $D = M_2(A)$. We fix an orthogonal involution τ on A. Then σ is the adjoint involution τ_h to a τ-hermitian form h on A^2. Let h_l be the hyperbolic hermitian form on A^{2l}. Let σ' be the orthogonal involution on $A' = M_{2+2l}(A)$ which is adjoint to $h \perp h_l$. A suitable choice of an isomorphism of the centre of $C(A', \sigma')$ with K defines a class $\xi' \in H^1(k, G_0^{\mathrm{ad}})$ with $\delta_0(\xi') = x$ ([DLT], Proposition 3).

By the general argument given at the beginning of the proof, we conclude that any group of type ${}^{1,2}D_m$, $m \geq 4$ is isotropic.

§3 Groups of multiplicative type

§3.1 Tori and groups of multiplicative type : reminders.

We briefly recollect results from [CTS1] and [CTS2]. Given a finite group G, a G-lattice M is called coflasque if $H^1(H, M) = 0$ for all subgroups $H \subset G$. A G-lattice M is called flasque if the G-lattice $M^0 := \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ is coflasque. If H is a normal subgroup of G, and if M is coflasque, then the G/H-lattice M^H is coflasque. If M is a coflasque G/H-module, it is a coflasque G-module. The notions thus extend to lattices equipped with a continuous action of a profinite group. Obvious examples of flasque and coflasque modules are the direct factors of permutation modules. If G is a metacyclic group, i.e. is a finite group all Sylow subgroups of which are cyclic, then a basic result of Endo and Miyata (cf. [CTS1], Prop. 2 p. 184) states that any flasque or coflasque G-lattice is a direct factor of a permutation module.

Given a G-module M of finite type over $\mathbf{Z}$, there exists an exact sequence of G-modules

$$0 \to M_2 \to M_1 \to M \to 0,$$

where M_1 is a G-permutation module and M_2 is a coflasque G-lattice, and the isomorphism class of the G-lattice M_2 is determined up to addition of a G-permutation module.

Given a G-module M of finite type over $\mathbf{Z}$, there exists an exact sequence of G-modules

$$0 \to M_4 \to M_3 \to M \to 0,$$

with M_4 a G-permutation module and M_3 a flasque G-lattice, and the isomorphism class of M_3 is determined up to addition of a G-permutation module.

Let now k be a field. A k-group of multiplicative type is an algebraic k-group which after a separable field extension of k becomes isomorphic to a closed subgroup of a product of copies of the multiplicative group $\mathbf{G}_m$. The map which to a k-group of multiplicative type associates its character group (over a separable closure of k) defines an antiduality between k-groups of multiplicative type and finitely generated discrete Galois modules. In this antiduality, tori correspond to torsionfree Galois modules. If the character group of a k-torus is a permutation module, then the k-torus is quasitrivial : it is a product of Weil restrictions of scalars of the group $\mathbf{G}_m$. A torus is called flasque, resp. coflasque, if its character group is flasque, resp. coflasque. Flasque k-tori F satisfy the following basic property : given any smooth connected k-variety X and any nonempty open set $U \subset X$, the restriction map $H^1_{\text{ét}}(X, F) \to H^1_{\text{ét}}(U, F)$ is onto ([CTS1]). One can actually define flasque tori over an arbitrary base scheme, and the same surjectivity property holds over regular schemes ([CTS2], Thm. 2.2).

Given any k-torus T, there exists an exact sequence of k-tori

$$1 \to F \to P \to T \to 1,$$

with P a quasitrivial k-torus and F a flasque k-torus. Such a sequence induces an isomorphism $T(k)/R \simeq H^1(k,F)$ ([CTS1]).

Given any k-group of multiplicative type μ, there exists an exact sequence of k-groups of multiplicative type

$$1 \to \mu \to F \to P \to 1,$$

with F a flasque k-torus and P a quasitrivial torus ([CTS2], Prop. 1.3). Such a sequence induces an isomorphism $H^1(k,\mu)/R \simeq H^1(k,F)$ (cf. [Gi2]).

The following lemma should have been recorded along with Lemme 2 of [CTS1]. Recall that for a finite group G, with group ring $\mathbf{Z}[G]$, one writes $N_G = \sum_{g\in G} g \in \mathbf{Z}[G]$ and one denotes by I_G the ideal of the group ring $\mathbf{Z}[G]$ generated by all $g-1$ for $g \in G$.

Lemma 3.1 *Let G be a finite group and $H \subset G$ a normal subgroup. Let M be a flasque G-module. Then the G/H-module M/I_HM is naturally isomorphic to the $\mathbf{Z}$-free G/H-module N_HM, and it is a flasque G/H-module.*

Proof The G-module $R = \mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z})$ is coflasque. By [CTS1], Lemme 2, the G/H-module R^H is a coflasque G/H-module. Now $R^H = \mathrm{Hom}_H(M,\mathbf{Z}) = \mathrm{Hom}_{\mathbf{Z}}(M/I_HM,\mathbf{Z})$. Multiplication by N_H on M induces a natural surjective map of G/H-modules from M/I_HM to the $\mathbf{Z}$-torsionfree submodule N_HM of M. The kernel of this map is $\hat{H}^{-1}(H,M)$, the torsion subgroup of M/I_HM. The hypothesis that M is flasque implies that this map is an isomorphism. The G/H-module $\mathrm{Hom}_{\mathbf{Z}}(M/I_HM,\mathbf{Z})$ is G/H-coflasque, hence the $\mathbf{Z}$-free G/H-module $M/I_HM = N_HM$ is flasque.

§3.2 Finiteness results

In [CTS1] and [CTS2], finiteness of $H^1(K,F)$ for F a flasque torus over a function field K over k (i.e. a field K finitely generated over the ground field k), was proved when k is of one of the following types : a finite field, a number field, a p-adic field, a real closed field, a separably closed field, with immediate application to the finiteness of $T(K)/R$ and of $H^1(K,\mu)/R$ for T an arbitrary K-torus and μ an arbitrary K-group of multiplicative type.

In this section, we establish the finiteness of $H^1(K,F)$ for F a flasque torus over some more fields K, thereby proving the finiteness of $T(K)/R$ for T a K-torus and of $H^1(K,\mu)/R$ for μ a K-group of multiplicative type over such fields.

Note the following easy remark. If $H^1(K,F)$ is finite for all flasque K-tori F, then for any finite separable field extension L/K, and any flasque L-torus F, $H^1(L,F)$ is finite. The point is that the K-torus $R_{L/K}F$ is then a flasque K-torus and $H^1(F,R_{L/K}T) \simeq H^1(L,T)$ for any L-torus T.

Theorem 3.2 *Let k be a field of characteristic zero, and let $K = k((t))$ be the Laurent series field. If $H^1(k,F)$ is finite for any flasque torus F over k, then $H^1(K,F)$ is finite for any flasque torus F over K.*

Proof Let F be a flasque $k((t))$-torus. There exists a finite Galois extension l/k and an integer $n > 0$ such that the field $M = l((t^{1/n}))$ defines a Galois extension of K which splits the K-torus F. Let $L = l((t)) \subset M$. Let $G = \mathrm{Gal}(M/K)$. Let $H = \mathrm{Gal}(M/L) \simeq \mathbf{Z}/n$ be the inertia group. This is a normal subgroup of G, and $G/H = \mathrm{Gal}(l/k)$. Let $U = l[[t^{1/n}]]^*$. Valuation defines a G-equivariant exact sequence

$$0 \to U \to M^* \to \mathbf{Z} \to 0.$$

Evaluation at $t = 0$ defines a G-equivariant, G-split exact sequence

$$0 \to U_1 \to U \to l^* \to 1.$$

The groupe U_1 of Einseinheiten is uniquely divisible.

The character group $\hat{F}$ of the K-torus F is a finitely generated, $\mathbf{Z}$-free G-module. By standard arguments, $H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, M^*)) = H^1(K, F)$.

The first exact sequence is $\mathbf{Z}$-split, hence we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, U) \to \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, M^*) \to \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, \mathbf{Z}) \to 0.$$

The G-cohomology of this sequence yields the exact sequence

$$H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, U)) \to H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, M^*)) \to H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, \mathbf{Z})).$$

The finite group $H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, \mathbf{Z}))$ is zero since the K-torus F is flasque.

Let us now study $H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, U))$. The second exact sequence is a split exact sequence of G-modules and the left term is uniquely divisible. We thus get an isomorphism

$$H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, U)) \xrightarrow{\simeq} H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, l^*)).$$

We have the restriction-inflation sequence associated to the normal subgroup H of G :

$$0 \to H^1(G/H, \mathrm{Hom}_H(\hat{F}, l^*)) \to H^1(G, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, l^*)) \to H^1(H, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, l^*))^{G/H}.$$

Since H acts trivially on l, we have $\mathrm{Hom}_H(\hat{F}, l^*) = \mathrm{Hom}_{\mathbf{Z}}(\hat{F}/I_H\hat{F}, l^*)$. Since $\hat{F}$ is G-flasque, Lemma 3.1 implies that $\hat{F}/I_H\hat{F}$ is a (torsionfree) flasque G/H-module. Let F_0 be the flasque k-torus with character group $\hat{F}/I_H\hat{F}$. We have

$$H^1(G/H, \mathrm{Hom}_H(\hat{F}, l^*)) = H^1(G/H, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}/I_H\hat{F}, l^*)) = H^1(k, F_0),$$

and by the hypothesis of the theorem this group is finite.

Let us now study the right hand side group $H^1(H, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, l^*))^{G/H}$ in the restriction-inflation exact sequence. Since $\hat{F}$ is G-flasque, hence H-flasque, and since H is cyclic, the theorem of Endo and Miyata implies that the H-module $\hat{F}$ is a direct factor of a permutation H-module, i.e. of a direct sum of modules $\mathbf{Z}[H/H_1]$ for various subgroups H_1 of H. Now

$$H^1(H, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[H/H_1], l^*)) \simeq H^1(H, \mathbf{Z}[H/H_1] \otimes l^*) \simeq H^1(H_1, l^*) = \mathrm{Hom}(H_1, l^*)$$

and this last group is clearly finite. Hence $H^1(H, \mathrm{Hom}_{\mathbf{Z}}(\hat{F}, l^*))$ is finite, which concludes the proof.

Remarks

1) The case where F has good reduction over K, i.e. comes from a $k[[t]]$-torus, hence from a k-torus, necessarily flasque, is clear, since in this case flasqueness of F implies $H^1(k[[t]], F) = H^1(k((t)), F)$, and then $H^1(k[[t]], F) = H^1(k, F)$.

2) Inspection of the proof shows that if one assumes finiteness of $H^1(k, T)$ for arbitrary k-tori, then one has finiteness of $H^1(K, T)$ for any K-torus T which becomes flasque after an unramified extension of K.

3) Again, inspection of the proof shows that if one assumes $H^1(k, T)$ finite for an arbitrary k-torus T, and one assumes that for each finite extension l of k, each quotient l^*/l^{*n} is finite (both assumptions being covered by the mere assumption $H^1(k, M)$ finite for any k-group M of multiplicative type), then $H^1(k((t)), T)$ is finite for any $k((t))$-torus T.

4) For the K-torus $T = R^1_{L/K}\mathbf{G}_m$, with $L = k((t^{1/2}))$, one easily computes $H^1(K, T) = k^*/k^{*2}$.

Let us say that a field k has *finite cohomology* if for any finite continuous Galois module M over the absolute Galois group of k, the Galois cohomology groups $H^n(k, M)$ are finite for all n. Examples of such fields are the separably closed or real closed fields, the finite fields, finite extensions of a p-adic field $\mathbf{Q}_p$. If k satisfies the property, then $k((t))$ also satisfies it.

Proposition 3.3 *Let k be a field of characteristic zero with finite cohomology. Let A be an integral domain with fraction field K. Assume that A is of one of the following types*

(a) a k-algebra of finite type over k ;

(b) an excellent, henselian, local domain with residue field k.

Let $U \subset$ Spec(A) be a nonempty open set and let T be a U-torus. Then the image of the restriction map $H^1_{\text{ét}}(U, T) \to H^1_{\text{ét}}(K, T)$ is finite.

Proof (suggested by O. Gabber) To prove the theorem, one may shrink U. We may thus assume that U is regular and of finite type over Spec(A). By considering character groups, one may easily produce an exact sequence

$$1 \to F \to P \to T \to 1$$

over U with P quasitrivial and F a U-torus (see [CTS2], Prop. 1.3), all being trivialized by a finite, connected, étale, Galois cover $U' \to U$ with group G. Going over to character groups, and using the semisimplicity of $\mathbf{Q}[G]$, one sees that there exists a homomorphism of k-tori $T \to P$ such that the composite $T \to P \to T$ (the last arrow being as in the exact sequence above) is multiplication by some positive integer n. The image of $H^1_{\text{ét}}(U, P) \to H^1_{\text{ét}}(K, P)$ is trivial, since $H^1_{\text{ét}}(K, P) = 0$ by Hilbert's theorem 90. To prove the result it is thus enough to know the finiteness of $H^2_{\text{ét}}(U, {}_nT)$, where ${}_nT$ denotes the n-torsion subgroup of T. The finite group scheme ${}_nT/U$ is a twisted locally constant group, split by the Galois cover $U' \to U$. Here U' is an open set of finite type of Spec(A'), where A' is the integral closure of K in a finite Galois (field) extension K' of K. Such a ring has the same property as A. Use of the Hochschild-Serre spectral sequence reduces the proof to that of the finiteness of the groups $H^1_{\text{ét}}(U', \mu_n)$ and $H^2_{\text{ét}}(U', \mu_n)$ for U' regular open in Spec(A'). In case (a), we recognize a standard finiteness property of étale cohomology of k-varieties ([SGA 4], XVI, Thm. 5.1). In case (b), the finiteness follows from [SGA 4] XIX, Thm. 5.1.

Theorem 3.4 *Let k be a field of characteristic zero with finite cohomology.*

Let A be an integral domain with fraction field K. Assume that A is of one of the following types

(a) a k-algebra of finite type ;

(b) an excellent, henselian, local domain with residue field k.

Then

(i) For any flasque torus F over K, the group $H^1(K, F)$ is finite.

(ii) For any K-torus, the groupe $T(K)/R$ is finite.

(iii) For any K-group of multiplicative type μ, the quotient $H^1(K, \mu)/R$ is finite.

Proof According to the reminders in §3.1, it is enough to prove (i). In case (a), there exists an integral regular affine k-variety X, with fraction field K, over which the flasque torus F extends to a flasque torus F/X. In case (b), there exists a nonempty regular open set $X \subset$ Spec(A) over which F extends to a flasque torus. Because F/X is flasque and X regular, the restriction map $H^1_{\text{ét}}(X, F) \to H^1_{\text{ét}}(K, F)$ is onto ([CTS2], Thm. 2.2). The result now follows from the previous proposition.

Remark If K is a function field over a number field, and F a flasque K-torus, then one may find a regular $\mathbf{Z}$-algebra A of finite type over $\mathbf{Z}$ with fraction field K. Let us localize this algebra by inverting the primes dividing n. By a theorem of Deligne, the étale cohomology groups $H^r_{\text{ét}}(A, \mu_n)$ are finite. The same proof as above now gives the finiteness of $H^1(K, F)$. The original proofs of that result ([CTS1], [CTS2]) used reduction to finite generation of the unit groups (Dirichlet) and finite generation of the Picard group, which if the transcendence degree of K/k is not zero involves the Mordell-Weil theorem.

§3.3. Weak approximation and Hasse principle for tori

In §4 and §5, weak approximation and the Hasse principle will be discussed for arbitrary linear algebraic groups over fields of type (gl) or (ll), i.e. in an essentially 2-dimensional situation. The case of tori is however particularly simple, and some of the results hold without the 2-dimensional restriction.

Let K be a field and S a finite set of distinct rank one discrete valuations on K. Let K_v denote the completion of K at $v \in S$. Let T be a K-torus and let

$$1 \to F \to P \to T \to 1$$

be a flasque resolution of T. We have the obvious diagram

$$\begin{array}{ccccccc} P(K) & \to & T(K) & \to & H^1(K,F) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_{v\in S} P(K_v) & \to & \prod_{v\in S} T(K_v) & \to & \prod_{v\in S} H^1(K_v,F) & \to & 0, \end{array}$$

The quasitrivial torus P is a Zariski open set of an affine space, hence it satisfies weak approximation. Since the morphism $P \to T$ is smooth, the induced maps on topological groups $P(K_v) \to T(K_v)$ are open ([Se1], Part II, Lie Groups, Chap. 3, §10.2). In particular, the image of $P(K_v)$ in $T(K_v)$ is open.

If we equip each $H^1(K_v,F)$ with the discrete topology, each map $T(K_v) \to H^1(K_v,F)$ is continuous, and so is the induced map

$$\varphi : \prod_{v\in S} T(K_v) \to \prod_{v\in S} H^1(K_v,F)/\delta(H^1(K,F)),$$

where δ denotes the diagonal map $H^1(K,F) \to \prod_{v\in S} H^1(K_v,F)$, and the quotient group $\prod_{v\in S} H^1(K_v,F)/\delta(H^1(K,F))$ is again equipped with the discrete topology. We have the straightforward :

Proposition 3.5 *Let $A_S(T)$ denote the quotient of the product $\prod_{v\in S} T(K_v)$ by the closure of the image of $T(K)$ under the diagonal map. The map φ induces an isomorphism between the (discrete) groups $A_S(T)$ and $\prod_{v\in S} H^1(K_v,F)/\delta(H^1(K,F))$.*

Let K be a function field (of arbitrary transcendence degree) over the ground field k. For each smooth projective model X/k of K, let Ω_X be the set of discrete valuations on K associated to the codimension 1 points on X. Let Ω be the union of all Ω_X.

Theorem 3.6 *Let k be a field of characteristic zero with finite cohomology. Let K/k be a function field. Let*

$$1 \to F \to P \to T \to 1$$

be an exact sequence of K-tori, with P quasitrivial and F flasque. Then

(a) For any finite set $S \subset \Omega$, the induced map $A_S(T) \to \prod_{v\in S} H^1(K_v,F)/\delta(H^1(K,F))$ is an isomorphism of finite groups.

(b) For any smooth model X/k of K/k, and any K-torus T, the kernel $Ш^1_X(K,T)$ of the diagonal map $H^1(K,T) \to \prod_{v\in\Omega_X} H^1(K_v,T)$ is a finite group. It embeds naturally into the kernel $Ш^2_X(K,F)$ of the map $H^2(K,F) \to \prod_{v\in\Omega_X} H^2(K_v,F)$.

(c) The kernel $Ш^1(K,T)$ of the map $H^1(K,T) \to \prod_{v\in\Omega} H^1(K_v,T)$ is finite. It embeds naturally into the kernel $Ш^2(K,F)$ of the map $H^2(K,F) \to \prod_{v\in\Omega} H^2(K_v,F)$.

Proof For any $v \in S$, the field K_v is isomorphic to the field $k(Y)((t))$ of Laurent series over the function field of a k-variety Y of dimension one less than that of X. Combining Theorems 3.2 and 3.4 (a), we see that for any flasque torus F over K, and any v, the group $H^1(K_v,F)$ is finite (by Theorem 3.4, the group $H^1(K,F)$ itself is also finite.) The rest of statement (a) is a special case of Proposition 3.5.

To prove (b), which clearly implies (c), we may restrict X to a smooth affine open set U over which the torus T extends to a U-torus. The kernel of the above map is contained in the image of $H^1_{\text{ét}}(U,T) \to H^1(K,T)$ (this well-known purity statement follows from the exact sequence at the bottom of page 163 of [CTS2]). Finiteness then follows from Proposition 3.3. The embedding statements in (b) and (c) are clear, since P being quasitrivial implies $H^1(K,P)=0$ and $H^1(K_v,P)=0$.

We now consider the local henselian case. Let k be a field of characteristic zero. Let A be an excellent, henselian, local domain with residue field k. Let K be its field of fractions. For a regular integral scheme X equipped with a proper birational map $X \to \mathrm{Spec}(A)$, we let Ω_X be the set of discrete valuations on K associated to the codimension 1 points on X. We let Ω be the union of all Ω_X.

Theorem 3.7 *Let k be a field of characteristic zero with finite cohomology. Let A be a two-dimensional, excellent, henselian, local domain with residue field k and let K be its fraction field. Let*

$$1 \to F \to P \to T \to 1$$

be an exact sequence of K-tori, with P quasitrivial and F flasque. Then

(a) For any finite set $S \subset \Omega$, the induced map $A_S(T) \to \prod_{v\in S} H^1(K_v,F)/\delta(H^1(K,F))$ is an isomorphism of finite groups.

(b) For any regular proper model $X \to \mathrm{Spec}(A)$, and any K-torus T, the kernel $Ш^1_X(K,T)$ of the diagonal map $H^1(K,T) \to \prod_{v\in\Omega_X} H^1(K_v,T)$ is a finite group. It embeds naturally into the kernel $Ш^2_X(K,F)$ of the map $H^2(K,F) \to \prod_{v\in\Omega_X} H^2(K_v,F)$.

(c) The kernel $Ш^1(K,T)$ of the map $H^1(K,T) \to \prod_{v\in\Omega} H^1(K_v,T)$ is finite. It embeds naturally into the kernel $Ш^2(K,F)$ of the map $H^2(K,F) \to \prod_{v\in\Omega} H^2(K_v,F)$.

(d) Assume moreover that k is algebraically closed. Then there is an isomorphism of finite groups between the kernel $Ш^1(K,T)$ of the map $H^1(K,T) \to \prod_{v\in\Omega} H^1(K_v,T)$ and the kernel $Ш^2(K,F)$ of the map $H^2(K,F) \to \prod_{v\in\Omega} H^2(K_v,F)$.

(e) Under the same assumptions as in (d), if T/K is split by a metacyclic extension, more generally if T is a birational direct factor of a K-rational variety, then for any finite set $S \subset \Omega$, we have $A_S(T)=0$, and we have $Ш^1(K,T)=0$.

Proof For $v \in S$, the field K_v is isomorphic to a Laurent series field $L((t))$ over a field L which is either a function field in one variable over k or is itself a Laurent series field $k((u))$. Combining Theorems 3.2 and 3.4, we see that for any flasque torus F over K, and any v, the group $H^1(K_v,F)$ is finite. By Theorem 3.4, the group $H^1(K,F)$ itself is also finite. The rest of statement (a) is a special case of Proposition 3.5. To prove (b), which clearly implies (c), we may restrict X to a smooth affine open set over which the torus T extends to a U-torus. By the purity statement referred to in the previous proof, the kernel of the above map is contained in the image of $H^1_{\text{ét}}(U,T) \to H^1(K,T)$. Finiteness then follows from Proposition

3.3. The embedding statements in (b) and (c) are clear, since P being quasitrivial implies $H^1(K,P)=0$ and $H^1(K_v,P)=0$. To prove (d), it is enough to know that $Ш^2(K,P)=0$ and that is an immediate consequence of Corollary 1.10 (c) of [CTOP] (Theorem 1.6 of the present paper). The case of an arbitrary quasitrivial torus reduces immediately to that statement. If T is split by a metacyclic extension, then one may assume that the flasque resolution of T has the same property, hence by the Endo-Miyata theorem, there exists a K-torus F' such that $F\times_K F'$ is a quasitrivial torus Q. This implies (Hilbert's theorem 90) that the torsor $P\to T$ has a section over an open set, hence that $T\times_K F$ is K-birational to P. Since weak approximation holds for P, it holds for T. We have $Ш^2(K,F)\oplus Ш^2(K,F')=Ш^2(K,Q)$ and this last group is trivial by the argument given in the proof of (d). Statement (c) then yields $Ш^1(K,T)=0$.

Remark Assume that k is algebraically closed, i.e. one is in the (ll) case. For $X\to \mathrm{Spec}(A)$ a regular model of A, and F a flasque K-torus, for almost all points $v\in\Omega_X$, one has $H^1(K_v,F)=0$. Indeed F extends to a flasque torus over an open set $U\subset X$. For any $v\in U^{(1)}$, let O_v be the completion of the local ring of X at v and let $H^1(O_v,F)=H^1_{\text{ét}}(\mathrm{Spec}(O_v),F)$. Then the restriction map $H^1(O_v,F)\to H^1(K_v,F)$ is surjective. By Hensel's lemma, $H^1(O_v,F)\simeq H^1(k(v),F_{k(v)})$, where $k(v)$ denotes the residue field at v ([SGA3], Exp. XXIV, Prop. 8.2.(ii). But any such field $k(v)$ is a field of cohomological dimension one, hence $H^1(k(v),M)=0$ for any $k(v)$-torus M.

For any K-torus T over K, with flasque resolution $1\to F\to P\to T\to 1$ and any regular model $X\to\mathrm{Spec}(A)$, this enables us to produce the *finite group*

$$A_X(T)=\mathrm{Coker}[H^1(K,F)\to\prod_{v\in\Omega_X}H^1(K_v,F)]$$

which measures the lack of weak approximation with respect to all places in Ω_X. But if we consider the union Ω of all Ω_X, we have no such result. It seems hard to define an analogue of the Brauer-Manin obstruction to weak approximation in the present context.

§3.4. A counterexample to weak approximation

In the number field case, examples of K-tori with $A_S(T)\neq 0$ or with $Ш^1(K,T)\neq 0$ have been known and discussed for a long time. One may wonder whether such examples exist in our context. The next proposition handles weak approximation.

Proposition 3.8 *Let $A=\mathbf{C}[[x,y]]\subset B=\mathbf{C}[[x^{1/2},y^{1/2}]]$, and let $K\subset L$ be the inclusion of fraction fields. Let T be the K-torus $R^1_{L/K}\mathbf{G}_m$. Let A_x, resp. A_y, be the discrete valuation ring which is the completion of A at the prime ideal x, resp. y; let K_x, resp. K_y, be the associated (topological) fraction field. The group $T(K)$ of K-rational points is dense in each of $T(K_x)$ and $T(K_y)$, but its image under the diagonal embedding $T(K)\subset T(K_x)\times T(K_y)$ is not dense.*

Proof Let $G=\mathrm{Gal}(L/K)$. Proposition 15 of [CTS1] states that for any Galois extension L/K with group G, for the K-torus $R^1_{L/K}\mathbf{G}_m$ there is a natural isomorphism $T(K)/R\simeq \hat{H}^{-1}(G,L^*)={}^NL^*/I_GL^*$, where ${}^NL^*$ denotes the kernel of the norm map $N:L^\times\to K^\times$. In the particular case under study, namely $G\simeq \mathbf{Z}/2\times\mathbf{Z}/2$, a transfer argument shows that the group $\hat{H}^{-1}(G,L^*)$ is killed by 2.

The diagonal map $T(K)\to T(K_x)\times T(K_y)$ induces the natural maps $\hat{H}^{-1}(G,L^*)\to \hat{H}^{-1}(G,L^*_x)$ and $\hat{H}^{-1}(G,L^*)\to\hat{H}^{-1}(G,L^*_y)$. We shall show that each of $\hat{H}^{-1}(G,L^*)$, $\hat{H}^{-1}(G,L^*_x)$ and $\hat{H}^{-1}(G,L^*_y)$ is isomorphic to $\mathbf{Z}/2$, each of the maps $\hat{H}^{-1}(G,L^*)\to\hat{H}^{-1}(G,L^*_x)$ and $\hat{H}^{-1}(G,L^*)\to\hat{H}^{-1}(G,L^*_y)$ being an isomorphism. The proposition will then follow from the quoted result of [CTS1].

The ring B is a unique factorization domain. Hence the map which to an element in L^* associates its divisor defines a short exact sequence of G-modules

$$1 \to B^* \to L^* \to \mathrm{Div}(B) \to 0.$$

Since $\mathrm{Div}(B)$ is a direct sum of G-permutation modules, one has $\hat{H}^{-1}(G, \mathrm{Div}(B)) = 0$, hence a surjection from $\hat{H}^{-1}(G, B^*)$ to $\hat{H}^{-1}(G, L^*)$. The G-module B^* is the direct product of $\mathbf{C}^*$ (with trivial action) and of the multiplicative group of series with value 1 at the origin, and the latter group is clearly uniquely divisible. Thus the inclusion $\mathbf{C}^* \subset B^*$ induces an isomorphism $\hat{H}^{-1}(G, \mathbf{C}^*) = \hat{H}^{-1}(G, B^*)$. One easily checks that $\hat{H}^{-1}(G, \mathbf{C}^*) = {}^N\mathbf{C}^*/I_G\mathbf{C}^* = \mu_4$, the generator being the class of a primitive 4-th root of unity i. All in all, the inclusion $\mathbf{C}^* \subset L^*$ induces a surjection $\mu_4 \to \hat{H}^{-1}(G, L^*)$, the last group being of order at most 2, spanned by the class of i.

We have $K_x = \mathbf{C}((y))((x))$, and $L_x := L \otimes_K K_x = \mathbf{C}((y^{1/2}))((x^{1/2}))$. This is a field extension of group $G \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$. Let $G =< \sigma, \tau >$, with $\sigma(x^{1/2}) = -x^{1/2}, \sigma(y^{1/2}) = y^{1/2}$, $\tau(x^{1/2}) = x^{1/2}, \tau(y^{1/2}) = -y^{1/2}$. The multiplicative group of $L_x = \mathbf{C}((y^{1/2}))((x^{1/2}))$ may be written as a direct product $\mathbf{C}^*.(x^{1/2})^{\mathbf{Z}}.(y^{1/2})^{\mathbf{Z}}.U$ with U a uniquely divisible G-module (namely the subgroup spanned by the Einseinheiten of $\mathbf{C}((y^{1/2}))[[x^{1/2}]]^*$ and the Einseinheiten of $\mathbf{C}[[y^{1/2}]]^*$). As a G-module, L_x^* is the direct product of its two subgroups $\mathbf{C}^*.(x^{1/2})^{\mathbf{Z}}.(y^{1/2})^{\mathbf{Z}}$ and U.

One then has

$$\hat{H}^{-1}(G, \mathbf{C}((y^{1/2}))((x^{1/2}))^*) = \hat{H}^{-1}(G, \mathbf{C}^*.(x^{1/2})^{\mathbf{Z}}.(y^{1/2})^{\mathbf{Z}}).$$

One easily computes ${}^N(\mathbf{C}^*.(x^{1/2})^{\mathbf{Z}}.(y^{1/2})^{\mathbf{Z}}) = \mu_4 \subset \mathbf{C}^*$ and $I_G(\mathbf{C}^*.(x^{1/2})^{\mathbf{Z}}.(y^{1/2})^{\mathbf{Z}}) =< -1 >\subset \mathbf{C}^*$, so that $\hat{H}^{-1}(G, \mathbf{C}^*.(x^{1/2})^{\mathbf{Z}}.(y^{1/2})^{\mathbf{Z}}) = \mathbf{Z}/2$ spanned by the class of $i \in \mathbf{C}^*$.

It is then clear that the map $\hat{H}^{-1}(G, L^*) \to \hat{H}^{-1}(G, L_x^*)$ is the identity on $\mathbf{Z}/2$ induced by the identity on $\mathbf{Z}/2\{i\} = \hat{H}^{-1}(G, \mathbf{C}^*)$. Permuting x and y yields the same result for $\hat{H}^{-1}(G, L^*) \to \hat{H}^{-1}(G, L_y^*)$.

The map $T(K)/R \to T(K_x)/R \times T(K_y)/R$ therefore reads as the diagonal map $\mathbf{Z}/2 \to \mathbf{Z}/2 \oplus \mathbf{Z}/2$, and weak approximation fails for T. More precisely, the pair $(i, 1)$ is not in the closure of $T(K)$ (we use the fact, clear from a flasque resolution, that the set of points R-equivalent to 1 on $T(K_x)$ is an open subgroup, and similarly for $T(K_y)$).

There remains the following open question (to which we expect a negative answer) :

Question Let A be a two-dimensional, excellent, henselian, local domain with algebraically closed residue field k and let K be its fraction field. Let T be a K-torus. Is the group $Ш^1(K, T)$ trivial ? That is, over such a field K, and with respect to the valuations in Ω, does the Hasse principle hold for principal homogeneous spaces under tori ?

Remarks

1) A closely related question is whether there exists a field K as above and a finite commutative K-group μ such that $Ш^2(K, \mu) \neq 0$.

2) Jaworski [Ja] has an interesting, explicit counterexample to the Hasse principle for $H^1(K, \mathbf{Z}/2) = K^*/K^{*2}$. His (slightly generalized) example is as follows. Let

$$A = \mathbf{C}[[x, y, z]]/(z^2 - (x^2 - y^p)(y^2 - x^q)),$$

where the integers p, q satisfy $pq \geq 5$. Let K be the field of fractions of the normal ring A. Then each of the classes $(x^2 - y^p), (y^2 - x^q)$ is a square in any completion K_v of K at a discrete, rank one valuation. But one easily checks that neither of these is a square in K. In [CTOP], Remark

3.1.2, yet another (less explicit) example is given. O. Gabber points out that this example can be made explicit, for instance by taking $A = \mathbf{C}[[x,y,z]]/(xyz + x^4 + y^4 + z^4)$.

3) For the case considered in Proposition 3.8, using the triviality of the Picard group of B, one sees that $Ш^1(K,T) = 0$.

§4. R-equivalence and weak approximation on linear algebraic groups

In this section, unless otherwise mentioned, the ground field is assumed to be of characteristic 0. We first recall a series of basic results on simply connected semisimple groups.

Let K be a field and G/K be a simply connected group of type ${}^1A_n$. Let B/K be a central simple algebra such that $G = \mathbf{SL}_{1,B}$. Let $B^{*1} \subset B^*$ be the group of elements of reduced norm 1. The commutator subgroup $[B^*, B^*] \subset B^{*1}$ is a normal subgroup of B^{*1}, all elements of which are R-equivalent to $1 \in G(K)$. We therefore have a clearly surjective homomorphism $\varphi : SK_1(B) := B^{*1}/[B^*, B^*] \to G(K)/R$, and both groups are abelian. This homomorphism is functorial in the field K.

Theorem 4.1 *Let $G = \mathbf{SL}_{1,B}$ be a simply connected K-group of type ${}^1A_n$. With notation as above :*

(i) (Voskresenskiĭ, [Vo1], [Vo2] p. 186) The homomorphism φ is an isomorphism.

(ii) (Yanchevskiĭ, [Ya2]) Suppose that over any finite field extension E of K, and any central simple algebra C/E, the reduced norm map Nrd : $C^* \to E^*$ *is onto. Then $B^{*1}/[B^*, B^*] = 1$, hence also $G(K)/R = 1$.*

(iii) If $\mathrm{cd}(K) \leq 2$*, then $B^{*1}/[B^*, B^*] = 1$, hence also $G(K)/R = 1$.*

Statement (iii) is a consequence of (ii) and the Merkurjev-Suslin theorem ([Su], Thm. 24.8).

Let K be a field and G/K be a group of type ${}^2A_n$. Then $G = SU(B,\tau)$ where L/K is a quadratic, separable, field extension of K, and B/L is a central simple algebra equipped with an involution of the second kind τ. We have the natural exact sequence of algebraic groups over K :

$$1 \to SU(B,\tau) \to U(B,\tau) \to R^1_{L/K}\mathbf{G}_m \to 1.$$

The homomorphism $B^* \to L^*$ defined by $b \mapsto \mathrm{Nrd}(b)/\tau\mathrm{Nrd}(b)$ defines another exact sequence of algebraic groups over K :

$$1 \to H \to R_{L/K}(\mathbf{GL}_{1,B}) \to R^1_{L/K}\mathbf{G}_m \to 1.$$

The group $H(K)$ of K-rational points of H is the preimage in B^* of $K^* \subset L^*$ under the reduced norm map $B^* \to L^*$. Any τ-symmetric element in B^* belongs to $H(K)$, and it is R-equivalent to 1 on H since $(B^*)^\tau$ is the set of k–rational points of an open set of an affine space. Let $\Sigma \subset H(K)$ be the (normal) subgroup spanned by the τ-symmetric elements. There is an obvious surjective homomorphism $\psi : H(K)/\Sigma \to H(K)/R$, and that homomorphism is functorial in the field K.

The following theorem is the result of successive efforts by Yanchevskiĭ, Monastyrniĭ, Merkurjev, Chernousov (see [CM]).

Theorem 4.2 *Let $G = SU(B,\tau)$ be a simply connected K-group of type ${}^2A_n$. With notation as above :*

(i) The underlying K-varieties of G and H are stably K-birationally equivalent.

(ii) There is a natural group isomorphism $H(K)/R \simeq G(K)/R$; that isomorphism is functorial in the field K.

(iii) The surjective homomorphism $\psi : H(K)/\Sigma \to H(K)/R$ is an isomorphism.

(iv) Suppose that over any finite field extension E of K, and any central simple algebra C/E, the reduced norm map $\mathrm{Nrd} : C^* \to E^*$ *is onto. Then $H(K)/\Sigma = 1$, hence also $G(K)/R = 1$.*

(v) If $\mathrm{cd}(K) \leq 2$, *then $H(K)/\Sigma = 1$, hence also $G(K)/R = 1$.*

Proof Results (i) to (iii) may be read off from [CM]. A key property is that over *any* field extension F of K, the maps induced on F-points by $U(B,\tau) \to R^1_{L/K}\mathbf{G}_m$ and by $R_{L/K}(\mathbf{GL}_{1,B}) \to R^1_{L/K}\mathbf{G}_m$ have the same image (a result of Merkurjev, see [CM], Prop. 5.2). Denote by M the K-group which is the fibre product of $U(B,\tau)$ and $R_{L/K}(\mathbf{GL}_{1,B})$ over $R^1_{L/K}\mathbf{G}_m$. One then has two natural exact sequences of K-groups

$$1 \to SU(B,\tau) \to M \to R_{L/K}(\mathbf{GL}_{1,B}) \to 1$$

and

$$1 \to H \to M \to U(B,\tau) \to 1,$$

and the above property implies that each of the morphisms $M \to R_{L/K}(\mathbf{GL}_{1,B})$ and $M \to U(B,\tau)$ admits a section over an open set. Each of the K-groups $R_{L/K}(\mathbf{GL}_{1,B})$ and $U(B,\tau)$ is a K-rational variety. This immediately gives statement (i). It also implies that each of the homomorphisms $SU(B,\tau) \to M$ and $H \to M$ induces group isomorphisms $SU(B,\tau)(K)/R \to M(K)/R$ and $H(K)/R \to M(K)/R$. This is statement (ii) in the theorem. For (iii), see [CM], Lemma 5.1 (in the isotropic case, see [Ya3], Remark p. 537). Result (iv) is due to Yanchevskiĭ ([Ya1], Thm. 1). Result (v) follows from (iv) by the Merkurjev-Suslin results.

Theorem 4.3 (Chernousov-Platonov) *Let K be a field of characteristic zero with* $\mathrm{cd}(K) \leq 2$. *Assume that over any finite field extension of K, index and exponent coincide for 2-primary and for 3-primary central simple algebras. Let G/K be a simply connected semisimple group without factor of type A_n. If G contains a factor of type E_8, assume* $\mathrm{cd}(K^{ab}) \leq 1$. *Then the K-variety G is K-rational, i.e. birational to affine space over K.*

Proof In [CP], Chernousov and Platonov state the theorem for p-adic and totally imaginary number fields. Let us indicate how their arguments (easily) extend to the present situation. For L/K a finite field extension and X an L-rational L-variety, the K-variety $R_{L/K}X$ is K-rational. By the standard reduction, we may thus assume that G is absolutely almost simple, simply connected, and not of type A_n.

Case B_n ($n \geq 2$). Then G is the spinor group of a nondegenerate quadratic form q of rank $2n+1 \geq 5$ over K. Under our assumption on K, any quadratic form of rank at least 5 is isotropic (Theorem 1.1 (f)). The statement is now a special case of the general result : Over any field F of characteristic not 2, for any isotropic quadratic form q over F the group $\mathrm{Spin}(q)$ is an F-rational variety (Platonov). This result may be proved in a number of ways ([CP], Prop. 4; [Me2], Prop. 6.1). One may give a proof by induction. Let us write the underlying vector space V as an orthogonal sum $W \perp K.v$, in such a manner that $a = q(v) \neq 0$ and the restriction q_0 of q to W is still isotropic. Let X be the affine quadric defined by $q(x) = a$. The group $\mathrm{Spin}(q)$ acts on X, the isotropy group of $v \in X$ being $\mathrm{Spin}(q_0)$. The isotropy assumption ensures that over *any* field extension F of K this action is transitive on F-points. This implies that the morphism $g \mapsto g.v$ admits a rational section, and $\mathrm{Spin}(q)$ is F-birational to the product of $\mathrm{Spin}(q_0)$ and the F-rational quadric X.

Case C_n. Over any field F, any simply connected group of type C_n is a (connected) unitary group, which is F-rational (Cayley transform) ([CP], Lemma 5).

Case of nontrialitarian D_n ($n \geq 4$). Over any field F, any simply connected group of this type can be realized as the spinor group $\mathrm{Spin}(D,h)$ associated to a nondegenerate hermitian form over a central division algebra D of degree d with an orthogonal involution of the first kind. Chernousov and Platonov prove that if the index of D is at most 4 and G is isotropic, then G

is F-rational ([CP], Cor. 5, Prop. 8 for inner forms, Prop. 10 and Prop. 11 for outer forms). Over the field K, the algebra D, which is of exponent 2, is of index at most 2, and the group G is isotropic by Theorem 2.1 (b) (see also Prop. 2.12). The result of Chernousov and Platonov therefore applies.

Case of trialitarian D_n. Over any field F, any isotropic group of that type is F-rational ([CP], Prop. 13). Over the field K, any such group is isotropic (Theorem 2.1).

Cases E_6 and E_7. Over any field F, any F-isotropic group of type E_6 or E_7 with Tits index as described in Theorem 2.1 is F-rational ([CP], Prop. 14 for inner forms of E_6, Prop. 16 for outer forms of E_6, Prop. 17 for E_7). Over the field K, Theorem 2.1 says that either G is quasisplit, or it has the quoted Tits index, hence in particular is isotropic.

Cases F_4, G_2, E_8. In these cases, the centre is trivial, hence all forms of the split group G are classified by $H^1(K, G)$, which by Theorem 1.2 is trivial. Thus any such group over K is K-split, hence K-rational.

Remark Chernousov and Platonov also investigate the rationality properties of arbitrary absolutely almost simple groups. In most cases, over p-adic and totally imaginary number fields, they establish the rationality when the group is not of type A_n. Presumably most of their results hold for our more general fields. Beware however of an inaccuracy in [CP] : In the main theorem, one must exclude adjoint outer forms of type D_n related to a quaternion division algebra A and a skew-hermitian form h over A whose anisotropic part has dimension 3 (this may occur over a totally imaginary number field).

Corollary 4.4 *Let K be a field of type (ll), (sl) or (gl). Let G/K be a simply connected group without factor of type A_n. In the (gl) case, assume that G contains no E_8-factor. Then the underlying K-variety of the group G is K-rational, i.e. it is K-birational to an affine space over K.*

Collecting the previous results, we obtain a general statement regarding R-equivalence on simply connected groups.

Theorem 4.5 *Let G/K be a semisimple, simply connected group over a field K of characteristic zero. Assume* $\mathrm{cd}(K) \leq 2$. *Assume that over any finite field extension of K, index and exponent coincide for 2-primary and for 3-primary central simple algebras. If G/K contains a factor of type E_8, assume* $\mathrm{cd}(K^{ab}) \leq 1$. *Then $G(K)/R = 1$.*

Proof The assumptions on the ground field k are stable by finite field extension. If L/K is a finite field extension and H/L is a linear algebraic group, then there is a natural isomorphism of groups $R_{L/K}(K)/R \simeq H(L)/R$. By the standard reduction, we may thus assume that G is an absolutely almost simple group. In the A_n case, $G(K)/R = 1$ by Theorems 4.1 and 4.2. In the other cases, G/K is a K-rational variety by Theorem 4.3, hence $G(K)/R = 1$.

Corollary 4.6 *Let G/K be a semisimple, simply connected group over a field K of type (ll), (sl) or (gl). In the (gl) case, assume that G has no simple factor of type E_8. Then $G(K)/R$ is trivial.*

Remarks

(i) Note the big difference between simply connected groups of type A_n and other (absolutely almost simple) simply connected groups over fields of one of our three types (excluding E_8 in case (gl)). For a group G not of type A_n, R-equivalence is universally trivial, i.e. $G(F)/R = 1$ for *any* field extension F/K. For a group of type A_n, we claim $G(F)/R = 1$ only for finite extensions of K. Merkurjev [Me1] has given examples with $G(F)/R \neq 1$ for G/K of type ${}^1A_n$ and F/K a suitable function field.

(ii) For all we know, Theorem 4.5 might hold under the assumption $\mathrm{cd}(K) \leq 3$. Indeed the cohomological invariants on $SK_1(B) = SL_{1,B}(K)/R$ studied by Suslin and later Merkurjev and Rost take their values in cohomology groups $H^i(K,.)$ for $i \geq 4$. The question whether $SK_1(B) = 1$ if $\mathrm{cd}(K) \leq 3$ was explicitly raised by Suslin.

(iii) Theorem 4.5 applies to p-adic fields : for such fields, the result had already been proved by Voskresenskiĭ. The theorem also applies to totally imaginary number fields. This algebraic proof of the triviality of $G(K)/R$ for G simply connected should be compared with the available proof of finiteness of $G(K)/R$ when K is an arbitrary number field (see [Gi1]) : that proof appeals to an ergodic result of Margulis.

We now consider the problem of weak approximation, and restrict attention to fields K of type (gl) or (ll). Recall from §1 that in each of these cases, there is an associated set Ω of places of K. For each place $v \in \Omega$, the completion K_v is of the shape $l((t))$, where l is either a function field in one variable over the algebraically closed field k, or is a Laurent series field $k((u))$. For each $v \in \Omega$, the topology on the local field K_v induces a topology on the group $G(K_v)$.

Theorem 4.7 *Let G/K be a simply connected group over a field K which is of type (gl) or (ll). In the (gl) case, assume that G has no E_8-factor. Then for any finite set of places $S \subset \Omega$, the diagonal map $G(K) \to \prod_{v \in S} G(K_v)$ has dense image.*

Proof The standard reduction allows us to assume that G/K is absolutely almost simple. If G is not of type A_n, then under our assumptions, the K-variety G is K-rational and weak approximation follows.

Suppose that G is of type ${}^1A_n$, i.e. $G = \mathbf{SL}_{1,B}$ for B/K a central simple algebra. Let $\{g_v\}_{v \in S} \in \prod_{v \in S} B^{*1}_{K_v}$. Each field K_v is a C_2-field ([Pf], Chap. 5, Thm. 2.32 p. 70). By Yanchevskiĭ's results (Theorem 4.1.(ii)), each g_v is a product of commutators in $B^*_{K_v}$. Since there are only finitely many places $v \in S$, there is an integer $n > 0$ such that each g_v for $v \in S$ is a product of exactly n commutators in $B^*_{K_v}$. Since B^* is the set of K-points of an open subvariety of an affine space over K, the diagonal map $B^* \to \prod_{v \in S} B^*_{K_v}$ has dense image. Approximating all entries in the n commutators, we conclude that $G(K)$ is dense in $\prod_{v \in S} G(K_v)$.

Suppose that G is of type ${}^2A_n$, i.e. $G = SU(B,\tau)$ where L/K is a quadratic, separable, field extension of K, and B/L is a central simple algebra equipped with an involution of the second kind τ. We refer to the discussion before Theorem 4.2 for the definition of the algebraic group H/K. Since G and H are stably K-birationally equivalent K-varieties, weak approximation for one of them is equivalent to weak approximation for the other. Let v be a place of K. Let $\{g_v\}_{v \in S} \in \prod_{v \in S} H(K_v)$. By Theorem 4.2 (iv) applied to the completion K_v, any element of $H(K_v)$ is a product of τ-symmetric elements in $B^*_{K_v}$. Since S is finite, there is an integer $n > 0$ such that each g_v for $v \in S$ is a product of exactly n τ-symmetric elements in $B^*_{K_v}$. The set of τ-symmetric elements in B^* is the set of K-points of a Zariski open set in an affine space over K. Thus weak approximation holds for such elements, and we conclude that $H(K)$ is dense in $\prod_{v \in S} H(K_v)$.

Remark In the ${}^2A_n$ case, the above proof essentially goes back to Yanchevskiĭ ([Ya3], Proposition following Thm. 5, p. 545). Use of [CM] enables one to streamline it.

We now discuss arbitrary connected linear algebraic groups. Recall the basic theorem ([Gi2], Thm. 6 p. 308).

Theorem 4.8 *Let K be a field of characteristic zero such that* $\mathrm{cd}(K) \leq 2$. *Let*

$$1 \to \mu \to G' \to G \to 1$$

be a K-isogeny of connected linear algebraic groups, where G' is the product of a semisimple simply connected group and a quasitrivial torus. Let

$$1 \to \mu \to F \to P \to 1$$

be a flasque resolution of μ (the torus F is flasque and the torus P is quasitrivial). The associated Galois cohomology sequences induce an exact sequence

$$G'(K)/R \longrightarrow G(K)/R \longrightarrow H^1(K,F) \longrightarrow 1.$$

We may now conclude :

Theorem 4.9 *Let K be a field of characteristic zero such that $\mathrm{cd}(K) \leq 2$. Assume that over any finite field extension of K, index and exponent coincide for 2-primary and for 3-primary central simple algebras. Let*

$$1 \to \mu \to G' \to G \to 1$$

be a K-isogeny of connected linear algebraic groups, where G' is the product of a semisimple simply connected group and a quasitrivial torus. Let

$$1 \to \mu \to F \to P \to 1$$

be a flasque resolution of μ (the torus F is flasque and the torus P is quasitrivial). If G' contains a factor of type E_8, assume $\mathrm{cd}(K^{ab}) \leq 1$. Then the Galois cohomology sequences induce an isomorphism :

$$G(K)/R \xrightarrow{\sim} H^1(K,F).$$

Proof A quasitrivial K-torus T is a K-rational variety, hence $T(K)/R = 1$. For G'/K as above, we have $G'(K)/R = 1$ by Theorem 4.5.

Corollary 4.10 *Let K be a field of characteristic zero such that $\mathrm{cd}(K) \leq 2$. Assume that over any finite field extension of K, index and exponent coincide for 2-primary and for 3-primary central simple algebras. Let G/K be a connected linear algebraic group. If G contains a factor of type E_8, assume $\mathrm{cd}(K^{ab}) \leq 1$. Then the group $G(K)/R$ is abelian.*

Proof By a known reduction, one may assume that G is reductive. By Lemme 1.10 of [Sa], there then exists an isogeny $1 \to \mu \to G' \to G^m \times T \to 1$, with T a quasitrivial K-torus, $m > 0$ and G' the product of a quasitrivial torus by a simply connected group with the same (absolute) simple factors as G. Let $1 \to \mu \to F \to P \to 1$ be a flasque resolution of μ. By Theorem 4.9, we have a group isomorphism $(G(K)/R)^m = (G(K)/R)^m \times T(K)/R \simeq H^1(K,F)$, hence $G(K)/R$ is an abelian group.

Corollary 4.11 *Let K be as in Theorem 4.9 and let G/K be a semisimple group. Under any of the following assumptions :*

(i) G is simply connected,

(ii) G is adjoint,

(iii) G is absolutely almost simple,

(iv) G is an inner form of a group which is split by a metacyclic extension of K,

we have $G(K)/R = 1$.

Proof Recall that a finite field extension L/K is called metacyclic if it is Galois and each Sylow subgroup of the Galois group is cyclic. We follow Sansuc's argument [Sa, Cor. 5.4]. Case (i) has already been handled (Theorem 4.5). If G is adjoint, then it is a product of restrictions of scalars of groups of type (iii) over finite extensions of K. Thus (ii) reduces to (iii). An absolutely almost simple K-group is an inner form of an absolutely almost simple quasisplit K-group. The latter is split by a metacyclic extension, the automorphism group of the Dynkin diagram being either 0, $\mathbf{Z}/2$ or $\mathfrak{S}_3$. One is thus reduced to case (iv). Let μ be the fundamental group of G. There exists a flasque resolution $1 \to \mu \to F \to P \to 1$ where F and P are split by the minimal splitting field of the multiplicative group μ (which by definition is the splitting field of the character group of μ). If a semisimple group is split, then its fundamental group μ is split (that is, the absolute Galois group acts trivally on the character group of μ). An inner twist does not affect the fundamental group. Thus F is split by a metacyclic extension, hence is a direct factor of a quasitrivial torus (theorem of Endo and Miyata), hence $H^1(K,F) = 0$. The triviality of $G(K)/R$ now follows from Theorem 4.9.

Specializing to the fields of main interest in this paper, we have

Theorem 4.12 *Let K be a field of type (ll), (sl) or (gl). Let G/K be a connected linear algebraic group. In the (gl) case, assume G has no E_8 factor. Then*

(i) The quotient $G(K)/R$ is a finite abelian group.

(ii) Suppose the group G has a presentation

$$1 \to \mu \to G' \to G \to 1$$

where G' is the product of a semisimple simply connected group and a quasitrivial torus, and $G' \to G$ is an isogeny with kernel μ. Let

$$1 \to \mu \to F \to P \to 1$$

be a flasque resolution of μ (the torus F is flasque and the torus P is quasitrivial). Then the Galois cohomology sequences induce an isomorphism of finite groups $G(K)/R \simeq H^1(K,F)$.

Proof For part (ii), simply combine Theorems 4.6, 4.8 and the finiteness results 3.2 and 3.4. To prove the finiteness of $G(K)/R$ in the general case, proceeding as in Corollary 4.10 above, one shows that for some $m > 0$, $(G(K)/R)^m$ is finite.

We now consider weak approximation for fields of type (ll) or (gl), and the associated set Ω of discrete valuations.

Theorem 4.13 *Let K be a field of type (ll) or (gl). Let $S \subset \Omega$ be a finite set. Let G/K be a connected linear group. In the case (gl), assume that G has no factor of type E_8.*

(i) The closure $\overline{G(K)}$ of the image $G(K)$ under the diagonal map $G(K) \to \prod_{v \in S} G(K_v)$ is a normal subgroup, and the quotient $A_S(G) = \prod_{v \in S} G(K_v)/\overline{G(K)}$ is a finite abelian group.

(ii) Suppose the group G has a presentation $1 \to \mu \to G' \to G \to 1$ where G' is the product of a semisimple simply connected group and a quasitrivial torus, and $G' \to G$ is an isogeny with kernel μ. Let $1 \to \mu \to F \to P \to 1$ be a flasque resolution of μ (the torus F is flasque and the torus P is quasitrivial). Then the composite maps $G(K) \to H^1(K,\mu) \to H^1(K,F)$ and $G(K_v) \to H^1(K_v,\mu) \to H^1(K_v,F)$ induce isomorphisms of finite abelian groups

$$A_S(G) \simeq \mathrm{Coker}[H^1(K,F) \to \prod_{v \in S} H^1(K_v,F)]$$

and

$$A_S(G) \simeq \mathrm{Coker}[G(K)/R \to \prod_{v \in S} G(K_v)/R].$$

Proof The arguments due to Kneser [Kn2] and Sansuc ([Sa], §3) (for the last isomorphism, see also [N]) in the number field context may be adapted to our context, one key remark being that they do not involve any class field theory (for a minor difference between the two set-ups, see remark after the proof).

Using Proposition 3.2 and Lemme 1.10 of [Sa], one reduces the proof of (i) to that of (ii). Let H be the group $F \overset{\mu}{\times} G'$, i.e. the contracted product of H and G' relatively to μ. There is a commutative diagram of exact sequences of algebraic groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mu & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & F & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & P & = & P & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The middle vertical sequence induces the following diagram, where horizontal maps are diagonal maps :

$$\begin{array}{ccc}
G'(K) & \longrightarrow & \prod_{v \in S} G'(K_v) \\
\downarrow & & \downarrow \\
H(K) & \longrightarrow & \prod_{v \in S} H(K_v) \\
\downarrow & & \downarrow \\
P(K) & \longrightarrow & \prod_{v \in S} P(K_v) \\
\downarrow & & \\
H^1(K, G') & &
\end{array}$$

By our assumptions, $H^1(K, G') = 1$. Since P is a quasitrivial torus, the bottom horizontal map has dense image. By Theorem 4.7, the top horizontal map has dense image. The vertical maps on the right hand side are continuous. Since the K-morphism $H \to P$ is smooth, each projection map $H(K_v) \to P(K_v)$ is open ([Se1], Part II, Lie Groups, Chap. 3, §10.2). All these statements put together imply that the diagonal map $H(K) \to \prod_{v \in S} H(K_v)$ has dense image.

We have exact sequences of pointed sets

$$H^1(K, G') \to H^1(K, H) \to H^1(K, P)$$

and

$$H^1(K_v, G') \to H^1(K_v, H) \to H^1(K_v, P).$$

Since P is quasitrivial, $H^1(K,P)=0$ and $H^1(K_v,P)=0$. By our assumptions on G, hence G', and on the field K, we have $H^1(K,G')=1$ and $H^1(K_v,G')=0$. Thus $H^1(K,H)=1$ and $H^1(K_v,H)=1$ for any $v\in\Omega$.

From the initial diagram we deduce another commutative diagram, where the vertical maps are diagonal maps :

$$\begin{array}{ccccccc} H(K) & \to & G(K) & \to & H^1(K,F) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_{v\in S} H(K_v) & \to & \prod_{v\in S} G(K_v) & \to & \prod_{v\in S} H^1(K_v,F) & \to & 0, \end{array}$$

where the right hand side zeroes have just been explained.

By Theorems 3.4 and 3.2, the groups $H^1(K,F)$ and $\prod_{v\in S} H^1(K_v,F)$ are finite. Let $\mathcal{A}_S(\mu)=\mathrm{Coker}[H^1(K,F)\to\prod_{v\in S} H^1(K_v,F)]$. Let $\varphi:\prod_{v\in S} G(K_v)\to\mathcal{A}_S(\mu)$ be the obvious composite homomorphism. We have shown that the diagonal map $H(K)\to\prod_{v\in S} H(K_v)$ has dense image. An easy diagram chase now shows that the closure of $G(K)$ in $\prod_{v\in S} G(K_v)$ coincides with the subgroup $\varphi^{-1}(1)$, which is clearly normal, and we have $\prod_{v\in S} G(K_v)/\overline{G(K)}\simeq\mathcal{A}_S(\mu)$.

Remark The above proof also works for a totally imaginary number field K and a finite set of completions K_v. In the number field case ([Sa], §3), one usually identifies $A_S(G)$ with the cokernel of $H^1(K,\mu)\to\prod_{v\in S} H^1(K_v,\mu)$. This is all right because for a (usual) local field $H^1(K_v,\mu)$ is finite. But for a field K_v with residue field a function field in one variable over an algebraically closed field, a group $H^1(K_v,\mu)$ need not be finite. The replacement of $H^1(K_v,\mu)$ by $H^1(K_v,F)$ does not however enable a total analogy. In the number field case, we have $H^1(K_v,F)=0$ for almost all v. This enables one to define a finite abelian group $A(G)$ which covers the defect of weak approximation at any finite set of places. In the (gl) and (ll) case, for a given model X, $H^1(K_v,F)=0$ for almost all $v\in\Omega_X$ (remark after Theorem 3.7), hence a finite group measures the lack of weak approximation with respect to all places in Ω_X. As for the whole set Ω, we have the same comments as in the quoted remark.

The same arguments as in Corollary 4.11 above now yield :

Corollary 4.14 *Let K be a field of type (ll) or (gl). Let G/K be a semisimple group. In the case (gl), assume that G has no factor of type E_8.*
Under any of the following assumptions :
(i) G is simply connected,
(ii) G is adjoint,
(iii) G is absolutely almost simple,
(iv) G is an inner form of a group which is split by a metacyclic extension of K,
for any finite $S\subset\Omega$, we have $A_S(G)=0$: weak approximation holds for G.

Remark In the (ll) case, starting from the example in Proposition 3.8, one may easily produce a finite abelian K-group μ and a sequence $1\to\mu\to F\to P\to 1$ such that for suitable $S\subset\Omega$, the quotient $\mathcal{A}_S(\mu)$ is not trivial. Starting from μ, one may then produce a semisimple group G such that $A_S(G)\neq 0$.

§5. Hasse principle

The following theorem is an analogue of a result of Borel for number fields.

Theorem 5.1 *Let K be a field of type (ll) or (gl). Let Ω be its associated set of places. Let G/K be a connected linear algebraic group. If K is of type (gl), assume that G has no E_8-factors. The set*

$$Ш^1(K,G) = \ker[H^1(K,G) \to \prod_{v\in\Omega} H^1(K_v,G)]$$

is a finite set.

Proof One reduces to the case where G is reductive. Using Lemme 1.10 of [Sa], one is reduced to the case of a group G equipped with an isogeny

$$1 \to \mu \to G' \to G \to 1$$

with G' the product of a semisimple simply connected group (with the same factors as G) by a quasitrivial torus. By Theorems 1.3 and 1.4 we have $H^1(K,G') = 1$, and this also holds true for all the twisted forms of G', hence we have an injective map $Ш^1(K,G) \hookrightarrow Ш^2(K,\mu)$.

In the (gl) case, let X/k be an integral surface with function field K. In the (ll) case, let A be the defining strictly henselian local domain for K, and let $X = \mathrm{Spec}(A)$. In both cases, over a suitable nonempty regular open set $U \subset X$ of finite type over X, there exists a commutative, finite étale group scheme μ_U/U whose generic fibre is our given μ. The elements of $H^2(K,\mu)$ with trivial restriction to $H^2(K_v,\mu)$ for each $v \in U^1$ belong to the image of the restriction map $H^2(U,\mu_U) \to H^2(K,\mu)$ (purity, [SGA4] XVI 3.7 and XIX 3.2) and the group $H^2(U,\mu_U)$ is finite (see the proof of Prop. 3.3).

The following theorem is an analogue of results of Sansuc ([Sa], Cor. 5.4 and Cor. 5.9) for number fields.

Theorem 5.2 *Let K be a field of type (ll) and Ω be its associated set of places. Let G/K be a connected semisimple group of fundamental group μ.*

a) The boundary $H^1(K,G) \to H^2(K,\mu)$ induces a bijection $Ш^1(K,G) \xrightarrow{\sim} Ш^2(K,\mu)$.

b) In either of the following cases :

- *(i) G is adjoint or G is absolutely almost simple,*
- *(ii) G is semisimple and is a direct factor of a K-rational variety, i.e. there exists a K-variety Y such that $G \times_K Y$ is K-birational to affine space over K,*

we have $Ш^1(K,G) = 1$.

Proof *a*) Since G is semisimple, we have the standard sequence

$$1 \to \mu \to G' \to G \to 1$$

with G' simply connected. By Theorems 2.1 and 2.2, the map $H^1(K,G) \to H^2(K,\mu)$ is bijective, and the same hold for the K_v. So $Ш^1(K,G) \to Ш^2(K,\mu)$ is bijective.

b) Let us discuss case (i). By the standard reduction, we may assume that G is absolutely almost simple. As explained in the proof of Corollary 4.11, for such a G, the finite abelian K-group μ is then split by a metacyclic extension. The last property implies that we can find a sequence

$$1 \to \mu \to F \to P \to 1$$

with P quasitrivial and F a direct factor of a quasitrivial torus : there exists a K-torus F' such that $F \times_K F' \simeq Q$ with $Q = \prod_i R_{K_i/K}\mathbf{G}_m$. We then have

$$Ш^1(K,G) \hookrightarrow Ш^2(K,\mu) \subset Ш^2(K,F) \subset \prod_i Ш^2(K_i,\mathbf{G}_m).$$

By Corollary 1.10 of [CTOP] (Theorem 1.6 in the present paper), the latter group vanishes. This completes the proof in case (i).

Let us consider case (ii). Let X be a smooth compactification of G. Let $\overline{K}$ be an algebraic closure of K and $\Gamma = \mathrm{Gal}(\overline{K}/K)$. Because G is semisimple, the natural map $\overline{K}^* \to \overline{K}[G]^*$ is a bijection (the only units are the constants). On the other hand, for the semisimple group G, there is a natural Γ-isomorphism between the character group $\hat{\mu}$ and the Picard group of $\overline{G} = G \times_K \overline{K}$. The natural sequence of Γ-modules :

$$0 \to \overline{K}[G]^*/\overline{K}^* \to \mathrm{Div}_\infty(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{G}) \to 0,$$

where $\mathrm{Div}_\infty(\overline{X})$ is the permutation module on the irreducible components of the complement of $\overline{G}$ in $\overline{X}$ therefore reads

$$0 \to \mathrm{Div}_\infty(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \hat{\mu} \to 0.$$

Under assumption (ii), there exists a Γ-module M, such that the Γ-module $\mathrm{Pic}(\overline{X}) \oplus M$ is Γ-isomorphic to a permutation Γ-module ([CTS3], Prop. 2.A.1 p. 461). Dualizing the exact sequence above, we find an exact sequence

$$1 \to \mu \to F \to P \to 1$$

with P a quasitrivial torus and F a direct factor (as a torus) of a quasitrivial torus. As above, this implies $Ш^2(K, \mu) = 0$, hence $Ш^1(K, G) = 1$.

Remarks

(i) It is likely that $Ш^1(K, G) = 1$ holds for an arbitrary K-rational group. In the number field case, this is a result of Sansuc ([Sa], Cor. 9.7).

(ii) For an arbitrary connected linear algebraic group G over K of type (ll), one does not expect $Ш^1(K, G) = 1$ to always hold. But we have not been able to produce a counterexample (this is very closely connected with the Question and Remarks following Prop. 3.8).

We now follow Borovoi's paper [Bo1] extremely closely. The aim is to show how Theorem 2.1 (conjecture 5.3 of [CTOP]) implies Conjecture 5.4 of [CTOP] (Corollary 5.7 below).

Borovoi's paper elaborates on earlier work of Springer and of Douai. One preprint quoted in [Bo1] has since appeared ([Bo2]). A useful complement to Borovoi's paper is the article [FSS].

Let k be a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. Let $\overline{G}$ be a connected linear $\overline{k}$-group and $L = (\overline{G}, \kappa)$ a k-kernel (otherwise known as k-band or k-lien). To $\overline{G}$ one associates a $\overline{k}$-torus $\overline{G}^{\mathrm{tor}}$ ([Bo1], Notation p. 218).

To the k-kernel L one associates the second cohomology set $H^2(k, L)$. This set may or may not be empty. Inside this set one defines special classes called neutral classes ([Bo1] 1.6). There may exist none, one, or more such classes. If $\overline{G}$ is reductive, then $H^2(k, L)$ contains a neutral element ([Bo1] Prop. 3.1). In general, if we let $\overline{G}^{\mathrm{red}}$ be the quotient of $\overline{G}$ by its unipotent radical, then there are an induced k-kernel $L^{\mathrm{red}} = (\overline{G}^{\mathrm{red}}, \kappa^{\mathrm{red}})$ and a canonical map of sets $r : H^2(k, L) \to H^2(k, L^{\mathrm{red}})$. An element $\eta \in H^2(k, L)$ is neutral if and only $r(\eta) \in H^2(k, L^{\mathrm{red}})$ is neutral ([Bo1], Prop. 4.1). When $\overline{G}$ is reductive, one defines an abelian group $H^2_{ab}(k, L)$ ([Bo1], 5.1) and an abelianization map $ab^2 : H^2(k, L) \to H^2_{ab}(k, L)$ ([Bo1], 5.3). This map sends the neutral classes to zero. For an arbitrary connected linear $\overline{k}$-group, and a k-kernel L, one sets $H^2_{ab}(k, L) = H^2_{ab}(k, L^{\mathrm{red}})$ and one defines the abelianization map to be the compositum

$$ab^2 : H^2(k, L) \to H^2(k, L^{\mathrm{red}}) \to H^2_{ab}(k, L^{\mathrm{red}}) = H^2_{ab}(k, L).$$

Neutral elements are sent to zero.

Proposition 5.3 *Let $L = (\overline{G}, \kappa)$ be a connected k-kernel. Assume $\mathrm{cd}(k) \leq 2$, and assume that for central simple algebras over finite field extensions of k, index and exponent coincide. Then an element $\eta \in H^2(k, L)$ is neutral if and only if $ab^2(\eta) = 0$.*

Proof See the proof of Theorem 5.5 of [Bo1] (p. 229, Section 5.8). In our context, the rôle of [Bo1], Lemma 5.7 for number fields (surjectivity of the boundary map) is played by our Theorem 2.1.

Remark As in [Bo1], Cor. 5.6, when $\overline{G}$ is semisimple simply connected, this implies that any element of $H^2(k, L)$ is neutral.

There is a natural homomorphism of $\overline{k}$-algebraic groups $t_* : \overline{G} \to \overline{G}^{\mathrm{tor}}$, where $\overline{G}^{\mathrm{tor}}$ is the biggest quotient torus of $\overline{G}$. The k-kernel L induces a k-kernel $L^{\mathrm{tor}} = (\overline{G}^{\mathrm{tor}}, \kappa^{\mathrm{tor}})$ on the (commutative) $\overline{k}$-torus $\overline{G}^{\mathrm{tor}}$: this defines a k-torus G^{tor}, k-form of $\overline{G}^{\mathrm{tor}}$ ([Bo1] 1.4, 1.7, 6.1). There is an identification $H^2(k, L^{\mathrm{tor}}) = H^2(k, G^{\mathrm{tor}})$, where the latter set is the usual second abelian cohomology group. There is a natural map

$$t_* : H^2(k, L) \to H^2(k, L^{tor}) = H^2(k, G^{\mathrm{tor}}),$$

referred to as the canonical map.

As explained in [Bo1], 6.1, for $L = (\overline{G}, \kappa)$ with $\overline{G}$ connected, there is an exact sequence ([Bo1], (6.1.1))

$$H^3(k, \mu) \to H^2_{ab}(k, L) \stackrel{t_{ab}}{\to} H^2(k, G^{\mathrm{tor}}),$$

where μ is a finite abelian k-group (in the notation of [Bo1], section 5,1, μ is the kernel of the homomorphism of k-groups $Z^{(\mathrm{sc})} \to Z$) and the composite map $H^2(k, L) \to H^2_{ab}(k, L) \to H^2(k, G^{\mathrm{tor}})$ is the canonical map t_*.

We now have the analogue of Proposition 6.2 of [Bo1].

Proposition 5.4 *Assume $\mathrm{cd}(k) \leq 2$, and assume that for central simple algebras over finite field extensions of k, index and exponent coincide. Let $L = (\overline{G}, \kappa)$ be a connected k-kernel. Then an element $\eta \in H^2(k, L)$ is neutral if and only if $t_*(\eta) = 0$, where t_* is the canonical map.*

Proof Since k has cohomological dimension 2, we have $H^3(k, \mu) = 0$, hence the map $H^2_{ab}(k, L) \to H^2(k, G^{\mathrm{tor}})$ is injective. Thus $t_*(\eta) = 0$ implies that the image of η under $ab^2 : H^2(k, L) \to H^2_{ab}(k, L)$ is zero. By Proposition 5.3 this implies that η is neutral. The converse statement is obvious.

Let X be a smooth variety over k which is a right homogeneous space of a semisimple simply connected k-group H. Let $\overline{G}$ be the isotropy group of a $\overline{k}$-point x of $X(\overline{k})$, and assume that this isotropy group is connected. Since the homogeneous space X is defined over k, there is an associated connected k-kernel $L = (\overline{G}, \kappa)$ ([Bo1], 7.1), and an associated k-torus G^{tor}. These data are functorial in the ground field k. By cocycles computations, one may show that up to k-isomorphism the k-torus G^{tor} does not depend on the choice of the point x. If X has a k-point x, and G_x is its stablizer, then G^{tor} is the maximal torus quotient of G_x, i.e. the character group of G^{tor} coincides with the Galois module of characters of $G_x \times_k \overline{k}$.

Associated to the homogeneous space X there is a class $\eta = \eta(X) \in H^2(k, L)$ ([Bo1], 7.7) which is neutral if and only if the homogeneous space X comes from a right principal homogeneous space under H, i.e. there exists such a space Y and a H-equivariant map $Y \to X$.

We are now in a position to prove the analogue of a result of Borovoi ([Bo1], Thm. 7.3).

Theorem 5.5 *Let A be an excellent henselian two-dimensional local domain, let K be its field of fractions and k its residue field. Assume that k is algebraically closed of characteristic zero. Let Ω be the set of all rank one discrete valuations on K. For $v \in \Omega$, let K_v be the* *completion of K at v. Let X be a smooth variety over K which is a right homogeneous space of a semisimple simply connected group H over K. Assume that the geometric stabilizers are connected. Let G^{tor} be the associated K-torus. Assume that one of the following holds :*

(i) $Ш^2(K, G^{\mathrm{tor}}) = 0$;

(ii) the torus G^{tor} is the kernel of a surjective map of direct factors of quasitrivial tori ;

(iii) the torus G^{tor} is split by a metacyclic extension of K ;

(iv) the torus G^{tor} has dimension at most one.

Then the Hasse principle holds for X : if for each $v \in \Omega$ the set $X(K_v)$ is not empty, then X has a K-rational point.

Proof Let $L = (\overline{G}, \kappa)$ be the K-kernel associated to the homogeneous space X. Consider the class $\eta = \eta(X) \in H^2(K, L)$. By Proposition 5.4 and Theorem 2.1, η is neutral if and only if $t_*(\eta) \in H^2(K, G^{\mathrm{tor}})$ is trivial. For any place v, the assumption $X(K_v) \neq \emptyset$ implies that η_v is neutral, hence $t_{v*}(\eta) \in H^2(K_v, G^{\mathrm{tor}})$ is trivial. Thus $t_*(\eta)$ lies in $Ш^2(K, G^{\mathrm{tor}}) = \mathrm{Ker}[H^2(K, G^{\mathrm{tor}}) \to \prod_{v \in \Omega} H^2(K_v, G^{\mathrm{tor}})$.

Over any field, any 1-dimensional torus is split by a quadratic extension of the ground field. Over any field, for a K-torus T which is split by a metacyclic extension of K, there exists an exact sequence of K-tori

$$1 \to T \to P_1 \to P_2 \to 1,$$

where P_1 and P_2 are direct factors of quasitrivial tori (this uses the Endo-Miyata theorem, see §3). For the torus $T = G^{\mathrm{tor}}$ over the field K, this sequence induces an embedding $Ш^2(K, G^{\mathrm{tor}}) \hookrightarrow Ш^2(K, P_1)$ and the latter group vanishes by Theorem 1.6 (applied to finite field extensions of K).

Thus η is neutral, hence X comes from a right principal homogeneous space Y under H. But $H^1(K, H)$ is trivial (Theorem 1.4), i.e. any principal homogeneous space under H has a K-point. Hence $Y(K) \neq \emptyset$, hence also $X(K) \neq \emptyset$.

The following lemma appears in the literature, but it is hard to locate a proof.

Lemma 5.6 *Let k be a field and X a smooth projective k-variety which is a homogeneous space of a semisimple simply connected k-group H. With notation as above, the k-torus G^{tor} is a quasitrivial torus.*

Proof Let $F = k(X)$ be the function field of X. Since X is absolutely irreducible, the field k is algebraically closed in F. Let L denote the field $F \otimes_k \overline{k}$. The extension L/F is Galois, and the natural map $\mathrm{Gal}(L/F) \to \mathrm{Gal}(\overline{k}/k) = \Gamma$ is an isomorphism. Let $G_\eta \subset H_F$ denote the stabilizer of the generic point of X, viewed as an F-point of X. By the functoriality mentioned before Theorem 5.5, the character group of the F-torus $G^{\mathrm{tor}} \times_k F$, which splits over the field L, coincides, as a Γ-module, with the character group $\hat{G}_\eta(L)$ of the F-group G_η. The map $h \mapsto h.\eta$ makes H_F into a torsor over X_F under the group G_F. There is an associated exact sequence of Γ-modules ([Sa], Prop. 6.10)

$$0 \to L[X]^* \to L[H]^* \to \hat{G}_\eta(L) \to \mathrm{Pic}(X_L) \to \mathrm{Pic}(H_L).$$

Since H is semisimple, $L^* = L[H]^*$, and since H is simply connected, $\mathrm{Pic}(H_L) = 0$. The exact sequence above reduces to an isomorphism $\hat{G}_\eta(L) \simeq \mathrm{Pic}(X_L)$. The Galois module $\mathrm{Pic}(X \times_k \overline{k})$ is a permutation module : it is the permutation module with basis the points of codimension 1 on $X \times_k \overline{k}$ which are in the complement of the big cell, which is an affine space over $\overline{k}$ (see [H2], proof of Satz 1.3.1 p. 118). This implies that the natural Γ-equivariant map $\mathrm{Pic}(X \times_k \overline{k}) \to \mathrm{Pic}(X_K \times_K L)$ is an isomorphism. We conclude that the character group of G^{tor} is a permutation module.

The following result is the analogue of a result of Harder over number fields ([H1]; [Bo1], Cor. 7.5). It was put forward as a conjecture in §5 of [CTOP].

Corollary 5.7 *Let K be as in Theorem 5.5 and let X be a smooth projective K-variety which is a homogeneous space of a connected linear algebraic K-group H. Then the Hasse principle holds for X.*

Proof Let $R \subset H$ be the radical of H, i.e. the maximal connected solvable normal subgroup of H. Let $\overline{G}$ be the isotropy point of a $\overline{K}$-point of X. This is a parabolic group, hence it is connected. Its contains $\overline{R}$. It follows that the action of H on X factorizes through the semisimple group $H^{\mathrm{ss}} = H/R$. Thus X is a homogeneous space under H^{ss}. Let $\lambda : H^{\mathrm{sc}} \to H^{\mathrm{ss}}$ be the universal covering of H. As λ is a surjective morphism, X is a homogeneous space under H^{sc}. We may thus assume that H is a semisimple simply connected group. The result now follows from Theorem 5.5 (ii) and Lemma 5.6.

We also have the analogue of Corollary 7.6 of [Bo1] (a generalization of a result of Rapinchuk) :

Corollary 5.8 *Let K be as above and let X be a symmetric homogeneous space of an absolutely simple simply connected K-group H (i.e. the geometric isotropy group is the group of invariants of an involution of $H \times_K \overline{K}$). Then the Hasse principle holds for X.*

Proof As Borovoi remarks, under the above assumption, the dimension of G^{tor} is at most one, hence the result by Theorem 5.5 (iv).

References

[A] M. Artin, Two-dimensional orders of finite representation type, Manuscripta mathematica **58** (1987) 445-471.

[BFP] E. Bayer-Fluckiger and R. Parimala, Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2, Invent. math. **122** (1995) 195-229.

[Bo1] M. V. Borovoi, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. **72** (1993) 217-239.

[Bo2] M. Borovoi, *Abelian Galois cohomology of reductive groups*, Memoirs of the American Mathematical Society **132** (Number 626) (1998).

[BT] F. Bruhat et J. Tits, Groupes algébriques sur un corps local III. Applications à la cohomologie galoisienne, J. Fac. Sci. Univ. Tokyo **34** (1987) 671-698.

[Ch] V. I. Chernousov, The kernel of the Rost invariant, Serre's Conjecture II and the Hasse principle for quasi-split groups of type ${}^{3,6}D_4$, E_6, E_7, preprint.

[CM] V. I. Chernousov and A. S. Merkurjev, R-equivalence and special unitary groups, J. Algebra **209** (1998) 175-198.

[CP] V. I. Chernousov and V. P. Platonov, The rationality problem for semisimple group varieties, J. reine angew. Math. (Crelle) **504** (1998) 1-28.

[CTG] J.-L. Colliot-Thélène, Exposant et indice d'algèbres simples centrales non ramifiées, avec un appendice par O. Gabber, L'Enseignement Mathématique, **48** (2002) 127-146.

[CTOP] J.-L. Colliot-Thélène, M. Ojanguren and R. Parimala, Quadratic forms over fraction fields of two-dimensional henselian rings and Brauer groups of related schemes, to appear in *Algebra, Arithmetic and Geometry*, Proceedings of the Mumbai Colloquium 2000.

[CTGP] J.-L. Colliot-Thélène, P. Gille et R. Parimala, Arithmétique des groupes algébriques linéaires sur certains corps géométriques de dimension deux. Comptes rendus de l'Académie des Sciences de Paris, Série I, Mathématique **333** (No. 9) (nov. 2001) 827-832.

[CTS1] J.-L. Colliot-Thélène et J.-J. Sansuc, La R-équivalence sur les tores, Ann. Sci. École Norm. Sup. 4ème série **10** (1977) 175-229.

[CTS2] J.-L. Colliot-Thélène et J.-J. Sansuc, Principal Homogeneous Spaces under Flasque Tori : Applications, J. Algebra **106** (1987) 148-205.

[CTS3] J.-L. Colliot-Thélène et J.-J. Sansuc, La descente sur les variétés rationnelles, II. Duke Math. J. **54** (1987) 375-492.

[DLT] I. Dejaiffe, D. W. Lewis, J.-P. Tignol, Witt equivalence of central simple algebras with involution, Rend. circ. mat. Palermo (Ser. 2) **49** (2000) 325-342.

[EGA IV_1] A. Grothendieck et J. Dieudonné, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, I. Inst. Hautes Études Sci. Publ. Math. **20** (1964).

[EGA IV_2] A. Grothendieck et J. Dieudonné, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, II. Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[FSS] Y. Z. Flicker, C. Scheiderer and R. Sujatha, Grothendieck's theorem on non-abelian H^2 and local-global principles, J. Amer. Math. Soc. **11** (1998) 731-750.

[Ga] R. Skip Garibaldi, Isotropic trialitarian algebraic groups, J. Algebra **210** (1998) 385-418.

[Gi1] P. Gille, La R-équivalence sur les groupes algébriques réductifs définis sur un corps global. Inst. Hautes Études Sci. Publ. Math. **86** (1997) 199-235.

[Gi2] P. Gille, Cohomologie des groupes quasi-déployés sur des corps de dimension cohomologique ≤ 2, Compositio math. **125** (2001) 283-325.

[Gr] A. Grothendieck, Le groupe de Brauer I, II, III, in *Dix exposés sur la cohomologie des schémas*, Masson (Paris) et North-Holland (Amsterdam), 1968.

[H1] G. Harder, Bericht über neuere Resultate der Galoiskohomologie halbeinfacher Gruppen, Jahresbericht der DMV **70** (1968) 182-216.

[H2] G. Harder, Halbeinfache Gruppenschemata über vollständigen Kurven, Invent. Math. **6** (1968) 107-149.

[Ja] P. Jaworski, On the strong Hasse principle for fields of quotients of power series rings in two variables, Math. Z. **236** (2001) 531-566.

[Kn1] M. Kneser, *Lectures on Galois cohomology of classical groups*, notes by P. Jothilingam, Tata Institute of Fundamental Research, Bombay, 1969.

[Kn2] M. Kneser, Schwache Approximation in algebraischen Gruppen, in *Colloque sur la théorie des groupes algébriques*, Bruxelles, CBRM (1962), 41-52.

[KMRT] M. A. Knus, A. S. Merkurjev, M. Rost, J.-P. Tignol, *The book of involutions.* Avec une préface par J. Tits. American Mathematical Society Colloquium Publications **44**, American Mathematical Society, Providence, RI (1998).

[Ma] H. Matsumura, *Commutative Algebra*, second edition, Mathematics Lecture Note Series **56**, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[Me1] A. S. Merkurjev, Generic element in SK_1 for simple algebras, K-Theory **7** (1993) 1-3.

[Me2] A. S. Merkurjev, K-Theory and Algebraic groups, Progress in Mathematics **169** (1998) 43-72.

[MPW] A. S. Merkurjev, I. A. Panin, A.R. Wadsworth, Index reduction formulas for twisted flag varieties, I, K-Theory **10** (1996) 517-596 ; II, K-Theory **14** (1998) 101-196.

[MT] A. S. Merkurjev and J.-P. Tignol, The multipliers of similitudes and the Brauer group of homogeneous varieties, J. reine angew. Math. (Crelle) **461** (1995) 13-47.

[N] Nguyễn Q. Thắng, Weak approximation, R-equivalence and Whitehead groups, in *Algebraic K-theory* (Toronto, ON, 1996), 345-354, Fields Inst. Commun. **16** (1997) 345-354.

[Pf] A. Pfister, *Quadratic forms with applications to algebraic geometry and topology*, London Mathematical Society Lecture Note Series **217**, Cambridge University Press, Cambridge, 1995.

[PR] V. P. Platonov and A. S. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press (1994).

[Sa] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. reine angew. Math. (Crelle) **327** (1981) 12-80.

[Sc] W. Scharlau, *Quadratic and hermitian forms*, Grundlehren der mathematischen Wissenschaften **270**, Springer (1985).

[Se1] J.-P. Serre, *Lie algebras and Lie groups*, Lectures given at Harvard University, 1964, W. A. Benjamin, Inc., New York-Amsterdam 1965.

[Se2] J.-P. Serre, *Cohomologie galoisienne*, Lecture Notes in Mathematics **5**, cinquième édition (1994), Springer-Verlag.

[Se3] J.-P. Serre, Cohomologie galoisienne : progrès et problèmes, in *Séminaire Bourbaki* **783** (1993-1994), Astérisque **227** (1995).

[Su] A. A. Suslin, Algebraic K-theory and norm residue homomorphism, Journal of Soviet mathematics **30** (1985) 2556-2611.

[SGA 3] M. Demazure et A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics **151**, **152**, **153**, Springer-Verlag, Berlin-New York (1962-1964).

[SGA 4] M. Artin, A. Grothendieck et J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Mathematics **269**, **270**, **305**, Springer-Verlag, Berlin-New York (1972-1973).

[Ti1] J. Tits, Classification of Algebraic Semisimple Groups, in *Algebraic Groups and Discontinuous Sugroups*, Proceedings of Symposia in Pure Mathematics **9**, American Mathematical Society, 1966, p. 33-62.

[Ti2] J. Tits, Strongly inner anisotropic forms of simple algebraic groups, J. Algebra **131** (1990) 648-677.

[Vo1] V. E. Voskresenkiĭ, O privedennoi gruppe Uaitheda prostoi algebry (On the reduced Whitehead group of a simple algebra), Uspekhi Mat. Nauk **32** (1977), 247–248.

[Vo2] V. E. Voskresenkiĭ, *Algebraic groups and their birational invariants*, Translated from the Russian manuscript by Boris Kunyavskiĭ. Translations of Mathematical Monographs **179**. American Mathematical Society, Providence, RI, 1998.

[Ya1] V. I. Yanchevskiĭ, Simple algebras with involutions and unitary groups, Mat. Sbornik **93** (135) (1974) 368-380, 487 ; English translation, Math USSR Sbornik **22** (1974) 372-385.

[Ya2] V. I. Yanchevskiĭ, The commutator subgroups of simple algebras with surjective reduced norm, Dokl. Akad. Nauk SSSR **221** (1975) 1056-1058 ; English translation, Soviet Mat. Doklady **16** (1975) 492-495.

[Ya3] V. I. Yanchevskiĭ, Reduced unitary K-theory. Applications to algebraic groups, Dokl. Akad. Nauk SSSR **110** (1976) 579-596 ; English translation, Math. USSR Sbornik **38** (1981) 533-548.

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