CONSTRUCTION AND CLASSIFICATION OF SOME GALOIS MODULES

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Abstract. In our previous paper we describe the Galois module structures of pth-power class groups $K^\times/K^{\times p}$, where K/F is a cyclic extension of degree p over a field F containing a primitive pth root of unity. Our description relies upon arithmetic invariants associated with K/F. Here we construct field extensions K/F with prescribed arithmetic invariants, thus completing our classification of Galois modules $K^\times/K^{\times p}$.

Let F be a field of characteristic not p containing a primitive pth root of unity ξ_p. For a cyclic field extension K/F with Galois group $\mathrm{Gal}(K/F)$ of order p, let $J = K^\times/K^{\times p}$, and let N denote the norm map from K to F.

In [MS], we proved that the structure of the $\mathbb{F}_p[\mathrm{Gal}(K/F)]$-module J is determined by the following three arithmetic invariants:

- $d = d(K/F) := \dim_{\mathbb{F}_p} F^\times/N(K^\times)$,
- $e = e(K/F) := \dim_{\mathbb{F}_p} N(K^\times)/F^{\times p}$, and
- $\Upsilon(K/F) := 1$ or 0 according to whether $\xi_p \in N(K^\times)$ or not.

Now if $G = \mathbb{Z}/p\mathbb{Z}$, then J may be considered an $\mathbb{F}_p[G]$-module via any isomorphism $G \cong \mathrm{Gal}(K/F)$, and the module structure of J is independent of the choice of isomorphism. It is a fundamental problem to classify the isomorphism classes of modules J for all K/F in our context. This problem is solved in Theorem 1 below. Corollaries 1 and 2 of Theorem 1 describe all modules J in an explicit way.

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In the following theorem we determine the sets of invariants (d, e, Υ) which may be realized by an extension K/F and in so doing classify all $\mathbb{F}_p[G]$-modules $J = J(K/F)$ up to isomorphism.

Theorem 1. *Let p be a prime number. For arbitrary cardinal numbers d, e, and for $\Upsilon \in \{0, 1\}$, there exists a cyclic field extension K/F of degree p containing a primitive pth root of unity with invariants (d, e, Υ) if and only if*

- *if $\Upsilon = 0$, then $1 \le d$,*
- *if $p > 2$ then $1 \le e$, and*
- *if $p = 2$ and $\Upsilon = 1$ then $1 \le e$.*

From the theorem above and from [MS, Theorem 3 and Corollary 2] we immediately obtain the following corollaries. We denote by $M_{i,j}$ the jth cyclic module $\mathbb{F}_p[G]$ such that $\dim_{\mathbb{F}_p} M_{i,j} = i$, where j is a suitable index.

Corollary 1. *Let $p > 2$ be a prime number, and let G be a cyclic group of order p. Then an $\mathbb{F}_p[G]$-module J is realizable as an $\mathbb{F}_p[\mathrm{Gal}(K/F)]$-module $K^\times/K^{\times p}$ for some cyclic G-extension K/F such that F contains a primitive pth root of unity if and only if there exist cardinal numbers d, e, and $\Upsilon \in \{0, 1\}$ such that*

(i) *If $\Upsilon = 0$, then $1 \le d$;*
(ii) *$1 \le e$; and*
(iii)

$$J = \left(\bigoplus_{j \in \mathfrak{K}_1} M_{1,j}\right) \bigoplus \left(\bigoplus_{j \in \mathfrak{K}_2} M_{2,j}\right) \bigoplus \left(\bigoplus_{j \in \mathfrak{K}_p} M_{p,j}\right),$$

where

(1) *$|\mathfrak{K}_1| + 1 = 2\Upsilon + d$,*
(2) *$|\mathfrak{K}_2| = 1 - \Upsilon$, and*
(p) *$|\mathfrak{K}_p| + 1 = e$.*

The invariants d, e, and Υ determine the module J uniquely.

For $p = 2$ using [MS, Theorem 3 and Corollary 3], along with our theorem above, we obtain the next corollary.

Corollary 2. *Now let G be a cyclic group of order 2. Then an $\mathbb{F}_2[G]$-module J is realizable as an $\mathbb{F}_2[\mathrm{Gal}(K/F)]$-module $K^\times/K^{\times 2}$ for some* *quadratic extension* K/F *with its arithmetic invariants* $d(K/F)$*,* $e(K/F)$*, and* $\Upsilon(K/F)$ *coinciding with* d*,* e*, and* $\Upsilon \in \{0,1\}$*, respectively, if and only if* d*,* e*, and* Υ *satisfy the conditions below.*

- *if* $\Upsilon = 0$*, then* $1 \leq d$*,*
- *if* $\Upsilon = 1$*, then* $1 \leq e$*.*

In this case

$$J = \left(\bigoplus_{j\in\mathfrak{K}_1} M_{1,j}\right) \bigoplus \left(\bigoplus_{j\in\mathfrak{K}_2} M_{2,j}\right)$$

where

(1) $|\mathfrak{K}_1| + 1 = 2\Upsilon + d$ *and*
(2) $|\mathfrak{K}_2| + \Upsilon = e$*.*

Moreover, such a module J *is determined uniquely by the invariants* $2\Upsilon + d$ *and by* $e - \Upsilon$ *if* e *is finite and by* e *alone if* e *is infinite.*

If $p > 2$ then two $\mathbb{F}_p[G]$-modules are isomorphic if and only if their invariants d, e, and Υ are the same, by [MS, Corollary 2]. Thus we see in particular that if $p > 2$, then the arithmetic invariants of J depend only upon the isomorphism type of the $\mathbb{F}_p[G]$-module J.

In the case $p = 2$, we see from [MS, Corollary 2] again that the arithmetic invariants d, e, and Υ determine our module $\mathbb{F}_2[G]$, but two isomorphic $\mathbb{F}_2[G]$-modules may have different arithmetic invariants; see [MS, Corollary 3]. Here is a very simple, concrete example illustrating this possibility. Let K_1/F_1 be a quadratic extension of finite fields of characteristic not 2, $F_2 = \mathbb{R}((t))$ be a field of power series with coefficients in real numbers $\mathbb{R}$, and $K_2 = F_2(\sqrt{-1})$. Then both modules $K_1^\times/K_1^{\times 2}$ and $K_2^\times/K_2^{\times 2}$ are isomorphic to a trivial $\mathbb{F}_2[G]$-module $\mathbb{F}_2$, but their arithmetic invariants (d_i, e_i, Υ_i) are $(0,1,1)$ for $i = 1$ and $(2,0,0)$ for $i = 2$.

1. Notation and Strategy

In all that follows F denotes a field, $F^\times = F \setminus \{0\}$ the multiplicative group of F, p a prime number, and $F^\times/F^{\times p}$ the group of pth-power classes of F. For each $f \in F^\times$ we denote by $[f]$ the class of f in $F^\times/F^{\times p}$. For each subset A of $F^\times$ we denote by $[A]$ the set of classes $\{[a] \mid a \in A\}$ and by $\langle [A] \rangle$ the subgroup of $F^\times/F^{\times p}$ generated by $[A]$.

We denote by ξ_p a primitive pth root of unity in F. (Some fields will be assumed to contain such a primitive pth root; for the other fields in this paper, we will prove that a primitive pth root is contained in the field.) Observe that our assumption that there exists a primitive pth root of unity implies that $\operatorname{char}(F) \neq p$.

For a Galois extension K/F, $\operatorname{Gal}(K/F)$ denotes the Galois group and $N_{K/F}$ denotes the norm map from K to F. We denote by F^s the separable closure of F and G_F the absolute Galois group $\operatorname{Gal}(F^s/F)$. As usual, $H^i(G_F, \mathbb{F}_p)$ are Galois cohomology groups of F with coefficients in $\mathbb{F}_p$. Since all absolute Galois groups will be pro-p-groups, all considered $\mathbb{F}_p$ modules are trivial. Finally, let $|B|$ be the cardinal number of a set B.

First observe that the conditions on d, e, and Υ listed in our theorem above are necessary:

(1) If $\Upsilon = 0$, then $\xi_p \notin N_{K/F}(K^\times)$ and hence
$$d = \dim_{\mathbb{F}_p} F^\times/N_{K/F}(K^\times) \geq 1;$$
(2) If $p > 2$ and $K = F(\sqrt[p]{a})$ for a suitable $a \in F^\times \setminus F^{\times p}$, then $a = N_{K/F}(\sqrt[p]{a})$ and hence
$$e = \dim_{\mathbb{F}_p} N_{K/F}(K^\times) > 0.$$
(3) If $p = 2$, $\Upsilon = 1$, and $K = F(\sqrt{a})$ for a suitable $a \in F^\times \setminus F^{\times 2}$, then $-1 \in N_{K/F}(K^\times)$ since $\Upsilon = 1$. Consequently $a \in N_{K/F}(K^\times)$, and thus $1 \leq e$.

Therefore in order to prove Theorem 1 when $p > 2$, it is sufficient to show, for each cardinal numbers d, e as above, for each $\Upsilon \in \{0, 1\}$, and for each prime number $p > 2$, the existence of a field F such that:

- F contains a primitive pth root ξ_p;
- $F^\times/F^{\times p}$ decomposes intro a direct sum of subgroups
$$F^\times/F^{\times p} = D \oplus \langle [a] \rangle \oplus E,$$
where $\dim_{\mathbb{F}_p}(\langle [a] \rangle \oplus E) = e$ and, setting $K = F(\sqrt[p]{a})$,
  (1) $[N_{K/F}(K^\times)] = \langle [a] \rangle \oplus E$;
  (2) $\dim_{\mathbb{F}_p}(F^\times/N_{K/F}(K^\times)) = \dim_{\mathbb{F}_p} D = d$; and
  (3) $\Upsilon = 0$ if and only if $\xi_p \notin N_{K/F}(K^\times)$.

In the case $p = 2$ and $\Upsilon = 1$ we use the same conditions as above, and if $p = 2$ and $\Upsilon = 0$ we require instead that $e = \dim_{\mathbb{F}_2} E$ and

$d = \dim_{\mathbb{F}_2}(D \oplus \langle [a] \rangle)$. The latter condition is imposed because $-1 \notin N_{F(\sqrt{a})/F}(F(\sqrt{a})^\times)$ if and only if $a \notin N_{F(\sqrt{a})/F}(F(\sqrt{a})^\times)$.

Our strategy is to interpret the required conditions on $F^\times/F^{\times p}$ above in terms of Galois cohomology. We then observe that these conditions are satisfied if G_F is a free product, in the category of pro-p-groups, of suitable pro-p-groups G_1 and G_2, and finally we use the very nice theorem proved by Efrat and Haran which guarantees the existence of a field with G_F above. This is one of the key results used in our paper.

Theorem 2. *(Efrat-Haran; see* [EH, Proposition 1.3]*) Let $F_1, \ldots, F_n$ be fields of equal characteristic such that $G_{F_1}, \ldots, G_{F_n}$ are pro-p-groups. Then there exists a field F of the same characteristic such that*

$$G_F \cong G_{F_1} \star \cdots \star G_{F_n},$$

where the product is free in the category of pro-p-groups.

In order to apply this theorem, we show the existence of the fields F_1 and F_2 such that G_{F_1} and G_{F_2} are prescribed Galois groups G_1 and G_2. We use the techniques of henselian valuations and formal power series to construct fields F_1 and F_2.

2. Lemmas

2.1. Valued fields F_1 with prescribed residue field F_0 and valuation group Γ.

Let v be a valuation on a field F_1, written additively. Then we denote by A_v the valuation ring $\{f \in F_1 \mid v(f) \geq 0\}$; by M_v the unique maximal ideal $\{f \in A_v \mid v(f) > 0\}$ of A_v; by F_v the residue field A_v/M_v of v; by Γ the valuation group $v(F_1^\times)$ of v; and by U the group $A_v \setminus M_v$ of units of v.

The following lemma is well known and we shall omit its straightforward proof.

Lemma 1. *Let F_1 be a valued field with valuation v, valuation group $\Gamma = v(F_1^\times)$, and group of units U. For each prime $p \neq \operatorname{char}(F_1)$ there exists an isomorphism*

$$\varphi : F_1^\times/F_1^{\times p} \longrightarrow U/U^p \oplus \Gamma/p\Gamma.$$

In particular

$$\dim_{\mathbb{F}_p} F_1^\times/F_1^{\times p} = \dim_{\mathbb{F}_p} U/U^p + \dim_{\mathbb{F}_p} \Gamma/p\Gamma.$$

It is well-known that for each field F_0 and for each totally ordered abelian group Γ, there exists a field F_1 with a valuation $v : F_1 \to \Gamma \cup \{\infty\}$ such that the residue field F_v is isomorphic to F_0 and the valuation group is Γ.

In order to construct such a field, set

$$F_1 = F_0((\Gamma)) := \{f : \Gamma \to F_0 \mid \operatorname{supp}(f) \text{ is well-ordered}\}.$$

Thus a typical element $f \in F_1$ can be written as a formal sum $f = \sum_{g\in\Gamma} a_g t^g$ such that the set $\operatorname{supp}(f) := \{g \in G \mid a_g \neq 0\}$ is a well-ordered subset of G. The valuation v on f is defined as: $v(0) = \infty$ and $v(f) = \min \operatorname{supp}(f)$ for $f \neq 0$. An important property of the valued field F_1 as above is the fact that it is henselian. (See for example [Rib, (1.3)].) In what follows we will identify F_v with F_0. We will also assume that char $F_0 \neq p$.

We will be particularly interested in controlling the pth-power classes of such a field. To do so, we choose particular groups Γ for our valuation groups. These groups will be direct sums of

$$\mathbb{Z}_{(p)} := \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0; \text{ if } a \neq 0 \text{ then } (a, b) = 1, p \nmid b \right\}.$$

Observe that $\mathbb{Z}_{(p)}$ is the valuation ring of a p-adic valuation on $\mathbb{Q}$. Let I be any non-empty, well-ordered set. Then set

$$\Gamma = \mathbb{Z}_{(p)}^{(I)} := \left\{ \gamma : I \to \mathbb{Z}_{(p)} \;\middle|\; |\operatorname{supp}(\gamma)| < \infty \right\}.$$

Thus Γ is a direct sum of $|I|$ copies of $\mathbb{Z}_{(p)}$. Observe that $\mathbb{Z}_{(p)}$ carries a natural ordering induced from $\mathbb{Q}$, and then we may order Γ lexicographically, as follows. Let $\gamma_1 \neq \gamma_2 \in \Gamma$. Then $\gamma_1 < \gamma_2$ if and only if $\gamma_1(i) < \gamma_2(i)$ for the least element $i \in I$ such that $\gamma_1(i) \neq \gamma_2(i)$. Then Γ is a linearly ordered abelian group. Recall that each non-empty set can be well-ordered. (See [La, Appendix 2, Theorem 4.1].)

We choose Γ as above because $G_{F_0((\Gamma))}$ will be pro-p (see Lemma 3 below) and because we may control the pth-power classes with the following lemma. This well-known lemma follows from Lemma 1 and the fact that the valued field F_1 is henselian. It is also an immediate consequence of [W, Lemma 1.4]. Therefore we shall omit its proof.

Lemma 2. *Let $F_1 = F_0((\Gamma))$ as above. Then*

$$\begin{aligned}\dim_{\mathbb{F}_p} F_1^\times/F_1^{\times p} &= \dim_{\mathbb{F}_p} F_0^\times/F_0^{\times p} + \dim_{\mathbb{F}_p} \Gamma/p\Gamma \\ &= \dim_{\mathbb{F}_p} F_0^\times/F_0^{\times p} + |I|.\end{aligned}$$

Finally, we record a criterion for G_{F_1} being pro-p:

Lemma 3. *Let* $F_1 = F_0((\Gamma))$ *as above with* $\mathrm{char}(F_0) = 0$ *and* G_{F_0} *pro-p. Then* G_{F_1} *is pro-p as well.*

Proof. From basic valuation theory, nicely summarized in [K, pages 3 and 4], and the fact that F_1 above is henselian,

$$G_{F_1} \cong T \rtimes G_{F_0},$$

where the action of G_{F_0} on T is uniquely determined by the cyclotomic character mapping G_{F_0} into a group of automorphisms of a group of roots of unity contained in F_0^s, and $T \cong \mathbb{Z}_p^I$, the topological product of $|I|$ copies of $\mathbb{Z}_p$. In particular, if G_{F_0} is a pro-p-group, so is G_{F_1}. □

2.2. $H^2(G_{F_1}, \mathbb{F}_p)$ for henselian valued fields F_1.

Now we study $H^2(G_{F_1}, \mathbb{F}_p)$ for our henselian valued fields F_1. The next lemma, taken from [W], will be used in the proof of Theorem 1 to show that norm groups of cyclic p-extensions of F_1 are not too large.

Suppose that F_1 is a field endowed with a henselian valuation v with valuation group $\Gamma = v(F_1^\times)$. Let F_1^{nr} denote the maximal unramified extension of F_1 in its separable closure F_1^s. Then $G_{F_0} \cong G_{F_1}/\operatorname{Gal}(F_1^s/F_1^{nr})$. Therefore, after identifying these groups, we have the inflation map

$$\inf = \inf{}_{F_0}^{F_1} : H^*(G_{F_0}, \mathbb{F}_p) \longrightarrow H^*(G_{F_1}, \mathbb{F}_p).$$

(See [W, page 483].)

Moreover, from basic Kummer theory we have the canonical isomorphism

$$\varphi_F : F^\times/F^{\times p} \longrightarrow H^1(G_F, \mathbb{F}_p),$$

as well as the corresponding canonical isomorphisms φ_{F_i}, $i = 1, 2$. We will denote by $(f)_F$ or $(f_i)_{F_i}$ the images $\varphi_F([f])$ or $\varphi_{F_i}([f_i])$. If the context is clear we will omit the subscript.

Assume next that $\{\pi_j,\ j \in \mathcal{J}\}$ is a set of elements of $F_1^\times$ such that their images in $\Gamma/p\Gamma$ form a basis of $\Gamma/p\Gamma$ over $\mathbb{F}_p$. Then we have the following lemma, obtained as a special case of a theorem of Wadsworth.

Lemma 4. [W, Theorem 3.6, page 483]. *Let* F_1 *and* F_0 *be as above. Then*

$$H^2(G_{F_1}, \mathbb{F}_p) = \inf(H^2(G_{F_0}, \mathbb{F}_p)) \oplus_{j \in \mathcal{J}} \left(\inf(H^1(G_{F_0}, \mathbb{F}_p)) \cup (\pi_j)\right) \oplus_{\{j_1, j_2\} \subset \mathcal{J},\ j_1 \neq j_2} \left((\pi_{j_1}) \cup (\pi_{j_2})\right).$$

Moreover, for each $j \in \mathcal{J}$,

$$\inf(H^1(G_{F_0}, \mathbb{F}_p)) \cong \inf(H^1(G_{F_0}, \mathbb{F}_p)) \cup (\pi_j)$$

and for each $j_1, j_2 \in \mathcal{J}$ such that $j_1 \neq j_2$, we have $(\pi_{j_1}) \cup (\pi_{j_2}) \neq 0$.

Note that in the last summand of Lemma 4 the sum ranges over subsets $\{j_1, j_2\}, j_1 \neq j_2$ of $\mathcal{J}$ and a choice between $(\pi_{j_1}) \cup (\pi_{j_2})$ and $(\pi_{j_2}) \cup (\pi_{j_1})$ is arbitrary but fixed.

2.3. **Residue fields F_0 with prescribed absolute Galois group.**

In our construction of F we choose a residue field F_0 depending on Υ and p. If $\Upsilon = 1$ we will simply put $F_0 = \mathbb{C}$, but when $\Upsilon = 0$ we require some special properties of F_0. In particular, in order that our cyclic extension $K = F(\sqrt[p]{a})$ have the desired invariant $\Upsilon = 0$, we require that K does not embed in a cyclic Galois extension L over F with degree $[L : F] = p^2$, for this is equivalent to $\xi_p \notin N_{K/F}(K^\times)$ by [A, Theorem 3].

To ensure that this nonembeddability condition holds, as well as to ensure that a certain nonabelian group of order p^3 does not occur as a Galois group over the field, we choose residue fields F_0 with absolute Galois groups taking a special form, and it is also convenient to require that $|F_0^\times / F_0^{\times p}|$ is small. As it turns out, we may choose some suitable algebraic infinite extension of $\mathbb{Q}$. Finitely generated pro-p-absolute Galois groups over $\mathbb{Q}$ and more generally any global field, were nicely classified in [E2]. (See also [E1] and [JP] for related results and techniques.)

The extensions we will need for $p > 2$ are given in the following

Lemma 5. [E2, page 84] *For each prime $p > 2$ there exists an algebraic extension $F_{0,p}$ of $\mathbb{Q}$ such that*

$$G_{F_{0,p}} = \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^{p+1} \rangle_{\text{pro-}p}$$

where the presentation is in the category of pro-p-groups.

Observe that the maximal abelian extension $F_{0,p}^{ab}$ of $F_{0,p}$ has $G_{F_{0,p}}^{ab} := \mathrm{Gal}(F_{0,p}^{ab}/F_{0,p})$ equal to

$$G_{F_{0,p}}^{ab} \cong \langle \bar{\sigma}, \bar{\tau} \mid \bar{\tau}^p = 1 \rangle = \mathbb{Z}_p \times \mathbb{Z}/p\mathbb{Z}$$

for all primes $p > 2$.

2.4. **Field arithmetic and free pro-p products.**

In this section we collect lemmas giving information about a field F derived from the structure of G_F, especially when G_F is a free pro-p product of two pro-p groups G_{F_1} and G_{F_2}.

First we record a lemma detecting the presence of primitive pth roots of unity in a field F, based only on the structure of G_F.

Lemma 6. *Suppose that $p > 2$ and that F is a field with $\mathrm{char}(F) \neq p$ and G_F pro-p. Then $\xi_p \in F^\times$.*

Proof. Because $\mathrm{char}(F) \neq p$, there exists a primitive pth root ξ_p of unity in F^s. If $\xi_p \in F^s \setminus F$ then $F(\xi_p)/F$ is a nontrivial Galois extension of degree $[F(\xi_p) : F] < p$. Therefore G_F has a nontrivial finite quotient of order coprime with p. This contradicts our assumption that G_F is a pro-p-group. Hence $\xi_p \in F^\times$ as asserted. □

Now suppose that $G_F = G_{F_1} \star G_{F_2}$ for pro-p absolute Galois groups G_F, G_{F_1}, and G_{F_2}, where the free product is taken in the category of pro-p-groups. From [N, (4.3) Satz] we see that the restriction homomorphism

$$\mathrm{res} : H^1(G_F, \mathbb{F}_p) \longrightarrow H^1(G_{F_1}, \mathbb{F}_p) \oplus H^1(G_{F_2}, \mathbb{F}_p) \qquad (1)$$

is an isomorphism. Now given $(f)_F$ in $H^1(G_F, \mathbb{F}_p)$, we denote the image $\mathrm{res}(f)_F$ by

$$\mathrm{res}(f)_F = (f)_{G_{F_1}} \oplus (f)_{G_{F_2}}.$$

This notation distinguishes, then, between $(f)_{F_1}$, which denotes $\varphi_{F_1}(f)$ for $f \in F_1^\times$, and $(f)_{G_{F_1}}$, which denotes the projection of $\mathrm{res}\,\varphi_F(f)$ onto the first summand.

One way of interpreting this restriction map is with the following

Lemma 7. *Let $G_F = G_{F_1} \star G_{F_2}$ be pro-p absolute Galois groups of fields containing a pth root of unity, and suppose that we have the following sequence:*

$$G_F \xrightarrow{\mathrm{can}} G_{F_1} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

Here the canonical map can *is an identity on G_{F_1} and contains G_{F_2} in its kernel.*

Then the right-hand surjection and the composed surjection correspond to fields $K_1 = F_1(\sqrt[p]{a_1})$ and $K = F(\sqrt[p]{a})$, respectively, where $(a)_{G_{F_1}} = (a_1)_{F_1}$ and $(a)_{G_{F_2}} = 0$.

Proof. The surjections are continuous homomorphisms, hence elements of $H^1(G_{F_1}, \mathbb{F}_p)$ and $H^1(G_F, \mathbb{F}_p)$, respectively, and the right-hand surjection is clearly the restriction of the composed surjection. The remainder follows by Kummer theory. □

Lemma 8. *Let $G_F = G_{F_1} \star G_{F_2}$ be pro-p absolute Galois groups and suppose that $(a)_F$ and $(b)_F$ satisfy $(a)_{G_{F_1}} = (b)_{G_{F_2}} = 0$. Then*

$$(a)_F \cup (b)_F = 0 \in H^2(G_F, \mathbb{F}_p).$$

The lemma follows from [N, (4.1) Satz] and from [Ris, Prop. 7.3, page 191]. However, we prove our lemma by translating the cup products into obstructions to basic Galois embedding problems, yielding an interesting Galois-theoretic variant of the proof.

Proof. If $(a) = 0$ or $(b) = 0$, we are done. Otherwise, the conditions $(a)_{G_{F_1}} = (b)_{G_{F_2}} = 0$ imply that (a) and (b) are linearly independent in $H^1(G_F, \mathbb{F}_p)$.

Now let H_{p^3} be the Heisenberg group of order p^3:

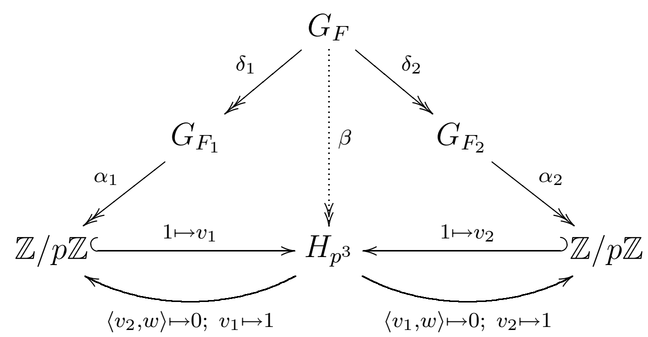
$$H_{p^3} = \langle v_1, v_2, w \mid v_1^p = v_2^p = w^p = 1, v_2 v_1 = w v_1 v_2, \\ [v_1, w] = [v_2, w] = 1 \rangle$$

In the case $p = 2$, H_8 is the familiar dihedral group D_4.

By [M, Corollary, page 523 and Theorem 3(A)], if (a) and (b) are linearly independent, then $(a) \cup (b) = 0$ if and only if H_{p^3} is the Galois group $\mathrm{Gal}(M/F)$ of a Galois extension M of F containing $F(\sqrt[p]{a}, \sqrt[p]{b})$ in such a way that

$$H_{p^3}/\langle v_1, w\rangle = \mathrm{Gal}(F(\sqrt[p]{a})/F) \text{ and } H_{p^3}/\langle v_2, w\rangle = \mathrm{Gal}(F(\sqrt[p]{b})/F).$$

Now consider the commutative diagram



Let $a_2 \in F_2^\times$ and $b_1 \in F_1^\times$ satisfy $(a_2)_{F_2} = (a)_{G_{F_2}}$ and $(b_1)_{F_1} = (b)_{G_{F_1}}$. Then set $K_1 = F_1(\sqrt[p]{b_1})$ and $K_2 = F_2(\sqrt[p]{a_2})$; these are $\mathbb{Z}/p\mathbb{Z}$-extensions of F_1 and F_2, respectively. We may then identify the left-hand $\mathbb{Z}/p\mathbb{Z}$ in the diagram with $\mathrm{Gal}(K_1/F_1)$ so that α_1 is the surjection of Galois theory. Similarly, the right-hand $\mathbb{Z}/p\mathbb{Z}$ may be identified with $\mathrm{Gal}(K_2/F_2)$ with α_2 the surjection of Galois theory. Finally, the topmost surjections δ_1 and δ_2 are canonical.

By Lemma 7, the surjections $\delta_1\alpha_1$ and $\delta_2\alpha_2$ correspond to fields $F(\sqrt[p]{b})$ and $F(\sqrt[p]{a})$, respectively. Now because $G_F = G_{F_1} \star G_{F_2}$, there exists a homomorphism $\beta : G_F \to H_{p^3}$, and because v_1 and v_2 generate H_{p^3}, β is a surjection.

Hence H_{p^3} is a Galois group over F corresponding to a normal subgroup H of G_F. Consider the smallest normal subgroup H_1 of G_F containing H and G_{F_2}. Then by the diagram, G_F/H_1 is the left-hand $\mathbb{Z}/p\mathbb{Z}$, which corresponds to $F(\sqrt[p]{b})$, and H_1/H is $\langle v_2, w\rangle$. Now consider the smallest normal subgroup H_2 of G_F containing H and G_{F_1}. Then by the diagram, G_F/H_2 is the right-hand $\mathbb{Z}/p\mathbb{Z}$, which corresponds to $F(\sqrt[p]{a})$, and H_2/H is $\langle v_1, w\rangle$.

Hence $(a) \cup (b) = 0$. □

Finally, we close with with a companion to Lemma 7. In Lemma 9 below, π denotes the canonical homomorphism of G to G_1 which is an identity on G_1, and is trivial on G_2.

Lemma 9. *Let $G = G_1 \star G_2$ be a free product of G_1 and G_2 in the category of pro-p-groups. Suppose that $A \cong \mathbb{Z}/p\mathbb{Z}$ is a factor group of G_1 such that the surjection $G_1 \to \mathbb{Z}/p\mathbb{Z}$ does not factor through $\mathbb{Z}/p^2\mathbb{Z}$. Then the following commutative diagram cannot occur:*

$$\begin{array}{ccc} G & \xrightarrow{\ \pi\ } & G_1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} & \longrightarrow & A. \end{array}$$

Proof. Suppose that contrary to our statement, such a diagram as the above exists. Then by passing to quotients by commutator subgroups we obtain

$$\begin{array}{ccc} G^{ab} & \xrightarrow{\pi^{ab}} & G_1^{ab} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\beta} & A. \end{array}$$

But $G^{ab} \cong G_1^{ab} \times G_2^{ab}$ and the canonical surjection onto G_1^{ab} is given by the projection map. Let δ be a splitting map of the projection map. Then $\gamma = \beta\alpha\delta$, contradicting the hypothesis. □

3. Proof of the Theorem

First we define fields F_0, F_1, F_2, and F using our given cardinal numbers d, e, and Υ, as well as the prime number p. Then we define the cyclic Galois extension K/F of degree p and check that the arithmetic invariants of K/F coincide with d, e, and Υ.

3.1. Constructing F_0, F_1, F_2, and F.

If $\Upsilon = 1$ then let $F_0 = \mathbb{C}$. If $\Upsilon = 0$ and $p = 2$, let $F_0 = \mathbb{R}$. (See Proposition 1 for alternative choices in these two cases.) If $\Upsilon = 0$ and $p > 2$ then let $F_{0,p}$ be the algebraic extension of $\mathbb{Q}$ of Lemma 5. In the first two cases we see trivially that $\xi_p \in F_0$, and in the last case $\xi_p \in F_0$ by Lemma 6. Observe that

$$\dim_{\mathbb{F}_p} F_0^\times / F_0^{\times p} = \begin{cases} 0, \text{ if } \Upsilon = 1; \\ 1, \text{ if } \Upsilon = 0, \ p = 2; \\ 2, \text{ if } \Upsilon = 0, \ p > 2. \end{cases} \tag{2}$$

We next construct the field F_1. Because $\Upsilon = 0$ implies $1 \le d$, for either choice of $\Upsilon \in \{0, 1\}$ there exists a well-ordered set I_1 such that $|I_1| + 1 = d + 2\Upsilon$. Let $\Gamma_1 = \mathbb{Z}_{(p)}^{(I_1)}$ be a direct sum of $|I_1|$ copies of $\mathbb{Z}_{(p)}$. Then Γ_1 is a linearly ordered abelian group. Finally set $F_1 := F_0((\Gamma_1))$. From Lemmas 2 and 3 it follows that G_{F_1} is a pro-p-group and

$$\dim_{\mathbb{F}_p} F_1^\times / F_1^{\times p} = \dim_{\mathbb{F}_p} F_0^\times / F_0^{\times p} + |I_1|.$$

Hence

$$\dim_{\mathbb{F}_p} F_1^\times / F_1^{\times p} = \begin{cases} d + 1, & \text{if } \Upsilon = 1; \\ d, & \text{if } \Upsilon = 0, \ p = 2; \\ d + 1, & \text{if } \Upsilon = 0, \ p > 2. \end{cases} \tag{3}$$

Similarly, we construct F_2 as follows. Because $p > 2$ and also $p = 2$ and $\Upsilon = 1$ implies $e > 0$, there exists a well-ordered set I_2 such that $1 + |I_2| = e$ in either of the cases $p > 2$ or $\Upsilon = 1$, $p = 2$, and $|I_2| = e$ in the case $\Upsilon = 0$, $p = 2$. Then again $\Gamma_2 = \mathbb{Z}_{(p)}^{(I_2)}$ is a linearly ordered abelian group. We set $F_2 := \mathbb{C}((\Gamma_2))$. Then from [K, pages 3 and 4] it follows that $G_{F_2} \cong \mathbb{Z}_p^{I_2}$, the topological product of $|I_2|$ copies of $\mathbb{Z}_p$. In particular

$$e = \begin{cases} \dim_{\mathbb{F}_p} F_2^\times / F_2^{\times p} + 1, & \text{if } p > 2 \text{ or } p = 2 \text{ and } \Upsilon = 1; \\ \dim_{\mathbb{F}_p} F_2^\times / F_2^{\times p}, & \text{if } p = 2,\ \Upsilon = 0. \end{cases} \tag{4}$$

From Theorem 2 we see that there exists a field F of characteristic zero, such that $G_F = G_{F_1} \star G_{F_2}$ is a free product of G_{F_1} and G_{F_2} in the category of pro-p-groups. In particular G_F is again a pro-p-group, and from Lemma 6 we see that F contains a primitive pth root of unity.

3.2. **Constructing K/F.**

We define the cyclic extension K/F of degree p as $K = F(\sqrt[p]{a})$ where $[a] \in F^\times / F^{\times p}$ is chosen via the isomorphism (1).

If $\Upsilon = 1$ then let $(a_1)_{F_1}$ be any nontrivial element in $H^1(G_{F_1}, \mathbb{F}_p)$, which is possible since $1 \leq \dim_{\mathbb{F}_p} H^1(G_{F_1}, \mathbb{F}_p) = d + 1$. Let $(a_2)_{F_2} = 0$ in $H^2(G_{F_2}, \mathbb{F}_p)$.

Now suppose $\Upsilon = 0$ and $p > 2$. We denote the fixed field of the factor $\mathbb{Z}_p$ in $G_{F_0}^{ab} \cong \mathbb{Z}_p \times \mathbb{Z}/p\mathbb{Z}$ acting on a maximal abelian extension F_0^{ab} of F_0 as $K_0 := F_0(\sqrt[p]{b})$. (See the discussion following Lemma 5.) Because $F_0^\times / F_0^{\times p}$ is naturally isomorphic to a subgroup of $F_1^\times / F_1^{\times p}$, we may set $(a_1)_{F_1} = \inf(b)_{F_0} \neq 0$ in $H^1(G_{F_1}, \mathbb{F}_p)$ and $(a_2)_{F_2} = 0$ in $H^2(G_{F_2}, \mathbb{F}_p)$.

Finally assume that $\Upsilon = 0$ and $p = 2$. We set $(a_1)_{F_1} = (-1)_{F_1} \neq 0$ in $H^1(G_{F_1}, \mathbb{F}_2)$ and $(a_2)_{F_2} = 0$ in $H^2(G_{F_2}, \mathbb{F}_2)$.

Now by (1) there exists $a \in F^\times$ such that $(a)_F \neq 0$ and $(a)_{G_{F_i}} = (a_i)_{F_i}$, $i = 1, 2$. Since we have already determined that $\xi_p \in F^\times$, we see that $K = F(\sqrt[p]{a})$ is a cyclic extension of degree p. Hence it remains to show that the arithmetic invariants $d(K/F)$, $e(K/F)$, and $\Upsilon(K/F)$ coincide with the prescribed cardinal numbers d, e, and Υ respectively.

3.3. **Determining $d(K/F)$ and $e(K/F)$ via annihilators.**

We first observe some relationships among cup products of elements in $H^1(G_F, \mathbb{F}_p)$. Recall that Lemma 8 tells us that if $c_1, c_2 \in F^\times$ with $(c_1)_{G_{F_1}} = (c_2)_{G_{F_2}} = 0$ then

$$(c_1)_F \cup (c_2)_F = 0 \in H^2(G_F, \mathbb{F}_p). \tag{5}$$

Moreover, recall that by [N, Satz 4.1]

$$H^2(G_F, \mathbb{F}_p) \cong H^2(G_{F_1}, \mathbb{F}_p) \times H^2(G_{F_2}, \mathbb{F}_p),$$

where the isomorphism is induced by the restriction maps

$$\mathrm{res}_i : H^2(G_F, \mathbb{F}_p) \to H^2(G_{F_i}, \mathbb{F}_p), i = 1, 2.$$

Because these restriction maps commute with cup product maps ([Ris, Proposition 7.3, page 191]), we see that if $c_1, c_2 \in F^\times$ such that $(c_1)_{G_{F_2}} = (c_2)_{G_{F_2}} = 0$ in $H^2(G_{F_2}, \mathbb{F}_p)$ then

$$(c_1)_F \cup (c_2)_F \neq 0 \quad \text{iff} \quad (c_1)_{G_{F_1}} \cup (c_2)_{G_{F_1}} \neq 0, \tag{6}$$

where the first cup product lies in $H^1(G_F, \mathbb{F}_p)$ and the second cup product lies in $H^1(G_{F_1}, \mathbb{F}_p)$. By symmetry the analogous statement with F_1 and F_2 exchanged holds as well.

Now we calculate $e(K/F)$. We adopt the following notation for an annihilator of $\phi \in H^1(G_F, \mathbb{F}_p)$:

$$\mathrm{ann}_{F_1} \phi := \{\eta \in H^1(G_F, \mathbb{F}_p) \mid \eta \cup \phi = 0\}.$$

For $x \in F^\times$ we have $x \in N_{K/F}(K^\times)$ if and only if $(x)_F \cup (a)_F = 0$ in $H^2(G_F, \mathbb{F}_p)$. Therefore, using (5) and (6) with the fact that in every case $(a)_{G_{F_2}} = 0$,

$$\begin{aligned} e(K/F) &= \dim_{\mathbb{F}_p} N(K^\times)/F^{\times p} \\ &= \dim_{\mathbb{F}_p} \{(x)_F \in H^1(G_F, \mathbb{F}_p) \mid (a)_F \cup (x)_F = 0\} \\ &= \dim_{\mathbb{F}_p} \{(y)_{F_1} \in H^1(G_{F_1}, \mathbb{F}_p) \mid (a)_{G_{F_1}} \cup (y)_{F_1} = 0\} \\ &\quad + \dim_{\mathbb{F}_p} H^1(G_{F_2}, \mathbb{F}_p) \\ &= \dim_{\mathbb{F}_p} \mathrm{ann}_{F_1} (a)_{G_{F_1}} + \dim_{\mathbb{F}_p} F_2^\times / F_2^{\times p}. \end{aligned}$$

If $\Upsilon = 1$ then $H^1(G_{F_0}, \mathbb{F}_p) = 0$, and so by Lemma 4 we have $\dim_{\mathbb{F}_p} \mathrm{ann}_{F_1} (a)_{G_{F_1}} = 1$.

If $p > 2$ and $\Upsilon = 0$ then by Lemma 4,

$$\dim_{\mathbb{F}_p} \mathrm{ann}_{F_1} (a)_{G_{F_1}} = \dim_{\mathbb{F}_p} \mathrm{ann}_{F_0} (b)_{F_0},$$

where b was chosen so that $K_0 = F_0(\sqrt[p]{b})$ was the fixed field of the factor $\mathbb{Z}_p$ in $G_{F_0}^{ab}$. Now because $(b)_{F_0} \cup (b)_{F_0} = 0$ as an identity in $H^2(G_F, \mathbb{F}_p)$ for $p > 2$,

$$1 \leq \dim_{\mathbb{F}_p} \operatorname{ann}_{F_0}(b)_{F_0} \leq 2.$$

On the other hand, let $c \in F^\times \setminus F^{\times p}$ be defined so that $F_0(\sqrt[p]{c})$ is contained in the fixed field of the factor $\mathbb{Z}/p\mathbb{Z}$ in $G_{F_0}^{ab}$. Then by (2), $\{(b)_{F_0}, (c)_{F_0}\}$ spans $H^1(G_{F_0}, \mathbb{F}_p)$. Furthermore, since by Lemma 5, H_{p^3} is not a quotient of $G_{F_0,p}$, $(b)_{F_0} \cup (c)_{F_0} \neq 0$ by [M, Corollary, page 523 and Theorem 3(A)]. We conclude that $\dim_{\mathbb{F}_p} \operatorname{ann}_{F_0}(b)_{F_0} = 1$.

Finally, if $p = 2$ and $\Upsilon = 0$ then again by Lemma 4

$$\dim_{\mathbb{F}_2} \operatorname{ann}_{F_1}(a)_{G_{F_1}} = \dim_{\mathbb{F}_2} \operatorname{ann}_{F_0}(-1)_{F_0}.$$

As $F_0 = \mathbb{R}$, $\dim_{\mathbb{F}_2} F_0^\times / F_0^{\times 2} = 1$ and $(-1)_{\mathbb{R}} \cup (-1)_{\mathbb{R}} \neq 0$, yielding $\dim_{\mathbb{F}_2} \operatorname{ann}_{F_1}(a)_{G_{F_1}} = 0$.

Combining our results with (4), we have that $e(K/F) = e$.

Now we turn to a similar calculation of $d(K/F)$. Again using (5) and (6) with the fact that in every case $(a)_{G_{F_2}} = 0$,

$$\begin{aligned} d(K/F) &= \dim_{\mathbb{F}_p}(H^1(G_F, \mathbb{F}_p)/\operatorname{ann}_F(a)_F) \\ &= \dim_{\mathbb{F}_p}(H^1(G_{F_1}, \mathbb{F}_p)/\operatorname{ann}_{F_1}(a)_{G_{F_1}}) \\ &= d. \end{aligned}$$

For this last equality we use (3) together with calculations of the dimension of $\operatorname{ann}_{F_1}(a)_{G_{F_1}}$ already achieved. Hence $d(K/F) = d$ in all cases.

3.4. **Determining Υ via quotients of G_F.**

It remains to show that $\Upsilon(K/F) = \Upsilon$. First consider the case $\Upsilon = 1$. Since $F_0 = \mathbb{C}$, ξ_p is a pth power in F_1, and $F_1(\sqrt[p]{a})$ embeds in a $\mathbb{Z}/p^2\mathbb{Z}$-extension $F_1(\sqrt[p^2]{a})$ of F_1. Then the surjection $G_{F_1} \to \operatorname{Gal}(K_1/F_1)$ factors through $\mathbb{Z}/p^2\mathbb{Z}$. Following the surjection with the canonical surjection $G_F \to G_{F_1}$, we see that $\mathbb{Z}/p^2\mathbb{Z}$ is a factor group of G_F. Moreover, by Lemma 7, the surjection $G_F \to \mathbb{Z}/p\mathbb{Z}$ corresponds to K. Hence K/F embeds in a $\mathbb{Z}/p^2\mathbb{Z}$-extension of F. By [A, Theorem 3], $\xi_p \in N_{K/F}(K^\times)$. Therefore $\Upsilon(K/F) = 1 = \Upsilon$.

Now consider the case $\Upsilon = 0$ and $p > 2$. Because $K_0 = F_0(\sqrt[p]{b})$ does not embed in a $\mathbb{Z}/p^2\mathbb{Z}$-extension of F_0, $\xi_p \notin N_{K_0/F_0}(K_0^\times)$. (See 3.2 for the definition of K_0.) Hence $(b)_{F_0} \cup (\xi_p)_{F_0} \neq 0$ in $H^1(G_{F_0}, \mathbb{F}_p)$, and, by Lemma 4, $(\inf(b)) \cup (\xi_p)_{F_1} \neq 0$ in $H^1(G_{F_1}, \mathbb{F}_p)$ as well. Choose $b_1 \in F_1^\times$ so that $(b_1)_{F_1} = \inf(b)_{F_0}$ and set $K_1 = F_1(\sqrt[p]{b_1})$. Then $\xi_p \notin N_{K_1/F_1}(K_1^\times)$ and by [A, Theorem 3] the field extension K_1/F_1 does not embed in a $\mathbb{Z}/p^2\mathbb{Z}$-extension of F_1. Therefore the surjection $G_{F_1} \to \mathrm{Gal}(F_1(\sqrt[p]{b})/F_1)$ does not factor through $\mathbb{Z}/p^2\mathbb{Z}$. From Lemmas 7 and 9 we see that surjection $G_F \to \mathrm{Gal}(K/F)$ does not factor through $\mathbb{Z}/p^2\mathbb{Z}$. Again by [A, Theorem 3] we conclude that $\xi_p \notin N_{K/F}(K^\times)$ and therefore $\Upsilon(K/F) = 0 = \Upsilon$.

Finally consider the case $\Upsilon = 0$ and $p = 2$. Because $-1 \notin N_{\mathbb{C}/\mathbb{R}}(\mathbb{C})$ from Lemma 4 we conclude that $-1 \notin N_{K_1/F_1}(K_1^\times)$. By [A, Theorem 3] the surjection $G_{F_1} \to \mathrm{Gal}(K_1/F_1)$ does not factor through $\mathbb{Z}/4\mathbb{Z}$. As in the previous case, by Lemmas 7 and 9 we see that K/F does not embed in a $\mathbb{Z}/4\mathbb{Z}$-extension of F. Again by [A, Theorem 3] we see that $-1 \notin N_{K/F}(K^\times)$ and therefore $\Upsilon(K/F) = 0 = \Upsilon$.

Hence we have checked in all cases that $\Upsilon(K/F) = \Upsilon$, and our proof of Theorem 1 is now complete. □

Remark. In [EH, Lemma 1.2 and Proposition 1.3] Efrat and Haran construct some fields with prescribed absolute Galois groups together with some bounds on the transcendence degrees of these fields. These bounds, together with the replacement of $\mathbb{C}$ by an algebraic closure of $\mathbb{Q}$ and of $\mathbb{R}$ by a real-closed algebraic number field R in the case when $p = 2$ and $\Upsilon = 0$ in our proof above, yield the following proposition.

Proposition 1. *Suppose that p is a prime number, and let d, e, and $\Upsilon \in \{0, 1\}$ be cardinal numbers such that if $\Upsilon = 0$ then $1 \leq d$, if $p > 2$ then $1 \leq e$, and if $p = 2$ and $\Upsilon = 1$ then $1 \leq e$. Then there exists a field F containing $\mathbb{Q}(\xi_p)$ and a cyclic Galois extension K of degree p over F such that*

$$e(K/F) = e, \quad d(K/F) = d, \quad \text{and} \quad \Upsilon(K/F) = \Upsilon.$$

Moreover

$$\mathrm{tr.deg}(F/\mathbb{Q}) \leq 1 + \max\{e, d + 1\}.$$

In particular if $d, e \in \mathbb{N} \cup \{0\}$, there exists a Galois cyclic extension K/F of degree p with prescribed invariants d, e, and Υ of finite transcendence degree over its prime field $\mathbb{Q}$.

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