KOSZUL COMPLEXES AND SYMMETRIC FORMS OVER THE PUNCTURED AFFINE SPACE

PAUL BALMER AND STEFAN GILLE

Abstract. Let X be a scheme which is not of equicharacteristic 2 and let $\mathbb{U}_X^n \subset \mathbb{A}_X^n$ be the punctured affine n-space over X. If $n \equiv \pm 1$ modulo 4, we show that there exists a ± 1-symmetric bilinear space $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$ over $\mathbb{U}_X^n$ which can not be extended to the whole affine space $\mathbb{A}_X^n$ and which is locally metabolic for $n \geq 2$. If X is regular, contains $\frac{1}{2}$ and is of finite Krull dimension, we show that the total Witt ring $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$ of $\mathbb{U}_X^n$ is a free $\mathrm{W}^{\mathrm{tot}}(X)$-module with two generators: the Witt classes of $< 1 >$ and of the above $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$. We describe $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$ similarly when n is even.

Introduction

Let X be scheme. We are studying the *(total) graded Witt ring*

$$\mathrm{W}^{\mathrm{tot}}(X) := \bigoplus_{i \in \mathbb{Z}} \mathrm{W}^i(X)$$

where the groups W^i are the derived Witt groups of Balmer [2, 3] and where the multiplicative structure is the one of Gille-Nenashev [10]. See more in Section 2.

We fix an integer $n \geq 1$ for the entire article. Consider the following open subset $\mathbb{U}_{\mathbb{Z}}^n \subset \mathbb{A}_{\mathbb{Z}}^n$ of the affine space $\mathbb{A}_{\mathbb{Z}}^n = \mathrm{Spec}(\mathbb{Z}[T_1, \ldots, T_n])$:

$$\mathbb{U}_{\mathbb{Z}}^n := \bigcup_{j=1}^{n} \mathrm{Spec}\left(\mathbb{Z}\big[T_1, \ldots, T_n, T_j^{-1}\big]\right) \subset \mathbb{A}_{\mathbb{Z}}^n .$$

For any scheme X, define by base-change the open subscheme $\mathbb{U}_X^n \subset \mathbb{A}_X^n$, called *the punctured affine space over* X, *i.e.* define $\mathbb{U}_X^n$ by the following pull-back square:

$$\begin{tikzcd} \mathbb{U}_X^n \arrow[r, "\sigma_X"] \arrow[d, "\upsilon_X"'] & X \arrow[d] \\ \mathbb{U}_{\mathbb{Z}}^n \arrow[r] & \mathrm{Spec}(\mathbb{Z}) . \end{tikzcd} \qquad (1)$$

Our main result is Theorem 7.14 below, which says in particular:

Theorem. *If the scheme X is regular, contains $\frac{1}{2}$ and has finite Krull dimension, there is a decomposition* $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n) = \mathrm{W}^{\mathrm{tot}}(X) \oplus \mathrm{W}^{\mathrm{tot}}(X) \cdot \varepsilon$ *for some Witt class* $\varepsilon = \varepsilon_X^{(n)}$ *in* $\mathrm{W}^{n-1}(\mathbb{U}_X^n)$. *If n=1, we have* $\varepsilon^2 = 1$. *If* $n \geq 2$, *we have* $\varepsilon^2 = 0$ *and an isomorphism*

$$\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n) \cong \frac{\mathrm{W}^{\mathrm{tot}}(X)\,[\varepsilon]}{\varepsilon^2}$$

of graded rings, with the generator ε in degree $n-1$.

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Our second goal is a "classical" description of the generator $\varepsilon_X^{(n)} \in \mathrm{W}^{n-1}(\mathbb{U}_X^n)$.

Recall a few facts. First, the derived Witt groups are 4-periodic: $\mathrm{W}^i = \mathrm{W}^{i+4}$. Secondly, W^0 and W^2 are naturally isomorphic to the usual Witt groups $\mathrm{W}^+_{\mathrm{us}}$ and $\mathrm{W}^-_{\mathrm{us}}$ of symmetric and skew-symmetric vector bundles respectively, as defined by Knebusch [12]. Thirdly, W^1 and $\mathrm{W}^3 = \mathrm{W}^{-1}$ are groups of formations, see Walter [15]. Therefore, if we want to describe in classical terms our generator $\varepsilon_X^{(n)}$ in W^{n-1}, we are bound to produce an explicit element of the above nature, *i.e.* a ± 1-symmetric form or formation, depending on the congruence of n modulo 4.

In this introduction, we focus on the case where n is odd and we write $n-1 = 2\ell$. In this case, we have to describe a $(-1)^\ell$-symmetric bundle

$$\big(\,\mathcal{E}_X^{(n)}\,,\,\varphi_X^{(n)}\,\big)$$

over $\mathbb{U}_X^n$ whose class in $\mathrm{W}^\pm_{\mathrm{us}}$ will be our generator $\varepsilon_X^{(n)}$. Let us stress that this will be a (skew-) symmetric space of classical nature, which does not involve triangulated categories. By the very naturality of the original problem, it suffices to construct this (skew-) symmetric bundle when $X = \mathrm{Spec}(\mathbb{Z})$ and then to pull it back over an arbitrary scheme X. Therefore, we start with a description of $(\mathcal{E}_{\mathbb{Z}}^{(n)}, \varphi_{\mathbb{Z}}^{(n)})$.

Let us denote by $A := \mathbb{Z}[T_1, \ldots, T_n]$ the polynomial ring in n variables and by

$$K_\bullet = K_\bullet(A, \underline{T})$$

the (homological) Koszul complex over A for the A-sequence $\underline{T} := (T_1, \ldots, T_n)$. There is a well-known isomorphism of complexes $\Theta_\bullet : K_\bullet \xrightarrow{\simeq} \mathrm{Hom}_A(K_\bullet, A)[n]$, see [7, Chap. 1.6] for instance. Since $K_\bullet|_{\mathbb{U}_{\mathbb{Z}}^n}$ is locally split, the $\mathcal{O}_{\mathbb{U}_{\mathbb{Z}}^n}$-module

$$\mathcal{E}_{\mathbb{Z}}^{(n)} \;:=\; \mathrm{Coker}\big(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1}\big)|_{\mathbb{U}_{\mathbb{Z}}^n} \;\simeq\; \mathrm{Ker}\, d_\ell|_{\mathbb{U}_{\mathbb{Z}}^n}$$

is locally free. From $n = 2\ell+1$, one easily sees that $\Theta_\ell \circ d_{\ell+1}$ induces an isomorphism

$$\varphi_{\mathbb{Z}}^{(n)} : \mathcal{E}_{\mathbb{Z}}^{(n)} \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_{\mathbb{U}_{\mathbb{Z}}^n}}(\mathcal{E}_{\mathbb{Z}}^{(n)}, \mathcal{O}_{\mathbb{U}_{\mathbb{Z}}^n})$$

which is $(-1)^\ell$-symmetric. For a scheme X, with the base-change morphism $\upsilon_X : \mathbb{U}_X^n \longrightarrow \mathbb{U}_{\mathbb{Z}}^n$ as in diagram (1), we define $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)}) := \upsilon_X^*(\mathcal{E}_{\mathbb{Z}}^{(n)}, \varphi_{\mathbb{Z}}^{(n)})$.

If n is even, we can construct similarly a complex of length 1, with a suitable (skew-) symmetric form, *i.e.* a formation, whose class in the Witt group $\mathrm{W}^{n-1}(X)$ is our wanted $\varepsilon_X^{(n)}$. This complex is also obtained by chopping off some parts of the above Koszul complex.

Putting things together, if we define the integer $-1 \le r \le 2$ by the equation $n-1 \equiv r \bmod 4$, we produce "short symmetric r-spaces" $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ (if n is odd this complex is concentrated in one degree and corresponds to $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$ above via the natural embedding $\mathrm{VB}_X \hookrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$), having the following properties (Theorems 7.13 and 8.2):

Theorem. *Let X be a scheme, not of equicharacteristic 2 (in particular $2 \neq 0$).*

(i) *The symmetric r-space $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ can not be extended to $\mathbb{A}_X^n$, i.e. there does not exist a symmetric r-space (P, ϕ) over $\mathbb{A}_X^n$ whose restriction $(P, \phi)\big|_{\mathbb{U}_X^n}$ is isometric (nor Witt equivalent) to $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$. In particular, $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ is not extended from X either. Hence, if n is odd, the same is true for the classical (skew-)symmetric space $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$.*

(ii) *Assume moreover that 2 is invertible in X and that $n \geq 2$. Then the space $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ is locally hyperbolic, i.e. for any $x \in \mathbb{U}_X^n$ we have $[(\mathcal{F}_X^{(n)}, \phi_X^{(n)})_x] = 0$ in $\mathrm{W}^r(\mathcal{O}_{\mathbb{U}_X^n, x})$, and moreover its square is Witt trivial, i.e. $[\mathcal{F}_X^{(n)}, \phi_X^{(n)}]^2$ is zero in $\mathrm{W}^{2r}(\mathbb{U}_X^n)$.*

This theorem says that the spaces $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ are quite specific to $\mathbb{U}_X^n$. They can not be extended to $\mathbb{A}_X^n$, not even up to Witt equivalence. In particular, these spaces are not metabolic on $\mathbb{U}_X^n$. On the other hand, they do become metabolic as soon as we localize them to some principal open given by $T_i \neq 0$, see 7.11.

There are two appendices. In the first one, for the sake of completeness, we show that when $n \geq 3$ our locally free $\mathcal{O}_{\mathbb{U}_X^n}$-module $\mathcal{E}_X^{(n)}$ can not be extended to a locally free $\mathcal{O}_{\mathbb{A}_X^n}$-module and in particular $\mathcal{E}_X^{(n)}$ is not free. The second appendix contains the compatibility between product and 4-periodicity, a fact which we use several times in this work.

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Contents

1. Conventions and notations

We collect here the notations which are kept unchanged in all sections.

First of all, recall that we have fixed an integer $n \geq 1$. We decompose it as

$$n = 4\,q + r + 1 \tag{2}$$

where $q \in \mathbb{N}$ and $r \in \{-1, 0, 1, 2\}$. Note that $n - 1 \equiv r \bmod 4$. We also baptize

$$\left[\frac{n}{2}\right] =: \ell\,. \tag{3}$$

Convention 1.1. Unless mentioned, a *ring* means a commutative ring with unit.

Convention 1.2. As always, when using a notation defined for schemes X in the affine case, $X = \mathrm{Spec}(R)$, we shall drop "Spec" as for instance: VB_R, $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_R)$, $\mathrm{W}^i(R)$ instead of $\mathrm{VB}_{\mathrm{Spec}(R)}$, $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_{\mathrm{Spec}(R)})$, $\mathrm{W}^i(\mathrm{Spec}(R))$, and so on. See 2.11.

Convention 1.3. We shall say that a scheme is *regular* if it is noetherian and separated and if all its local rings are regular.

Notation 1.4. Let X be a scheme. We denote by $\mathbb{A}^n_X$ the affine n-space and by $\mathbb{U}^n_X$ the punctured affine n-space over X. The obvious structure morphisms and base-change morphisms will be denoted as follows:

$$\begin{tikzcd}
\mathbb{U}^n_Y \arrow[r, "\iota_Y"'] \arrow[rr, bend left, "\sigma_Y"] \arrow[d, "\upsilon_f"'] & \mathbb{A}^n_Y \arrow[r, "\pi_Y"'] \arrow[d, "\alpha_f"'] & Y \arrow[d, "f"] \\
\mathbb{U}^n_X \arrow[r, "\iota_X"'] \arrow[d, "\upsilon_X"'] & \mathbb{A}^n_X \arrow[r, "\pi_X"'] \arrow[d, "\alpha_X"'] & X \arrow[d] \\
\mathbb{U}^n_{\mathbb{Z}} \arrow[r, "\iota_{\mathbb{Z}}"'] & \mathbb{A}^n_{\mathbb{Z}} \arrow[r, "\pi_{\mathbb{Z}}"'] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd} \tag{4}$$

for any morphism of schemes $f : Y \to X$.

2. Recalling derived Witt groups

This section is a quick course on triangular Witt groups over schemes, included only for the reader's convenience. Here, X is a scheme with structure bundle $\mathcal{O}_X$.

2.1. *Categories and dualities.*

We denote by the symbol VB_X the exact category of locally free $\mathcal{O}_X$-modules of finite rank, *i.e. vector bundles.* The usual duality on VB_X is abbreviated

$$(-)^\vee := \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)\,.$$

$\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ stands for the bounded derived category of VB_X. We use homological notations for complexes. The translation functor $\Sigma : \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X) \longrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$, also written $P_\bullet \longmapsto P_\bullet[1]$, is given by $(P_\bullet[1])_j = P_{j-1}$; as usual, Σ changes the sign of all differentials: $d^{P[1]}_j = -d^P_{j-1}$.

Let $P_\bullet = (P_\bullet, d^P_\bullet)$ be a complex in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$. Its *dual* $\mathcal{D}_X(P_\bullet)$ is the complex

$$\mathcal{D}_X(P_\bullet) := \qquad \begin{array}{ccccccc} \cdots \longrightarrow & P_{-j}{}^\vee & \xrightarrow{\;{d^P_{-j+1}}^\vee\;} & P_{-(j-1)}{}^\vee & \longrightarrow \cdots \\ & \deg\, j & & \deg\, (j-1) & \end{array}$$

and similarly for morphisms of complexes. In other words, $\mathcal{D}_X$ is the derived functor of $(-)^\vee = \mathcal{H}om_{\mathcal{O}_X}(\,-\,, \mathcal{O}_X)$. This defines a duality on $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ turning it into a triangulated category with duality in the sense of [2]. The isomorphism between the identity and the double dual, $\varpi : \mathrm{id}_{\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)} \xrightarrow{\simeq} \mathcal{D}_X\mathcal{D}_X$, is given in each degree j by the canonical (evaluation) isomorphism $\mathrm{can}_{P_j} : P_j \longrightarrow P_j{}^{\vee\vee}$. We consider VB_X

as a subcategory of $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ via the natural embedding $c_0 : \mathrm{VB}_X \longrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$, which maps a vector bundle $P \in \mathrm{VB}_X$ to the complex

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \underset{\deg 0}{P} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

and does the same on morphisms. The restriction of the duality $\mathcal{D}_X$ to this subcategory is the original duality of VB_X and the restriction of ϖ is the above can.

Definition 2.2. Let $P_\bullet$ be a complex in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$. Let $i \in \mathbb{Z}$, and $\phi : P_\bullet \longrightarrow \mathcal{D}_X(P_\bullet)[i]$ be a morphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$. We say that ϕ is an *symmetric i-form* on the complex $P_\bullet$ if

$$\mathcal{D}_X(\phi)[i] \cdot \varpi_{P_\bullet} = (-1)^{\frac{i(i+1)}{2}} \phi .$$

We then say that $(P_\bullet, \phi)$ is an *symmetric i-pair.* If ϕ is moreover an isomorphism we say that $(P_\bullet, \phi)$ is a symmetric *i-space* over X. Two symmetric i-pairs $(P_\bullet, \phi)$ and $(Q_\bullet, \psi)$ are called *isometric* if there exists in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ an *isometry* between them, that is, an isomorphism $h : P_\bullet \xrightarrow{\simeq} Q_\bullet$ such that $\phi = \mathcal{D}_X(h)[i] \cdot \psi \cdot h$.

Remark 2.3. Note that if $(P_\bullet, \phi)$ is a symmetric i-pair then $(P_\bullet[2], \phi[2])$ is a symmetric $(i+4)$-pair because $\mathcal{D}_X(P_\bullet)[1] = \mathcal{D}_X(P_\bullet[-1])$ for all $P_\bullet \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$.

Let $f : Y \longrightarrow X$ be a morphism of schemes. There is a natural isomorphism of functors $\eta_f : f^*\mathcal{D}_X \xrightarrow{\simeq} \mathcal{D}_Y f^*$ which is induced by the natural isomorphism of locally free $\mathcal{O}_Y$-modules $f^*\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_Y}(f^*P, \mathcal{O}_Y)$. If now $(P_\bullet, \phi)$ is a symmetric i-space over X then the isomorphism

$$f^*(P_\bullet) \xrightarrow{f^*\phi} f^*(\mathcal{D}_X(P_\bullet)[i]) = f^*(\mathcal{D}_X(P_\bullet))[i] \xrightarrow{\eta_{f,P}[i]} \mathcal{D}_Y(f^*P_\bullet)[i]$$

is a symmetric i-form and so $f^*(P_\bullet, \phi) := (f^*(P_\bullet),\ \eta_{f,P}[i] \cdot f^*(\phi))$ a symmetric i-space over Y.

2.4. *"Short" i-forms: Forms and formations.*

We present examples of symmetric i-pairs $(P_\bullet, \phi)$ in four cases $i = -1, 0, 1, 2$.

$\boxed{i=0}$ deg 0

$$\begin{tikzcd} \cdots 0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\phi_0=\phi_0^\vee"] & 0 \cdots \\ \cdots 0 \arrow[r] & P_0^\vee \arrow[r] & 0 \cdots \end{tikzcd}$$

$\boxed{i=1}$ deg 1 deg 0

$$\begin{tikzcd} \cdots 0 \arrow[r] & P_1 \arrow[r, "d"] \arrow[d, "\phi_1"'] & P_0 \arrow[r] \arrow[d, "-\phi_1^\vee"] & 0 \cdots \\ \cdots 0 \arrow[r] & P_0^\vee \arrow[r, "-d^\vee"'] & P_1^\vee \arrow[r] & 0 \cdots \end{tikzcd}$$

$\boxed{i=2}$ deg 1

$$\begin{tikzcd} \cdots 0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\phi_1=-\phi_1^\vee"] & 0 \cdots \\ \cdots 0 \arrow[r] & P_1^\vee \arrow[r] & 0 \cdots \end{tikzcd}$$

$\boxed{i=-1}$ deg 0 deg -1

$$\begin{tikzcd} \cdots 0 \arrow[r] & P_0 \arrow[r, "d"] \arrow[d, "\phi_0"'] & P_{-1} \arrow[r] \arrow[d, "\phi_0^\vee"] & 0 \cdots \\ \cdots 0 \arrow[r] & P_{-1}^\vee \arrow[r, "-d^\vee"'] & P_0^\vee \arrow[r] & 0 \cdots \end{tikzcd}$$

In each case, the complexes $P_\bullet$ and $P_\bullet^\vee$ are depicted horizontally and the symmetric i-form $\phi : P_\bullet \longrightarrow \mathcal{D}_X(P_\bullet)[i]$ vertically. The symmetric pairs of the left-hand column are classical symmetric and skew-symmetric forms embedded in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ via the functor c_0 (slightly pushed to the left for $i=2$). These symmetric i-pairs are *i-spaces* exactly when ϕ_0 and ϕ_1 is an isomorphism. The symmetric i-pairs of the right-hand column are *i-spaces* when ϕ is a quasi-isomorphism, *i.e.* when its cone is an exact complex; these are *formations*; we call them symmetric if $i=-1$ and skew-symmetric if $i=1$. The four types of i-form presented above will be called *short*, for the obvious reasons.

2.5. *Product of symmetric spaces.*

The precise definition of this product is given in [10], where the reader will also find an explanation for the existence of *two* different products – the left and the right one – which differ by signs. To fix the ideas, *we will use here the left product.* Let $(P_\bullet,\phi)$ be a symmetric i-form and $(Q_\bullet,\psi)$ a symmetric j-form. The product

$$(P_\bullet,\phi)\star(Q_\bullet,\psi)$$

is then a symmetric $(i+j)$-form on the tensor product (of complexes) $P_\bullet\otimes_{\mathcal{O}_X}Q_\bullet$ and we denote it by $(P_\bullet\otimes_{\mathcal{O}_X}Q_\bullet\,,\,\phi\star\psi)$. Up to signs and identifications like for instance $P_\bullet\otimes_{\mathcal{O}_X}(Q_\bullet^\vee[j])\simeq(P_\bullet\otimes_{\mathcal{O}_X}Q_\bullet^\vee)[j]$, the morphism of complexes $\phi\star\psi$ is equal to the tensor product $\phi\otimes\psi$. Via c_0, this product coincides on short 0-spaces with the usual tensor product of symmetric spaces as defined in Knebusch [12].

2.6. *Symmetric cones.*

We now recall the important *cone construction.* Let $\phi:P_\bullet\longrightarrow\mathcal{D}_X(P_\bullet)[i]$ be a symmetric i-form (maybe not an isomorphism). Let $Q_\bullet$ be the mapping cone of ϕ. Then, there exists an isomorphism ψ such that the following diagram commutes:

$$\begin{tikzcd}
P_\bullet \arrow[r, "\phi"] \arrow[d, "\simeq"', "(-1)^{\frac{i(i+1)}{2}}\cdot\varpi_P"] & \mathcal{D}_X(P_\bullet)[i] \arrow[r, "u"] \arrow[d, "="] & Q_\bullet \arrow[r, "v"] \arrow[d, "\psi"', "\simeq"] & P_\bullet[1] \arrow[d, "(-1)^{\frac{i(i+1)}{2}}\cdot\varpi_P[1]"', "\cong"] \\
\mathcal{D}_X\mathcal{D}_X(P_\bullet) \arrow[r, "\mathcal{D}_X(\phi)[i]"'] & \mathcal{D}_X(P_\bullet)[i] \arrow[r, "-\mathcal{D}_X(v)[i+1]"'] & \mathcal{D}_X(Q_\bullet)[i+1] \arrow[r, "(-1)^i\mathcal{D}_X(u)[i+1]"'] & \mathcal{D}_X\mathcal{D}_X(P_\bullet)[1]
\end{tikzcd}$$

If the isomorphism ψ is moreover a *symmetric* $(i+1)$-form, we call such a diagram *a cone diagram* (over ϕ) and we say that $(Q_\bullet,\psi)$ is *a symmetric cone* of the pair $(P_\bullet,\phi)$.

Note that both rows of the diagram are exact triangles in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$: The upper one by definition and the lower one is the dual of the upper row, shifted i times. Assume for a moment that 2 is invertible over our scheme X. Then we can always choose the isomorphism ψ to be a symmetric $(i+1)$-form, see [2]. Moreover, if $(Q'_\bullet,\psi')$ is another symmetric cone of ϕ, then there exists an isometry $(Q_\bullet,\psi)\simeq(Q'_\bullet,\psi')$. We say then that $(Q_\bullet,\psi)$ is *the* symmetric cone of ϕ, in symbols:

$$(Q_\bullet,\psi)=\operatorname{cone}\phi=\operatorname{cone}(P_\bullet,\phi)\,.$$

2.7. *Witt groups.*

The *usual Witt group* of symmetric (respectively skew-symmetric) spaces $\mathrm{W}_{\mathrm{us}}(X)$ (respectively $\mathrm{W}^-_{\mathrm{us}}(X)$) classifies these spaces up to isometry and modulo metabolic ones. More information about these Witt groups can be found in the fundamental paper of Knebusch [12]. The i-th *derived Witt group* $\mathrm{W}^i(X)$ classifies symmetric

i-spaces up to isometry and modulo neutral spaces, *i.e.* spaces with Lagrangian (*cf.* [2], Sect. 2). In fact, a symmetric i-space is neutral exactly if it is a symmetric cone of some symmetric $(i-1)$-form, as described above. Observe that this definition does not require 2 to be invertible in X. We denote by $[P_\bullet,\phi]$ the Witt class of $(P_\bullet,\phi)$.

The Witt groups are contravariant functors. If $f: Y \longrightarrow X$ is a morphism of schemes then the assignment $[P_\bullet,\phi] \longmapsto [f^*(P_\bullet,\phi)]$ defines a homomorphism $f^*: \mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(Y)$ for all $i \in \mathbb{Z}$.

2.8. *Periodicity.*

The derived Witt groups are 4-periodic. The shift by two: $P_\bullet \longmapsto P_\bullet[2]$ induces an isomorphism $\tau : \mathrm{W}^i(X) \xrightarrow{\simeq} \mathrm{W}^{i+4}(X)$ for all $i \in \mathbb{Z}$ and all schemes X. The same periodicity applies to the Witt groups with support defined below.

2.9. *Agreement and localization (with $\frac{1}{2}$).*

We assume now that "X contains $\frac{1}{2}$", *i.e.* that X is a $\mathbb{Z}[1/2]$-scheme, *i.e.* 2 is invertible in the ring $\Gamma(X,\mathcal{O}_X)$. The main result of [3] is that the functor c_0 : $\mathrm{VB}_X \longrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ induces isomorphisms:

$$\mathrm{W}(X) = \mathrm{W}_{\mathrm{us}}(X) \xrightarrow{\simeq} \mathrm{W}^0(X) \qquad [P,\phi] \longmapsto [c_0(P), c_0(\phi)]$$

and

$$\mathrm{W}^-(X) = \mathrm{W}^-_{\mathrm{us}}(X) \xrightarrow{\simeq} \mathrm{W}^2(X) \qquad [Q,\psi] \longmapsto [c_0(Q)[1]\,,\, c_0(\psi)[1]\,]\,.$$

Other Witt groups appearing in this work are the *Witt groups with support.* For a complex $P_\bullet \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ let

$$\operatorname{supp} P_\bullet := \{x \in X \mid H_j(P_\bullet)_x \neq 0 \quad \text{for at least one } j\ \}\,,$$

be its (homological) support. Let Z be a closed subscheme of X with open complement U. The full triangulated subcategory of $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ which consists of complexes with support contained in Z is denoted $\mathbb{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$. The restriction of the duality $\mathcal{D}_X$ to $\mathbb{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ is again a duality, turning $\mathbb{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ into a triangulated category with duality. The corresponding triangular Witt groups $\mathrm{W}^i_Z(X)$ $(i \in \mathbb{Z})$ are called the derived Witt groups of X with support in Z. They appear in the localization sequence of Balmer [2]. If X is a regular scheme then there is an exact sequence

$$\cdots \longrightarrow \mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(U) \xrightarrow{\partial} \mathrm{W}^{i+1}_Z(X) \longrightarrow \mathrm{W}^{i+1}(X) \longrightarrow \cdots.$$

The connecting morphism ∂ comes from the cone construction 2.6 as follows. Let $w \in \mathrm{W}^i(U)$. Then $\partial(w) = [\mathrm{cone}(P_\bullet,\phi)]$, where $(P_\bullet,\phi)$ is any symmetric i-pair over X with $[(P_\bullet,\phi)|_U] = w$ (the existence of $(P_\bullet,\phi)$ is guaranteed by regularity of X). The Witt groups with support are natural and so is the localization sequence.

2.10. *The graded Witt ring.*

The (left) product of symmetric spaces of 2.5 yields a product structure

$$\star : \ \mathrm{W}^i(X) \times \mathrm{W}^j_Z(X) \ \longrightarrow \ \mathrm{W}^{i+j}_Z(X) \quad (\,[P_\bullet,\phi]\,,\,[Q_\bullet,\psi]\,) \longmapsto \big[(P_\bullet,\phi)\star(Q_\bullet,\psi)\big]$$

for any $i,j \in \mathbb{Z}$, any scheme X and closed subset $Z \subseteq X$. Via this pairing, $\mathrm{W}^{\mathrm{tot}}(X) := \bigoplus_{i\in\mathbb{Z}} \mathrm{W}^i(X)$ is a graded skew-commutative associative $\mathrm{W}^0(X)$-algebra, the *graded Witt ring* of X and $\mathrm{W}^{\mathrm{tot}}_Z(X) := \bigoplus_{i\in\mathbb{Z}} \mathrm{W}^i_Z(X)$ is a graded $\mathrm{W}^{\mathrm{tot}}(X)$-module.

This product is compatible with the connecting homomorphism ∂ in the localization sequence by [10, Thm. 2.11], *i.e.* we have a commutative diagram

$$\begin{tikzcd} \mathrm{W}^i(X) \times \mathrm{W}^j(U) \arrow[r, "\star"] \arrow[d, "\mathrm{id}\times\partial"'] & \mathrm{W}^{i+j}(U) \arrow[d, "\partial"] \\ \mathrm{W}^i(X) \times \mathrm{W}_Z^{j+1}(X) \arrow[r, "\star"] & \mathrm{W}_Z^{i+j+1}(X)\,. \end{tikzcd} \tag{5}$$

In words, the connecting homomorphism is a left $\mathrm{W}^{\mathrm{tot}}(X)$-linear map. But note that it is not right $\mathrm{W}^{\mathrm{tot}}(X)$-linear:

$$\partial(y \star x) \;=\; (-1)^{ij}\partial(x \star y) \overset{(5)}{=} (-1)^{ij} x \star \partial(y) \;=\; (-1)^i \partial(y) \star x \tag{6}$$

(the first and last equation by skew-commutativity), where $x \in \mathrm{W}^i(X)$ and $y \in \mathrm{W}^j(U)$.

Remark 2.11. Of course, Convention 1.2 applies here as well. For instance, if $X = \mathrm{Spec}(R)$ is affine and $Z \subset X$ is defined by the ideal I we might say that a complex "has support in the ideal I" and we shall write $\mathrm{W}_I^i(R)$ instead of $\mathrm{W}_Z^i(X)$.

3. Basic facts about Koszul complexes

In this section, A is a ring and $\underline{T} = (T_1, \ldots, T_n)$ is any sequence in A. As before, we write the dual as $M^\vee := \mathrm{Hom}_A(M, A)$, for any A-module M.

We first recall the definition of the Koszul complex

$$K_\bullet(A, \underline{T}) \;=:\; (K_\bullet\,,\, d_\bullet)\,.$$

Let $e_1, e_2, \ldots, e_n$ be a basis of the free A-module $A^n = \bigoplus_{i=1}^n A \cdot e_i$. The A-module $K_i = K_i(A, \underline{T}) := \bigwedge^i A^n$ is by definition the i-th exterior power of A^n. As is well-known, the module K_i is free with basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i} \,\big|\, 1 \le j_1 < \ldots < j_i \le n\}$. The differential $d_i = d_i(A, \underline{T}) : K_i \longrightarrow K_{i-1}$ is given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \;\longmapsto\; \sum_{k=1}^{i} (-1)^{k-1}\, T_{j_k} \cdot e_{j_1} \wedge \ldots \widehat{e_{j_k}} \ldots \wedge e_{j_i}\,,$$

where the symbol $\widehat{e_{j_k}}$ indicates that e_{j_k} has been omitted. We consider this (homological) *Koszul complex* $K_\bullet(A, \underline{T})$:

$$\cdots\, 0 \longrightarrow K_n(A, \underline{T}) \xrightarrow{d_n(A,\underline{T})} K_{n-1}(A, \underline{T}) \longrightarrow \cdots \xrightarrow{d_1(A,\underline{T})} K_0(A, \underline{T}) \longrightarrow 0 \,\cdots$$

as an element of $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ with $K_j(A, \underline{T})$ in degree j.

There is a structure of symmetric n-space on $K_\bullet(A, \underline{T})$, that we now give in an economic way; see more details in Remark 3.3. For each $i = 1, \ldots, n$, let $K_\bullet(A, T_i) \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ be the short Koszul complex for the one-element sequence (T_i), *i.e.*

$$K_\bullet(A, T_i) = \qquad \cdots \longrightarrow 0 \longrightarrow \underset{\deg 1}{A} \xrightarrow{\cdot T_i} \underset{\deg 0\,.}{A} \longrightarrow 0 \longrightarrow \cdots$$

This complex can be equipped with the following symmetric 1-form (see 2.2):

$$\begin{tikzcd}
K_\bullet(A,T_i) = \arrow[d, "\Theta(A,T_i)"', ":="] & \cdots 0 \arrow[r] & A \arrow[r, "\cdot T_i"] \arrow[d, "\mathrm{id}"'] & A \arrow[r] \arrow[d, "-\mathrm{id}"] & 0 \cdots \\
\mathcal{D}_A(K_\bullet(A,T_i))[1] = & \cdots 0 \arrow[r] & A \arrow[r, "\cdot(-T_i)"] & A \arrow[r] & 0 \cdots \\
& & \deg 1 & \deg 0\,, &
\end{tikzcd}$$

where we identify $A = \mathrm{Hom}_A(A,A)$ as usual. This is of course a symmetric 1-space since $\Theta(A,T_i)$ is an isomorphism of complexes, hence a quasi-isomorphism. It is easily checked that the tensor product of complexes $K_\bullet(A,T_1)\otimes_A \ldots \otimes_A K_\bullet(A,T_n)$ is equal to the Koszul complex $K_\bullet(A,\underline{T})$ of the sequence $\underline{T} = (T_1,\ldots,T_n)$ and therefore we can give the following:

Definition 3.1. With the above notations, we define a symmetric n-form

$$\Theta(A,\underline{T}) \,:\, K_\bullet(A,\underline{T}) \longrightarrow \mathcal{D}_A(K_\bullet(A,\underline{T}))[n]$$

as the product (see 2.5)

$$\big(K_\bullet(A,\underline{T})\,,\,\Theta(A,\underline{T})\big) \;:=\; \big(K_\bullet(A,T_1),\Theta(A,T_1)\big)\star\ldots\star\big(K_\bullet(A,T_n),\Theta(A,T_n)\big)\,.$$

This defines a symmetric n-space $\big(K_\bullet(A,\underline{T})\,,\,\Theta(A,\underline{T})\big)$ which we call the *canonical space* on the Koszul complex $K_\bullet(A,\underline{T})$.

We have the following functorial property.

Lemma 3.2. *Let $f : A \longrightarrow B$ be a morphism of rings. Then, there is a natural isometry*

$$f^*\big(K_\bullet(A,\underline{T}),\Theta(A,\underline{T})\big) \xrightarrow{\simeq} \big(K_\bullet(B,f(\underline{T})),\Theta(B,f(\underline{T}))\big)\,.$$

Proof. The natural isomorphism $K_\bullet(A,\underline{T})\otimes_A B \xrightarrow{\simeq} K_\bullet(B,f(\underline{T}))$ is an isometry. □

Remark 3.3. To define this canonical space on $K_\bullet(A,\underline{T})$, it is not necessary to use the product structure of the derived Witt groups. The advantage of this approach is that we see at once that the canonical n-space is a symmetric n-space, but for calculations in the sequel it might be useful to have a good description of the symmetric n-form $\Theta(A,\underline{T})$. We define an isomorphism

$$\rho \,:\, K_\bullet(A,\underline{T}) \xrightarrow{\simeq} \mathcal{D}_A(K_\bullet(A,\underline{T}))[n]$$

following [7], Sect. 1.6. We fix for this an isomorphism $\omega : \bigwedge^n(A^n) \xrightarrow{\simeq} A$, and define an A-bilinear pairing

$$b_i \,:\, K_i(A,\underline{T}) \times K_{n-i}(A,\underline{T}) \;\longrightarrow\; A$$

by $(x,y) \longmapsto \omega(x\wedge y)$ for all $0\le i\le n$. This b_i induces a homomorphism ϱ_i : $K_i(A,\underline{T}) \longrightarrow \mathrm{Hom}_A(K_{n-i}(A,\underline{T}),A) = K_{n-i}(A,\underline{T})^\vee$ which is an isomorphism for all $0\le i\le n$. It is straightforward (although a little cumbersome) to check that

$$d_{n-(i-1)}(A,\underline{T})^\vee\cdot\varrho_i = (-1)^{i-1}\varrho_{i-1}\cdot d_i(A,\underline{T})\,.$$

Consider the family of morphisms $(\rho_i)_{i\in\mathbb{Z}}$ defined by $\rho_i := (-1)^{\frac{i(i+1)}{2}+\frac{n(n-1)}{2}}\cdot\varrho_i$ for $0\le i\le n$ and by $\rho_i = 0$ otherwise. This defines an isomorphism of complexes $\rho = \rho_\bullet \,:\, K_\bullet(A,\underline{T}) \longrightarrow \mathcal{D}_A(K_\bullet(A,\underline{T}))[n]$, which coincides with the morphism of complexes $\Theta(A,\underline{T})$ as a thrilling calculation using [10, Ex. 1.4, Rem. 1.9] shows.

By the following lemma this is easier to see if $\underline{T}$ is a regular sequence, which is the only interesting case for us here.

Lemma 3.4. *Assume that $\underline{T}$ is a regular sequence, i.e. $I := \sum_{i=1}^{n} AT_i \neq A$ and T_i is not a zero divisor in $A/\sum_{j=1}^{i-1} AT_j$ for all $i = 1, \ldots, n$. Identify $A \cong \mathrm{Hom}_A(A, A)$ as usual. Then the following properties hold:*

(i) *The Koszul complex $K_\bullet(A, \underline{T})$ and its n-dual $\mathcal{D}_A(K_\bullet(A, \underline{T}))[n]$ are A-free resolutions of A/I.*

(ii) *We have $H_0(\Theta(A, \underline{T})) = (-1)^{\frac{n(n-1)}{2}} \,\mathrm{id}_{A/I}$.*

(iii) *For any morphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ between the Koszul complex and its n-dual*
$$\varsigma \;:\; K_\bullet(A, \underline{T}) \longrightarrow \mathcal{D}_A(K_\bullet(A, \underline{T}))[n]\,,$$
there exists an $s \in A$ such that $\varsigma = s \cdot \Theta(A, \underline{T})$ in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$.

(iv) *In (iii), ς is an isomorphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ if and only if $s + I$ is a unit in the quotient ring A/I.*

Proof. For (i), $\underline{T} = (T_1, \ldots, T_n)$ is a regular sequence by assumption and so the complex $K_\bullet(A, \underline{T})$ is an A-free resolution of A/I by [7, Cor. 1.6.14]. For (ii), see Remark 3.3 above. Point (iii) follows from (i) and (ii). Point (iv) is immediate from (iii). □

Remark 3.5. It is clear that the restriction of $K_\bullet(A, \underline{T})$ becomes zero in the derived category $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_{A[T_j^{-1}]})$ for all $1 \leq j \leq n$. Hence the complex $K_\bullet(A, \underline{T})$ has support in the closed subset of $\mathrm{Spec}(A)$ defined by the ideal $I := \sum_{j=1}^{n} A\,T_j$. Therefore the symmetric n-space $\big(K_\bullet(A, \underline{T})\,,\, \Theta(A, \underline{T})\big)$ defines an element in
$$\big[K_\bullet(A, \underline{T})\,,\, \Theta(A, \underline{T})\big] \in \mathrm{W}^n_I(A)\,.$$

Proposition 3.6. *Let $1 \leq i \leq n$. Define the ideal $I_i := \sum_{k \neq i} A\,T_k$ of A. Then $\big[K_\bullet(A, \underline{T})\,,\, \Theta(A, \underline{T})\big] = 0$ in $\mathrm{W}^n_{I_i}(A)$.*

Proof. The group $\mathrm{W}^{n-1}_{I_i}(A)$ obviously contains the element
$$\begin{aligned} y \;:=\; & [K_\bullet(A, T_1), \Theta(A, T_1)] \star \ldots \star [K_\bullet(A, T_{i-1}), \Theta(A, T_{i-1})] \\ & \star [K_\bullet(A, T_{i+1}), \Theta(A, T_{i+1})] \star \ldots \star [K_\bullet(A, T_n), \Theta(A, T_n)]\,. \end{aligned}$$
Recall from 2.10 that the product also gives
$$\star : \mathrm{W}^1(A) \times \mathrm{W}^{n-1}_{I_i}(A) \longrightarrow \mathrm{W}^n_{I_i}(A)\,.$$
Since the product is skew-commutative, we have
$$[K_\bullet(A, T_i), \Theta(A, T_i)] \star y = (-1)^{i-1} \big[K_\bullet(A, \underline{T})\,,\, \Theta(A, \underline{T})\big]\,,$$
where we consider $[K_\bullet(A, T_i), \Theta(A, T_i)]$ as an element of $\mathrm{W}^1(A)$. Therefore the result follows from the observation that this element is indeed zero in $\mathrm{W}^1(A)$. In fact, the complex $c_0(A) \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ is a Lagrangian (*cf.* [2], Sect. 2) of the symmetric 1-space $(K_\bullet(A, T_i), \Theta(A, T_i))$:
$$\begin{tikzcd} c_0(A) = \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[r] \arrow[d] & \cdots \\ K_\bullet(A, T_i) = & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\cdot T_i"] & A \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

and so $[K_\bullet(A,T_i),\Theta(A,T_i)]=0$ in $\mathrm{W}^1(A)$. □

In the above proof, note that the class $y\in \mathrm{W}^{n-1}_{I_i}(A)$ does not belong to $\mathrm{W}^{n-1}_{I}(A)$ and thus the argument can not be used to deduce that $[K_\bullet(A,\underline{T})\,,\,\Theta(A,\underline{T})\,]=0$ in $\mathrm{W}^n_I(A)$. On the contrary, see Theorem 7.2.

Corollary 3.7. *$\big[K_\bullet(A,\underline{T})\,,\,\Theta(A,\underline{T})\big]=0$ in $\mathrm{W}^n(A)$.*

Proof. Clear since $\mathrm{W}^n_I(A)\longrightarrow \mathrm{W}^n(A)$ factors via $\mathrm{W}^n_{I_1}(A)$ for instance. □

4. Koszul cut in two

We want to "split" the Koszul complex of Section 3 into two pieces, dual to each other. This is easy to understand but a little tricky to write. Recall our running conventions of Section 1 that $r+4\,q=n-1$, see (2), and that $\ell:=[\frac{n}{2}]$, see (3). Now, more precisely, we want to define a symmetric r-pair $\big(\,C_\bullet(A,\underline{T})\,,\,\Xi(A,\underline{T})\,\big)$, such that there is an isometry

$$\mathrm{cone}(C_\bullet(A,\underline{T}),\Xi(A,\underline{T}))\;\simeq\;\big(\,K_\bullet(A,\underline{T}),\Theta(A,\underline{T})\,\big)\,[-2q]\,.$$

We abbreviate the canonical form on $K_\bullet(A,\underline{T})=:K_\bullet$ by

$$\Theta:=\Theta(A,\underline{T})\ ,$$

and set

$$E\;=\;E(A,\underline{T})\;:=\;\mathrm{Coker}\,\Big(\,K_{\ell+2}(A,\underline{T})\xrightarrow{d_{\ell+2}(A,\underline{T})}K_{\ell+1}(A,\underline{T})\,\Big)\,. \tag{7}$$

Let $\mathrm{pr}_E=\mathrm{pr}_{E(A,\underline{T})}\;:\;K_{\ell+1}\longrightarrow E=\mathrm{Coker}\,d_{\ell+2}$ be the projection. Since $d_{\ell+1}d_{\ell+2}=0$ there exists a unique morphism $\bar d_{\ell+1}=\bar d_{\ell+1}(A,\underline{T}):E\longrightarrow K_\ell$, such that

$$d_{\ell+1}(A,\underline{T})\;=\;\bar d_{\ell+1}(A,\underline{T})\cdot\mathrm{pr}_E\ . \tag{8}$$

For each $j=0,\dots,n$, we have $\mathrm{rank}_A(K_j)=\binom{n}{j}$. In particular, if $n=2\ell+1$ is odd, we have $\mathrm{rank}_A K_\ell=\mathrm{rank}_A K_{\ell+1}$ and life will be easy. When $n=2\ell$ is even, K_ℓ has maximal (even) rank $\binom{2\ell}{\ell}$ and we need some preparatory considerations. In this case, the symmetric n-form $\Theta_\bullet:K_\bullet\xrightarrow{\simeq}\mathcal{D}_A(K_\bullet)[n]$ gives an isomorphism

$$\Theta_\ell\;:\;K_\ell\longrightarrow K_\ell^\vee\;=\;\mathrm{Hom}_A(K_\ell,A)$$

which is symmetric if ℓ is even and skew-symmetric otherwise.

Lemma 4.1. *If $n=2\ell$ is even, there exists two totally isotropic subspaces L and M of (K_ℓ,Θ_ℓ), of same rank $\frac12\binom{2\ell}{\ell}$, such that $K_\ell=L\oplus M$ and such that Θ_ℓ becomes*

$$\Theta_\ell\;=\;\begin{pmatrix}0 & (-1)^\ell\lambda^\vee\,\mathrm{can}_M\\ \lambda & 0\end{pmatrix}\quad:\qquad K_\ell=L\oplus M\longrightarrow L^\vee\oplus M^\vee=K_\ell^\vee\ ,$$

where $\lambda:L\xrightarrow{\simeq}M^\vee$ is an isomorphism. Moreover, we have

$$(-1)^\ell d_{\ell+1}^\vee\cdot(\mathrm{pr}_L)^\vee\cdot\lambda^\vee\cdot\mathrm{can}_M\cdot\mathrm{pr}_M\cdot d_{\ell+1}\;+\;d_{\ell+1}^\vee\cdot(\mathrm{pr}_M)^\vee\cdot\lambda\cdot\mathrm{pr}_L\cdot d_{\ell+1}\;\;=\;0\,, \tag{9}$$

where $\mathrm{pr}_L:K_\ell\longrightarrow L$ and $\mathrm{pr}_M:K_\ell\longrightarrow M$ denote the projections.

Proof. Let $e_1, \ldots, e_n$ be a basis of A^n. The complementary subspaces of $K_\ell = \bigwedge^{\ell} A^n$

$$L := \bigoplus_{2 \le i_2 < i_3 < \ldots < i_\ell \le n} A \cdot e_1 \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots \wedge e_{i_\ell}$$

and

$$M := \bigoplus_{2 \le i_1 < i_2 < \ldots < i_\ell \le n} A \cdot e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_\ell}$$

both have rank $\binom{2\ell-1}{\ell-1} = \binom{2\ell-1}{\ell} = \frac{1}{2}\binom{2\ell}{\ell}$. Now use the description of Θ_ℓ given in Remark 3.3. Let $\omega : \bigwedge^{n} A^n \xrightarrow{\simeq} A$ be the isomorphism which sends $e_1 \wedge \ldots \wedge e_n$ to $1 \in A$. Then $\Theta_\ell(x)(y) = \pm\omega(x \wedge y)$. From this we easily get that both subspaces are totally isotropic: for L it is because $e_1 \wedge e_1 = 0$ and for M it is because two subsets with ℓ elements in $\{2, \ldots, n\}$ must intersect. Since Θ_ℓ is a $(-1)^\ell$-symmetric isomorphism, its decomposition in $L \oplus M$ must be as claimed in the lemma.

Note that $\Theta : K_\bullet \longrightarrow \mathcal{D}_A(K_\bullet)[n]$ is a morphism of complexes and we have

$$d^\vee_{\ell+1} \cdot \Theta_\ell \cdot d_{\ell+1} = 0 \ : \qquad K_{\ell+1} \xrightarrow{d_{\ell+1}} K_\ell \xrightarrow{\Theta_\ell} K^\vee_\ell \xrightarrow{d^\vee_{\ell+1}} K^\vee_{\ell+1} \,.$$

In the decomposition $K_\ell = L \oplus M$, the morphism $d_{\ell+1} : K_{\ell+1} \longrightarrow K_\ell$ becomes $\begin{pmatrix} \mathrm{pr}_L \cdot d_{\ell+1} \\ \mathrm{pr}_M \cdot d_{\ell+1} \end{pmatrix}$. Replacing this in $d^\vee_{\ell+1} \cdot \Theta_\ell \cdot d_{\ell+1} = 0$ gives equation (9) by a direct matrix multiplication. $\square$

Definition 4.2. As the above discussion shows, we will have to distinguish the cases where n is odd from those where n is even and the definition extends over 4.3 - 4.6 below. We shall consider a sign

$$\epsilon_n \in \{-1, 1\}$$

which will be fixed later on, see 6.3.

We start with $n = 2\ell + 1$ odd.

4.3. *Case* $\underline{r = 0}$.

Then $\ell = 2q$ is even and $\big(C_\bullet(A, \underline{T}), \Xi(A, \underline{T})\big)$ is defined to be the following symmetric 0-pair:

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r, "d_n"] \arrow[d] & \cdots \arrow[r] & K_{\ell+2} \arrow[r, "d_{\ell+2}"] \arrow[d] & K_{\ell+1} \arrow[r] \arrow[d, "\epsilon_n \Theta_\ell \cdot d_{\ell+1}"'] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & K^\vee_{\ell+1} \arrow[r, "d^\vee_{\ell+2}"] & K^\vee_{\ell+2} \arrow[r] & \cdots \arrow[r, "d^\vee_n"] & K^\vee_n \arrow[r] & 0
\end{tikzcd}$$

deg ℓ deg 0.

If $\underline{T}$ is a regular sequence then the Koszul complex $K_\bullet$ is exact and so we have the following quasi-isomorphism

$$\begin{tikzcd}
C_\bullet(A,\underline{T}) = \arrow[d, "p=p(A,\underline{T})"', ":="] & 0 \arrow[r] & K_n \arrow[r, "d_n"] \arrow[d] & \cdots \arrow[r] & K_{\ell+2} \arrow[r, "d_{\ell+2}"] \arrow[d] & K_{\ell+1} \arrow[r] \arrow[d, "\mathrm{pr}_E"'] & 0 \arrow[r] \arrow[d] & {} \\
c_0(E) = & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & E \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

deg ℓ deg 0.

4.4. *Case* $\underline{r=2}$.

Then $\ell = 2q+1$ is odd and $\big(C_\bullet(A,\underline{T})\,,\,\Xi(A,\underline{T})\big)$ is defined to be the following symmetric 2-pair:

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r, "d_n"] \arrow[d] & \cdots \arrow[r] & K_{\ell+2} \arrow[r, "d_{\ell+2}"] \arrow[d] & K_{\ell+1} \arrow[r] \arrow[d, "\epsilon_n \Theta_\ell \cdot d_{\ell+1}"'] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & K_{\ell+1}^\vee \arrow[r, "d_{\ell+2}^\vee"] & K_{\ell+2}^\vee \arrow[r] & \cdots \arrow[r, "d_n^\vee"] & K_n^\vee \arrow[r] & 0
\end{tikzcd}$$

deg $\ell+1$ deg 1.

As above if $\underline{T}$ is a regular sequence the projection $\mathrm{pr}_E : K_{\ell+1} \longrightarrow E$ induces a quasi-isomorphism of complexes $p = p(A,\underline{T}) : C_\bullet(A,\underline{T}) \longrightarrow c_0(E)[1]$.

Now let $n = 2\ell$ be even.

We fix two totally isotropic subspaces L and M of K_ℓ and an isomorphism $\lambda : L \longrightarrow M^\vee$ as in Lemma 4.1 and keep notations as there. We set

$$h := \lambda^\vee \cdot \mathrm{can}_M \cdot \mathrm{pr}_M \cdot d_{\ell+1} \quad : \quad K_{\ell+1} \longrightarrow L^\vee\,.$$

We now define the space $\big(C_\bullet(A,\underline{T})\,,\,\Xi(A,\underline{T})\big)$ for n even. It follows from equation (9) in Lemma 4.1 that both squares in the middle of the two diagrams below commute and so the morphism $\Xi(A,\underline{T})$ is really a morphism of complexes.

4.5. *Case* $\underline{r=-1}$.

Then $\ell = 2q$ is even and $\big(C_\bullet(A,\underline{T})\,,\,\Xi(A,\underline{T})\big)$ is defined to be the following symmetric (-1)-pair:

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r, "d_n"] \arrow[d] & \cdots \arrow[r] & K_{\ell+2} \arrow[r, "d_{\ell+2}"] \arrow[d] & K_{\ell+1} \arrow[r, "\mathrm{pr}_L\, d_{\ell+1}"] \arrow[d, "\epsilon_n h"'] & L \arrow[r] \arrow[d, "\epsilon_n h^\vee\, \mathrm{can}_L"] & 0 \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & L^\vee \arrow[r, "-(\mathrm{pr}_L\, d_{\ell+1})^\vee"'] & K_{\ell+1}^\vee \arrow[r, "-d_{\ell+2}^\vee"'] & \cdots \arrow[r, "-d_n^\vee"'] & K_n^\vee \arrow[r] & 0
\end{tikzcd}$$

deg $\ell-1$ deg 0 deg -1

If the sequence $\underline{T}$ is regular the homology of $C_\bullet(A,\underline{T})$ is not concentrated in one degree (as in the case n odd) but there exists a “short” complex $F_\bullet(A,\underline{T})$ defined as follows and which is quasi-isomorphic to $C_\bullet(A,\underline{T})$:

$$\begin{tikzcd}
C_\bullet(A,\underline{T}) \; = \arrow[d, "p=p(A,\underline{T})"', ":="] & \cdots \arrow[r] & K_{\ell+2} \arrow[r, "d_{\ell+2}"] \arrow[d] & K_{\ell+1} \arrow[r, "\mathrm{pr}_L \cdot d_{\ell+1}"] \arrow[d, "\mathrm{pr}_E"] & L \arrow[r] \arrow[d, "="] & 0 \arrow[r] \arrow[d] & \cdots \\
F_\bullet(A,\underline{T}) \; := & \cdots \arrow[r] & 0 \arrow[r] & E \arrow[r, "\mathrm{pr}_L \cdot \bar{d}_{\ell+1}"'] & L \arrow[r] & 0 \arrow[r] & \cdots \\
 & & & \text{deg } 0 & \text{deg } -1. & &
\end{tikzcd}$$

4.6. *Case* $\underline{r=1}$.

Then $\ell = 2q+1$ is odd and $\big(C_\bullet(A,\underline{T})\,,\,\Xi(A,\underline{T})\big)$ is defined to be the following symmetric 1-pair:

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r, "d_n"] \arrow[d] & \cdots \arrow[r] & K_{\ell+2} \arrow[r, "d_{\ell+2}"] \arrow[d] & K_{\ell+1} \arrow[r, "\mathrm{pr}_L\, d_{\ell+1}"] \arrow[d, "\epsilon_n h"'] & L \arrow[r] \arrow[d, "-\epsilon_n h^\vee \mathrm{can}_L"] & 0 \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & L^\vee \arrow[r, "-(\mathrm{pr}_L\, d_{\ell+1})^\vee"'] & K_{\ell+1}^\vee \arrow[r, "-d_{\ell+2}^\vee"'] & \cdots \arrow[r, "-d_n^\vee"'] & K_n^\vee \arrow[r] & 0 \\
 & \text{deg } \ell & & & \text{deg } 1 & \text{deg } 0 & & &
\end{tikzcd}$$

As in the case $r=-1$, when $\underline{T}$ is a regular sequence, we have a quasi-isomorphism $p = p(A,\underline{T}) : C_\bullet(A,\underline{T}) \longrightarrow F_\bullet(A,\underline{T})$, where $F_\bullet(A,\underline{T})$ is now the complex

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & E \arrow[r, "\mathrm{pr}_L \cdot \bar{d}_{\ell+1}"] & L \arrow[r] & 0 \arrow[r] & \cdots \\
 & & \text{deg } 1 & \text{deg } 0. & &
\end{tikzcd}$$

Lemma 4.7. *Let $f : A \longrightarrow B$ be a morphism of rings. There is a natural isometry*

$$f^*\big(C_\bullet(A,\underline{T})\,,\,\Xi(A,\underline{T})\big) \xrightarrow{\simeq} \big(C_\bullet(B,\underline{f(T)})\,,\,\Xi(B,\underline{f(T)})\big)\,.$$

Proof. Straightforward, *cf.* Lemma 3.2. $\square$

Lemma 4.8. *The mapping cone of the morphism $\Xi(A,\underline{T})$ is isomorphic (as a complex) to $K_\bullet(A,\underline{T})[-2q]$.*

Proof. This is an easy direct computation, which we leave to the reader. It is clear in the cases where n is odd and it requires Lemma 4.1 for n even. In all four cases, we use the isomorphism Θ to replace the $K_j^\vee$ by K_{n-j} for $j \geq \ell+1$. $\square$

Remark 4.9. Note that we do not claim that the *symmetric* cone of the *symmetric* r-form $\Xi(A,\underline{T})$ is the Koszul complex *with its canonical form* $\Theta(A,\underline{T})$. This would be true, however, with a suitable choice of the sign ϵ_n. Instead of going into these computations, we shall use a simplifying trick: see 6.3.

5. The Koszul symmetric space $\mathbf{K}_X^{(n)}$ over $\mathbb{A}_X^n$

Let R be a ring. We apply the constructions of Section 3 to $A := R[T_1, \ldots, T_n]$, the polynomial ring in n variables over R, and to the sequence $\underline{T} := (T_1, \ldots, T_n)$. The reader can think of $R = \mathbb{Z}$ or $R = \mathbb{Z}[1/2]$, since these are the important cases, from which the rest will be deduced.

Definition 5.1. The *Koszul symmetric n-space* $\mathbf{K}_R^{(n)} = \big(\mathrm{K}_R^{(n)}, \Theta_R^{(n)}\big)$ over $\mathbb{A}_R^n$ is the symmetric n-space where $\mathrm{K}_R^{(n)} := K_\bullet(A, \underline{T})$ will be called *the Koszul complex over* $\mathbb{A}_R^n$ and where the symmetric n-form $\Theta_R^{(n)} := \Theta(A, \underline{T})$ is the one of Definition 3.1.

Remark 5.2. Pay attention: $\mathbf{K}_R^{(n)}$ is a symmetric n-space defined over the ring $A = R\,[T_1, \ldots, T_n]$ and not over the ring R, as the notation might suggest.

It is clear that the Koszul symmetric n-space behaves well with respect to base-change. More precise, let $f : R \longrightarrow S$ be a morphism of rings and let

$$\alpha_f \ : \ R\,[T_1, \ldots, T_n] \longrightarrow S\,[T_1, \ldots, T_n]$$

be the obvious induced morphism. Then, by Lemma 3.2 there is a natural isometry

$$\alpha_f^*(\mathbf{K}_R^{(n)}) \xrightarrow{\simeq} \mathbf{K}_S^{(n)} .$$

In particular, $\mathbf{K}_R^{(n)}$ is extended from $\mathbf{K}_{\mathbb{Z}}^{(n)}$. This justifies the following extension of Definition 5.1.

Definition 5.3. Let X be a scheme. We define the symmetric n-space

$$\mathbf{K}_X^{(n)} := \alpha_X^*(\mathbf{K}_{\mathbb{Z}}^{(n)})$$

where $\alpha_X : \mathbb{A}_X^n \longrightarrow \mathbb{A}_{\mathbb{Z}}^n$ is the base-change morphism, see 1.4. We call $\mathbf{K}_X^{(n)}$ the *Koszul symmetric n-space* over $\mathbb{A}_X^n$. Like before, we denote the underlying complex of free $\mathcal{O}_{\mathbb{A}_X^n}$-modules and its symmetric n-form by

$$\mathrm{K}_X^{(n)} = \alpha_X^*\big(\mathrm{K}_{\mathbb{Z}}^{(n)}\big) \qquad \text{and} \qquad \Theta_X^{(n)} = \alpha_X^*\big(\Theta_{\mathbb{Z}}^{(n)}\big) .$$

Remark 5.4. It is obvious from the definition that for any morphism of schemes $f : Y \longrightarrow X$ we have an isometry $\alpha_f^*\big(\mathbf{K}_X^{(n)}\big) \simeq \mathbf{K}_Y^{(n)}$ over $\mathbb{A}_Y^n$

Definition 5.5. By Remark 3.5, the complex $\mathrm{K}_X^{(n)}$ has support in the closed subset $\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n$ of $\mathbb{A}_X^n$. Therefore, the symmetric n-space $\mathbf{K}_X^{(n)}$ represents a Witt class

$$\kappa_X^{(n)} := [\mathbf{K}_X^{(n)}] \quad \in \mathrm{W}^n_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}(\mathbb{A}_X^n) .$$

Remark 5.6. Note that there are several choices of signs in the definition of the symmetric n-space $\mathbf{K}_X^{(n)}$. Other sign conventions give the same space or its negative, but the results of this work are clearly independent of such choices.

6. The half-Koszul symmetric space $\mathbf{B}_X^{(n)}$ over $\mathbb{U}_X^n$

Definition 6.1. Let R be a ring. We now apply the splitting of Section 4 to the space $\mathbf{K}_R^{(n)}$ of Section 5. As above, we put $A := R\,[T_1, \dots, T_n]$ and $\underline{T} := (T_1, \dots, T_n)$. We define

$$\mathrm{C}_R^{(n)} := C_\bullet(A, \underline{T}) \qquad \text{and} \qquad \Xi_R^{(n)} := \Xi(A, \underline{T})$$

as defined in 4.3 to 4.6. For any scheme X we define

$$\mathrm{C}_X^{(n)} := \alpha_X^*(\mathrm{C}_\mathbb{Z}^{(n)}) \qquad \text{and} \qquad \Xi_X^{(n)} := \alpha_X^*(\Xi_\mathbb{Z}^{(n)})$$

where $\alpha_X : \mathbb{A}_X^n \longrightarrow \mathbb{A}_\mathbb{Z}^n$ is the base-change morphism. This coincides with the above in the affine case by Lemma 4.7. For all $n \in \mathbb{N}$, the symmetric r-pair $(\mathrm{C}_X^{(n)}, \Xi_X^{(n)})$ on $\mathbb{A}_X^n$ will be denoted by $\mathbf{C}_X^{(n)}$.

Remark 6.2. We have the following facts:

1. The pair $\mathbf{C}_X^{(n)}$ is a symmetric r-pair (see Definition 2.2) indeed.
2. If $f : Y \longrightarrow X$ is a morphism of schemes then there is a natural isometry

$$\alpha_f^*(\mathbf{C}_X^{(n)}) \xrightarrow{\simeq} \mathbf{C}_Y^{(n)}\,.$$

6.3. *The symmetric cone of* $\mathbf{C}_X^{(n)}$.

Instead of calculating $\mathrm{cone}(\mathbf{C}_X^{(n)})$ directly (which is possible, but cumbersome) we take full advantage of Lemma 3.4. More precisely, we use the fact that any quasi-isomorphism $K_\bullet(\mathbb{Z}, \underline{T}) \longrightarrow \mathcal{D}_{\mathbb{Z}[T_1,\dots,T_n]}(K_\bullet(\mathbb{Z}, \underline{T}))[n]$ is equal to the symmetric n-form $\pm\Theta_\mathbb{Z}^{(n)}$ in $\mathbb{D}^\mathrm{b}(\mathrm{VB}_{\mathbb{Z}[T_1,\dots,T_n]})$.

So let for a moment $R = \mathbb{Z}$ and $A = \mathbb{Z}[T_1, \dots, T_n]$. We abbreviate $K_\bullet := K_\bullet(A, \underline{T})$ and $\Theta = \Theta(A, \underline{T})$. We get from Lemma 4.8 the following commutative diagram (where $\mathcal{D} = \mathcal{D}_A$)

$$\begin{tikzcd}
\mathrm{C}_\mathbb{Z}^{(n)} \arrow[r, "\Xi_\mathbb{Z}^{(n)}"] \arrow[d, "(-1)^{\frac{r(r+1)}{2}}\varpi_{\mathrm{C}_\mathbb{Z}^{(n)}}"] & \mathcal{D}(\mathrm{C}_\mathbb{Z}^{(n)})[r] \arrow[r, "u"] \arrow[d, "="] & K_\bullet[-2q] \arrow[r, "v"] & \mathrm{C}_\mathbb{Z}^{(n)}[1] \arrow[d, "(-1)^{\frac{r(r+1)}{2}}\varpi_{\mathrm{C}_\mathbb{Z}^{(n)}}[1]"'] \\
\mathcal{D}\mathcal{D}(\mathrm{C}_\mathbb{Z}^{(n)}) \arrow[r, "\mathcal{D}(\Xi_\mathbb{Z}^{(n)})[r]"'] & \mathcal{D}(\mathrm{C}_\mathbb{Z}^{(n)})[r] \arrow[r, "-\mathcal{D}(v)[r+1]"'] & \mathcal{D}(K_\bullet[-2q])[r+1] \arrow[r, "(-1)^r\mathcal{D}(u)[r+1]"'] & \mathcal{D}\mathcal{D}(\mathrm{C}_\mathbb{Z}^{(n)})[1]
\end{tikzcd} \tag{10}$$

whose rows are exact triangles for all $n \in \mathbb{N}$ (the bottom row is the dual of the upper one, shifted r times). By the very basic properties of triangulated categories there exists an isomorphism

$$\varsigma \,:\, K_\bullet[-2q] \longrightarrow \mathcal{D}(K_\bullet[-2q])[r+1] = \big(\mathcal{D}(K_\bullet)[n]\big)[-2q]\,,$$

in $\mathbb{D}^\mathrm{b}(\mathrm{VB}_A)$ such that diagram (10) commutes. By Lemma 3.4 the isomorphism ς is equal to $\pm\Theta[-2q]$. Replacing if necessary $\Xi_\mathbb{Z}^{(n)}$ by $-\Xi_\mathbb{Z}^{(n)}$, *i.e.* replacing ϵ_n by $-\epsilon_n$ in the definition of $\mathbf{C}_R^{(n)}$, we can assume that $\varsigma = \Theta[-2q]$ for all $n \in \mathbb{N}$, *i.e.* $(K_\bullet[-2q], \varsigma) = \mathbf{K}_\mathbb{Z}^{(n)}[-2q]$ for all $n \in \mathbb{N}$.

We fix ϵ_n as explained above.

We can now calculate $\mathrm{cone}(\mathbf{C}^{(n)}_X)$ for any scheme X and any $n \in \mathbb{N}$. The pull-back via the base-change morphism $\alpha_X : \mathbb{A}^n_X \longrightarrow \mathbb{A}^n_{\mathbb{Z}}$ of diagram (10) above is a cone diagram for the symmetric r-form $\alpha^*_X(\mathbf{C}^{(n)}_{\mathbb{Z}})$. By Lemma 4.7 we have an isometry $\mathbf{C}^{(n)}_X \simeq \alpha^*_X(\mathbf{C}^{(n)}_{\mathbb{Z}})$ and so we get $\mathrm{cone}(\mathbf{C}^{(n)}_X) \simeq \alpha^*_X(\mathbf{K}^{(n)}_{\mathbb{Z}}[-2q]) \simeq \alpha^*_X(\mathbf{K}^{(n)}_{\mathbb{Z}})[-2q]$ (*cf.* Lemma B.1 for the later isometry). We have proven:

Theorem 6.4. *With this choice of ϵ_n, the cone of the symmetric pair $\mathbf{C}^{(n)}_X$ is the Koszul symmetric space shifted as follows:*

$$\mathrm{cone}(\mathbf{C}^{(n)}_X) \,=\, \mathbf{K}^{(n)}_X[-2q]\,.$$

□

In particular, $\Xi^{(n)}_X|_{\mathbb{U}^n_X}$ is an isomorphism because the homology of $K_\bullet|_{\mathbb{U}^n_X}$ vanishes.

Definition 6.5. Let X be a scheme. The symmetric r-space

$$\mathbf{B}^{(n)}_X := \mathbf{C}^{(n)}_X|_{\mathbb{U}^n_X}$$

will be called *the half-Koszul space* over the scheme X. Its Witt class is denoted by

$$\varepsilon^{(n)}_X := [\mathbf{B}^{(n)}_X] \quad \in \mathrm{W}^r(\mathbb{U}^n_X)\,.$$

The following is obvious (*cf.* Remark 6.2).

Lemma 6.6. *Let $f : Y \longrightarrow X$ be a morphism of schemes. Then there is a natural isometry*

$$\upsilon^*_f(\varepsilon^{(n)}_X) \xrightarrow{\simeq} \varepsilon^{(n)}_Y\,.$$

□

By the main result of [3] we know that $\mathbf{B}^{(n)}_X$ is Witt equivalent to a space living on a short complex (see 2.4). In fact, $\mathbf{B}^{(n)}_X$ is not only Witt equivalent, but isometric to such a space on a "short complex".

We use the notation of 4.3–4.6 with $R = \mathbb{Z}$, *i.e.* $A = \mathbb{Z}[T_1, \dots, T_n]$ is the polynomial ring in n variables over $\mathbb{Z}$, $\underline{T} = (T_1, \dots, T_n)$, and $K_\bullet = K_\bullet(A, \underline{T})$ is the Koszul complex of the sequence $\underline{T}$ over A. As in 4.3–4.6 we denote the differential of this Koszul complex by $d_\bullet$ and set

$$E = E(A, \underline{T}) = \mathrm{Coker}\left(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \right).$$

Note that $\underline{T}$ is a regular sequence and so $K_\bullet$ is a finite free resolution of $\mathbb{Z} \simeq A/I$, where I is the ideal generated by $\underline{T}$. It follows that $K_\bullet(A, \underline{T})|_{\mathrm{Spec}\, A[T_i^{-1}]}$ is a split exact sequence and so

$$\mathcal{E}^{(n)}_{\mathbb{Z}} \;:=\; \mathrm{Coker}\left(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \right)\Big|_{\mathbb{U}^n_{\mathbb{Z}}} \;=\; E\big|_{\mathbb{U}^n_{\mathbb{Z}}} \simeq \mathrm{Ker}\, d_\ell\big|_{\mathbb{U}^n_{\mathbb{Z}}} \tag{11}$$

is a locally free $\mathcal{O}_{\mathbb{U}^n_{\mathbb{Z}}}$-module of rank $\sum\limits_{i=0}^{\ell}(-1)^i\binom{n}{\ell-i} \;=\; \binom{n-1}{n-\ell-1}$. Clearly the same is true for the pull-back

$$\mathcal{E}^{(n)}_X \;:=\; \upsilon^*_X(\mathcal{E}^{(n)}_{\mathbb{Z}})\,, \tag{12}$$

where $\upsilon_X : \mathbb{U}^n_X \longrightarrow \mathbb{U}^n_{\mathbb{Z}}$ is induced by base change, see (4). Note that we have

$$\mathcal{E}^{(n)}_X \;=\; \mathrm{Coker}\left(\mathrm{K}^{(n)}_{X,\ell+2} \;\longrightarrow\; \mathrm{K}^{(n)}_{X,\ell+1}\right)\Big|_{\mathbb{U}^n_X} \;\simeq\; \mathrm{Ker}\left(\mathrm{K}^{(n)}_{X,\ell} \;\longrightarrow\; \mathrm{K}^{(n)}_{X,\ell-1}\right)\Big|_{\mathbb{U}^n_X}\,,$$

where $\mathrm{K}^{(n)}_X = \mathrm{K}^{(n)}_{X,\bullet} = \alpha^*_X(\mathrm{K}^{(n)}_{\mathbb{Z}})$ (see Definition 5.3).

We consider now the case n odd and n even separately.

6.7. *The space* $\mathbf{B}_X^{(n)}$ *if* $n = 2\ell+1$ *is odd, i.e.* $r = 0$ *or* $r = 2$.

Since the functor $(-)^\vee = \mathrm{Hom}_A(-\,,A)$ is left exact we have $E^\vee = \mathrm{Ker}\,\big(K_{\ell+1}^\vee \xrightarrow{d_{\ell+2}^\vee} K_{\ell+2}^\vee\big)$ and hence a well defined homomorphism

$$\varphi_{\mathbb{Z}}^{(n)} \;:=\; \epsilon_n\cdot\theta_\ell\cdot \bar d_{\ell+1}\big|_{\mathbb{U}^n_{\mathbb{Z}}} \;:\; \mathcal{E}_{\mathbb{Z}}^{(n)} \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{U}^n_{\mathbb{Z}}}}(\mathcal{E}_{\mathbb{Z}}^{(n)},\mathcal{O}_{\mathbb{U}^n_{\mathbb{Z}}}) \;=\; E^\vee\big|_{\mathbb{U}^n_{\mathbb{Z}}}$$

which is $(-1)^\ell$-symmetric, where $\bar d_{\ell+1}$ is the unique morphism $K_{\ell+1}\longrightarrow E$, such that $d_{\ell+1} = \bar d_{\ell+1}\cdot \mathrm{pr}_E$ (*cf.* (8)). We set

$$(\mathcal{F}_{\mathbb{Z}}^{(n)},\phi_{\mathbb{Z}}^{(n)}) \;:=\; \begin{cases} c_0(\mathcal{E}_{\mathbb{Z}}^{(n)},\varphi_{\mathbb{Z}}^{(n)}) & \text{if } \ell \text{ is even} \\[2ex] c_0(\mathcal{E}_{\mathbb{Z}}^{(n)},\varphi_{\mathbb{Z}}^{(n)})[1] & \text{if } \ell \text{ is odd.} \end{cases}$$

This is a symmetric r-pair. Recall now from 4.3–4.6 that the projection $\mathrm{pr}_E : K_{\ell+1}\longrightarrow E$ induces a quasi-isomorphism (hence an isomorphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$)

$$p|_{\mathbb{U}^n_{\mathbb{Z}}} \;=\; p(\mathbb{Z},\underline{T})|_{\mathbb{U}^n_{\mathbb{Z}}} \;:\; \mathrm{C}_{\mathbb{Z}}^{(n)} \xrightarrow{\;\simeq\;} c_0(\mathcal{E}_{\mathbb{Z}}^{(n)}) \qquad (\text{respectively } \mathrm{C}_{\mathbb{Z}}^{(n)} \xrightarrow{\;\simeq\;} c_0(\mathcal{E}_{\mathbb{Z}}^{(n)})[1]),$$

which is easily seen to be an isometry $\mathrm{B}_{\mathbb{Z}}^{(n)} \xrightarrow{\;\simeq\;} (\mathcal{F}_{\mathbb{Z}}^{(n)},\phi_{\mathbb{Z}}^{(n)})$. It follows that $\phi_{\mathbb{Z}}^{(n)}$ is an isomorphism and so $(\mathcal{F}_{\mathbb{Z}}^{(n)},\phi_{\mathbb{Z}}^{(n)})$ a symmetric r-space. In particular $(\mathcal{E}_{\mathbb{Z}}^{(n)},\varphi_{\mathbb{Z}}^{(n)})$ is a $(-1)^\ell$-symmetric space over $\mathbb{U}^n_{\mathbb{Z}}$. Applying the pull-back υ_X^* we get:

(i) *The pair*
$$(\mathcal{E}_X^{(n)},\varphi_X^{(n)}) \;:=\; \upsilon_X^*(\mathcal{E}_{\mathbb{Z}}^{(n)},\varphi_{\mathbb{Z}}^{(n)})$$
is a $(-1)^\ell$*-symmetric space over* $\mathbb{U}^n_X$.

(ii) *The half Koszul space* $\mathbf{B}_X^{(n)}$ *is isometric to the short symmetric* r*-space:*

$$\mathbf{E}_X^{(n)} \;:=\; (\mathcal{F}_X^{(n)},\phi_X^{(n)}) \;:=\; \upsilon_X^*(\mathcal{F}_{\mathbb{Z}}^{(n)},\phi_{\mathbb{Z}}^{(n)}) \;=\; \begin{cases} c_0(\mathcal{E}_X^{(n)},\varphi_X^{(n)}) & \textit{if } \ell \textit{ is even} \\[2ex] c_0(\mathcal{E}_X^{(n)},\varphi_X^{(n)})[1] & \textit{if } \ell \textit{ is odd.} \end{cases}$$

Example 6.8. If $n=1$ then $(\mathcal{E}_X^{(n)},\varphi_X^{(n)})$ is the symmetric space

$$\mathcal{O}_{\mathbb{U}^n_X} \xrightarrow{\;\cdot T\;} \mathcal{O}_{\mathbb{U}^n_X} = \mathcal{H}om_{\mathcal{O}_{\mathbb{U}^n_X}}(\mathcal{O}_{\mathbb{U}^n_X},\mathcal{O}_{\mathbb{U}^n_X})\,.$$

6.9. *The space* $\mathbf{B}_X^{(n)}$ *if* $n = 2\ell$ *is even, i.e.* $r=-1$ *or* $r = 1$.

We fix $L, M \subset K_\ell$ and $\lambda : L \xrightarrow{\;\simeq\;} M^\vee$ as in Lemma 4.1 (with $R=\mathbb{Z}$), and let $\mathrm{pr}_L : K_\ell \longrightarrow L$ and $\mathrm{pr}_M : K_\ell \longrightarrow M$ be the respective projections. We denote $\mathcal{L} := L|_{\mathbb{U}^n_{\mathbb{Z}}}$ and $\mathrm{pr}_{\mathcal{L}} := \mathrm{pr}_L|_{\mathbb{U}^n_{\mathbb{Z}}} : K_\ell|_{\mathbb{U}^n_{\mathbb{Z}}} \longrightarrow \mathcal{L}$.

On the complex $\mathcal{F}_{\mathbb{Z}}^{(n)} := F_\bullet(A,\underline{T})\big|_{\mathbb{U}^n_{\mathbb{Z}}}$ we have the following symmetric r-form:

$$\begin{tikzcd}
\mathcal{F}_{\mathbb{Z}}^{(n)} = \arrow[d, "\phi_{\mathbb{Z}}^{(n)}"', ":="] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{E}_{\mathbb{Z}}^{(n)} \arrow[rr, "\mathrm{pr}_{\mathcal{L}}\cdot\bar d_{\ell+1}|_{\mathbb{U}^n_{\mathbb{Z}}}"] \arrow[d, "\epsilon_n\bar h|_{\mathbb{U}^n_{\mathbb{Z}}}"'] & & \mathcal{L} \arrow[r] \arrow[d, "(-1)^\ell\epsilon_n(\bar h|_{\mathbb{U}^n_{\mathbb{Z}}})^\vee\,\mathrm{can}_{\mathcal{L}}"'] & {} \\
\mathcal{D}_{\mathbb{U}^n_{\mathbb{Z}}}(\mathcal{F}_{\mathbb{Z}}^{(n)})[(-1)^{\ell+1}] = & \cdots \arrow[r] & 0 \arrow[r] & \mathcal{L}^\vee \arrow[rr, "(\mathrm{pr}_{\mathcal{L}}\cdot\bar d_{\ell+1}|_{\mathbb{U}^n_{\mathbb{Z}}})^\vee"'] & & (\mathcal{E}_{\mathbb{Z}}^{(n)})^\vee \arrow[r] & {}
\end{tikzcd}$$

if $r=-1$	deg 0	deg -1
if $r\;=\;1$	deg 1	deg 0

where $\bar{h} = \lambda \cdot \mathrm{can}_M \cdot \mathrm{pr}_M \cdot \bar{d}_{\ell+1}$. Since $d_{\ell+1}|_{\mathbb{U}^n_{\mathbb{Z}}} = (\bar{d}_{\ell+1} \cdot \mathrm{pr}_E)|_{\mathbb{U}^n_{\mathbb{Z}}}$ we see that the quasi-isomorphism $p|_{\mathbb{U}^n_{\mathbb{Z}}} : \mathcal{F}^{(n)}_{\mathbb{Z}} \xrightarrow{\simeq} \mathrm{C}^{(n)}_{\mathbb{Z}}$ is an isometry $\mathbf{B}^{(n)}_{\mathbb{Z}} \xrightarrow{\simeq} (\mathcal{F}^{(n)}_{\mathbb{Z}}, \phi^{(n)}_{\mathbb{Z}})$, and so $\phi^{(n)}_{\mathbb{Z}}$ is an isomorphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_{\mathbb{Z}})$. Therefore $(\mathcal{F}^{(n)}_{\mathbb{Z}}, \phi^{(n)}_{\mathbb{Z}})$ is a symmetric r-space over $\mathbb{U}^n_{\mathbb{Z}}$. Applying the pull-back υ^*_X we get:

The half Koszul space $\mathbf{B}^{(n)}_X$ is isometric to the short symmetric r-space:

$$\mathbf{E}^{(n)}_X := (\mathcal{F}^{(n)}_X, \phi^{(n)}_X) := \upsilon^*_X(\mathcal{F}^{(n)}_{\mathbb{Z}}, \phi^{(n)}_{\mathbb{Z}}).$$

Remark 6.10. We give in Appendix A a proof of the following fact. If $n \geq 3$ then the locally free $\mathcal{O}_{\mathbb{U}^n_X}$-module $\mathcal{E}^{(n)}_X$ can not be extended to a locally free $\mathcal{O}_{\mathbb{A}^n_X}$-module, and hence in particular is not extended from X. Of course this is wrong for $n = 1, 2$, since then $\ell = 0$ and therefore $\mathcal{E}^{(n)}_{\mathbb{Z}} \simeq K_n(\mathbb{Z}, \underline{T})\big|_{\mathbb{U}^n_{\mathbb{Z}}}$ which is free.

It follows already from this that it is impossible to extend the symmetric r-space $\mathbf{E}^{(n)}_X$ to $\mathbb{A}^n_X$ as long as $n \geq 3$. We will see in Theorem 8.2 that even more is true. It is impossible to extend $\mathbf{E}^{(n)}_X$ up to Witt equivalence to $\mathbb{A}^n_X$, *i.e.* $[\mathbf{E}^{(n)}_X]$ is not in the image of $\mathrm{W}^r(\mathbb{A}^n_X) \xrightarrow{\iota^*_X} \mathrm{W}^r(\mathbb{U}^n_X)$, and this for any $n \geq 1$.

7. Witt groups of the punctured affine space

Recall the notations of Section 1, like formula (4), defining $r \in \{-1, 0, 1, 2\}$ by $n = 4\,q + r + 1$. We begin with an easy application of triangular Witt theory:

Theorem 7.1. *Let X be a regular scheme containing $\frac{1}{2}$. There exists a split short exact sequence*

$$0 \longrightarrow \mathrm{W}^i(X) \xrightarrow{\sigma^*_X} \mathrm{W}^i(\mathbb{U}^n_X) \xrightarrow{\partial} \mathrm{W}^{i+1}_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X) \longrightarrow 0,$$

for all $i \in \mathbb{Z}$, where ∂ is the connecting homomorphism of the localization $\mathbb{U}^n_X \subset \mathbb{A}^n_X$. This sequence is natural in X in the obvious way. Moreover, a left inverse to σ^*_X is given by $\gamma^* : \mathrm{W}^i(\mathbb{U}^n_X) \longrightarrow \mathrm{W}^i(X)$ for any X-point $\gamma : X \to \mathbb{U}^n_X$, i.e. any morphism $\gamma : X \to \mathbb{U}^n_X$ such that $\sigma_X \circ \gamma = \mathrm{id}_X$.

Proof. Consider the commutative (plain) diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \mathrm{W}^i_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X) \arrow[r] & \mathrm{W}^i(\mathbb{A}^n_X) \arrow[r, "\iota^*_X"] & \mathrm{W}^i(\mathbb{U}^n_X) \arrow[r, "\partial"] \arrow[dl, dashed, bend left, "\gamma^*"] & \mathrm{W}^{i+1}_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X) \arrow[r] & \cdots \\
& & \mathrm{W}^i(X) \arrow[u, "\pi^*_X", "\simeq"'] \arrow[ur, "\sigma^*_X"] & & &
\end{tikzcd}$$

The long sequence is exact by localization, see 2.9, and the homomorphism π^*_X is an isomorphism by homotopy invariance [4, Thm. 3.1]. Now, for any X-point $\gamma : X \to \mathbb{U}^n_X$, for instance $(1, \ldots, 1)$, since $\sigma_X \circ \gamma = \mathrm{id}_X$, the homomorphism σ^*_X is split injective with the wanted left inverse. Hence the homomorphism ι^*_X is also split injective and the unlabeled morphism in the above diagram must be equal to zero, for all $i \in \mathbb{Z}$, in particular for $i + 1$. This, in turn, gives the surjectivity of ∂ and the result follows easily. $\square$

We want to apply "dévissage" to the relative groups $\mathrm{W}^{i+1}_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X)$ and we will need

Theorem 7.2. *Let X be a regular $\mathbb{Z}[1/2]$-scheme of finite Krull dimension. Consider the structure morphism $\pi_X : \mathbb{A}^n_X \longrightarrow \operatorname{Spec} X$. Then, the homomorphism*

$$\vartheta^{(n)}_X \ : \ \mathrm{W}^{i-n}(X) \longrightarrow \mathrm{W}^i_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X)\,, \qquad w \longmapsto \pi_X{}^*(w) \ \star \ \kappa^{(n)}_X$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proof. The affine case $X = \operatorname{Spec}(R)$ is [9, Thm. 9.3].

Since the homomorphism $\vartheta^{(n)}_X$ is given by the product with a "universal" Witt class $\kappa^{(n)}_X := [\mathbf{K}^{(n)}_X]$, we can deduce the global statement from the affine case by Mayer-Vietoris as follows. If a regular scheme

$$V_0 = V_1 \cup V_2$$

is covered by two open subschemes V_1 and V_2, we also have a covering of the corresponding affine spaces $\mathbb{A}^n$, in a compatible way with the closed subsets $\mathbb{A}^n \smallsetminus \mathbb{U}^n$. So, we get a Mayer-Vietoris long exact sequence [4] with supports, given in the first row of the diagram below, where we use the abbreviations $\mathbb{Y}^n(X) := \mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X$ for all schemes X and $V_3 := V_1 \cap V_2$:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{W}^i_{\mathbb{Y}^n(V_1)}(\mathbb{A}^n_{V_1}) \oplus \mathrm{W}^i_{\mathbb{Y}^n(V_2)}(\mathbb{A}^n_{V_2}) \arrow[r] & \mathrm{W}^i_{\mathbb{Y}^n(V_3)}(\mathbb{A}^n_{V_3}) \arrow[r, "\partial"] \arrow[dr, phantom, "\ddagger"] & \mathrm{W}^{i+1}_{\mathbb{Y}^n(V_0)}(\mathbb{A}^n_{V_0}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{W}^{i-n}(V_1) \oplus \mathrm{W}^{i-n}(V_2) \arrow[r] \arrow[u, "\vartheta^{(n)}_{V_1} \oplus \vartheta^{(n)}_{V_2}"] & \mathrm{W}^{i-n}(V_3) \arrow[r, "(-1)^n \partial"'] \arrow[u, "\vartheta^{(n)}_{V_3}"] & \mathrm{W}^{i-n+1}(V_0) \arrow[r] \arrow[u, "\vartheta^{(n)}_{V_0}"'] & \cdots
\end{tikzcd}$$

The second line is exact by the Mayer-Vietoris exact sequence for $V_0 = V_1 \cup V_2$. We claim that the diagram commutes. This is easy to check for the unmarked squares. For the square marked (‡) this follows from the following calculation. Let $w \in \mathrm{W}^{i-n}(V_3)$, then

$$\partial\Big(\pi^*_{V_3}(w) \star [\mathbf{K}^{(n)}_{V_3}]\Big) \ = \ \partial\Big(\pi^*_{V_3}(w) \star [\mathbf{K}^{(n)}_{V_0}]\Big) \ = \ (-1)^n \partial\big(\pi^*_{V_3}(w)\big) \star [\mathbf{K}^{(n)}_{V_0}]$$

The first equality holds without ∂ and uses the following fact : before computing the product of the class $[\mathbf{K}^{(n)}_{V_0}] \in \mathrm{W}^n_{\mathbb{Y}^n(V_0)}(\mathbb{A}^n_{V_0})$ with $\pi^*_{V_3}(w) \in \mathrm{W}^{i-n}(\mathbb{A}^n_{V_3})$, we can as well restrict $[\mathbf{K}^{(n)}_{V_0}]$ to $\mathbb{A}^n_{V_3}$ and then multiply; this restriction of $[\mathbf{K}^{(n)}_{V_0}]$ is precisely $[\mathbf{K}^{(n)}_{V_3}]$ by Remark 5.4. The second equality is a consequence of (6). It then suffices to apply naturality of the localization sequence to replace $\partial\big(\pi^*_{V_3}(w)\big)$ by $\pi^*_{V_0}\big(\partial(w)\big)$ and we have the claimed commutativity of (‡). The usual Five-Lemma gives the statement by induction on the number of open subschemes in an affine covering of the regular scheme X (recall Convention 1.3). □

Remark 7.3. We do not now whether $\vartheta^{(n)}_X$ is an isomorphism for more general schemes X, like *e.g.* regular (affine) schemes of infinite Krull dimension. The proof of [9, Thm. 9.3] uses coherent Witt groups and therefore only applies to regular rings of finite Krull dimension.

Theorem 7.4. *Let X be a $\mathbb{Z}[1/2]$-scheme. Let $1 \leq i \leq n$ be an integer and consider the X-point $\gamma_i : X \to \mathbb{U}^n_X \subset \mathbb{A}^n_X$ corresponding to $T_i = 1$ and $T_j = 0$ for all $j \neq i$.*

If $n \geq 2$, then the evaluation at this point of the Witt class $\varepsilon_X^{(n)} \in \mathrm{W}^r(\mathbb{U}_X^n)$ of the half-Koszul space is zero

$$\gamma_i^*(\varepsilon_X^{(n)}) = 0 \quad in \quad \mathrm{W}^r(X)\,.$$

Proof. Also denote by $\gamma_i : \mathrm{Spec}(\mathbb{Z}[1/2]) \longrightarrow \mathbb{U}^n_{\mathbb{Z}[1/2]}$ the "same" point. We have a commutative diagram

$$\begin{tikzcd} \mathrm{W}^r(\mathbb{U}^n_{\mathbb{Z}[1/2]}) \arrow[r, "\gamma_i^*"] \arrow[d, "\upsilon_X^*"'] & \mathrm{W}^r(\mathbb{Z}[1/2]) \arrow[d] \\ \mathrm{W}^r(\mathbb{U}^n_X) \arrow[r, "\gamma_i^*"'] & \mathrm{W}^r(X) \end{tikzcd}$$

with the obvious morphisms and we know from 6.6 that $\upsilon_X^*(\varepsilon^{(n)}_{\mathbb{Z}[1/2]}) = \varepsilon_X^{(n)}$. Therefore, it suffices to prove the result for $\mathbb{Z}[1/2]$.

If $r \neq 0$ this is trivially true because in this case $\mathrm{W}^r(\mathbb{Z}[1/2]) = 0$. In fact, since $\mathbb{Z}[1/2]$ is a Dedekind domain, we have by [6, Thm. 10.1] that $\mathrm{W}^{-1}(\mathbb{Z}[1/2]) = \mathrm{W}^2(\mathbb{Z}[1/2]) = 0$ and $\mathrm{W}^1(\mathbb{Z}[1/2]) \simeq \mathrm{Coker}\left(\mathrm{W}(\mathbb{Z}[1/2]) \xrightarrow{\sum \partial_p} \bigoplus_{p\neq 2} \mathrm{W}(\mathbb{Z}/\mathbb{Z}p)\right)$, where ∂_p is a second residue homomorphism associated with the prime number p. But this cokernel is also zero by the classical calculation of the Witt group of $\mathbb{Q}$, *cf. e.g.* [14, Thm. VI.6.11].

If $r = 0$, *i.e.* $n = 2\ell+1$ with $\ell \neq 0$ even this follows from the following lemma. $\square$

Lemma 7.5. *Let R be a ring and $n = 2\ell + 1$ with $\ell \geq 2$ even. Then $\gamma_i^*(\mathbf{B}_R^{(n)}) \simeq \gamma_i^*(\mathcal{E}_R^{(n)}, \varphi_R^{(n)})$ is trivial in $\mathrm{W}(R) \simeq \mathrm{W}^0(R)$.*

Proof. After renumbering we may assume $i = 1$. Let $K_\bullet = K_\bullet(A, \underline{T})$ be the Koszul complex of the regular sequence $\underline{T} = (T_1, \ldots, T_n)$ over $A = R[T_1, \ldots, T_n]$, and $\Theta = \Theta(A, \underline{T}) : K_\bullet \xrightarrow{\simeq} \mathcal{D}_A(K_\bullet)[n]$ the canonical symmetric n-form. Recall that the differential $d_s : K_s \longrightarrow K_{s-1}$ is then given by

$$e_{i_1} \wedge \ldots \wedge e_{i_s} \longmapsto \sum_{j=1}^{s} (-1)^{j-1} T_{i_j} \cdot e_{i_1} \wedge \ldots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \ldots \wedge e_{i_s}\,,$$

where $e_1, \ldots, e_n$ constitute a basis of A^n. We denote (*cf.* 1.4) ι_R the open immersion $\mathbb{U}^n_R \hookrightarrow \mathbb{A}^n_R$. Then we have

$$(C', \Xi') := \gamma_1^*(\mathbf{B}_R^{(n)}) = \gamma_1^* \iota_R^*(\mathbf{C}_R^{(n)})$$

and $K'_\bullet = \gamma_1^* \iota_R^*(K_\bullet)$ is the Koszul complex $K_\bullet(R, (1, 0, \ldots, 0))$ for the sequence $(1, 0, \ldots, 0) \subset R$. We denote the differential of this Koszul complex by d'. Note that this complex is split exact. The isomorphism of complexes $\Theta' := \gamma_1^* \iota_R^*(\Theta)$ is a symmetric n-form on $K'_\bullet$ and (C', Ξ') is the following symmetric 0-space:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & K'_n \arrow[r, "d'_n"] \arrow[d] & K'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & K'_{\ell+2} \arrow[r, "d'_{\ell+2}"] \arrow[d] & K'_{\ell+1} \arrow[r] \arrow[d, "\epsilon_n \Theta'_\ell \cdot d'_{\ell+1}"'] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & K'^{\vee}_{\ell+1} \arrow[r, "d'^{\vee}_{\ell+2}"'] & K'^{\vee}_{\ell+2} \arrow[r] & \cdots \arrow[r, "d'^{\vee}_n"'] & K'^{\vee}_n \arrow[r] & 0 \end{tikzcd}$$

deg ℓ deg 0

over R, where we have set $K_i'^\vee = \mathrm{Hom}_R(K_i', R)$. We give now a direct summand E' of $K'_{\ell+1}$, such that the projection $K'_{\ell+1} \longrightarrow E'$ induces a quasi-isomorphism $C'_\bullet \xrightarrow{\simeq} c_0(E')$ ($c_0 : \mathrm{VB}_R \longrightarrow \mathbb{D}^\mathrm{b}(\mathrm{VB}_R)$ the natural embedding).

The elements $v_i = 1 \otimes e_i$ ($i = 1, \ldots, n$) are a basis of $R \otimes_A A^n = \gamma_1^* \iota^*(A^n)$, and so the exterior products $v_{i_1} \wedge \ldots \wedge v_{i_s}$ ($1 \le i_1 < \ldots < i_s \le n$) are free generators of $K'_s \simeq \wedge^s R^n$. The differential d'_s acts on them as follows:

$$d'_s(v_{i_1} \wedge \ \ldots \ \wedge v_{i_s}) \ = \ \begin{cases} 0 & 2 \le i_1 < \ \ldots \ < i_s \le n \\ v_{i_2} \wedge \ldots \wedge v_{i_s} & 1 = i_1 < i_2 < \ \ldots \ < i_s \le n\,, \end{cases}$$

therefore the R-module $E' \ := \ v_1 \wedge \big(\bigwedge^{\ell} R^n \big) \subset K'_{\ell+1}$ is isomorphic to $\mathrm{Coker}\, d'_{\ell+2}$. Hence $c_0(E') \simeq C'_\bullet$ because $C'_\bullet$ has non vanishing homology only in degree 0, and we have an isometry $c_0(E', \varphi') \simeq (C', \Xi')$, where $\varphi' := \epsilon_n(\Theta'_l \cdot d'_{\ell+1})|_{E'}$.

Consider now the following free submodule of E':

$$M' \ := \ (v_1 \wedge v_2) \wedge \big(\bigwedge^{\ell-1} R^n \big)\,.$$

We claim that M' is a totally isotropic subspace of (E', φ'). From this the lemma follows because $\mathrm{rank}\, M' = \frac{1}{2} \mathrm{rank}\, E'$ and so (E', φ') is hyperbolic by [1, I Thm. 4.6].

To see this we use the description of Θ given in Remark 3.3. Let $\omega : \wedge^n A^n \xrightarrow{\simeq} A$ be as in this remark and $\omega' := \mathrm{id}_R \otimes \omega$. Then $\omega'(v_1 \wedge \ldots \wedge v_n) = 1$ and $\Theta'_\ell(x)(y) = \pm\omega'(x \wedge y)$ for all $x \in K'_\ell$ and $y \in K'_{\ell+1}$. If now $x, y \in M'$ then $y = v_1 \wedge v_2 \wedge y'$ and $d'_{\ell+1}(x) = v_2 \wedge x'$ for some $x', y' \in \wedge^{\ell-1} R^n$, and so $\pm\varphi'(x)(y) = \omega'(d'_{\ell+1}(x) \wedge y) = 0$ since $v_2 \wedge v_2 = 0$. ☐

Theorem 7.6. *Let X be a regular $\mathbb{Z}[1/2]$-scheme. The composition of the connecting homomorphism with the 4-periodicity isomorphism:*

$$\mathrm{W}^r(\mathbb{U}^n_X) \xrightarrow{\ \partial\ } \mathrm{W}^{r+1}_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X) \xrightarrow[\simeq]{\ \tau^q\ } \mathrm{W}^n_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X)\,,$$

maps the Witt class $\varepsilon^{(n)}_X \in \mathrm{W}^r(\mathbb{U}^n_X)$ of the half-Koszul space to the Witt class $\kappa^{(n)}_X \in \mathrm{W}^n_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X)$ of the Koszul space over $\mathbb{A}^n_X$.

Proof. Recall that we always have $n = 4\,q + r + 1$. The statement is a direct consequence of Theorem 6.4, using the definition of the connecting homomorphism ∂ via the symmetric cone 2.9 and the fact that $\varepsilon^{(n)}_X = \big[\mathbf{C}^{(n)}_X|_{\mathbb{U}^n_X}\big]$ by Definition 6.5. ☐

Theorem 7.7. *Let X be a regular $\mathbb{Z}[1/2]$-scheme. For all $i \in \mathbb{Z}$, define the following homomorphism:*

$$\rho^{(n)}_X : \ \mathrm{W}^{i-r}(X) \longrightarrow \mathrm{W}^i(\mathbb{U}^n_X)\,, \qquad w \longmapsto \sigma_X{}^*(w) \ \star \ \varepsilon^{(n)}_X\,.$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{W}^{i-r}(X) & \xleftarrow[\simeq]{\ \tau^q\ } & \mathrm{W}^{i+1-n}(X) \\ {\scriptstyle \rho^{(n)}_X}\big\downarrow & & \big\downarrow{\scriptstyle \simeq\ \vartheta^{(n)}_X} \\ \mathrm{W}^i(\mathbb{U}^n_X) & \xrightarrow{\ \partial\ } & \mathrm{W}^{i+1}_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X) \end{array}$$

for all $i \in \mathbb{Z}$, where the isomorphism $\vartheta^{(n)}_X$ is the one of Theorem 7.2 and where τ is the 4-periodicity isomorphism.

Proof. Recall of course that $1-n+4\,q=r$ by (2). We have to show that

$$\partial\rho_X^{(n)}([x]) \;=\; \vartheta_X^{(n)}([\,x[-2q]\,]) \tag{13}$$

for all $[x]\in \mathrm{W}^{i-r}(X)$. Using the fact that $\sigma_X:\mathbb{U}_X^n \longrightarrow \operatorname{Spec} X$ composes

$$\mathbb{U}_X^n \xrightarrow{\iota_X} \mathbb{A}_X^n \xrightarrow{\pi_X} \operatorname{Spec} X\,,$$

we get:

$$\begin{aligned}\partial\rho_X^{(n)}([x]) &= \pi_X^*([x])\star\partial(\varepsilon_X^{(n)}) && \text{by equation (5)}\\ &= \pi_X^*([x])\star\tau^{-q}(\kappa_X^{(n)}) && \text{by Theorem 6.4.}\end{aligned}$$

But this is equal to the right hand side of (13) because:

$$\begin{aligned}\vartheta_X^{(n)}([\,x[-2q]\,]) &= \pi_X^*([x])[-2q]\;\star\,\kappa_X^{(n)} && \text{by Lemma B.1}\\ &= \pi_X^*([x])\;\star\,\tau^{-q}(\kappa_X^{(n)}) && \text{by Lemma B.3.}\end{aligned}$$

□

Corollary 7.8. *Let X be a regular $\mathbb{Z}[1/2]$-scheme. We have an isomorphism*

$$(\sigma_X^*,\rho_X^{(n)})\;:\;\mathrm{W}^i(X)\oplus\mathrm{W}^{i-r}(X)\xrightarrow{\simeq}\mathrm{W}^i(\mathbb{U}_X^n)\,.$$

for all $i\in\mathbb{Z}$.

Proof. From Theorem 7.1, it suffices to show that the homomorphism

$$\rho_X^{(n)}:\mathrm{W}^{i-r}(X)\longrightarrow\mathrm{W}^i(\mathbb{U}_X^n)$$

has the two following properties: first $\partial\cdot\rho_X^{(n)}$ is an isomorphism; secondly that $\gamma^*\cdot\rho_X^{(n)}$ is zero. The first one follows from Theorem 7.7 and the second one from the definition of $\rho_X^{(n)}$ and Theorem 7.4, since γ^* is a morphism of graded rings by [10, Thm. 3.2]. □

Remark 7.9. Note that this result generalizes the well-known calculation of the Witt group of a Laurent ring (*cf. e.g.* [13]).

As for the Laurent ring it is likely that the result is true for a bigger class of rings, *e.g.* all regular rings. But already in the Laurent ring case it fails to be true for all (commutative) rings as *loc. cit.* Examples 8.1 and 8.2 show.

To understand the ring structure on $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$, we need some properties of the symmetric spaces $\mathbf{K}_X^{(n)}$ and $\mathbf{B}_X^{(n)}$, which can be proven for non necessarily regular schemes as well. The case $n=1$, *i.e.* the "Laurent scheme" case, is well-known, so we have to deal with $n\geq 2$.

Theorem 7.10. *Let X be a $\mathbb{Z}[1/2]$-scheme. If $n\geq 2$ then the symmetric r-space $\mathbf{B}_X^{(n)}$ is locally trivial, i.e. for any $x\in\mathbb{U}_X^n$ we have $[(\mathbf{B}_X^{(n)})_x]=0$ in $\mathrm{W}^r(\mathcal{O}_{\mathbb{U}_X^n,x})$.*

Proof. Define for all $i\in\{1,\ldots,n\}$ the principal open $\mathbb{V}_X^n(i)$ of $\mathbb{A}_X^n$ given by the equation $T_i\neq 0$. Let $J\subseteq\{1,\ldots,n\}\subset\mathbb{N}$. We define

$$\mathbb{V}_X^n(J)\;:=\;\bigcup_{j\in J}\mathbb{V}_X^n(j)\;\subseteq\;\mathbb{U}_X^n=\mathbb{V}_X^n(\{1,\ldots,n\})$$

and denote $\iota_J:\mathbb{V}_X^n(J)\longrightarrow\mathbb{U}_X^n$ the corresponding open immersion. Since $n\geq 2$, we can cover $\mathbb{U}_X^n$ with the open subschemes $\mathbb{V}_X^n(J)$ with $|J|\leq n-1$. So it suffices to prove the following stronger result. □

Theorem 7.11. *With the above notations, if $|J| \leq (n-1)$ then $[\mathbf{B}_X^{(n)}|_{\mathbb{V}_X^n(J)}] = 0$ in $\mathrm{W}^r(\mathbb{V}_X^n(J))$.*

Proof. We easily reduce to the case $X = \operatorname{Spec}\mathbb{Z}[1/2]$. In this case we argue as follows. For brevity we set $R := \mathbb{Z}[1/2]$.

For J empty, the result is trivial since $\mathbb{V}_R^n(J) = \varnothing$ and its Witt group is zero. So we assume that $J \neq \varnothing$. Consider the closed complement $\mathbb{Y}_R(J) := \mathbb{A}_R^n \smallsetminus \mathbb{V}_R^n(J)$ of $\mathbb{V}_R^n(J) \subset \mathbb{U}_R^n$. Note that $\mathbb{A}_R^n \smallsetminus \mathbb{U}_R^n \subset \mathbb{Y}_R(J)$. By Theorem 7.1, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{W}^r(R) \arrow[r, "\sigma_R^*"] \arrow[d, "="] & \mathrm{W}^r(\mathbb{U}_R^n) \arrow[r, "\partial"] \arrow[d, "\iota_J^*"] & \mathrm{W}^{r+1}_{\mathbb{A}_R^n \smallsetminus \mathbb{U}_R^n}(\mathbb{A}_R^n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{W}^r(R) \arrow[r, "\iota_J^*\sigma_R^*"] & \mathrm{W}^r(\mathbb{V}_R^n(J)) \arrow[r, "\partial"] & \mathrm{W}^{r+1}_{\mathbb{Y}_R(J)}(\mathbb{A}_R^n) \arrow[r] & 0\,.
\end{tikzcd}$$

We get from the right-hand commutative square, from Theorem 7.6, and from Proposition 3.6 that $\partial\big(\iota_J^*(\varepsilon_R^{(n)})\big) = 0$. Therefore, by exactness of the above second row, there exists a unique class $w \in \mathrm{W}^r(R)$ such that $\iota_J^*(\mathbf{B}_R^{(n)}) = \iota_J^*(\sigma_R^*(w))$. In fact, $w = \gamma^*(\iota_J^*(\mathbf{B}_R^{(n)}))$ for any R-point $\gamma : \operatorname{Spec}(R) \to \mathbb{V}_R^n(J)$, which exists by assumption $J \neq \varnothing$. Choose $j \in J$ and define the R-point $\gamma : \operatorname{Spec}(R) \to \mathbb{V}_R^n(J)$ to be given by $T_j = 1$ and $T_i = 0$ for $i \neq j$. Since $w = \gamma^*(\iota_J^*(\mathbf{B}_R^{(n)})) = (\iota_J \cdot \gamma)^*(\mathbf{B}_R^{(n)}))$ and since $\iota_J \cdot \gamma : \operatorname{Spec}(R) \to \mathbb{U}_R^n$ is simply the R-point γ_j of Theorem 7.4, we conclude from it that $w = 0$. Hence $\iota_J^*(\mathbf{B}_R^{(n)}) = 0$ as wanted. $\square$

Remark 7.12. The statement of Theorem 7.10 is obviously not true for $n = 1$. The proof fails for $n = 1$ because then $\mathbb{V}_X^n(J) = \varnothing$ for any J such that $|J| \leq n - 1 = 0$ and hence we can not cover $\mathbb{U}_X^n$ with these.

It follows from this theorem above and [5, Thm. 4.2] that if $n \geq 2$ the space $\varepsilon_X^{(n)}$ is nilpotent in $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$. We prove a more precise result.

Theorem 7.13. *Let X be a $\mathbb{Z}[\frac{1}{2}]$-scheme. Assume that $n \geq 2$ then*

$$(\varepsilon_X^{(n)})^2 \;=\; \varepsilon_X^{(n)} \star \varepsilon_X^{(n)} \;=\; 0$$

in $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$. If $n = 1$ then $(\varepsilon_X^{(n)})^2 = 1$ in $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$.

Proof. If $n = 1$ this is an obvious consequence of Example 6.8.

So let $n \geq 2$. Since $\alpha_X^* : \mathrm{W}^{\mathrm{tot}}(\mathbb{Z}[1/2]) \longrightarrow \mathrm{W}^{\mathrm{tot}}(X)$ is a morphism of graded rings (*cf.* [10, Thm. 3.2]) it is enough to prove this for the affine scheme $X = \operatorname{Spec}\mathbb{Z}[1/2]$.

Because we assume $n \geq 2$ there exists non-empty subsets $J_1, J_2 \subset \{1, \dots, n\}$ with $J_1 \neq J_2$ and $J_1 \cup J_2 = \{1, \dots, n\}$. We define $\mathbb{V}^n_{\mathbb{Z}[1/2]}(J_i) \subseteq \mathbb{U}^n_{\mathbb{Z}[1/2]}$ as in the proof of Theorem 7.10 above and let $\mathbb{Y}^n_{\mathbb{Z}[1/2]}(J_i) := \mathbb{U}^n_{\mathbb{Z}[1/2]} \setminus \mathbb{V}^n_{\mathbb{Z}[1/2]}(J_i)$ be the complement ($i = 1, 2$). Note that $J_1 \cup J_2 = \{1, \dots, n\}$ implies $\mathbb{Y}^n_{\mathbb{Z}[1/2]}(J_1) \cap \mathbb{Y}^n_{\mathbb{Z}[1/2]}(J_2) = \emptyset$.

By Theorem 7.11 we know that $[\mathbf{B}^{(n)}_{\mathbb{Z}[1/2]}|_{\mathbb{V}^n_{\mathbb{Z}[1/2]}(J_i)}] = 0$ for $i = 1, 2$. Therefore by the localization sequence there exists $x_i \in \mathrm{W}^r_{\mathbb{Y}^n_{\mathbb{Z}[1/2]}(J_i)}(\mathbb{U}^n_{\mathbb{Z}[1/2]})$ with $x_i = \varepsilon^{(n)}_{\mathbb{Z}[1/2]}$ in $\mathrm{W}^r(\mathbb{U}^n_{\mathbb{Z}[1/2]})$ for $i = 1, 2$, and so $x_1 \star x_2 = (\varepsilon^{(n)}_{\mathbb{Z}[1/2]})^2$ in $\mathrm{W}^{2r}(\mathbb{U}^n_{\mathbb{Z}[1/2]})$. But the space $x_1 \star x_2$ lives on a complex with support in $\mathbb{Y}^n_{\mathbb{Z}[1/2]}(J_1) \cap \mathbb{Y}^n_{\mathbb{Z}[1/2]}(J_2) = \emptyset$ and so $(\varepsilon^{(n)}_{\mathbb{Z}[1/2]})^2 = 0$. $\square$

Denote $\mathrm{W}^{\mathrm{tot}}(X)[\varepsilon]$ the graded skew polynomial ring in one variable ε of degree r over the graded ring $\mathrm{W}^{\mathrm{tot}}(X)$. Recall that this means that $c\cdot\varepsilon=(-1)^{r\deg c}(\varepsilon\cdot c)$ for a homogeneous element $c\in\mathrm{W}^{\mathrm{tot}}(X)$. We have a homogeneous homomorphism of graded rings given by

$$\mathrm{W}^{\mathrm{tot}}(X)[\varepsilon]\longrightarrow\mathrm{W}^{\mathrm{tot}}(\mathbb{U}^n_X)\qquad\sum_{i=0}^{m}c_i\varepsilon^i\longmapsto\sum_{i=0}^{m}\sigma_X^*(c_i)\star(\varepsilon_X^{(n)})^i\,.$$

Using this morphism we can restate Corollary 7.8 and Theorem 7.13 as follows

Theorem 7.14. *Let X be a regular scheme of finite Krull dimension over $\mathbb{Z}[1/2]$. Then we have an isomorphism of graded rings*

$$\left.\begin{array}{ll}\text{if } n\geq 2: & \mathrm{W}^{\mathrm{tot}}(X)[\varepsilon]\Big/(\varepsilon^2)\\[1ex] \text{if } n=1: & \mathrm{W}^{\mathrm{tot}}(X)[\varepsilon]\Big/(\varepsilon^2-1)\end{array}\right\}\xrightarrow{\simeq}\mathrm{W}^{\mathrm{tot}}(\mathbb{U}^n_X)\,.$$

8. Witt non-triviality of the (half) Koszul spaces

Theorem 8.1. *Let X be a scheme which is not of equicharacteristic 2. Then the Witt class of the symmetric n-space $\mathbf{K}_X^{(n)}$ is non-trivial in the Witt group with support $\mathrm{W}^n_{\mathbb{A}^n_X\smallsetminus\mathbb{U}^n_X}(\mathbb{A}^n_X)$.*

Proof. By assumption, there is a point $x\in X$ whose residue field $k(x)$ has characteristic different from 2. By specialization at x (see Remark 5.4 for naturality), it suffices to prove the result for the regular $\mathbb{Z}[1/2]$-scheme $X:=\mathrm{Spec}(k(x))$. Here, we apply Theorem 7.2 with $i:=n$ and $w:=1\in\mathrm{W}^0(X)$, the unit of the Witt ring. □

Theorem 8.2. *Let X be any scheme which is not of equicharacteristic 2. Then the Witt class $\varepsilon_X^{(n)}$ of the symmetric r-space $\mathbf{B}_X^{(n)}$ is not in the image of the natural homomorphism $\mathrm{W}^r(\mathbb{A}^n_X)\longrightarrow\mathrm{W}^r(\mathbb{U}^n_X)$. In particular, $\mathbf{B}_X^{(n)}$ can not be extended to the whole affine space $\mathbb{A}^n_X$.*

Proof. Again, by specialization at a point x with $\mathrm{char}(k(x))\neq 2$, we are reduced to prove the result for the $\mathbb{Z}[1/2]$-regular scheme $X:=\mathrm{Spec}(k(x))$. In this case, the following composition vanishes:

$$\mathrm{W}^r(\mathbb{A}^n_X)\xrightarrow{\iota_X^*}\mathrm{W}^r(\mathbb{U}^n_X)\xrightarrow{\partial}\mathrm{W}^r_{\mathbb{A}^n_X\smallsetminus\mathbb{U}^n_X}(\mathbb{A}^n_X)\,.$$

Here, the connecting homomorphism ∂ is, for instance, as in Theorem 7.6, where we proved that $\partial(\varepsilon_X^{(n)})$ coincides with $[\mathbf{K}_X^{(n)}]$, up to 4-periodicity. So, $\varepsilon_X^{(n)}$ can not be extended to $\mathbb{A}^n_X$ since $[\mathbf{K}_X^{(n)}]\neq 0\in\mathrm{W}^n_{\mathbb{A}^n_X\smallsetminus\mathbb{U}^n_X}(\mathbb{A}^n_X)$ by Theorem 8.1. Note that we have to pass via the regular case otherwise the connecting homomorphism ∂ is not defined. □

Appendix A. The locally free module $\mathcal{E}_X^{(n)}$

We use the notation of the main part of the text. We want to prove:

Theorem A.1. *Let X be a noetherian scheme and $n \geq 3$. Then there does not exist a locally free $\mathcal{O}_{\mathbb{A}^n_X}$-module $\mathcal{F}$, such that*

$$\mathcal{F}\Big|_{\mathbb{U}^n_X} \simeq \mathcal{E}_X^{(n)} .$$

In particular $\mathcal{E}_X^{(n)}$ is not a free $\mathcal{O}_{\mathbb{U}^n_X}$-module.

Let $x \in X$ and $\operatorname{Spec} k(x) \xrightarrow{f} X$ be the corresponding point. If there exists a locally free $\mathcal{O}_{\mathbb{A}^n_X}$-module $\mathcal{F}$, such that $\mathcal{F}|_{\mathbb{U}^n_X} \simeq \mathcal{E}_X^{(n)}$, then

$$\mathcal{E}_{k(x)}^{(n)} \simeq \upsilon_f^*(\mathcal{E}_X^{(n)}) \simeq \upsilon_f^*(\mathcal{F}|_{\mathbb{U}^n_X}) \simeq \alpha_f^*(\mathcal{F})|_{\mathbb{U}^n_{k(x)}} ,$$

and so it is enough to show the theorem for $X = \operatorname{Spec} R$ with R a field. Similarly, localizing $R[T_1, \ldots, T_n]$ at the origin, we are reduced to the local case which follows from the following result of commutative algebra.

Theorem A.2. *Let $(A, \mathfrak{m})$ be a regular local ring, $\underline{T} = (T_1, \ldots, T_n)$ a regular system of parameters (see* [7, Def. 2.2.1]*), and $\mathbb{U} = \bigcup_{i=1}^{n} \operatorname{Spec} A_{T_i} = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ the punctured spectrum of A. Assume that $\dim A = n \geq 3$. Then*

$$\mathcal{S}_j := \operatorname{Ker}\big(K_j(A, \underline{T}) \xrightarrow{d_j(A,\underline{T})} K_{j-1}(A, \underline{T}) \big)\Big|_{\mathbb{U}}$$

can not be extended to a free A-module if $n > j \geq 2$.

Let in the following $\mathcal{I}_j = \operatorname{Ker} d_j(A, \underline{T})$, i.e. $\mathcal{S}_j = \mathcal{I}_j|_{\mathbb{U}}$. Recall also that $(-)^\vee = \operatorname{Hom}_A(-, A)$. For the proof we need:

Proposition A.3. (i) *Let $j \geq 2$. Then the A-module $\mathcal{I}_j$ is reflexive, i.e. the natural morphism* $\mathrm{can} : \mathcal{I}_j \longrightarrow \mathcal{I}_j^{\vee\vee}$ *is an isomorphism.*

(ii) *If M and N are finitely generated A-modules, such that $M|_{\mathbb{U}} \simeq N|_{\mathbb{U}}$, and both $M|_{\mathbb{U}}$ and $N|_{\mathbb{U}}$ are locally free, then $M^\vee \simeq N^\vee$.*

Proof. By assumption $\mathcal{I}_j$ is a second Syzygie and so (i) is a consequence of [8, Thm. 3.6]. For (ii), by [11, Thm. 6.9.17] there exists $c \geq 0$, such that the given isomorphism $M|_{\mathbb{U}} \xrightarrow{\simeq} N|_{\mathbb{U}}$ is the restriction of a morphism $\mathfrak{m}^c M \longrightarrow N$. Therefore we can assume that there exists $g : M \longrightarrow N$, such that $g|_{\mathbb{U}}$ is an isomorphism, *i.e.* $\operatorname{Ker} g$ and $\operatorname{Coker} g$ have finite length.

Now we use the following fact (see [7, Thm. 1.2.8]). Since $\dim A \geq 2$ and A is regular (hence in particular Cohen-Macaulay) we have $\operatorname{Ext}^i_A(G, A) = 0$ for any finite length module G and $i = 0, 1$.

This and the exact sequences

$$0 \longrightarrow \operatorname{Ker} g \longrightarrow M \longrightarrow \operatorname{Im} g \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} g \longrightarrow N \longrightarrow \operatorname{Coker} g \longrightarrow 0$$

give $M^\vee \simeq (\operatorname{Im} g)^\vee \simeq N^\vee$. ∎

Proof. (of Theorem A.2) Assume that P is a free A-module, such that $P|_{\mathbb{U}} \simeq \mathcal{S}_j$. We have $\mathcal{I}_j|_{\mathbb{U}} \simeq \mathcal{S}_j$, too, and so $\mathcal{I}_j^{\vee} \simeq P^{\vee}$ by Proposition A.3 (ii). Part (i) of this proposition tells us that $\mathcal{I}_j$ is reflexive and hence $\mathcal{I}_j \simeq \mathcal{I}_j^{\vee\vee} \simeq P^{\vee\vee}$ is free. But this is impossible, because

$$\mathrm{Tor}^A_{n-j}(\mathcal{I}_j, A/\mathfrak{m}) \simeq \mathrm{Tor}^A_n(A/\mathfrak{m}, A/\mathfrak{m}) \simeq A/\mathfrak{m} \neq 0$$

(note that we need here $j < n = \dim A$). We are done. □

Appendix B. The product and 4-periodicity

We have used in this work the fact that the product commutes with the translation. This has not been established in [10]. For the sake of completeness we give here a proof but refer to *loc. cit.* for unexplained notations and definitions.

To start with let $\mathcal{A}^{(0)} = (\mathcal{A}, \mathcal{D}_{\mathcal{A}}, \delta_{\mathcal{A}}, \varpi^{\mathcal{A}})$ and $\mathcal{B}^{(0)} = (\mathcal{B}, \mathcal{D}_{\mathcal{B}}, \delta_{\mathcal{B}}, \varpi^{\mathcal{B}})$ be triangulated categories with $\delta_{\mathcal{A}}$- respectively $\delta_{\mathcal{B}}$-exact duality (like *e.g.* $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ with the usual 1-exact duality as in the main part of this work). We denote the shift functor in these triangulated categories by $\Sigma_{\mathcal{A}}$ respectively $\Sigma_{\mathcal{B}}$ (to distinguish we do not use $X \mapsto X[1]$). A symmetric i-space in $\mathcal{A}^{(0)}$ is a pair (X, ψ) consisting of an object $X \in \mathcal{A}$ and a symmetric i-form $X \xrightarrow{\psi} \Sigma^i_{\mathcal{A}} \mathcal{D}_{\mathcal{A}} X$ which is an isomorphism, the symmetry of an i-form reads $\Sigma^i_{\mathcal{A}} \mathcal{D}_{\mathcal{A}}(\psi) \cdot \varpi^{\mathcal{A}}_X = (-1)^{\frac{i(i+1)}{2}} \delta^i_{\mathcal{A}} \cdot \psi$. As in the case of derived categories if (X, ψ) is a symmetric i-form then $(\Sigma^2_{\mathcal{A}} X, \Sigma^2_{\mathcal{A}}(\psi))$ is a symmetric $(i+4)$-form.

Let $(F, \rho) : \mathcal{A}^{(0)} \longrightarrow \mathcal{B}^{(0)}$ be a duality preserving functor, *i.e.* $\rho : F\mathcal{D}_{\mathcal{A}} \xrightarrow{\simeq} \mathcal{D}_{\mathcal{B}} F$ is an isomorphism of functors satisfying some compatibility axioms. We will only use the following. Since F is a covariant exact functor between triangulated categories there exists a family of isomorphisms of functors $\theta^{(i)} : F\Sigma^i_{\mathcal{A}} \xrightarrow{\simeq} \Sigma^i_{\mathcal{B}} F$ ($i \in \mathbb{Z}$) which are related by the following formulas:

$$\theta^{(i+j)} = \Sigma^i_{\mathcal{B}}(\theta^{(j)}) \cdot \theta^{(i)}_{\Sigma^j_{\mathcal{A}}} \tag{14}$$

$(i, j \in \mathbb{Z})$. Then we have

$$\mathcal{D}_{\mathcal{B}} \Sigma^{-1}_{\mathcal{B}} (\theta^{(1)}_{\Sigma^{-1}_{\mathcal{A}}}) \cdot \rho_{\Sigma^{-1}_{\mathcal{A}}} \;=\; (\delta_{\mathcal{A}} \delta_{\mathcal{B}}) \cdot \Sigma_{\mathcal{B}}(\rho) \cdot \theta^{(1)}_{\mathcal{D}_{\mathcal{A}}} \tag{15}$$

(*cf. loc. cit.* Definition 1.8). This axioms are made such that if (X, ψ) is a symmetric i-space in $\mathcal{A}^{(0)}$ then

$$(F, \rho)_*(X, \psi) := \big(FX \,,\, (\delta_{\mathcal{A}} \delta_{\mathcal{B}})^i \, \Sigma^i_{\mathcal{B}}(\rho_X) \cdot \theta^{(i)}_{\mathcal{D}_{\mathcal{A}} X} \cdot F(\psi) \big)$$

is a symmetric i-space in $\mathcal{B}^{(0)}$.

Lemma B.1. *Let (X, ψ) be a symmetric i-space in $\mathcal{A}$. Then there is an isometry*

$$(F, \rho)_*(\Sigma^2_{\mathcal{A}} X, \Sigma^2_{\mathcal{A}}(\psi)) \xrightarrow{\simeq} \Sigma^2_{\mathcal{B}}((F, \rho)_*(X, \psi)) .$$

Proof. We claim that $\theta^{(2)}_X : F\Sigma^2_{\mathcal{A}} X \simeq \Sigma^2_{\mathcal{B}} FX$ is an isometry, *i.e.* we have to show

$$\begin{aligned} &(\delta_{\mathcal{A}} \delta_{\mathcal{B}})^{i+4} \, \Sigma^{i+4}_{\mathcal{B}}(\rho_{\Sigma^2_{\mathcal{A}} X}) \cdot \theta^{(i+4)}_{\mathcal{D}_{\mathcal{A}} \Sigma^2_{\mathcal{A}} X} \cdot F\Sigma^2_{\mathcal{A}}(\psi) \\ &\quad = (\delta_{\mathcal{A}} \delta_{\mathcal{B}})^i \, \Sigma^{i+4}_{\mathcal{B}} \mathcal{D}_{\mathcal{B}}(\theta^{(2)}_X) \cdot \Sigma^{i+2}_{\mathcal{B}}(\rho_X) \cdot \Sigma^2_{\mathcal{B}}(\theta^{(i)}_{\mathcal{D}_{\mathcal{A}} X}) \cdot \Sigma^2_{\mathcal{B}} F(\psi) \cdot \theta^{(2)}_X \end{aligned} \tag{16}$$

We observe first that

$$\Sigma^2_{\mathcal{B}} F(\psi)\cdot \theta^{(2)}_X = \theta^{(2)}_{\Sigma^i_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}X}\cdot F\Sigma^2_{\mathcal{A}}(\psi)$$

since $\theta^{(2)}$ is a natural transformation. By using (14) three times we get (recall that by definition $\Sigma_{\mathcal{A}}\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}\Sigma^{-1}_{\mathcal{A}}$)

$$\theta^{(i+4)}_{\mathcal{D}_{\mathcal{A}}\Sigma^2_{\mathcal{A}}X} \;=\; \Sigma^{i+3}_{\mathcal{B}}(\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma^2_{\mathcal{A}}X})\cdot\Sigma^{i+2}_{\mathcal{B}}(\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}X})\cdot\Sigma^2_{\mathcal{B}}(\theta^{(i)}_{\mathcal{D}_{\mathcal{A}}X})\cdot\theta^{(2)}_{\Sigma^i_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}X}$$

and so (16) is equivalent to

$$\Sigma^{i+4}_{\mathcal{B}}(\rho_{\Sigma^2_{\mathcal{A}}X})\cdot\Sigma^{i+3}_{\mathcal{B}}(\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma^2_{\mathcal{A}}X})\cdot\Sigma^{i+2}_{\mathcal{B}}(\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}X}) \;=\; \Sigma^{i+4}_{\mathcal{B}}\mathcal{D}_{\mathcal{B}}(\theta^{(2)}_X)\cdot\Sigma^{i+2}_{\mathcal{B}}(\rho_X)\,.$$

Hence the result follows from the following calculation:

$$\begin{aligned}
&\Sigma^{i+3}_{\mathcal{B}}(\Sigma_{\mathcal{B}}(\rho_{\Sigma^2_{\mathcal{A}}X})\cdot\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma^2_{\mathcal{A}}X})\cdot\Sigma^{i+2}_{\mathcal{B}}(\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}X})\\
&\quad= (\delta_{\mathcal{A}}\delta_{\mathcal{A}})\;\Sigma^{i+3}_{\mathcal{B}}(\mathcal{D}_{\mathcal{B}}\Sigma^{-1}_{\mathcal{B}}(\theta^{(1)}_{\Sigma_{\mathcal{A}}X})\cdot\rho_{\Sigma_{\mathcal{A}}X})\cdot\Sigma^{i+2}_{\mathcal{B}}(\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}X}) \quad \text{by Eq. (15)}\\
&\quad= (\delta_{\mathcal{A}}\delta_{\mathcal{A}})\;\Sigma^{i+4}_{\mathcal{B}}(\mathcal{D}_{\mathcal{B}}\theta^{(1)}_{\Sigma_{\mathcal{A}}X})\cdot\Sigma^{i+2}_{\mathcal{B}}(\Sigma_{\mathcal{B}}(\rho_{\Sigma_{\mathcal{A}}X})\cdot\theta^{(1)}_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}X})\\
&\quad= \Sigma^{i+4}_{\mathcal{B}}\mathcal{D}_{\mathcal{B}}(\Sigma_{\mathcal{B}}\theta^{(1)}_X\cdot\theta^{(1)}_{\Sigma_{\mathcal{A}}X})\cdot\Sigma^{i+2}_{\mathcal{B}}(\rho_X) \quad \text{by Eq. (15)}\\
&\quad= \Sigma^{i+4}_{\mathcal{B}}\mathcal{D}_{\mathcal{B}}(\theta^{(2)}_X)\cdot\Sigma^{i+2}_{\mathcal{B}}(\rho_X)
\end{aligned}$$

since $\theta^{(2)}_X = \Sigma_{\mathcal{B}}\theta^{(1)}_X\cdot\theta^{(1)}_{\Sigma_{\mathcal{A}}X}$ by Eq. (14). □

Assume now we have a third triangulated category with duality, say $\mathcal{C}^{(0)} = (\mathcal{C},\mathcal{D}_{\mathcal{C}},\delta_{\mathcal{C}},\varpi^{\mathcal{C}})$, and a dualizing pairing (*loc. cit.* Definition 1.11)

$$\boxtimes\,:\;\mathcal{A}^{(0)}\times\mathcal{B}^{(0)}\longrightarrow\mathcal{C}^{(0)}\,.$$

Example B.2. Let X be a scheme and $Z\subseteq X$ a closed subset. Then the (derived) tensor product

$$\otimes_{\mathcal{O}_X}\;:\;\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)\times\mathbb{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)\longrightarrow\mathbb{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$$

is a dualizing pairing. Note that in this case $\delta_{\mathcal{A}}=\delta_{\mathcal{B}}=\delta_{\mathcal{C}}=1$.

Let (X,ψ) be a symmetric i-space in $\mathcal{A}^{(0)}$ and (Y,ϕ) a symmetric j-space in $\mathcal{B}^{(0)}$. The left product $(X,\psi)\star_l(Y,\phi)$ is then defined by considering $X\boxtimes -$ as duality preserving functor with the aid of a duality transformation $\mathcal{L}(\psi)$ which depends on ψ, *i.e.* the left product $(X,\psi)\star_l(Y,\phi)$ of these spaces is by definition $(X\boxtimes -,\mathcal{L}(\psi))_*(Y,\phi)$. The right product $\star_r$ is defined analogous by making the functor $-\boxtimes Y$ duality preserving using the symmetric j-form ϕ. Both products are related by the following isometry:

$$(X,\psi)\star_l(Y,\phi)\;\simeq\;(\delta_{\mathcal{A}}\delta_{\mathcal{C}})^j\cdot(\delta_{\mathcal{B}}\delta_{\mathcal{C}})^i\cdot(-1)^{ij}\;\cdot\big((X,\psi)\star_r(Y,\phi)\big)\,. \qquad (17)$$

(*loc. cit.* Theorem 2.9). From this we easily deduce

Lemma B.3. *There is an isometry*

$$(\Sigma^2_{\mathcal{A}}X,\Sigma^2_{\mathcal{A}}(\psi))\star_l(Y,\phi)\;\simeq\;(X,\psi)\star_l(\Sigma^2_{\mathcal{B}}Y,\Sigma^2_{\mathcal{B}}(\phi))\,,$$

and the same is true for the right product.

Proof. From Lemma B.1 we get isometries $(X,\psi)\star_l(\Sigma^2_{\mathcal{B}}Y,\Sigma^2_{\mathcal{B}}(\phi))\simeq\Sigma^2_{\mathcal{C}}\big((X,\psi)\star_l(Y,\phi)\big)$ and $(\Sigma^2_{\mathcal{A}}X,\Sigma^2_{\mathcal{A}}(\psi))\star_r(Y,\phi)\simeq\Sigma^2_{\mathcal{C}}\big((X,\psi)\star_r(Y,\phi)\big)$. Hence the theorem follows by applying (17) twice. □

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Paul Balmer, D-Math, ETH Zentrum, 8092 Zürich, Switzerland
E-mail address: balmer@math.ethz.ch
URL: http://www.math.ethz.ch/~balmer

Stefan Gille, SFB 478, Univ. Münster, Hittorfstr. 27, 48149 Münster, Germany
E-mail address: gilles@math.uni-muenster.de