Irreducible Subgroups of $GL_1(D)$ Satisfying Group Identities

R. Ebrahimian, D. Kiani, M. Mahdavi-Hezavehi
ebrahimian@mehr.sharif.edu
kiani@mehr.sharif.edu
mahdavih@sharif.edu
Department of Mathematical Sciences
Sharif University of Technology
P. O. Box 11365-9415
Tehran, Iran

Abstract

Let D be a finite dimensional F-central division algebra, and G be an irreducible subgroup of $D^* := GL_1(D)$. Here we investigate the structure of D under various group identities on G. In particular, it is shown that when $[D : F] = p^2$, p a prime, then D is cyclic if and only if D^* contains a nonabelian subgroup satisfying a group identity.

1 Introduction

Let D be an F-central division algebra of degree n. We define D to be *quasi-crossed product* (QCP for short) if it contains a maximal subfield L with a chain of fields $F \subsetneqq K \subseteq L$ such that K/F is Galois and L/K is abelian Galois. If $L = K$, then D is crossed product. Also, D is said to be *soluble* QCP if $Gal(K/F)$ is soluble. We also recall that a subgroup G of D^* is *irreducible* if the F-linear hull of G, $F[G] = D$. Irreducible soluble subgroups of the multiplicative group of a divison ring were first studied by Suprunenko in [11]. Assume that D is a noncommutative finite dimensional F-central division algebra. It is clear that D^* is an improper irreducible subgroup of D^*. More generally, if N is a noncentral normal subgroup of D^*, then, by Cartan-Brauer-Hua Theorem, N is irreducible. But N cannot satisfy a group identity since it is known that N contains a noncyclic free subgroup [3] . So, we are interested in proper irreducible subgroups of D^* that are not normal. To construct such subgroups of D^*, take a basis $\{g_i\}_{1\leq i\leq n}$ of D/F and consider the subgroup $G = \langle g_i, 1 \leq i \leq n\rangle$. Since G is finitely generated it is not normal in D^* (cf. [6]) and we clearly have $F[G] = D$. Therefore, we can always find nonnormal proper irreducible subgroups in D^*. We shall see later on that if a nonnormal proper irreducible subgroup G of D^* satisfies a group identity, then we may be able to determine some information about the structure of D. For example, it is shown that if D^* contains an irreducible subgroup satisfying a group identity, then D is QCP. In particular, when $[D : F] = p^2$, p a prime, any nonabelian subgroup of D^* is irreducible. In this case, it is proved that D is cyclic if and only if D^* contains a nonabelian subgroup satisfying a group identity.

2 Notations and conventions

Let D be a division ring with center F and G be a subgroup of D^*. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated by elements of G over F. We shall say that G is *irreducible* if $D = F[G]$. For any group G we denote its center by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the *normalizer* of H in G, and $\langle H, K\rangle$ the group generated by H and K, where K is a subgroup of G. We shall say that H is *abelian-by-finite* if there is an abelian normal subgroup K of H such that H/K is finite. H is called *center-by-finite* if $H/Z(H)$ is finite. Let S be a subset of D, then the *centralizer* of S in D is denoted by $C_D(S)$. For results related to central simple algebras see [8].

3 Irreducible Subgroups of D^*

This section deals with a few results on division algebras whose multiplicative groups contain certain subgroups. Using these results we eventually prove our main theorem which asserts that if D^* contains an irreducible soluble subgroup, then D is QCP. We then examine the structure of division algebras whose multiplicative groups contain irreducible abelian-by-finite subgroups. Using the results, it is shown that if D has a nonzero characteristic and D^* contains an irreducible subgroup satisfying a group identity, then D is QCP. Finally, a criterion is given for a division algebra D of prime degree to be cyclic. To be more precise, it is shown that D is cyclic if and only if D^* contains a nonabelian subgroup satisfying a group identity. We begin our study with the following lemma, for a proof see [2].

LEMMA A. *Let D be a finite dimensional F-central division algebra. If D is soluble crossed product, then D^* contains an irreducible soluble subgroup.*

We shall also need the following easy lemma from group theory.

LEMMA 3.1. *Let G be a group in which $Z(G)$ is a maximal abelian normal subgroup. Then $Z(G/Z(G))$ is trivial.*

PROOF. Assume on the contrary that $x \in G \setminus Z(G)$ with $Z(G)x \in Z(G/Z(G))$. Using the fact that for any element $y \in G$ we have $Z(G)xy = Z(G)yx$, we obtain $yxy^{-1}x^{-1} \in Z(G)$. Thus, for any $y \in G$ we have $yxy^{-1} \in \langle Z(G), x\rangle$. Hence, $\langle Z(G), x\rangle$ is an abelian normal subgroup of G properly containing $Z(G)$. This contradiction completes the proof. $\square$

We are now in a position to prove the following:

THEOREM 3.2. *Let D be a noncommutative finite dimensional F-central division algebra. Assume that D^* contains an irreducible soluble subgroup. Then there exists an irreducible soluble subgroup S and a noncentral abelian normal subgroup A of S such that S/A is finite.*

PROOF. By Lemma 3 of [5], we know that any soluble subgroup S of D^* is abelian-by-finite, i.e., there is an abelian normal subgroup A in S of finite index. We may take in our considerations A maximal when we deal with irreducible subgroups of D^* which are clearly nonabelian. Hence, to prove the theorem we must show that there exists an irreducible soluble subgroup S that is not center-by-finite. On the contrary, assume that the set Σ of all irreducible soluble subgroups of D^* are center-by-finite. Since D is of finite dimension over F we may view D^* as a linear group in $GL_n(F)$, where $n = [D : F]$, and consequently each element of Σ is a linear group. Thus, by Huppert-Zassenhaus Theorem (cf. [1, p. 104]), the soluble length l of each element $H \in \Sigma$ is at most $2n$. Using this fact and Zorn's Lemma, we conclude that every element of Σ is contained in a maximal element of Σ. Let G be a maximal element of Σ. It is clear that $F^* \subseteq G$. By our assumption F^* is a maximal abelian normal subgroup in G of finite index. We claim that if N is normal in G, then $Z(F[N]) = F$. Using the fact that G normalizes N we conclude that G normalizes $F[N]$. Obviously $Z(F[N])^*G$ is an irreducible soluble subgroup of D^*. Also it is easily seen that $Z(F[N])^*$ is a normal abelian subgroup of $Z(F[N])^*G$ of finite index. Therefore, by our assumption we have $Z(F[N]) = F$. Now, G/F^* as a soluble group must contain a nontrivial abelian normal subgroup. Let N/F^* be a maximal abelian normal subgroup of G/F^*. We claim that $C_G(N) = F^*$. To see this, put $M = NC_G(N)$. If $M = N$, then $C_G(N) \subseteq N$ and hence $C_G(N) = Z(N) = F^*$. Now suppose that $M \neq N$. Considering the fact that M is normal in G and that N/F^* is a maximal abelian normal subgroup of G/F^*, we conclude that M/F^* is not abelian. The group M/N is a nontrivial soluble subgroup of G/N. Thus, by looking at the derived series of M/N we can find a nontrivial abelian normal subgroup T/N of M/N such that T is normal in G. By the choice of N/F^*, we obtain that T/F^* is not abelian. Therefore, there exist $z, t \in T$ such that $ztz^{-1}t^{-1} \notin F^*$. Using the definition of M, we can find $n, m \in N$ and $p, q \in C_G(N)$ such that $z = np$, $t = mq$. Therefore, we have $ztz^{-1}t^{-1} = npmqp^{-1}n^{-1}q^{-1}m^{-1} = nmn^{-1}m^{-1}pqp^{-1}q^{-1}$. On the other hand we have $ztz^{-1}t^{-1} \in N$ since T/N is abelian. Thus, $pqp^{-1}q^{-1} \in N$ and hence $pqp^{-1}q^{-1} \in N \cap C_G(N) = Z(N) = F^*$. Now, using the fact that N/F^* is abelian we obtain $nmn^{-1}m^{-1} \in F^*$ and so $ztz^{-1}t^{-1} \in F^*$. This contradiction shows that if N/F^* is a maximal abelian normal subgroup of G/F^*, then $C_G(N) = F^*$ which establishes the claim. Now, set $D_1 = F[N]$ and, by the above observation, we note that $Z(D_1) = F$. Since G normalizes D_1 we see that for any $g \in G$ we may define a natural homomorphism $f_g : D_1 \longrightarrow D_1$, given by $f_g(x) = gxg^{-1}$, for any $x \in D_1$. Hence, by Skolem-Noether Theorem there is an element $a_g \in D_1^*$ such that $f_g = f_{a_g}$. If $u, v \in D_1$ satisfy $f_u = f_v$, then for any $x \in D_1$ we have $uxu^{-1} = vxv^{-1}$. Therefore, $u^{-1}v \in Z(D_1) = F$, which shows that u, v are equal modulo F^*, i.e., $F^*u = F^*v$. Now, for any $x \in D_1$ we have $gxg^{-1} = a_gxa_g^{-1}$, and hence $b_g = a_g^{-1}g \in C_D(D_1)$. The fact that b_g commutes with a_g implies that a_g, g, and b_g pairwise commute. Set $A = \bigcup_{g\in G} F^*a_g$ and $B = \bigcup_{g\in G} F^*b_g$. We claim that both A, B are groups. To see this, it is enough to show that for any $g, h \in G$ we have $F^*a_{g^{-1}} = F^*a_g^{-1}$, $F^*a_ha_g = F^*a_{hg}$, $F^*b_{g^{-1}} = F^*b_g^{-1}$, and $F^*b_hb_g = F^*b_{hg}$. We have $b_g = a_g^{-1}g$. For any $x \in D_1$, $f_{a_{g^{-1}}}(x) = f_{g^{-1}}(x) = g^{-1}xg = (a_gb_g)^{-1}x(a_gb_g) = a_g^{-1}xa_g = f_{a_g^{-1}}(x)$. Therefore, $F^*a_g^{-1} = F^*a_{g^{-1}}$. Also we have $f_{a_{hg}}(x) = f_{hg}(x)$. Hence $f_{a_{hg}}(x) = hgxg^{-1}h^{-1} = ha_gxa_g^{-1}h^{-1} = a_ha_gxa_g^{-1}a_h^{-1} = (a_ha_g)x(a_ha_g)^{-1} = f_{a_ha_g}(x)$. Therefore, $F^*a_{hg} = F^*a_ha_g$ which shows that A is a group. Next considering the fact that $a_g \in D_1$ and $b_h \in C_D(D_1)$ we obtain $b_hb_g = b_ha_g^{-1}g = a_g^{-1}b_hg = a_g^{-1}a_h^{-1}hg = (a_ha_g)^{-1}hg$. Thus, since A is a group we conclude that $F^*b_hb_g = F^*(a_ha_g)^{-1}hg = F^*a_{hg}^{-1}hg = F^*b_{hg}$, $F^*b_g^{-1} = F^*a_gg^{-1} = F^*a_{g^{-1}}^{-1}g^{-1} = F^*b_{g^{-1}}$. Therefore, B is also a group. We claim that B is a soluble group that is normalized by G. To see this, consider the epimorphism $\theta : G \longrightarrow B/F^*$ given by $\theta(g) = F^*b_g$ for all $g \in G$. Hence B/F^* as a homomorphic image of a soluble group is soluble, and so B is also soluble. Furthermore, since $a_g \in D_1$ and $B \subseteq C_D(D_1)$ for any $g \in G$ we have $gBg^{-1} = a_gb_gBb_g^{-1}a_g^{-1} = a_gBa_g^{-1} = B$, which establishes our claim. Therefore, BG is an irreducible soluble subgroup of D^*. By maximality of G we conclude that $B \subseteq G$. Now, we have $B \subseteq C_D(D_1) \cap G \subseteq C_G(D_1) = C_G(N) = F^*$. Since $g = a_gb_g$ we obtain $G \subseteq D_1$ and hence $D_1 = D$, i.e., N is irreducible. By our assumption we conclude that F^* is a maximal abelian normal subgroup of N. Therefore, by Lemma 3.1, $Z(N/F^*)$ is trivial, which is a contradiction and so the result follows. $\square$

To prove our main theorems, we shall need the following results, for a proof see [2].

Lemma B. *Let D be a finite dimensional F-central division algebra. Suppose that K is a subfield of D containing F. If G is an irreducible subgroup of D^* such that $K^* \lhd G$, then K/F is Galois and $G/C_G(K^*) \simeq Gal(K/F)$.*

Lemma C. *Let D be a finite dimensional F-central division algebra and let G be an irreducible subgroup of D^*. If K is a subfield of D containing F such that $[G : C_G(K^*)] = [K : F]$, then $C_D(K) = F[C_G(K^*)]$.*

Theorem D. *A noncommutative finite dimensional F-central division algebra D is abelian crossed product if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is abelian. Equivalently D^* contains an irreducible metabelian subgroup.*

We are now prepared to prove one of our main results of this section in the following form:

Theorem 3.3. *Let D be a noncommutative finite dimensional F-central division algebra. If D^* contains an irreducible soluble subgroup, then D is soluble QCP.*

Proof. As in Theorem 3.2, assume that Σ is the set of all irreducible soluble subgroups of D^*. It is clear that every element of Σ is contained in a maximal element. Suppose that G is a maximal element of Σ. It is easily checked that $F^*G \in \Sigma$, and by maximality of G we have $F^*G = G$. Hence $F^* \subseteq G$. Now, by Theorem 3.2, there exists an irreducible soluble subgroup S of D^* containing a noncentral abelian normal subgroup B of finite index. Put $T = F(B)^*S$. Since S normalizes B one can easily show that T is an irreducible soluble subgroup of D^*. Therefore, T is contained in a maximal element G of Σ. Now, by Lemma 3 of [5], there exists a maximal abelian normal subgroup A in G of finite index. It is clear that $[F(A)^* : F^*]$ is infinite since otherwise F would be a finite field (cf. [4, p. 213]), hence by Wedderburn's Theorem, we conclude that D is a field, a contradiction. Therefore, G/F^* is infinite. Since A is a maximal abelian normal subgroup of finite index in G we conclude that $F^* \neq A$. Set $K = F(A)$. One may easily show that $G \subseteq N_{D^*}(K^*)$ and that K^*G is an irreducible soluble subgroup of D^*. Therefore, by maximality of G we have $K^* \subseteq G$. Thus, $A \subseteq K^* \lhd G$, which shows that $A = K^*$. Now, by Lemma B, K/F is Galois and $G/C_G(K^*) \simeq Gal(K/F)$. Therefore, K/F is a soluble Galois extension. If $C_D(K) = K$, one may easily conclude that K is a maximal subfield of D, hence $L = K$ and so D is a soluble crossed product division algebra. So, we may assume that $C_D(K) \neq K$. By Lemma C, we conclude that $F[C_G(K^*)] = C_D(K)$. Put $D_1 = C_D(K)$. We have $F[C_G(K^*)] = D_1$, hence D_1 contains an irreducible soluble subgroup. On the other hand by Centralizer Theorem we have $Z(D_1) = K$. Now, by Lemma 11 of [11] D_1^* contains an irreducible metabelian subgroup. Therefore, by Lemma D, we conclude that D_1 is an abelian crossed product division algebra. Thus, there exists a maximal subfield L of D_1 such that L/K is abelian Galois. Now, K/F is Galois and $Gal(K/F)$ is soluble. To complete the proof, it is enough to show that L is a maximal subfield of D. To see this, we have $C_D(L) \subseteq C_D(K) = D_1$, and hence $C_D(L) \subseteq C_{D_1}(L) = L$, which completes the proof of the theorem. □

As a special case of Theorem 3.3, we obtain the following corollary, which is the main result of [7].

COROLLARY 3.4. *Let D be an F-central division algebra of prime degree p. Then D is cyclic if and only if D^* contains a nonabelian soluble subgroup.*

PROOF. the "only if" part is clear with Lemma A. By Theorem 3.3, D is soluble QCP. Thus, there exists a noncentral subfield K such that K/F is Galois and $Gal(K/F)$ is soluble. Now, for dimensional reasons, the proof is complete. □

The next result establishes the structure of division algebras whose multiplicative subgroups contain irreducible abelian-by-finite subgroups.

PROPOSITION 3.5. *Let D be a noncommutative finite dimensional F-central division algebra. Assume that D^* contains an irreducible abelian-by-finite subgroup G. If $Char F = p > 0$, then D is QCP.*

Proof. By our assumption, there exists an abelian normal subgroup A in G of finite index. Since G is nonabelian we may take A maximal. We now divide the proof into two cases:

Case 1: Suppose that A is noncentral in G. Therefore, A is also noncentral in D^*. Set $H = F(A)^*G$. Since G normalizes A one can easily see that H is an irreducible abelian-by-finite subgroup of D^*. Let $K = F(A)$. It is clear that H/K^* is finite and $K \neq F$. By Lemma B, we conclude that K/F is Galois and $H/C_H(K^*) \simeq Gal(K/F)$. If $C_D(K) = K$, we conclude that K is a maximal subfield of D and $L = K$, hence D is QCP. So, we may assume that $C_D(K) \neq K$. Now, using Lemma C, we obtain $F[C_H(K^*)] = C_D(K)$. Set $D_1 = C_D(K)$. We have that $C_H(K^*)/K^*$ is a finite group. On the other hand by the Centralizer Theorem, we have $Z(D_1) = K^*$. Hence $Z(C_H(K^*)) = K^*$. Now, by group theory, we know that the derived group $C_H(K^*)'$ is a finite group (cf. [9, p. 443]). On the other hand $Char F = p > 0$. Now, by a theorem of [4, p. 204], we conclude that $C_H(K^*)'$ is cyclic. Therefore, D_1^* contains an irreducible metabelian subgroup. Hence, by Theorem D, we conclude that D_1 is an abelian crossed product division algebra. Thus, there exists a maximal subfield L of D_1 such that L/K is Galois and $Gal(L/K)$ is abelian. Now, we have $C_D(L) \subseteq C_D(K) = D_1$, and so $C_D(L) \subseteq C_{D_1}(L) = L$, which shows that L is a maximal subfield of D. Therefore, D is QCP.

Case 2: Assume that $A = Z(G)$. Since G is irreducible we conclude that $Z(G)$ is central in D^*. Because A is maximal in G we conclude that every abelian normal subgroup of G is contained in F^*. Now, we know that $G/Z(G)$ is finite and hence the derived group G' is a finite group (cf. [9, p. 443]). On the other hand, $char F = p > 0$, by a theorem of [4, p. 204], we conclude that G' is cyclic. Therefore, G is an irreducible soluble subgroup of D^* and by Theorem 3.3, we obtain that D is QCP, which completes the proof. □

Corollary 3.6. *Let D be a noncommutative finite dimensional division algebra of nonzero characteristic. If D^* contains an irreducible subgroup satisfying a group identity, then D is QCP.*

Proof. Suppose that G is an irreducible subgroup of D^* which satisfies a group identity. Let $\{g_1, \ldots, g_n\} \subseteq G$ be a basis of the vector space D over the field $Z(D)$ and set $G_1 = \langle g_1 \ldots g_n \rangle$. Then G_1 is an irreducible subgroup of D^*, which is finitely generated and satisfies a group identity. By Tits' Alternative (cf. [12, p. 150]), we conclude that G_1 is soluble-by-finite. Now, Lemma 3 of [5] implies that G_1 is abelian-by-finite. Therefore, by Theorem 3.5, the result follows. □

Finally, using above results, a criterion is given for cyclicity of a division algebra of prime index in terms of subgroups of D^*. This criterion includes those given in [7] and provides us with various conditions on D to be cyclic.

THEOREM 3.7. *Let D be a finite dimensional F-central division algebra of prime degree p. Then D is cyclic if and only if D^* contains a nonabelian subgroup satisfying a group identity.*

PROOF. One way is clear from Lemma A. On the other hand, assume that D^* contains a nonabelian subgroup G satisfying a group identity. We may view G as a linear group over F. By a result of Platonov (cf. [12, p. 149]), we may conclude that there is a soluble normal subgroup S of finite index in G. If S is nonabelian, the result follows from Corollary 3.4. So we may assume that S is abelian. Set $T = F^*G$ and $A = F^*S$. It is easily seen that A is a normal abelian subgroup of finite index in T. We may take A maximal in T. Two cases may occur:

Case 1: If $A = F^*$, then T/F^* is a nontrivial finite subgroup of D^*/F^* and the result follows from the Theorem of [7].

Case 2: If $A \neq F^*$, then $K = F[A]$ is a maximal subfield of D. Since A is normal in T, by Lemma B, we conclude that K/F is Galois and it is clear that K/F is cyclic which completes the proof.

□

The above result provides us with various cyclicity conditions on subgroups of D^*. There are some conditions on subgroups of D^* that imply those subgroups satisfy a group identity. We collect some well-known ones in the following:

Corollary 3.8. *Let D be an F-centeral division algebra of prime degree p. Then D is cyclic if either of the following conditions holds:*

(a) D^* contains a nonabelian soluble subgroup.

(b) D^* contains a nonabelian subgroup of a bounded exponent.

(c) D^* contains a nonabelian finite subgroup.

The authors thank the referee for his constructive comments. They are also indebted to the Research Council of Sharif University of Technology for support.

References

[1] D. Dixon, The structure of linear groups, Van Nostrand, London, 1971.

[2] R. Ebrahimian, D. Kiani, M. Mahdavi-Hezavehi, *Supersoluble crossed product criterion for division algebras* , Preprint.

[3] J. Z. Goncalves, *Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings*, Canad. Math. Bull. 27(1984), 365-370.

[4] T. Y. Lam, A first course in noncommutative rings, Second edition,GTM, No. 131, Springer-Verlag, (2001).

[5] M. Mahdavi-Hezavehi, *Free subgroups in maximal subgroups of $GL_1(D)$*, J. Alg. 241 (2001) 720-730.

[6] M. Mahdavi-Hezavehi, M. G. Mahmudi, S. Yasamin, *Finitely generated subnormal subgroups of $GL_n(D)$ are central*, Journal of Algebra 225, 517-521 (2000).

[7] M. Mahdavi-Hezavehi and J. P. Tignol, *Cyclicity conditions for division algebras of prime degree*, Proc. Amer. Math. Soc. 131(2003), 3673-3676.

[8] L. Rowen, Ring theory, Volume II, Academic Press, INC, (1988).

[9] W. R. Scott, Group theory, Dover, New York 1987.

[10] M. Shirvani and B. A. F. Wehrfritz, Skew linear group, LMS Lecture Note Series, No. 118, 1986.

[11] D. A. Suprunenko, *On solvable subgroups of multiplicative groups of a field*, English Transl., Amer. Math. Soc. Transl., (2)46(1965), p. 153-161.

[12] A. E. Zalesskii, Linear group, in "Algebra IV", part II(A. I. Kostrikin I. R. Shafarevich, Eds.), Encyclopedia of Mathematics and Science. Springer Verlag, Berlin/New York, 1993.