Rationally isotropic quadratic spaces are locally isotropic

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Abstract

Let R be a regular local ring, K its field of fractions and (V, φ) a quadratic space over R. In the case of R containing a field of characteristic zero we show that if $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R.

1 Characteristic zero case

1.0.1 Theorem (Main). *Let R be a regular local ring, K its field of fractions and (V, φ) a quadratic space over R. Suppose R contains a field of characteristic zero. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R, that is there exists a unimordular vector $v \in V$ with $\varphi(v) = 0$.*

This theorem is trivial for all discrete valuation rings. The case of any two dimensional regular local ring is proved in [O]. To prove the Theorem we need certain auxiliary preliminaries.

1.0.2 Lemma. *Let $U = Spec(R)$ and let $\mathcal{X} \xrightarrow{p} U$ be a smooth projective morphism. Let $u \in U$ be the closed point of U and let $X = p^{-1}(u)$ be the closed fibre of p. Let $f : Y \to \mathcal{X}$ be a projective morphism of an essentially k-smooth scheme Y with $dim(Y) = dim(U)$. Suppose that f is transversal to the closed imbedding $i : X \hookrightarrow \mathcal{X}$. Then the morphism $q = p \circ f : Y \to U$ is finite etale.*

In fact, since f is transversal to i the scheme $f^{-1}(X)$ is a k-smooth scheme of dimension zero. Since $f^{-1}(X) = q^{-1}(u)$ the morphism q is a quasi-finite. Since q is projective it is finite and surjective. The schemes Y and U are both regular. Therefore q is flat. Since the scheme $q^{-1}(u)$ is a k-smooth scheme of dimension zero it is etale over $Spec(k)$ and hence it is etale over the point u. Since $q^{-1}(u)$ is the closed fibre of the flat morphism q the morphism q is etale.

The next lemma is a variant of the lemma [LM, Lemma 7.1].

1.0.3 Lemma. *Suppose that k is an infinite field admitting a resolution of singularities. Let W be a k-smooth scheme and let $i : Z \hookrightarrow W$ be a smooth closed subscheme. Then $CH_*(W)$ is generated by the elements of the form $f_*(1)$ where $f : Y \to W$ is a projective morphism transverse to i and $f_* : CH_*(Y) \to CH_*(W)$ is the push-forward.*

The proof of the lemma repeats the proof of Lemma [LM, Lemma 7.1] word by word except that the formal group law using in our case is the additive one.

The following theorem proved in [PR] is a generalization of Springer's Theorem

1.0.4 Theorem. *Let R be a local Noetherian domain which has an infinite residue field. Let $R \subset S$ be an integral extension which is etale over R. Let (V, φ) be a quadratic space over R such that the space $(V, \varphi) \otimes_R S$ contains an isotropic unimodular vector. If the degree of $[S : R]$ is odd then already the space (V, φ) contains a unimordular isotropic vector.*

Now we are ready to prove the Main Theorem. We prove only the geometric case of the theorem. The general case is reduced to the geometric one by standard aguments using a theorem of Popescu [P] (compare [OP]). A self-contained proof of this result of Popescu one can find in [Sw].

Proof of Theorem 1.0.1. To prove the geometric case consider the projective quadric $\mathcal{X} \subset \mathbf{P}(V)$ given by the equation $\varphi = 0$ in the projective space $\mathbf{P}(V) = Proj(S^{\cdot}(V^{\vee}))$. Let X be the closed fibre of the projection $p : \mathcal{X} \to U$. Let $\eta : Spec(K) \to U$ be the generic point of U and let $\mathcal{X}_\eta$ be the generic fibre of $\mathcal{X}$. Let m be a K-rational point of $\mathcal{X}_\eta$ and let $M \subset \mathcal{X}$ be its closure in $\mathcal{X}$ and let $[M] \in CH_*(\mathcal{X})$ be the class of M. By Lemma 1.0.3 there exist a finite family of integers n_i and a finite family of projective morphisms $f_i : Y_i \to \mathcal{X}$ (with k-smooth irreducible Y_i's) which are transversal to the closed fibre X and such that $\sum n_i f_{i,*}(1) = [M]$ in $CH_*(\mathcal{X})$. By Lemma 1.0.2 each the morphism $q_i = p \circ f_i : Y_i \to U$ is finite etale. Let $deg : CH_0(\mathcal{X}_\eta) \to \mathbb{Z}$ be the degree map. Since $deg(m) = 1$ and $\sum n_i f_{i,*}(1) = [M]$ there exists an index i such the degree of the finite etale morphism $q_i : Y_i \to U$ is odd. The existence of the Y_i-point $f_i : Y_i \to \mathcal{X}$ of $\mathcal{X}$ shows that we are under the hypotheses of Theorem 1.0.4. Hence there exists a U-point of the quadric $\mathcal{X}$. Theorem 1.0.1 is proved.

□

1.0.5 Corollary. *Let R be a regular local ring containing a field of characteristic zero. Let K be the field of fractions of R. Let (W, ψ) be a quadratic space over K which is unramified over R. Then there exists a quadratic space (V, φ) over R extending the space (W, ψ), i.e. the spaces $(V, \varphi) \otimes_R K$ and (W, ψ) are isomorphic.*

Proof. By the purity theorem [OP, Theorem A] there exists a quadratic space (V, φ) over R and an integer $n \geq 0$ such that $(V, \varphi) \otimes_R K \cong (W, \psi) \perp \mathbb{H}^n$, where $\mathbb{H}$ is a hyperbolic plane. If $n > 0$ then the space $(V, \varphi) \otimes_R K$ is isotropic. By the Main Theorem the space (V, φ) is isotropic too. Thus this space contains a hyperbolic summand and we can split it off. Repeating this procedure several times we may assume that $n = 0$, which means that $(V, \varphi) \otimes_R K \cong (W, \psi)$.

□

1.0.6 Corollary. *Let R be a regular local ring containing a field of characteristic zero. Let K be the field of fractions of R. Let (V, φ) be a quadratic space over R and let $u \in R^*$ be a unit. Suppose the equation $\varphi = u$ has a solution over K then it has a solution over R, t.e. there exists a unimodular vector $v \in V$ with $\varphi(v) = u$.*

Proof. Let $< -u >$ be a the rank one quadratic space over R corresponding the unit $-u$. The space $(V,\varphi)_K \perp < -u >_K$ is isotropic thus the space $(V,\varphi) \perp < -u >$ is isotropic by the Main Theorem. Hence there exists a unimodular vector $v \in V$ with $\varphi(v) = u$.

□

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