HOLES IN I^n

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ABSTRACT. Let F be a field with $\operatorname{char}(F) \neq 2$. We prove Vishik's conjecture stating that for any positive integer n and any anisotropic quadratic form ϕ over F lying in the n-th power I^n of the fundamental ideal I of the Witt ring of F, either $\dim(\phi) \geq 2^{n+1}$ or $\dim(\phi) = 2^{n+1} - 2^i$ for some $i \in \{1, 2, \ldots, n+1\}$. This provides a complete solution of an old-standing problem in the algebraic theory of quadratic forms. The proof is based on computations in the Chow groups of direct products of projective quadrics (involving the Steenrod operations); the method developed can be also applied to other types of algebraic varieties.

CONTENTS

1. INTRODUCTION

In this text F is a field with $\operatorname{char}(F) \neq 2$. Let I^n for some $n \geq 1$ be the n-th power of the fundamental ideal I (of the classes of the even-dimensional quadratic forms) of the Witt ring $W(F)$. An old-standing question in the algebraic theory of quadratic forms asks about the possible values of dimension of an anisotropic quadratic form ϕ over F such that $[\phi] \in I^n$, where $[\phi]$ is the class of ϕ in $W(F)$.

Examples with $\dim(\phi) = 2^{n+1} - 2^i$ for each $i \in \{1, 2, \ldots, n+1\}$ are easy to construct (see Remark 7.2). A classical theorem of J. Arason and A. Pfister [1, Hauptsatz] states that $\dim(\phi)$ is never between $0 = 2^{n+1} - 2^{n+1}$ and $2^n = 2^{n+1} - 2^n$. Also it is known that every value between 2^n and $2^n + 2^{n-1} = 2^{n+1} - 2^{n-1}$ is impossible (A. Pfister for $n = 3$, [18, Satz 14]; D. Hoffmann for $n = 4$, [6, main thm.]; A. Vishik for all n, [23], see also [22, thm. 1.5] [1]).

Date: December 2003.

Key words and phrases. Quadratic forms, Witt indices, Chow groups, Steenrod operations, correspondences. *2000 Mathematical Subject Classifications:* 11E04; 14C25.

[1] Alternative proofs are given in [14, thm. 4.4] and in [12].

Finally, A. Vishik has shown that all even values $\geq 2^{n+1}$ are possible ([22, thm. 4.12], see also §7 here) and suggested the following

Conjecture 1.1 (Vishik, [22, conject. 4.11]). [2] *If* $\phi \in I^n$ *and* $\dim(\phi) < 2^{n+1}$, *then* $\dim(\phi) = 2^{n+1} - 2^i$ *for some* $i \in \{1, 2, \dots, n+1\}$.

In the present text we prove this conjecture (see §6), obtaining a complete answer to the question about possible dimensions of anisotropic quadratic forms whose classes lie in I^n. The proof closely follows the method of [12], but involves essentially more computations. As [12] as well, it makes use of an important property of the quadratic forms satisfying the hypotheses of Conjecture 1.1 established by A. Vishik in [22]. Here we give an extended version of this result (see Proposition 4.28) with an elementary, complete, and self-contained (in particular, independent of [22]) proof.

In the proof of Conjecture 1.1 we work with projective quadrics rather than with quadratic forms themselves. The method of proof is explained in §3; it certainly applies to other types of algebraic varieties (in place of quadrics).

Acknowledgements. This work was completed during the last week of September 2003 when I was visiting the University of Alberta; an important part of this work was written down during the first week of November 2003 when I was visiting the Universität Göttingen, where I gave a course of lectures on the subject. I am very grateful to both institutions for hospitality and stimulating conditions. Besides I would like to mention that numerous remarks of A. S. Merkurjev improved very much the final version of this text.

2. Notation and preliminary observations

Everywhere in the text, X is a smooth projective quadric over F of an even dimension $D = 2d$ or of an odd dimension [3] $D = 2d + 1$ given by a non-degenerate quadratic form ϕ. We write X^r for the direct product $X \times \cdots \times X$ (over F) of r copies of X and we write $\bar{\mathrm{Ch}}(X^r)$ for the image of the restriction homomorphism $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ where $\bar{X} = X_{\bar{F}}$ with a fixed algebraic closure $\bar{F}$ of F and $\mathrm{Ch}(.)$ stays for the *modulo 2* total Chow group. We say that an element of $\mathrm{Ch}(\bar{X}^r)$ is *rational*, if it lies in the subgroup $\bar{\mathrm{Ch}}(X^r) \subset \mathrm{Ch}(\bar{X}^r)$.

A basis of the group $\mathrm{Ch}(\bar{X})$ (over $\mathbb{Z}/2\mathbb{Z}$) consists of h^i and l_i, $i \in \{0, 1, \dots, d\}$, where h stays for the class of a hyperplane section of $\bar{X}$ while l_i is the class of an i-dimensional linear subspace[4] lying on $\bar{X}$. Moreover, a basis of the group $\mathrm{Ch}(\bar{X}^r)$ for every $r \geq 1$ is given by the exterior products of the basis elements of $\mathrm{Ch}(\bar{X})$ (see, e.g., [11, §1] for an explanation why this is a basis). Speaking about a *basis* or a *basis element* (or a *basis*

[2] A. Vishik announced (for the first time in June 2002) that he has a proof of Conjecture 1.1; his proof however is not available.

[3] Only the even-dimensional case is important for our main purpose; the odd-dimensional case is included for the sake of completeness.

[4] In the case of even D and $i = d$ (and only in this case) the class l_i depends on the choice of the subspace: more precisely, there are two different classes of d-dimensional subspaces on $\bar{X}$ and no canonical choice of one of them is possible; we do not care about this however and we just choose one of them, call it l_d and "forget" about the other one which is equal to $h^d - l_d$.

cycle: we will often apply the word "cycle" to an element of a Chow group) of $\mathrm{Ch}(\bar{X}^r)$, we will always refer to the basis described above. By the *decomposition* of an element $\alpha \in \mathrm{Ch}(\bar{X}^r)$ we always mean its representation as a sum of basis cycles. We say that a basis cycle β is *contained* in the decomposition of α (or simply "is contained in α"), if β is a summand of the decomposition. More generally, for two cycles $\alpha', \alpha \in \mathrm{Ch}(\bar{X}^r)$, we say that α' *is contained* in α, or that α' is a *subcycle* of α (notation: $\alpha' \subset \alpha$), if every basis element contained in α' is also contained in α.

A basis element of $\mathrm{Ch}(\bar{X}^r)$ is called *non-essential*, if it is an exterior product of powers of h (including $h^0 = 1 = [\bar{X}]$); the other basis elements are called *essential*. An element of $\mathrm{Ch}(\bar{X}^r)$ which is a sum of non-essential basis elements, is called non-essential as well. Note that all non-essential elements are rational simply because h is rational.[5]

The multiplication table for the ring $\mathrm{Ch}(\bar{X})$ is determined by the rules $h^{d+1} = 0$, $h \cdot l_i = l_{i-1}$ ($i \in \{d, d-1, \dots, 0\}$; we adopt the agreement that $l_{-1} = 0$), and $l_d^2 = (D+1)\cdot(d+1)\cdot l_0$ (see [8, §2.1] and [9, §1.2.1]). The multiplication tables for the rings $\mathrm{Ch}(\bar{X}^r)$ (for all $r \geq 2$) follows by

$$(\beta_1 \times \cdots \times \beta_r) \cdot (\beta'_1 \times \cdots \times \beta'_r) = \beta_1\beta'_1 \times \cdots \times \beta_r\beta'_r \; .$$

The cohomological action of the topological Steenrod algebra on $\mathrm{Ch}(\bar{X}^r)$ (see [2] for the construction of the action of the topological Steenrod algebra on the Chow group of a smooth projective variety; originally Steenrod operations in algebraic geometry were introduced (in the wider context of motivic cohomology) by V. Voevodsky, [24]) is determined by the fact that the total Steenrod operation $S\colon \mathrm{Ch}(\bar{X}^r) \to \mathrm{Ch}(\bar{X}^r)$ is a (non-homogeneous) ring homomorphism, commuting with the exterior products and satisfying the formulae (see [11, §2 and cor. 3.3])

$$S(h^i) = h^i \cdot (1+h)^i, \quad S(l_i) = l_i \cdot (1+h)^{D-i+1}, \quad i \in \{0, 1, \dots, d\}$$

(in order to apply these formulae, one needs a computation of the binomial coefficients modulo 2, done, e.g., in [13, lemma 1.1]).

The group $\bar{\mathrm{Ch}}(X)$ is easy to compute. First of all one has

Lemma 2.1. *If the quadric X is anisotropic (that is, $X(F) = \emptyset$), then $\bar{\mathrm{Ch}}_0(X) \not\ni l_0$.*

Proof. If $l_0 \in \bar{\mathrm{Ch}}_0(X)$, then the variety X contains a closed point x of an odd degree $[F(x) : F]$. It follows that the quadratic form ϕ is isotropic over an odd degree extension of the base field (namely, over $F(x)$) and therefore, by the Springer-Satz (see [20, thm. 5.3]), is isotropic already over F. □

Corollary 2.2. *If X is anisotropic, then the group $\bar{\mathrm{Ch}}(X)$ is generated by the non-essential basis elements.*

Proof. If the decomposition of an element $\alpha \in \bar{\mathrm{Ch}}(X)$ contains an essential basis element l_i for some $i \neq D/2$, then $l_i \in \bar{\mathrm{Ch}}(X)$ because l_i is the i-dimensional homogeneous component of α (and $\bar{\mathrm{Ch}}(X)$ is a graded subring of $\mathrm{Ch}(\bar{X})$). If the decomposition of an

[5] There are at least two direct ways to show that h is rational: (1) h is the pull-back of the hyperplane class H with respect to the embedding of $\bar{X}$ into the projective space, and H is rational; (2) h is the first Chern class of $[\mathcal{O}_{\bar{X}}(1)] \in K_0(\bar{X})$, and $[\mathcal{O}_{\bar{X}}(1)] = \mathrm{res}([\mathcal{O}_X(1)])$ is rational.

element $\alpha \in \bar{\mathrm{Ch}}(X)$ contains the essential basis element l_i for $i = D/2$, then the $(D/2)$-dimensional homogeneous component of α is either $l_{D/2}$ or $l_{D/2} + h^{D/2}$ and we still have $l_i \in \bar{\mathrm{Ch}}(X)$. It follows that $l_0 = l_i \cdot h^i \in \bar{\mathrm{Ch}}(X)$, a contradiction with Lemma 2.1. $\square$

Now assume for a moment that the quadric X is isotropic but not completely split (that is, $\mathfrak{i}_0(X) \leq d$), write a for the Witt index $\mathfrak{i}_0(X)$ of X (defined as the Witt index $\mathfrak{i}_0(\phi)$ of ϕ, see [20, def. 5.10 of ch. 1]), and let X_0 be the projective quadric given by the anisotropic part ϕ_0 of ϕ (one has $\dim(X_0) = \dim(X) - 2a$; the case $\dim(X_0) = 0$ is possible). We consider a group homomorphism $pr\colon \mathrm{Ch}(\bar{X}) \to \mathrm{Ch}(\bar{X}_0)$ determined on the basis by the formulae $h^i \mapsto h^{i-a}$ and $l_i \mapsto l_{i-a}$ (here we adopt the agreement $h^i = 0$ and $l_i = 0$ for all negative i). Also we consider a backward group homomorphism $in\colon \mathrm{Ch}(\bar{X}_0) \to \mathrm{Ch}(\bar{X})$ determined by the formulae $h^i \mapsto h^{i+a}$ and $l_i \mapsto l_{i+a}$ for $i \in \{0, 1, \dots, d-a\}$.

Let r be a positive integer. For every length r sequence $i_1, \dots, i_r$ of integers satisfying $i_j \in [0,\ a] \cup [D-a+1,\ D]$, we define a group homomorphism

$$pr_{i_1 \dots i_r}\colon \mathrm{Ch}(\bar{X}^r) \to \mathrm{Ch}(\bar{X}_0^s)$$

with $s = \#\{i_j |\ i_j = a\}$, called *projection*. Let $\{j_1 < \dots < j_s\}$ be the set of indices such that $i_{j_s} = a$. We put $J_l = \{j |\ i_j < a\}$ and $J_h = \{j |\ i_j > a\}$. Then we define $pr_{i_1 \dots i_r}(\alpha_1 \times \dots \times \alpha_r)$ for a basis element $\alpha_1 \times \dots \times \alpha_r$ as $pr(\alpha_{j_1}) \times \dots \times pr(\alpha_{j_s})$ as far as $\alpha_j = l_{i_j}$ for any $j \in J_l$ and $\alpha_j = h^{D-i_j}$ for any $j \in J_h$; we set $pr_{i_1 \dots i_r}(\alpha_1 \times \dots \times \alpha_r) = 0$ otherwise.

Also we define a backward group homomorphism $in_{i_1 \dots i_r}\colon \mathrm{Ch}(\bar{X}_0^s) \to \mathrm{Ch}(\bar{X}^r)$, called *inclusion*, by

$$in_{i_1 \dots i_r}(\beta_1 \times \dots \times \beta_s) = \alpha_1 \times \dots \times \alpha_r$$

for a basis element $\beta_1 \times \dots \times \beta_s$, where $\alpha_j = l_{i_j}$ for $j \in J_l$, $\alpha_j = h^{D-i_j}$ for $j \in J_h$, and $\alpha_{j_k} = in(\beta_k)$ for $k = 1, 2, \dots, s$.

Proposition 2.3 (cf. [12, lemma 2.2])**.** *In the notation introduced right above, the direct sum*

$$\bigoplus_{(i_1,\dots,i_r)} pr_{i_1 \dots i_r}\colon \mathrm{Ch}(\bar{X}^r) \to \bigoplus_{(i_1,\dots,i_r)} \mathrm{Ch}(\bar{X}_0^{s(i_1,\dots,i_r)})$$

of all projections is an isomorphism with the inverse given by the sum of all inclusions. Under these isomorphisms, rational cycles correspond to rational cycles.

Proof. By the Rost motivic decomposition theorem for isotropic quadrics (original proof is in [19], generalizations are obtained in [10] and [4]), there is a motivic decomposition (in the category of the integral Chow motives)

$$(*) \qquad X \simeq \mathbb{Z} \oplus \mathbb{Z}(1) \oplus \dots \oplus \mathbb{Z}(a-1) \oplus X_0(a) \oplus \mathbb{Z}(D-a+1) \oplus \dots \oplus \mathbb{Z}(D)$$

(where $\mathbb{Z}$ is the motive of $\operatorname{Spec} F$, while $M(i)$ is the i-th Tate twist of a motive M). Rasing to the r-th power, we get a motivic decomposition of the variety X^r; each summand of this decomposition is a twist of the motive of X_0^s with s varying between 0 and r. If we numerate the summands of the decomposition $(*)$ by their twists, then the summands of the decomposition of X^r are numerated by the sequences

$$i_1, \dots, i_r \quad \text{with} \quad i_j \in [0,\ a] \cup [D-a+1,\ D]\ .$$

Moreover, the $(i_1, \dots, i_r)$-th summand is $X_0^s(i_1 + \dots + i_r)$, where $s = \#\{i_j |\ i_j = a\}$.

In order to finish the proof of Proposition 2.3, it suffices to show that the projection morphism to the $(i_1, \dots, i_r)$-th summand considered on the Chow group and over $\bar{F}$ coincides with $pr_{i_1 \dots i_r}$ while the inclusion morphism of the $(i_1, \dots, i_r)$-th summand considered on the Chow group and over $\bar{F}$ coincides with $in_{i_1 \dots i_r}$. Clearly it suffices to check this for $r = 1$ only. For $i \neq D/2$, this is particularly easy to do because of the relation $\dim_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Ch}_i(\bar{X})) \leq 1$. Indeed, $\mathrm{Ch}_i(\mathbb{Z}(k)) = 0$ for $k \neq i$. Therefore for any i with $a \leq i \leq D - a$, $i \neq D/2$, the projection and the inclusion between $\mathrm{Ch}_i(\bar{X})$ and $\mathrm{Ch}_{i-a}(\bar{X}_0)$ are isomorphisms and, as a consequence, they interchange the only non-zero elements of these two groups (which are l_i and l_{i-a} if $i < D/2$, or h^{D-i} and h^{D-i-a} if $i > D/2$). For $i < a$, the projection and the inclusion are isomorphisms between $\mathrm{Ch}_i(\bar{X})$ and $\mathbb{Z}/2\mathbb{Z} = \mathrm{Ch}_i(\mathbb{Z}(i))$ and the only non-zero element of the first Chow group is l_i. Finally, for $i > D - a$, the projection and the inclusion are isomorphisms between $\mathrm{Ch}_i(X)$ and $\mathbb{Z}/2\mathbb{Z}$ and the only non-zero element of the Chow group is h^{D-i}.

For $i = D/2$ (here we are in the case of even D, of course), the basis of the group $\mathrm{Ch}_i(\bar{X})$ is given by the elements h^d and l_d, while the basis of the group $\mathrm{Ch}_{i-a}(\bar{X}_0)$ is given by the elements h^{d-a} and l_{d-a}. The subgroups $\bar{\mathrm{Ch}}_d(X) \subset \mathrm{Ch}_d(\bar{X})$ and $\bar{\mathrm{Ch}}_{d-a}(X_0) \subset \mathrm{Ch}_{d-a}(\bar{X}_0)$, however, are 1-dimensional, generated by h^d and h^{d-a} (because $l_{d-a} \notin \bar{\mathrm{Ch}}(X_0)$ by Corollary 2.2). Since these subgroups are interchanged by the projection and the inclusion, h^d corresponds to h^{d-a}. Now there are only two possibilities for the element of $\mathrm{Ch}_d(\bar{X})$ corresponding to $l_{d-a} \in \mathrm{Ch}_{d-a}(\bar{X}_0)$: either this is l_d or this is $l_d + h^d$. Which one of these two possibilities takes place depends on the construction of the motivic decomposition $(*)$; but a given motivic decomposition can be always corrected in such a way that the first possibility takes place (one can simply use an automorphism of the variety X_0 interchanging l_{d-a} with $l_{d-a} + h^{d-a}$). □

The "most important" summand in the motivic decomposition of X^r is, of course, X_0^r. We introduce a special notation for the projection and the inclusion related to this summand: $pr^r = pr_{a \dots a}$ and $in^r = in_{a \dots a}$.

Corollary 2.4. *The (mutually semi-inverse) homomorphisms*

$$pr^r \colon \mathrm{Ch}(\bar{X}^r) \to \mathrm{Ch}(\bar{X}_0^r) \quad \textit{and} \quad in^r \colon \mathrm{Ch}(\bar{X}_0^r) \to \mathrm{Ch}(\bar{X}^r)$$

(for any $r \geq 1$) map rational cycles to rational cycles; moreover, the induced homomorphism $pr^r \colon \bar{\mathrm{Ch}}(X^r) \to \bar{\mathrm{Ch}}(X_0^r)$ is surjective. □

Now we get an extended version of Corollary 2.2 which reads as follows:

Corollary 2.5. *For an arbitrary quadric X (isotropic or not) and any integer i one has: $l_i \in \bar{\mathrm{Ch}}(X)$ if and only if $\mathfrak{i}_0(X) > i$ (where $\mathfrak{i}_0(X) = \mathfrak{i}_0(\phi)$ is the Witt index).*

Proof. The "if" part of the statement is trivial. Let us prove the "only if" part using an induction on i. The case of $i = 0$ is served by Lemma 2.1.

Now we assume that $i > 0$ and $l_i \in \bar{\mathrm{Ch}}(X)$. Since $l_i \cdot h = l_{i-1}$, the element l_{i-1} is rational as well, and by the induction hypothesis $\mathfrak{i}_0(X) \geq i$. If $\mathfrak{i}_0(X) = i$, then the image of $l_i \in \bar{\mathrm{Ch}}(X)$ under the map $pr^1 \colon \mathrm{Ch}(\bar{X}) \to \mathrm{Ch}(\bar{X}_0)$ equals l_0 and is rational by Corollary 2.4. Therefore, by Lemma 2.1, the quadric X_0 is isotropic, a contradiction. □

We recall that the *splitting pattern* $\mathrm{sp}(\phi)$ of an anisotropic quadratic form ϕ is defined as the set of integers

$$\mathrm{sp}(\phi) = \{\dim(\phi_E)_0 |\ E/F \text{ is a field extension}\}$$

(here ϕ_E stays for the quadratic form over the field E obtained from ϕ by extending the scalars; $(\phi_E)_0$ is the anisotropic part of ϕ_E).

The splitting pattern can be obtained using the *generic splitting tower* of M. Knebusch (arbitrary filed extensions of F are then replaced by concrete fields). To construct this tower, we put $F_1 = F(X)$, the function field of the projective quadric X given by ϕ. Then we put $\phi_1 = (\phi_{F_1})_0$ and write X_1 for the projective quadric (over the field F_1) given by the quadratic form ϕ_1. We proceed by setting $F_2 = F_1(X_1)$ and so on until we can (we stop on $F_{\mathfrak{h}}$ such that $\dim(\phi_{\mathfrak{h}}) \leq 1$). The tower of fields $F \subset F_1 \subset \cdots \subset F_{\mathfrak{h}}$ obtained this way is called the generic splitting tower of ϕ and (see [15])

$$\mathrm{sp}(\phi) = \{\dim(\phi), \dim(\phi_{F_1})_0, \dots, \dim(\phi_{F_{\mathfrak{h}}})_0\} = \{\dim(\phi), \dim(\phi_1), \dots, \dim(\phi_{\mathfrak{h}})\}$$

(the integer $\mathfrak{h} = \mathfrak{h}(\phi)$ is the *height* of ϕ; note that the elements of $\mathrm{sp}(\phi)$ are written down in the descending order).

An equivalent invariant of ϕ is called the *higher Witt indices* of ϕ and defined as follows. Let us write the set of integers $\{\mathfrak{i}_0(\phi_E)|\ E/F \text{ a field extension}\}$, where $\mathfrak{i}_0(\phi_E)$ is the usual Witt index of ϕ_E, in the form

$$\{0 = \mathfrak{i}_0(\phi) < \mathfrak{i}_1 < \mathfrak{i}_1 + \mathfrak{i}_2 < \cdots < \mathfrak{i}_1 + \mathfrak{i}_2 + \cdots + \mathfrak{i}_{\mathfrak{h}}\} .$$

The sequence of the positive integers $\mathfrak{i}_1, \dots, \mathfrak{i}_{\mathfrak{h}}$ is called the higher Witt indices of ϕ. For every $q \in \{0, 1, \dots, \mathfrak{h}\}$, we also set

$$\mathfrak{j}_q = \mathfrak{j}_q(\phi) = \mathfrak{i}_0 + \mathfrak{i}_1 + \cdots + \mathfrak{i}_q = \mathfrak{i}_0(\phi_{F_q}) .$$

Clearly, one has

$$\{0 = \mathfrak{j}_0, \mathfrak{j}_1, \dots, \mathfrak{j}_{\mathfrak{h}}\} = \{\mathfrak{i}_0(\phi_E)|\ E/F \text{ is a field extension}\}$$

(this set of integers is sometimes also called the splitting pattern of ϕ in the literature).

The following easy observation is crucial:

Theorem 2.6. *The splitting pattern as well as the higher Witt indices of an anisotropic quadratic form ϕ (of some given dimension) are determined by the group*

$$\bar{\mathrm{Ch}}(X^*) = \bigoplus_{r \geq 1} \bar{\mathrm{Ch}}(X^r) .$$

Proof. The pull-back homomorphism $g_1^* \colon \mathrm{Ch}(X^r) \to \mathrm{Ch}(X^{r-1}_{F(X)})$ with respect to the morphism of schemes $g_1 : X^{r-1}_{F(X)} \to X^r$ given by the generic point of, say, the first factor of X^r, is surjective. It induces an epimorphism $\bar{\mathrm{Ch}}(X^r) \twoheadrightarrow \bar{\mathrm{Ch}}(X^{r-1}_{F(X)})$, which is a restriction of the epimorphism $\mathrm{Ch}(\bar{X}^r) \twoheadrightarrow \mathrm{Ch}(\bar{X}^{r-1}_{F(X)})$ mapping each basis element of the shape $h^0 \times \beta$ with $\beta \in \mathrm{Ch}(\bar{X}^{r-1})$ to $\beta \in \mathrm{Ch}(\bar{X}^{r-1}_{F(X)})$ and killing all other basis elements. Therefore the group $\bar{\mathrm{Ch}}(X^*)$ determines the group $\bar{\mathrm{Ch}}(X^*_{F(X)})$. In particular, the group $\bar{\mathrm{Ch}}(X_{F(X)})$ is determined, so that we have reconstructed $\mathfrak{i}_0(X_{F(X)}) = \mathfrak{i}_1(X)$ (see Corollary 2.5). Moreover, by Corollary 2.4, the group $\bar{\mathrm{Ch}}(X^*_{F(X)})$ determines the group $\bar{\mathrm{Ch}}(X_1^*)$ (via the surjection pr^*; X_1 staying for the anisotropic part of $X_{F(X)}$), and we can proceed by induction. □

Remark 2.7. The proof of Theorem 2.6 makes it clear that the statement of this theorem can be made more precise in the following way. If for some $q \in \{1, 2, \dots, \mathfrak{h}\}$ the Witt indices $\mathfrak{i}_1, \dots, \mathfrak{i}_{q-1}$ are already reconstructed, then one determines $\mathfrak{i}_q = \mathfrak{j}_q - \mathfrak{i}_{q-1} - \dots - \mathfrak{i}_1$ by the formula

$$\mathfrak{j}_q = \max\{j |\text{ the product } h^0 \times h^{\mathfrak{j}_1} \times \dots \times h^{\mathfrak{j}_{q-1}} \times l_{j-1} \text{ is contained in a rational cycle}\} .$$

Remark 2.8. Concluding this section, we would like to underline that the role of the algebraic closure $\bar{F}$ in the definition of the group $\bar{\mathrm{Ch}}(X^*)$ is secondary: the group $\mathrm{Ch}(\bar{X}^*)$ (used in the definition of $\bar{\mathrm{Ch}}(X^*)$) has to be interpreted as the direct limit $\lim \mathrm{Ch}(X_E^*)$ taken over *all* field extensions E/F. The homomorphism $\mathrm{Ch}(\bar{X}^*) \to \lim \mathrm{Ch}(X_E^*)$ is an isomorphism. More generally, the homomorphism $\mathrm{Ch}(X_E^*) \to \lim \mathrm{Ch}(X_E^*)$ for a given E/F is an isomorphism if and only if the quadratic form ϕ_E is completely split (in particular, for any E/F with completely split ϕ_E, there is a canonical isomorphism $\mathrm{Ch}(X_E^*) = \mathrm{Ch}(\bar{X}^*)$, coinciding with the composition $\mathrm{res}_{\bar{E}/\bar{F}}^{-1} \circ \mathrm{res}_{\bar{E}/E}$, where $\bar{E}$ is a field containing E and $\bar{F}$).

3. Strategy of proof

As shown in Theorem 2.6, the group $\bar{\mathrm{Ch}}(X^*)$ determines the splitting pattern of the quadratic form ϕ. In its turn, the splitting pattern of ϕ determines the powers of the fundamental ideal of the Witt ring containing the class of ϕ. At least it is easy to prove

Lemma 3.1. *Let ϕ be an even-dimensional anisotropic quadratic form and let $n \geq 1$ be an integer. We write p for the least positive integer of* $\mathrm{sp}(\phi)$ *(note that p is a power of* 2, [20, thm. 5.4(i)]*). If $[\phi] \in I^n$, then $p \geq 2^n$.*

Proof. We assume that $[\phi] \in I^n(F)$. Let E/F be a field extension such that $\dim(\phi_E)_0 = p$. Since $0 \neq [(\phi_E)_0] \in I^n(E)$, we get that $p \geq 2^n$ by the Arason-Pfister theorem. □

Remark 3.2. It is not needed in this paper but nevertheless good to know that the converse statement to Lemma 3.1 is also true. This is a hard result, however. It is proved in [16, thm. 4.3].

Now we are able to describe the strategy of our proof of Conjecture 1.1. Let us consider a power I^n of the fundamental ideal. Let ϕ be an anisotropic quadratic form with $[\phi] \in I^n$, having some dimension prohibited by Conjecture 1.1. The group $\bar{\mathrm{Ch}}(X^*)$, where X is the projective quadric given by ϕ, should satisfy some restrictions listed bellow. This group is a subgroup of $\mathrm{Ch}(\bar{X}^*)$, the later one depends only on the dimension of ϕ. So, we prove Conjecture 1.1, if we check that every subgroup of $\mathrm{Ch}(\bar{X}^*)$, satisfying the list of restrictions, can not be $\bar{\mathrm{Ch}}(X^*)$ for the form ϕ (see Lemma 3.1 with Theorem 2.6). This is the way we prove Conjecture 1.1.

And here is the list of restrictions on $\bar{\mathrm{Ch}}(X^*)$ considered as a subset of $\mathrm{Ch}(\bar{X}^*)$ (a big part of this list is of course valid for an arbitrary smooth projective variety in place of the quadric X):

Proposition 3.3. *Assuming that the quadric X is anisotropic,*[6] *we have:*

[6] Anisotropy is important only for (4), (8), (9), and (10).

(1) $\bar{\mathrm{Ch}}(X^*)$ *is closed under addition and multiplication;*
(2) $\bar{\mathrm{Ch}}(X^*)$ *is closed under passing to the homogeneous components (with respect to the grading of the Chow group and to the $*$-grading);*
(3) $\bar{\mathrm{Ch}}(X^*)$ *contains* $h^0 = [\bar{X}]$ *and* $h^1 = h$ *(and therefore contains h^i for all $i \geq 0$);*
(4) [**Springer-Satz**]
$\bar{\mathrm{Ch}}(X^*)$ *does not contain* l_0*;*
(5) *for every $r \geq 1$,* $\bar{\mathrm{Ch}}(X^r)$ *is closed under the automorphisms of* $\mathrm{Ch}(\bar{X}^r)$ *given by the permutations of factors of X^r;*
(6) *for every $r \geq 1$,* $\bar{\mathrm{Ch}}(X^*)$ *is closed under push-forwards and pull-backs with respect to all r projections $X^r \to X^{r-1}$ and to all r diagonals $X^r \to X^{r+1}$ (taking in account the previous restriction, it is enough to speak only about the first projection*

$$x_1 \times x_2 \times \cdots \times x_r \mapsto x_2 \times \cdots \times x_r$$

and the first diagonal

$$x_1 \times x_2 \times \cdots \times x_r \mapsto x_1 \times x_1 \times x_2 \cdots \times x_r$$

here);
(7) $\bar{\mathrm{Ch}}(X^*)$ *is closed under the total Steenrod operation*

$$S\colon \mathrm{Ch}(\bar{X}^*) \to \mathrm{Ch}(\bar{X}^*)\ ;$$

(8) [A. Vishik, **"Size of binary correspondences"**]
if $\bar{\mathrm{Ch}}(X^2) \ni h^0 \times l_i + l_i \times h^0$ *for some integer $i \geq 0$, then the integer* $\dim(X) - i + 1$ *is a power of* 2*;*
(9) [**"Inductive restriction"**]
the image of $\bar{\mathrm{Ch}}(X^{*+1})$ *under the composition*

$$\mathrm{Ch}(\bar{X}^{*+1}) \xrightarrow{g_1^*} \mathrm{Ch}(\bar{X}^*_{F(X)}) \xrightarrow{pr^*} \mathrm{Ch}(\bar{X}^*_1)$$

(g_1^* is introduced in the proof of Theorem 2.6, pr^* in Corollary 2.4) should coincide with $\bar{\mathrm{Ch}}(X^*_1)$ *and therefore should satisfy all restrictions listed in this proposition (including the current one);*
(10) [**"Supplement to inductive restriction"**]
for any integer $r \geq 2$, any integer $s \in [1,\ r)$, and any projection $pr_{i_1 \dots i_r}$ of Proposition 2.3, the image of $\bar{\mathrm{Ch}}(X^r)$ *under* $pr_{i_1\dots i_r}\colon \mathrm{Ch}(\bar{X}^r) \to \mathrm{Ch}(\bar{X}_1^{s(i_1,\dots,i_r)})$ *is inside of* $\bar{\mathrm{Ch}}(X_1^{s(i_1,\dots,i_r)})$ *(reconstructed by* (9)*).*

Proof. Only the property (8) needs a proof. We note that this property does not seem to be a consequence of the others. It is proved in [13, thm. 5.1] by some computation in the *integral* Chow group of X^*, not in the modulo 2 Chow group (although involving modulo 2 Steenrod operations) (the original proof is in [7, thm. 6.1]; it makes use of higher motivic cohomology).

More precisely, the case of $i = 0$ is proved in [13, thm. 5.1]. In order to reduce the general case to the case of $i = 0$, we take an arbitrary subquadric $Y \subset X$ of codimension i and pullback the cycle $h^0 \times l_i + l_i \times h^0$ with respect to the imbedding $Y^2 \hookrightarrow X^2$. The result is $h^0 \times l_0 + l_0 \times h^0 \in \bar{\mathrm{Ch}}(Y^2)$. Therefore, $\dim(Y) + 1$ is a power of 2 by [13, thm.

5.1]. Since $\dim(Y) = \dim(X) - i$, it follows that the integer $\dim(X) - i + 1$ is the same power of 2. $\square$

Remark 3.4. Obviously, one can write down some additional restrictions on $\bar{\mathrm{Ch}}(X^*)$. However, all restrictions I know are consequences of the restrictions of Proposition 3.3. For instance, $\bar{\mathrm{Ch}}(X^*)$ should be stable with respect to the exterior products; but this is a consequence of the stability with respect to the internal products (1) and the pull-backs with respect to projections (6). Another example: the image of the total Chern class $c\colon K_0(\bar{X}^*) \to \mathrm{Ch}(\bar{X}^*)$ restricted to $K_0(X^*)$ (note that $K_0(X^*)$ is computed for quadrics [21] and, more generally, for all projective homogeneous varieties [17]) should be inside of $\bar{\mathrm{Ch}}(X^*)$; but it is already guarantied by the fact that $\bar{\mathrm{Ch}}(X^*)$ is closed under addition and multiplication (1) and contains h^0 and h^1 (3).[7] One more example: $\bar{\mathrm{Ch}}(X^2)$ should be closed under the composition of correspondences (see [5, §16] for the definition of composition of correspondences); but the operation of composition of correspondences is produced by pull-backs and push-forwards with respect to projections together with the operation of multiplication of cycles.

Remark 3.5. Let us remark that all operations involved in the list of restrictions of Proposition 3.3 are easy to compute in terms of the basis elements. The multiplication in $\mathrm{Ch}(\bar{X}^*)$ was described in the previous §; a formula for the total Steenrod operation was given already as well. Also the operations used in the inductive restriction are computed (see Corollary 2.4 and the proof of Theorem 2.6). As to the pull-backs and push-forwards with respect to the first projection $pr : X^{r+1} \to X^r$ and to the first diagonal $\delta : X^r \to X^{r+1}$, they are computed for basis elements $\beta_0, \beta_1, \dots, \beta_r \in \mathrm{Ch}(\bar{X})$ as follows:

$$\begin{aligned}
pr^*(\beta_1 \times \cdots \times \beta_r) &= h^0 \times \beta_1 \times \cdots \times \beta_r \,; \\
pr_*(\beta_0 \times \beta_1 \times \cdots \times \beta_r) &= \begin{cases} \beta_1 \times \cdots \times \beta_r \,, & \text{if } \beta_0 = l_0, \\ 0\,, & \text{otherwise;} \end{cases} \\
\delta^*(\beta_0 \times \beta_1 \times \cdots \times \beta_r) &= (\beta_0 \cdot \beta_1) \times \beta_2 \times \cdots \times \beta_r \,; \\
\delta_*(\beta_1 \times \cdots \times \beta_r) &= \big((\beta_1 \times h^0) \cdot \Delta\big) \times \beta_2 \times \cdots \times \beta_r = \\
& \big((h^0 \times \beta_1) \cdot \Delta\big) \times \beta_2 \times \cdots \times \beta_r \,;
\end{aligned}$$

where $\Delta \in \bar{\mathrm{Ch}}(X^2)$ is the class of the diagonal computed in Corollary 3.9.

Remark 3.6. One can obviously simplify a little bit the list of restrictions of Proposition 3.3. For instance, instead of stability under the push-forwards with respect to the diagonals, it suffices to require that the cycle $\sum_{i=0}^{d}(h^i \times l_i + l_i \times h^i)$, related to the diagonal, lies in $\bar{\mathrm{Ch}}(X^2)$ (see Remark 3.5 and Corollary 3.9). Also the inductive restriction is not so restrictive as it may seem to: the group $\bar{\mathrm{Ch}}(X_1^*)$ automatically satisfies most of the required restrictions.

Remark 3.7. Looking at the list of restrictions, it is easy to see that every group $\bar{\mathrm{Ch}}(X^r)$ determines $\bar{\mathrm{Ch}}(X^{<r})$. Moreover, one can show that $\bar{\mathrm{Ch}}(X^d)$ determines the whole group

[7] However the property with the Chern class can be a good replacement for (3) when transfering this theory to other algebraic varieties in place of the quadric X.

$\bar{\mathrm{Ch}}(X^*)$. [8] Since $\bar{\mathrm{Ch}}(X^d)$ is a subgroup of the finite group $\mathrm{Ch}(\bar{X}^d)$, it follows, in particular, that the invariant $\bar{\mathrm{Ch}}(X^d)$ (of the quadratic forms ϕ of a given dimension) has only a finite number of different values (this way one also sees that Conjecture 1.1 can be checked for any concrete dimension by computer).

We will often use the composition of correspondences, even for the cycles on bigger than 2 powers of X: this is a convenient way to handle the things. Namely, for $\alpha \in \bar{\mathrm{Ch}}(X^r)$ and $\alpha' \in \bar{\mathrm{Ch}}(X^{r'})$, we may consider α as a correspondence, say, from X^{r-1} to X, and we may consider α' as a correspondence from X to $X^{r'-1}$; then the composition $\alpha' \circ \alpha$ is a cycle in $\bar{\mathrm{Ch}}(X^{r+r'-2})$, and here is the formula for composing the basis elements (we put here this obvious formula because it will be used many times in our computations):

Lemma 3.8. *The composition $\beta' \circ \beta \in \mathrm{Ch}(\bar{X}^{r+r'-2})$ of two basis elements*

$$\beta = \beta_1 \times \cdots \times \beta_r \in \mathrm{Ch}(\bar{X}^r) \quad \textit{and} \quad \beta' = \beta'_1 \times \cdots \times \beta'_{r'} \in \mathrm{Ch}(\bar{X}^{r'})$$

is equal to $\beta_1 \times \cdots \times \beta_{r-1} \times \beta'_2 \times \cdots \times \beta'_{r'}$, if $\beta_r \cdot \beta'_1 = l_0$; otherwise the composition $\beta' \circ \beta$ is 0. □

Corollary 3.9. *For the diagonal class $\Delta \in \bar{\mathrm{Ch}}(X^2)$, one has:*

$$\Delta = \sum_{i=0}^{d} (h^i \times l_i + l_i \times h^i) + (D+1) \cdot (d+1) \cdot (h^d \times h^d)\ .$$

In particular, the sum $\sum_{i=0}^{d} (h^i \times l_i + l_i \times h^i)$ is always rational.

Proof. Using Lemma 3.8, it is straight-forward to verify that the cycle given by the above formula acts (by composition) trivially on any basis cycle of $\mathrm{Ch}^2(\bar{X}^2)$. □

4. Cycles on X^2

We are using the notation introduced in § 2. In particular, X is the projective quadric of an even dimension $D = 2d$ or of an odd dimension $D = 2d+1$ given by a quadratic form ϕ over the field F. In this § we assume that X is anisotropic. Let $\mathfrak{i}_1, \dots \mathfrak{i}_{\mathfrak{h}}$ be the higher Witt indices of ϕ (with $\mathfrak{h}$ being the height of ϕ). We write S for the set $\{1, 2, \dots, \mathfrak{h}\}$ and we set $\mathfrak{j}_q = \mathfrak{i}_1 + \mathfrak{i}_2 + \cdots + \mathfrak{i}_q$ for every $q \in S$.

An "important" as well as the "first interesting" part of the group $\bar{\mathrm{Ch}}(X^*)$ is the group $\bar{\mathrm{Ch}}(X^2)$ and especially $\bar{\mathrm{Ch}}_D(X^2) = \bar{\mathrm{Ch}}^D(X^2)$ (note that, due to the Rost nilpotence theorem ([19], see also [3]), the latter group detects all motivic decompositions of X). The group $\bar{\mathrm{Ch}}_D(X^2)$ was studied intensively by A. Vishik (see, e.g., [23]). In the next § we reproduce most of his results concerning this group. Actually we give an extended version of these results describing the structure of a bigger group, namely of the group $\bar{\mathrm{Ch}}^{\leq D}(X^2)$.

Originally, Vishik's results are formulated in terms of motivic decompositions of X; by this reason, their proofs use the Rost nilpotence theorem for quadrics, which is not used in the present text at all. Here we simplify the formulation; we also give a different complete self-contained proof and show that all these results are consequences of the restrictions on $\bar{\mathrm{Ch}}(X^*)$ listed in Proposition 3.3. More precisely, we start with some general results

[8] It would be interesting to rewrite all restrictions of Proposition 3.3 in terms of the group $\bar{\mathrm{Ch}}(X^d)$.

concerning an arbitrary anisotropic quadric X; the proofs of these results use neither the restriction provided by the Steenrod operation, nor the "size of binary correspondences" restriction; a summary of these results is given in Theorem 4.24. Proposition 4.28, appearing in the very end of the section, contains the result on the so called small quadrics (Definition 4.27), needed in the proof of Conjecture 1.1; its proof uses the "size of binary correspondences" restriction (the Steenrod operation does still not show up).

We start by the following easy observation:

Lemma 4.1. *For an even $D = \dim(X)$, the basis element $l_d \times l_d \in \bar{\mathrm{Ch}}_D(X^2)$ does not appear in the decomposition of a rational cycle.*

Proof. We assume the contrary. Let α be a cycle in $\bar{\mathrm{Ch}}_D(X^2)$ containing $l_d \times l_d$. Then the push-forward with respect to the projection onto the second factor $pr\colon X^2 \to X$ of the cycle $\alpha \cdot (h^d \times h^0)$ is rational and equals l_d or $h^d + l_d$ (because $pr_*\big(\beta \cdot (h^d \times h^0)\big)$ is l_d for $\beta = (l_d \times l_d)$, h^d for $\beta = l_d \times h^d$, and 0 for every other basis cycle $\beta \in \mathrm{Ch}_D(\bar{X}^2)$). Therefore the cycle l_d is rational, showing that X is hyperbolic (Corollary 2.5), a contradiction. □

Lemma 4.2. *If $\alpha_1, \alpha_2 \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$, then the cycle $\alpha_1 \cap \alpha_2$ is rational (where the notation $\alpha_1 \cap \alpha_2$ means the sum of the basis cycles contained simultaneously in α_1 and in α_2).*

Proof. Clearly, we may assume that α_1 and α_2 are homogeneous of the same dimension $D + i$ and do not contain any non-essential basis element. Then the intersection (modulo the non-essential elements) is computed as $\alpha_1 \cap \alpha_2 = \alpha_2 \circ \Big(\alpha_1 \cdot (h^0 \times h^i)\Big)$ (see Lemma 3.8). □

Definition 4.3. We write $\mathrm{Che}(\bar{X}^*)$ for the subgroup of $\mathrm{Ch}(\bar{X}^*)$ generated by the essential basis elements. We set $\mathrm{Che}(X^*) = \mathrm{Che}(\bar{X}^*) \cap \bar{\mathrm{Ch}}(X^*)$. Note that the group $\mathrm{Che}(X^*)$ is a subgroup of $\bar{\mathrm{Ch}}(X^*)$ isomorphic to the quotient of $\bar{\mathrm{Ch}}(X^*)$ by the subgroup of the non-essential elements.

Definition 4.4. A non-zero cycle of $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ is called *minimal*, if it does not contain any proper rational subcycle. Note that a minimal cycle does not contain non-essential basis elements, i.e., lies in $\mathrm{Che}(X^2)$. Also note that a minimal cycle is always homogeneous.

A very first structure result on $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ reads as follows:

Proposition 4.5. *The minimal cycles form a basis of the group $\mathrm{Che}^{\leq D}(X^2)$. Two different minimal cycles do not intersect each other (here we speak about the notion of intersection of cycles introduced in Lemma 4.2). The sum of the minimal cycles of dimension D is equal to the sum $\sum_{i=0}^{d} h^i \times l_i + l_i \times h^i$ of all D-dimensional essential basis elements (excluding $l_d \times l_d$ in the case of even D).*

Proof. The first two statements of Proposition 4.5 follow from Lemma 4.2. The third statement follows from previous ones together with the rationality of the diagonal cycle (see Corollary 3.9). □

Definition 4.6. Let α be a homogeneous cycle in $\mathrm{Ch}^{\leq D}(\bar{X}^2)$. For every i with $0 \leq i \leq \dim(\alpha) - D$, the products $\alpha \cdot (h^0 \times h^i)$, $\alpha \cdot (h^1 \times h^{i-1})$, and so on up to $\alpha \cdot (h^i \times h^0)$ will be called the (i-th order) *derivatives* of α. Note that all derivatives are still in $\mathrm{Ch}^{\leq D}(\bar{X}^2)$ and that all derivatives of a rational cycle are also rational.

Lemma 4.7. (1) *Each derivative of any essential basis element $\beta \in \mathrm{Che}^{\leq D}(\bar{X}^2)$ is an essential basis element.*

(2) *For any $r \geq 0$ and any essential basis cycles $\beta_1, \beta_2 \in \mathrm{Che}_{D+r}(\bar{X}^2)$, two derivatives $\beta_1 \cdot (h^{i_1} \times h^{j_1})$ and $\beta_2 \cdot (h^{i_2} \times h^{j_2})$ of β_1 and β_2 coincide only if $\beta_1 = \beta_2$, $i_1 = i_2$, and $j_1 = j_2$.*

Proof. (1) If $\beta \in \mathrm{Che}_{D+r}(\bar{X}^2)$ for some $r \geq 0$, then, up to transposition, $\beta = h^i \times l_{i+r}$ with $i \in [0,\ d-r]$. An arbitrary derivative of β is equal to $\beta \cdot (h^{j_1} \times h^{j_2}) = h^{i+j_1} \times l_{i+r-j_2}$ with some $j_1, j_2 \geq 0$ such that $j_1 + j_2 \leq r$. We have $0 \leq i + j_1 \leq d$ and therefore h^{i+j_1} is a basis element. We also have $d \geq i + r - j_2 \geq 0$ and therefore l_{i+r-j_2} is a basis element too.

Statement (2) is trivial. $\square$

Remark 4.8. For the sake of visualization, it is good to think of the basis cycles of $\mathrm{Ch}^{\leq D}(\bar{X}^2)$ (with $l_{D/2} \times l_{D/2}$ excluded by the reason of Lemma 4.1) as of points of the "pyramid"

$$\begin{array}{c}
\circ \\
\circ\ \ \circ \\
\circ\ \ \circ\ \ \circ \\
\circ\ \ \circ\ \ \circ\ \ \circ \\
*\ \ \circ\ \ \circ\ \ \circ\ \ \circ\ \ \circ\ \ * \\
*\ \ *\ \ \circ\ \ \circ\ \ \circ\ \ \circ\ \ *\ \ * \\
*\ \ *\ \ *\ \ \circ\ \ \circ\ \ \circ\ \ *\ \ *\ \ * \\
*\ \ *\ \ *\ \ *\ \ \circ\ \ \circ\ \ *\ \ *\ \ *\ \ * \\
*\ \ *\ \ *\ \ *\ \ *\ \ \circ\ \ *\ \ *\ \ *\ \ *\ \ *
\end{array}$$

($D = 8$ on the picture; for an odd D the pyramid has no "step", see, e.g., the picture of Remark 4.12), where the $\circ$-s stay for the non-essential basis elements while the $*$-s stay for the essential ones; the point on the top is $h^0 \times h^0$; for every $i \in \{0, 1, \ldots, D\}$, the i-th row of the pyramid represents the basis of $\mathrm{Ch}^i(\bar{X}^2)$ (in the case of even D, the D-th row is the basis without $l_d \times l_d$) ordered by the codimension of the first factors (starting with $h^0 \times ?$). For any $\alpha \in \mathrm{Che}^{\leq D}(\bar{X}^2)$, we can put a mark on the points representing basis elements contained in the decomposition of α; the set of marked points is the *diagram* of α. If α is homogeneous, the marked points lie in the same row. Now it is easy to interpret the derivatives of α: the diagram of an i-th order derivative is a projection of the marked points of the diagram of α to the i-th row bellow along some direction. The diagram of every derivative of α has the same number of marked points as the diagram of α (Lemma 4.7). The diagrams of two different derivatives of the same order are shifts (to the right or to the left) of each other.

Lemma 4.9. *The following conditions on a homogeneous cycle $\alpha \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$ are equivalent:*

(1) *α is minimal;*
(2) *all derivatives of α are minimal;*
(3) *at least one derivative of α is minimal.*

Proof. Derivatives of a proper subcycle of α are proper subcycles of the derivatives of α; therefore, (3) $\Rightarrow$ (1).

In order to show that (1) $\Rightarrow$ (2), it suffices to show that two first order derivatives $\alpha \cdot (h^0 \times h^1)$ and $\alpha \cdot (h^1 \times h^0)$ of a minimal cycle α are minimal. In the contrary case, replacing eventually α by its transposition, we come to the situation where the derivative $\alpha \cdot (h^0 \times h^1)$ of a minimal α is not minimal. It follows that the cycle $\alpha \cdot (h^0 \times h^i)$, where

$i = \dim(\alpha) - D$, is not minimal too; let α' be its proper subcycle. Taking the composition $\alpha \circ \alpha'$ and removing the non-essential summands, we get a proper subcycle of α (see Lemma 3.8). □

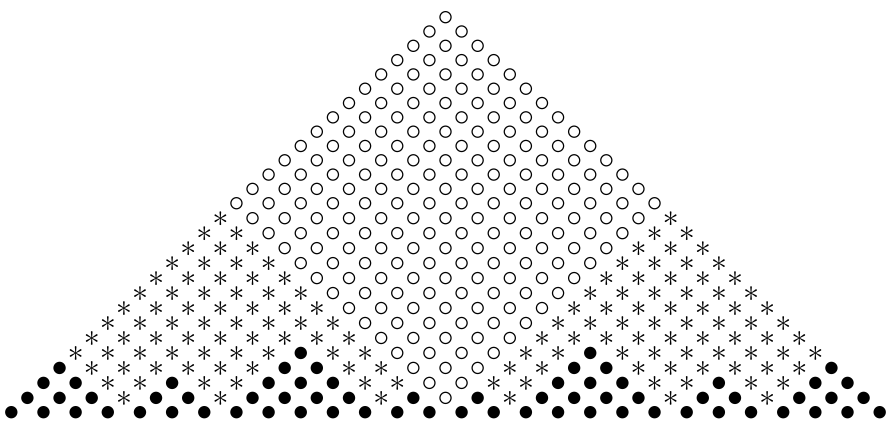
Corollary 4.10. *The derivatives of a minimal cycle are disjoint.*

Proof. The derivatives of a minimal cycle are minimal (Lemma 4.9) and pairwise different (Lemma 4.7). Two different minimal cycles are disjoint by Lemma 4.2 (see also Proposition 4.5). □

Lemma 4.11. *Let α be an element of $\bar{\mathrm{Ch}}_{D+k-1}(X^2)$ with some $k \geq 1$. For any $q \in \{1, \ldots, \mathfrak{h}\}$ and for any non-negative i with $\mathfrak{i}_q - k < i < \mathfrak{i}_q$, the cycle α contains neither the product $h^{\mathfrak{j}_{q-1}+i} \times l_{\mathfrak{j}_{q-1}+i+k-1}$ nor the transposition of this product.*

Proof. Let us assume the contrary: for some $k \geq 1$, some $q \in \{1, \ldots, \mathfrak{h}\}$, and some i with $\mathfrak{i}_q - k < i < \mathfrak{i}_q$, there exists a rational cycle α containing the product $h^{\mathfrak{j}_{q-1}+i} \times l_{\mathfrak{j}_{q-1}+i+k-1}$ or the transposition of this product. If α contains the transposition of the product, we replace α by the transposition of α. Passing to the $(q-1)$-th field of the generic splitting tower and using the projection of Corollary 2.4, we come to the situation where $q = 1$ and α contains the product $h^i \times l_{i+k-1}$ such that $\mathfrak{i}_1 - k < i < \mathfrak{i}_1$. The projection $pr_{D-i,\mathfrak{i}_1}(\alpha_{F(X)})$ is a rational cycle on $\bar{X}_1$ containing $l_{i+k-1-\mathfrak{i}_1}$ (note that $i + k - 1 - \mathfrak{i}_1 \geq 0$). We get a contradiction with Corollary 2.2. □

Remark 4.12. In order to "see" the statement of Lemma 4.11, it is helpful to mark by • the essential basis elements which are not "forbidden" by this lemma (we are speaking about the pyramid of basis cycles drawn in Remark 4.8). We will get isosceles triangles based on the lower row of the pyramid. For example, if X is a 29-dimensional quadric with the higher Witt indices $4, 3, 5, 2$ (such a quadric X does not exist in reality, but is convenient for the illustration), then the picture looks as follows:



Definition 4.13. The triangles of Remark 4.12 will be called the *shell triangles* (their bases are shells in the sense of A. Vishik). The shell triangles in the left half of the pyramid are counted from the left starting by 1. The shells triangles in the right half of the pyramid are counted from the right starting by 1 as well (so that the symmetric triangles have the same number; for any $q \in S$, the bases of the q-th triangles have (each) $\mathfrak{i}_q$ points). The rows of the shell triangles are counted from bellow starting by 0. The

points of rows of the shell triangles (of the left ones as well as of the right one) are counted from the left starting by 1.

Lemma 4.14. *For every rational cycle $\alpha \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$, the number of the essential basis cycles contained in α is even (that is, the number of the marked points in the diagram of any $\alpha \in \mathrm{Che}^{\leq D}(X^2)$ is even).*

Proof. We may assume that α is homogeneous, say, $\alpha \in \bar{\mathrm{Ch}}_{D+k}(X^2)$, $k \geq 0$. Let n be the number of the essential basis cycles contained in α. The pull-back $\delta^*(\alpha)$ of α with respect to the diagonal $\delta \colon X \to X^2$ produces $n \cdot l_k \in \bar{\mathrm{Ch}}(X)$. By Corollary 2.2, it follows that n is even. □

Lemma 4.15. *Let $\alpha \in \bar{\mathrm{Ch}}(X^2)$ be a cycle containing $\beta = h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q - 1}$ for some $q \in S = \{1, 2, \dots, \mathfrak{h}\}$ (this β is the top of the q-th left shell triangle). Then α also contains the transposition of β.*

Proof. Replacing F by the field F_{q-1} of the generic splitting tower of F, X by X_{q-1}, and α by $pr^2(\alpha_{F_{q-1}})$, we come to the situation where $q = 1$.

Then we replace α by its homogeneous component containing β and apply to it Lemma 4.11 (with $k = \mathfrak{i}_1$). Let us assume that the transposition of β is not contained in α.

By Lemma 4.11 α does not contain any of the essential basis cycles having h^i with $0 < i < \mathfrak{i}_1$ as a factor; therefore the number of the essential basis elements contained in α and the number of the essential basis elements contained in $pr^2(\alpha_{F(X)}) \in \bar{\mathrm{Ch}}(X_1^2)$ differ by 1. In particular, these two numbers have different parity. However, the number of the essential basis elements contained in α is even by Lemma 4.14. By the same lemma, the number of the essential basis elements contained in $pr^2(\alpha_{F(X)})$ is even too. □

Definition 4.16. A minimal cycle $\alpha \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$ is called *primordial*, if it is not a positive order derivative of another rational cycle.

Lemma 4.17. *Let $\alpha \in \bar{\mathrm{Ch}}(X^2)$ be a minimal cycle. Assume that for some $q \in S$, the cycle α contains $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q - 1}$. Then α is symmetric and primordial.*

Proof. The cycle $\alpha \cap t(\alpha)$ (where $t(\alpha)$ is the transposition of α; intersection of cycles is defined in Lemma 4.2) is symmetric, rational (Lemma 4.2), contained in α, and, by Lemma 4.15, still contains $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q - 1}$ (in particular, $\alpha \cap t(\alpha) \neq 0$). It coincides with α by the minimality of α.

It is easy to "see" that α is primordial looking at the picture of Remark 4.12 (because α contains the top point of some shell triangle). Nevertheless, let us do the proof by formulae. If there exists a rational cycle $\beta \neq \alpha$ such that α is a derivative of β, then there exists a rational cycle β' such that α is an order one derivative of β', that is, $\alpha = \beta' \cdot (h^0 \times h^1)$ or $\alpha = \beta' \cdot (h^1 \times h^0)$. In the first case β' should contain the basis cycle $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q}$, while in the second case β' contains $h^{\mathfrak{j}_{q-1}-1} \times l_{\mathfrak{j}_q - 1}$. However, these both cases are not possible by Lemma 4.11 (take $k = \mathfrak{i}_q + 1$ with $i = 0$ for the first case and $i = \mathfrak{i}_{q-1} - 1$ for the second case). □

It is easy to see that a cycle α with the property of Lemma 4.17 exists at least for $q = 1$:

Lemma 4.18. *There exists a cycle in $\bar{\mathrm{Ch}}_{D+\mathfrak{i}_1-1}(X^2)$ containing $h^0 \times l_{\mathfrak{i}_1 - 1}$.*

Proof. Take a preimage of $l_{\mathfrak{i}_1-1} \in \bar{\mathrm{Ch}}(X_{F(X)})$ under the surjection $\bar{\mathrm{Ch}}(X^2) \twoheadrightarrow \bar{\mathrm{Ch}}(X_{F(X)})$ given by the pull-back with respect to the morphism $X_{F(X)} \to X^2$ produced by the generic point of the first factor of X^2. □

The following lemma is proved already in [11] (under the name of "Vishik's principle"), but only for odd-dimensional quadrics and by a different as here method.

Lemma 4.19. *For any cycle $\rho \in \bar{\mathrm{Ch}}_D(X^2)$, any $q \in S$, and any $i \in [1,\ \mathfrak{i}_q]$, the element $h^{\mathfrak{j}_{q-1}+i-1} \times l_{\mathfrak{j}_{q-1}+i-1}$ is contained in ρ if and only if the element $l_{\mathfrak{j}_q-i} \times h_{\mathfrak{j}_q-i}$ is contained in ρ.*

Proof. Clearly, it is enough to prove Lemma 4.19 for $q = 1$ only. Let ρ be a counter-example to Lemma 4.19. Replacing ρ by its transposition, if necessary, we can come to the situation where $\rho \ni l_{\mathfrak{i}_1-i} \times h^{\mathfrak{i}_1-i}$ and $\rho \not\ni h^{i-1} \times l_{i-1}$ for some $i \in [1,\ \mathfrak{i}_1]$.

Let α be the cycle of Lemma 4.18. The composition $\beta = \big(t(\alpha) \cdot (h^{i-1} \times h^0)\big) \circ \rho$ (where $t(\alpha)$ is the transposition of α) is a rational homogeneous cycle containing $l_{\mathfrak{i}_1-i} \times h^0$ and not containing $h^{i-1} \times l_{\mathfrak{i}_1-1}$. Therefore $\gamma = t\big(\beta \cdot (h^{\mathfrak{i}_1-i} \times h^0)\big)$ is a rational cycle containing $h^0 \times l_0$ and not containing $l_{\mathfrak{i}_1-1} \times h^{\mathfrak{i}_1-1}$. It follows that the composition $\alpha \circ \gamma$ is a rational cycle containing $h^0 \times l_{\mathfrak{i}_1-1}$ and not containing $l_{\mathfrak{i}_1-1} \times h^0$, a contradiction with Lemma 4.15. □

To announce the result which follow, we prefer to use the language of picture rather then the language of formulae:

Corollary 4.20. *The diagram of an arbitrary $\alpha \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$ has the following property: for any $q \in S$ and any integers $i \geq 1$ and $k \geq 0$, the i-th point of the k-th row of the q-th* left *shell triangle is marked if and only if the i-th point of the k-th row of the q-th* right *shell triangle is marked (see Definition 4.13 for the agreement on counting the rows and the points of the shell triangles).*

Proof. The case of $k = 0$ is treated in Lemma 4.19 (while Lemma 4.15 treats the case of "maximal" k). The case of an arbitrary k is reduced to the case of $k = 0$ by taking a k-th order derivative of α. □

Remark 4.21. By Corollary 4.20, it follows that the diagram of a cycle in $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ is determined by, say, the left half of itself.

Example 4.22. As an application of the results on X^2 obtained by now (first of all, of Corollary 4.20), we give a short (simpler as the original) proof of the main result of [11], which can be stated as follows: if ϕ is an anisotropic quadratic form and 2^r is the biggest power of 2 dividing the difference $\dim(\phi) - \mathfrak{i}_1(\phi)$, then $\mathfrak{i}_1(\phi) \leq 2^r$. For the proof, assume that $\mathfrak{i}_1 = \mathfrak{i}_1(\phi) > 2^r$ and consider the Steenrod operation $S^{2^r}(\alpha)$ of a homogeneous cycle $\alpha \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$ containing $h^0 \times l_{\mathfrak{i}_1-1}$ (for the existence of α see Lemma 4.18; note that $S^{2^r}(\alpha)$ is still inside of $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ just because of the inequality $\mathfrak{i}_1 > 2^r$). Since

$$S^{2^r}(h^0 \times l_{\mathfrak{i}_1-1}) = h^0 \times S^{2^r}(l_{\mathfrak{i}_1-1}) = \binom{\dim(\phi) - \mathfrak{i}_1}{2^r} \cdot (h^0 \times l_{\mathfrak{i}_1-1-2^r})$$

and the binomial coefficient is odd, we get that $S^{2^r}(\alpha) \ni h^0 \times l_{\mathfrak{i}_1-1-2^r}$. On the other hand, $\alpha \not\ni l_{\mathfrak{i}_1-1+i} \times h^i$ for any $i \in [1, \mathfrak{i}_1 - 1]$ by Lemma 4.11; consequently, $S^{2^r}(\alpha) \not\ni l_{\mathfrak{i}_1-1-2^r+i} \times h^i$

for these i; in particular, this is so for $i = \mathfrak{i}_1 - 1$. Now, applying Corollary 4.20, we get that $S^{2^r}(\alpha) \not\ni h^0 \times l_{\mathfrak{i}_1-1-2^r}$, a contradiction.

The following lemma generalizes Lemma 4.18:

Lemma 4.23. *Let $q \in S$. Assume that the group $\bar{\mathrm{Ch}}_D(X^2)$ contains a cycle γ such that*

(1) *γ does not contain any $h^i \times l_i$ with $i < \mathfrak{j}_{q-1}$;*
(2) *γ contains $h^i \times l_i$ for some integer $i \in [\mathfrak{j}_{q-1},\ \mathfrak{j}_q)$ (note that the interval is semi-open).*

Then the group $\bar{\mathrm{Ch}}_{D+\mathfrak{i}_q-1}(X^2)$ contains a cycle α such that $\alpha \ni h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q-1}$ and $\alpha \not\ni h^i \times l_{i+\mathfrak{i}_q-1}$ for any $i < \mathfrak{j}_{q-1}$.

Proof. We use an induction on q. In the case of $q = 1$, the assumption of Lemma 4.23 is always true (think of $\gamma = \Delta$); the cycle α is constructed in Lemma 4.18. In the remaining part of the proof we assume that $q > 1$.

Let i be the smallest integer such that $\gamma \ni h^{\mathfrak{j}_{q-1}+i} \times l_{\mathfrak{j}_{q-1}+i}$. As a first step, we proof that the group $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ contains a cycle α' containing $h^{\mathfrak{j}_{q-1}+i} \times l_{\mathfrak{j}_q-1}$ and none of $h^j \times l_?$ with $j < \mathfrak{j}_{q-1} + i$ (if $i = 0$ then we can take $\alpha = \alpha'$ and finish the proof).

Applying the induction hypothesis to the quadric X_1 with the cycle $pr^2(\gamma_{F(X)}) \in \bar{\mathrm{Ch}}(X_1^2)$ (and using the inclusion homomorphism of Corollary 2.4), we get a cycle in $\bar{\mathrm{Ch}}_{D+\mathfrak{i}_q-1}(X^2_{F(X)})$ containing $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q-1}$. One of its derivatives is a homogeneous cycle in $\bar{\mathrm{Ch}}(X^2_{F(X)})$ containing $h^{\mathfrak{j}_{q-1}+i} \times l_{\mathfrak{j}_q-1}$. Therefore the group $\bar{\mathrm{Ch}}(X^3)$ contains a homogeneous cycle containing $h^0 \times h^{\mathfrak{j}_{q-1}+i} \times l_{\mathfrak{j}_q-1}$. Considering it as a correspondence of the middle factor of X^3 into the product of two outer factors, composing it with γ, and taking the pull-back with respect to the first diagonal $X^2 \to X^3$, we get the required cycle α'.

The highest order derivative $\alpha' \cdot (h^{\mathfrak{i}_q-1-i} \times h^0)$ of α' contains $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q-1}$. By Lemma 4.19, it also contains $l_{\mathfrak{j}_{q-1}} \times h^{\mathfrak{j}_{q-1}}$. Therefore its transposition contains $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_{q-1}}$. Replacing γ by the constructed rational cycle, we come to the situation with $i = 0$ (see the second paragraph of the proof), finishing the proof. □

We come to the main result of on the structure of $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ for an arbitrary anisotropic projective quadric X:

Theorem 4.24. *The set of the primordial (see Definition 4.16) cycles $\Pi \subset \mathrm{Che}^{\leq D}(X^2)$ has the following properties.*

(1) *All derivatives of all cycles of Π are minimal and pairwise different; they form a basis of $\mathrm{Che}^{\leq D}(X^2)$.*
(2) *Every cycle in Π is symmetric.*
(3) *For every $\pi \in \Pi$, there exists one and only one $q = f(\pi) \in S = \{1, 2, \ldots, \mathfrak{h}\}$ such that*
    (a) $\dim(\pi) = D + \mathfrak{i}_q - 1$;
    (b) *$\pi \not\ni h^i \times l_{i+\mathfrak{i}_q-1}$ for any $i < \mathfrak{j}_{q-1}$;*
    (c) $\pi \ni h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q-1}$.
(4) *The map $f\colon \Pi \to S$ thus obtained is injective, its image consists of $q \in S$ such that there exists a cycle $\alpha \in \bar{\mathrm{Ch}}^{\leq D}(X^2)$ satisfying $\alpha \ni h^i \times l_?$ for some $i \in [\mathfrak{j}_{q-1},\ \mathfrak{j}_q)$ and $\alpha \not\ni h^i \times l_?$ for any $i \in [0,\ \mathfrak{j}_{q-1})$ (in particular, $f(\Pi) \ni 1$).*

Proof. We construct a chain of subsets

$$\emptyset = \Pi_0 \subset \Pi_1 \subset \cdots \subset \Pi_{\mathfrak{h}}$$

of the set Π such that for every $q \in S$, all highest derivatives of all cycles of Π_q are minimal and pairwise different, and their sum contains $h^i \times l_i$ for all $i < \mathfrak{j}_q$. The procedure looks as follows. If for some $q \in S$ the set Π_{q-1} is already constructed, we decide whether we set $\Pi_q = \Pi_{q-1} \cup \{\pi\}$ with certain cycle π or we set $\Pi_q = \Pi_{q-1}$. To make this decision, we consider the sum α of all highest derivatives of all cycles of Π_{q-1}. We know that α contains $h^i \times l_i$ for all $i \in [0,\ \mathfrak{j}_{q-1})$. If α also contains $h^i \times l_i$ for all $i \in [\mathfrak{j}_{q-1},\ \mathfrak{j}_q)$, then we set $\Pi_q = \Pi_{q-1}$; otherwise the cycle $\gamma = \alpha + \Delta$ satisfies the hypothesis of Lemma 4.23, and we set $\Pi_q = \Pi_{q-1} \cup \{\pi\}$ with π being the minimal cycle containing $h^{\mathfrak{j}_{q-1}} \times l_{\mathfrak{j}_q - 1}$ (π exists and has Property (3b) by Lemma 4.23; π is primordial by Lemma 4.17).

The set $\Pi_{\mathfrak{h}}$ thus constructed has all properties claimed for Π in Theorem 4.24. Indeed the elements of $\Pi_{\mathfrak{h}}$ are symmetric by Lemma 4.17. The sum of all highest derivatives of all elements of $\Pi_{\mathfrak{h}}$ contains $h^i \times l_i$ for all i; therefore this sum also contains the remaining basis elements $l_i \times h^i$ for all i (see Lemma 4.19). It follows that every D-dimensional minimal cycle is a derivative of an element of $\Pi_{\mathfrak{h}}$. Consequently, every minimal cycle in $\bar{\mathrm{Ch}}^{\leq D}(X^2)$ is a derivative of of a cycle of $\Pi_{\mathfrak{h}}$. It follows that $\Pi_{\mathfrak{h}} = \Pi$. All minimal cycles form a basis according to Proposition 4.5. □

As easy as important information on relations between the primordial cycles on X^2 and on X_1^2 is as follows:

Proposition 4.25. *Let Π be the set of all primordial cycles for X; let Π_1 be the set of all primordial cycles for X_1. As usual we set $\mathfrak{i}_1 = \mathfrak{i}_1(X)$. One has:*

(1) $\#\Pi - 1 \leq \#\Pi_1$;
(2) *if* $\Pi \not\ni h^0 \times l_{\mathfrak{i}_1 - 1} + l_{\mathfrak{i}_1 - 1} \times h^0$, *then* $\#\Pi \leq \#\Pi_1$.

Proof. Let us extend the function $f\colon \Pi \to S$ on the set of all non-zero cycles in $\mathrm{Che}^{\leq D}(X^2)$, defining $f(\alpha)$ as the minimal $q \in S$ such that $\alpha \ni h^i \times l_?$ for some $i \in [\mathfrak{j}_{q-1},\ \mathfrak{j}_q)$ and $\alpha \not\ni h^i \times l_?$ for any $i \in [0,\ \mathfrak{j}_{q-1})$. By Item 4 of Theorem 4.24 (which is a consequence of Lemma 4.23), the image of the extended f coincides with $f(\Pi)$. Let $f_1\colon \mathrm{Che}^{\leq D}(X_1^2) \to S_1$ be the same map for the quadric X_1. We denote as Π' the set Π without the primordial cycle containing $h^0 \times l_{\mathfrak{i}_1 - 1}$ (this is the primordial cycle whose image under f is 1). For any $\pi \in \Pi'$ the cycle $pr^2(\pi) \in \mathrm{Che}(X_1^2)$ is non-zero and $f_1(pr^2(\pi)) = f(\pi) - 1$. It follows that $\#\Pi_1 = \#f_1(\Pi_1) = \#\,\mathrm{Im}(f_1) \geq \#f(\Pi') = \#\Pi' = \#\Pi - 1$, and the first statement of Proposition 4.25 is proved.

If now $\Pi \not\ni h^0 \times l_{\mathfrak{i}_1 - 1} + l_{\mathfrak{i}_1 - 1} \times h^0$, then $pr^2(\pi_{F(X)})$ is non-zero for *every* $\pi \in \Pi$. Note that for the cycle $\pi \in \Pi$ containing $h^0 \times l_{\mathfrak{i}_1 - 1}$, one has $f_1(pr^2(\pi)) \notin f_1(pr^2(\Pi'))$ (because π is disjoint with all derivatives of the cycles of Π' and, consequently, $pr^2(\pi)$ is disjoint with all derivatives of the cycles of $pr^2(\Pi')$). Therefore $\#\Pi \leq \#\Pi_1$, and the second statement of Proposition 4.25 is proved as well. □

We need some more notation.

Definition 4.26. For any $r \geq 1$, the symmetric group S_r acts on the group $\mathrm{Ch}(\bar{X}^r)$ by permutations of factors of $\bar{X}^r$. If $\alpha \in \mathrm{Ch}(\bar{X}^r)$, we write $\mathrm{Sym}(\alpha)$ for the "symmetrization" of α, that is,

$$\mathrm{Sym}(\alpha) = \sum_{s \in S_r} s(\alpha) .$$

Definition 4.27. A non-zero anisotropic quadratic form ϕ over F is said to be *small* if for some positive integer n (which is uniquely determined by $\dim(\phi)$ by the Arason-Pfister theorem) one has $\phi \in I^n$ while $\dim \phi < 2^{n+1}$. A projective quadric is *small* if so is the corresponding quadratic form.

The following result is an extended version of [22, thm. 4.1].

Proposition 4.28. *Let X be a small $2d$-dimensional quadric of the first Witt index $a = \mathfrak{i}_1(X)$. Then*

(1) *the integer a divides all the higher Witt indices $\mathfrak{i}_1, \dots, \mathfrak{i}_{\mathfrak{h}}$ of X; in particular, it divides $d + 1 = \mathfrak{i}_1 + \cdots + \mathfrak{i}_{\mathfrak{h}}$;*

(2) *the cycle*

$$\pi = \mathrm{Sym}\left(\sum_{i=1}^{(d+1)/a} h^{(i-1)a} \times l_{ia-1}\right) \in \mathrm{Ch}_{2d+a-1}(\bar{X}^2)$$

is rational;

(3) *moreover, for every $k \geq 0$ the Chow group $\bar{\mathrm{Ch}}_{2d+k}(X^2)$ is generated by the non-essential basis elements and the cycles*

$$\pi \cdot (h^{j-1} \times h^{a-k-j}) , \;\; j = 1, 2, \dots, a - k$$

(in particular, for $k \geq a$, this Chow group consists of the non-essential elements only).

Proof. Let Π be the set of primordial cycles. It suffices to show that $\#\Pi = 1$ (then the unique element $\pi \in \Pi$ automatically has the form and the property required). We prove it using an induction on $\mathfrak{h} = \mathfrak{h}(X)$. If $\mathfrak{h} = 1$, then $\#\Pi = 1$, since generally $1 \leq \#\Pi \leq \mathfrak{h}$.

Now we assume that $\mathfrak{h} \geq 2$. Let us consider the quadric X_1 (over the field $F(X)$) and let Π_1 be the set of primordial cycles for X_1. Then $\#\Pi_1 = 1$ by the induction hypothesis, and we get what we need by Item 2 of Proposition 4.25, if we check that the cycle $\mathrm{Sym}(h^0 \times l_{a-1})$ is not rational. By Item 8 of Proposition 3.3, this cycle can be rational only if the integer $2d - (a-1) + 1 = 2d - a + 2$ is a power of 2. Since however

$$2^n \leq \dim(\phi_1) = 2d + 2 - 2a < \underline{\underline{2d - a + 2}} < 2d + 2 = \dim(\phi) \leq 2^{n+1} ,$$

the integer $2d - a + 2$ is *not* a power of 2. □

Remark 4.29. Proposition 4.28 holds also for anisotropic ϕ with $[\phi] \in I^n$ and $\dim(\phi) = 2^{n+1}$ (the same proof is valid for such ϕ as well).

5. Cycles on X^3

Let ϕ be a small quadratic form and let n be the positive integer such that $[\phi] \in I^n$ while $\dim(\phi) < 2^{n+1}$. We recall that X stays for the projective quadric given by ϕ. Let us write down the dimension of ϕ as a sum of powers of 2:

$$\dim(\phi) = 2^n + 2^{n_1} + \cdots + 2^{n_m} , \;\; n > n_1 > \cdots > n_m \geq 1 .$$

In this section we *assume* that $m \geq 2$, that the height of ϕ is at least 3, and that the first two higher Witt indices of ϕ are as follows: $\mathfrak{i}_1(\phi) = 2^{n_m - 1}$ and $\mathfrak{i}_2(\phi) = 2^{n_{m-1}-1}$. To simplify the formulae which follow, we introduce the notation

$$a = \mathfrak{i}_1(\phi); \quad b = \mathfrak{i}_2(\phi); \quad c = \mathfrak{i}_3(\phi)$$

and

$$d = \dim(X)/2 = 2^{n-1} + 2^{n_1 - 1} + \cdots + 2^{n_m - 1} - 1 \ .$$

Here is our main construction:

Proposition 5.1 (cf. [12, prop. 2.7])**.** *The group* $\bar{\mathrm{Ch}}(X^3)$ *contains a homogeneous cycle*

$$\mu = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^0 \times h^{(i-1)b+a} \times l_{ib+a-1}\right) + \mu' \ ,$$

where μ' *is a sum of only those essential basis elements which contain neither* h^0 *nor* h^i *with* $a \nmid i$.

Proof. Let X_1 be the projective quadric (over the field $F(X)$) given by the anisotropic part of the form $\phi_{F(X)}$. Applying Item 2 of Proposition 4.28 to the quadric X_1 (taking in account that $\mathfrak{i}_1(X_1) = \mathfrak{i}_2(X) = b$ and $\dim(X_1) = 2(d-a)$), we , in particular, get that the group $\bar{\mathrm{Ch}}(X_1^2)$ contains the cycle

$$\beta' = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b} \times l_{ib-1}\right) \ .$$

Therefore (see Corollary 2.4), the group $\bar{\mathrm{Ch}}(X^2_{F(X)})$ contains the cycle

$$\beta = in^2(\beta') = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1}\right) \ .$$

The pull-back homomorphism $g_1^* \colon \bar{\mathrm{Ch}}(X^3) \to \bar{\mathrm{Ch}}(X^2_{F(X)})$ with respect to the morphism $g_1 : X^2_{F(X)} \to X^3$, given by the generic point of the first factor of X^3, is surjective. Therefore, there exists a homogeneous cycle $\mu \in \bar{\mathrm{Ch}}(X^3)$ such that $g_1^*(\mu) = \beta$. Note that g_1^* sends every basis cycle of the type $h^0 \times \zeta \times \xi$ to $\zeta \times \xi$ while killing the other basis elements. Consequently we have

$$\mu = h^0 \times \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1}\right) + \epsilon \ ,$$

where ϵ is a sum of some basis cycles which do not have h^0 on the first factor place.

We now proceed by transforming the cycle μ in such a way that μ does not leave the group $\bar{\mathrm{Ch}}(X^3)$ and $g_1^*(\mu)$ remains the same.

By Proposition 4.28 (now applied to X itself), the cycle

$$\gamma = \mathrm{Sym}\left(\sum_{i=1}^{(d+1)/a} h^{(i-1)a} \times l_{ia-1}\right) \cdot (h^0 \times h^{a-1}) = \sum_{i=1}^{(d+1)/a} (h^{(i-1)a} \times l_{(i-1)a} + l_{ia-1} \times h^{ia-1})$$

is in $\bar{\mathrm{Ch}}(X^2)$. Considering it as a correspondence, we replace μ by the composition $\mu \circ \gamma$, where $\mu \in \bar{\mathrm{Ch}}(X_1 \times X_2 \times X_3)$ is considered as a correspondence from X_1 to $X_2 \times X_3$ (all X_i are copies of X). Now a basis element $h^i \times ? \times ?$ occurs in the decomposition of μ only if i is divisible by a (see Lemma 3.8).

Considering μ as a correspondence from X_2 to $X_1 \times X_3$ and replacing it by the composition $\mu \circ \gamma$, [9] we come to the situation where a basis element $? \times h^i \times ?$ occurs in the decomposition of μ only if i is divisible by a (while all previously established properties of μ still hold).

Finally, considering μ as a correspondence from X_3 to $X_1 \times X_2$ and replacing it by the composition $\mu \circ \gamma$, we come to the situation where a basis element $? \times ? \times h^i$ occurs in the decomposition of μ only if i is divisible by a (while all the previously established properties of μ still hold).

The last change we apply to μ is as follows: we remove all non-essential basis cycles in the decomposition of μ. We claim that now our cycle μ has the required shape.

Let us write μ_0 for the sum of those summands in the decomposition of μ which have h^0 as at least one factor. To finish the proof of the proposition, it suffices to check that

$$\mu_0 = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^0 \times h^{(i-1)b+a} \times l_{ib+a-1}\right) .$$

First of all let us check that none of the 3 basis cycles obtained from $h^0 \times h^0 \times l_{b-1}$ by a permutation of factors appears in the decomposition of μ_0. We assume that the cycle $h^0 \times h^0 \times l_{b-1}$ does appear and we pull-back μ with respect to the morphism $g_{12}\colon X_{F(X \times X)} \to X^3$ given by the generic point of the product of the first two factors of X^3. We get

$$\bar{\mathrm{Ch}}(X_{F(X\times X)}) \ni g_{12}^*(\mu) = g_{12}^*(h^0 \times h^0 \times l_{b-1}) = l_{b-1}$$

showing that the Witt index of the quadric $X_{F(X\times X)}$ is at least b (see Corollary 2.5). However this Witt index coincides with $\mathfrak{i}_1(X) = a$ and a is smaller than b (actually $a \le b/2$). The contradiction obtained shows that the cycle $h^0 \times h^0 \times l_{b-1}$ does not appear in the decomposition of μ_0. For every permutation of $h^0 \times h^0 \times l_{b-1}$ we get the same result simply by changing in the appropriate way the numeration of the factors.

It follows that $\mu_0 = \mu_1 + \mu_2 + \mu_3$, where μ_i is the sum of summands in the decomposition of μ such that h^0 is their i-th factor. By the construction of μ we known that

$$\mu_1 = h^0 \times \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1}\right) ,$$

[9] Strictly speaking, this is $t_{12}\big(t_{12}(\mu) \circ \gamma\big)$, where t_{12} is the automorphism of $\bar{\mathrm{Ch}}(X^3)$ induced by the transposition of the first two factors of X^3.

and it suffices to check that $\mu_2 = t_{12}(\mu_1)$ and $\mu_3 = t_{13}(\mu_1)$ with t_{1i} staying for the automorphism of the Chow group $\mathrm{Ch}(\bar{X}^3)$ given by the transposition of the first and i-th factor of $\bar{X}^3$.

In order to see that $\mu_2 = t_{12}(\mu_1)$, we pull-back the cycle μ to X^2 with respect to the morphism

$$\delta_1 \colon X^2 \to X^3 \ , \quad x_1 \times x_2 \mapsto x_1 \times x_1 \times x_2$$

given by the diagonal map of the first factor of X^2 into the product of the first two factors of X^3. The decomposition of the homogeneous cycle $\delta_1^*(\mu) \in \bar{\mathrm{Ch}}_{2d+b-1}(X^2)$ does not contain any non-essential cycle. Therefore, since $b > a$, $\delta_1^*(\mu) = 0$ by Proposition 4.28. On the other hand, $\delta_1^*(\mu_1)$ contains $h^a \times l_{b+a-1}$ while neither $\delta_1^*(\mu_3)$ nor $\delta_1^*(\mu - \mu_0)$ do. It follows that $\delta_1^*(\mu_2)$ contains $h^a \times l_{b+a-1}$ as well and consequently μ_2 contains the basis cycle $h^a \times h^0 \times l_{b+a-1}$. Now we use the pull-back with respect to the morphism $g_2 \colon X^2_{F(X)} \to X^3$ given by the generic point of the second factor of X^3. The homogeneous cycle $g_2^*(\mu) = g_2^*(\mu_2)$ lies in $\bar{\mathrm{Ch}}(X^2_{F(X)})$, contains the basis cycle $h^a \times l_{b+a-1}$, and does not contain any non-essential basis element. Passing to the anisotropic part X_1 of $X_{F(X)}$ and using Corollary 2.4, we get a homogeneous cycle η in $\bar{\mathrm{Ch}}(X_1^2)$, namely $\eta = pr^2(g_2^*(\mu))$, which contains $h^0 \times l_{b-1}$ and does not contain any non-essential cycle. Note that $g_2^*(\mu_2)$ is in the image of $in^2 \colon \mathrm{Ch}(\bar{X}_1^2) \to \mathrm{Ch}(\bar{X}^2_{F(X)})$, so that μ_2 can be reconstructed from η.

By Proposition 4.28 it follows that

$$\eta = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b} \times l_{ib-1}\right) .$$

Consequently

$$g_2^*(\mu_2) = in^2(\eta) = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1}\right)$$

and $\mu_2 = t_{12}(\mu_1)$.

The equality $\mu_3 = t_{13}(\mu_1)$ is checked similarly. $\square$

We remark that the "defect part" μ' of the cycle μ does not appear in [12, prop. 2.7] when working with a small quadric of height 2. In our case here, the height of X is at least 3, μ' does really exist and represents an additional difficulty. The main observation which is crucial to overcome this difficulty is as follows:

Lemma 5.2. *Let μ' be as in Proposition 5.1. In the decomposition of μ' we consider the basis elements with h^a on the i-th factor place and write μ'_i for their sum. Then each of the cycles μ'_1, $t_{12}(\mu'_2)$, and $t_{13}(\mu'_3)$ is the sum of <u>some</u> of the following $(c-b)/a$ elements*

$$\chi_j = h^a \times \left(\mathrm{Sym}\left(\sum_{i=1}^{(d-b-a+1)/c} h^{(i-1)c+b+a} \times l_{ic+b+a-1}\right) \cdot \left(h^{(j-1)a} \times h^{c-b-ja}\right)\right)$$

where $j \in \{1, 2, \dots, (c-b)/a\}$.

Proof. Clearly, it suffices to prove the statement on μ'_1 (the statements on μ'_2 and μ'_3 are proved in the same way interchanging the roles of the three factors of X^3).

Let us go over the function field $F(X)$. We still have $\mu \in \bar{\mathrm{Ch}}(X^3_{F(X)})$. Therefore $pr^3_X(\mu) \in \bar{\mathrm{Ch}}(X^3_1)$, where X_1 is the anisotropic part of $X_{F(X)}$ and $pr^3_X : \mathrm{Ch}(\bar{X}^3_{F(X)}) \to \mathrm{Ch}(\bar{X}^3_1)$ is the projection of Corollary 2.4. We note that $pr^3_X(\mu) = pr^3_X(\mu')$. Moreover, μ' is in the image of the inclusion $in^3_X : \mathrm{Ch}(\bar{X}^3_1) \to \mathrm{Ch}(\bar{X}^3_{F(X)})$ because every h^i which is a factor of a basis element in the decomposition of μ' has $i \geq a$ and every l_i which is a factor of a basis element in the decomposition of μ' has $i \geq a$ as well (just look at the dimension of μ'). Therefore μ' can be reconstructed from its image under pr^3_X, namely, $\mu' = in^3_X(pr^3_X(\mu'))$.

Now we move from $\bar{\mathrm{Ch}}(X^3_1)$ to $\bar{\mathrm{Ch}}\left((X_1)^2_{F(X)(X_1)}\right)$ using g^*_1 (the pull-back with respect to the morphism given by the generic point of the first factor of X^3_1). Note that $g^*_1(pr^3_X(\mu')) = g^*_1(pr^3_X(\mu'_1))$ and the cycle $pr^3_X(\mu'_1)$ can be reconstructed from its image under g^*_1. Moreover the cycle $g^*_1(pr^3_X(\mu'_1))$ is in the image of the inclusion $in^2_{X_1} : \mathrm{Ch}(\bar{X}^2_2) \to \mathrm{Ch}\left((\bar{X}_1)^2_{F(X_1)}\right)$, where X_2 is the anisotropic part of $(X_1)_{F(X)(X_1)}$. In order to see it, we note that every basis cycle in the decomposition of $g^*_1(pr^3_X(\mu'_1))$ is equal to $h^{(i-1)a} \times l_{b+ia-1}$ for some $i \geq 1$. Clearly, such a basis cycle is in the image of $in^2_{X_1}$ if and only if $(i-1)a \geq b$. So, if the cycle $g^*_1(pr^3_X(\mu'_1))$ is not in the image of $in^2_{X_1}$, then its decomposition contains the basis element $h^{(i-1)a} \times l_{b+ia-1}$ with some i such that $(i-1)a < b$. It follows that the decomposition of the rational cycle

$$pr_{2d-2a-(i-1)a,a}(g^*_1(pr^3_X(\mu'_1))) \in \bar{\mathrm{Ch}}(X_2)$$

contains l_{ia-1}. This is a contradiction because X_2 is anisotropic and therefore the group $\bar{\mathrm{Ch}}(X_2)$ does not contains essential elements (Corollary 2.2).

So, one can reconstruct the cycle $g^*_1(pr^3_X(\mu'_1))$ from its image under the projection $pr^2_{X_1} : \bar{\mathrm{Ch}}\left((X_1)^2_{F(X)(X_1)}\right) \to \bar{\mathrm{Ch}}(X^2_2)$ and for our purposes it is sufficient to determine this image. To do so, we apply Proposition 4.28 to the quadric X_2 getting that the cycle

$$(pr^2_{X_1} \circ g^*_1 \circ pr^3_X)(\mu'_1) \in \bar{\mathrm{Ch}}_{2d-b-a-1}(X^2_2)$$

is the sum of some essential generators of the group $\bar{\mathrm{Ch}}_{2d-b-a-1}(X^2_2)$ indicated in Item 3 of Proposition 4.28 (we note that $\dim(X_2) = 2(d-b-a)$ so that $2d-b-a-1 = \dim(X_2)+b+a-1 \leq \dim(X_2)+c-1$). Finally, taking in account that h^i for a given i can be a factor of a basis element appearing in the decomposition of $(pr^2_{X_1} \circ g^*_1 \circ pr^3_X)(\mu'_1)$ only if i is divisible by a, we get the desired description of the cycle μ'_1. □

6. Proof of Conjecture 1.1

In this § we prove Conjecture 1.1. Suppose that this conjecture is not true, that is, over some field F and for some positive integer n, there exists a quadratic form ϕ over F with $[\phi] \in I^n$ and with $\dim(\phi)$ prohibited by Conjecture 1.1. Note that n is at least 4 (see §1). In the splitting pattern of the form ϕ, let us choose the smallest number $\dim(\phi_E)_0$ prohibited by Conjecture 1.1. Let us replace the form ϕ by this $(\phi_E)_0$ (and F by E) and write X for the projective quadric given by the new ϕ. Note that $\dim(\phi) > 2^n + 2^{n-1}$ ([23] (the original proof), or [14, thm. 4.4], or [12]). Moreover,

$$\dim(\phi_{F(X)})_0 = 2^n + 2^{n-1} + \cdots + 2^m = 2^{n+1} - 2^m$$

for some m with $3 \le m \le n-1$ and $\dim(\phi) = 2^n + 2^{n-1} + \cdots + 2^m + 2\mathfrak{i}_1$ where $\mathfrak{i}_1 = \mathfrak{i}_1(\phi)$ is the first Witt index of ϕ. Note that $\mathfrak{i}_1 < 2^{m-1}$ simply because $\dim(\phi) < 2^{n+1}$. Now it follows by [11, conject. 0.1] or by Item 1 of Proposition 4.28 (take in account that the highest Witt index of ϕ is 2^{n-1}) that $\mathfrak{i}_1 = 2^{p-1}$ for some integer p satisfying $1 \le p \le m-1$, and $\dim(\phi) = 2^n + 2^{n-1} + \cdots + 2^m + 2^p$. Since ϕ is a counter-example, p is not $m-1$, so that $1 \le p \le m-2$ in fact.

Finally, [11, conject. 0.1] (or Item 1 of Proposition 4.28) and the fact that all dimensions $\dim(\phi_E)_0 < \dim(\phi)$ are allowed by Conjecture 1.1, allows one to determine all further higher Witt indices of ϕ (compare with the proof of Corollary 6.2). They are as follows (starting from $\mathfrak{i}_2$): $2^{m-1}, 2^{m-2}, \dots, 2^{n-1}$, meaning that the splitting pattern of ϕ consists of the partial sums of the sum $2^n + 2^{n-1} + \cdots + 2^m + 2^p$. Therefore the hypothesis of the preceding § (and, in particular, the hypothesis of Proposition 5.1 and Lemma 5.2) are satisfied.

To simplify the formulae which follow, like we did in the previous §, we introduce the notation

$$a = \mathfrak{i}_1(\phi) = 2^{p-1};\ \ b = \mathfrak{i}_2(\phi) = 2^{m-1};\ \ c = \mathfrak{i}_3(\phi) = 2^m$$

and

$$d = \dim(X)/2 = 2^{n-1} + 2^{n-2} + \cdots + 2^{m-1} + 2^{p-1} - 1\ .$$

Let us consider the cycle $\mu \in \bar{\mathrm{Ch}}(X^3)$ of Proposition 5.1 as a correspondence from $\bar{X}$ to $\bar{X}^2$; let us consider the cycle $S^{2a}(\mu) \cdot (h^0 \times h^0 \times h^{b-1}) \in \bar{\mathrm{Ch}}(X^3)$ as a correspondence from $\bar{X}^2$ to $\bar{X}$ (where S^i stays for the degree i component of the total Steenrod operation S). Then we may take the composition of correspondences

$$\mu \circ \big(S^{2a}(\mu) \cdot (h^0 \times h^0 \times h^{b-1})\big) \in \bar{\mathrm{Ch}}(X^4)\ .$$

Let us additionally consider the morphism

$$\delta\colon X^2 = X_1 \times X_2 \to X_1 \times X_2 \times X_3 \times X_4 = X^4\ ,\ \ x_1 \times x_2 \mapsto x_1 \times x_2 \times x_1 \times x_2$$

(all X_i are copies of X) given by the product of the diagonals $X_1 \to X_1 \times X_3$ and $X_2 \to X_2 \times X_4$ (that is, δ is the diagonal morphism of X^2).

The following proposition contradicts to Proposition 4.28 (note that $\dim(\xi) = 2d + b - 2a - 1 \ge 2d + a$) and proves therefore Conjecture 1.1.

Proposition 6.1. *Let $\mu \in \bar{\mathrm{Ch}}(X^3)$ be the cycle of Proposition 5.1. Then the decomposition of the cycle*

$$\xi = \delta^*\Big(\mu \circ \big(S^{2a}(\mu) \cdot (h^0 \times h^0 \times h^{b-1})\big)\Big) \in \bar{\mathrm{Ch}}_{2d+b-2a-1}(X^2)$$

contains the basis cycle $h^a \times l_{b-a-1}$.

Proof. It is easy to see that each power of h which is a factor of a basis element involved in the decomposition of the cycle $\mu \circ \big(S^{2a}(\mu) \cdot (h^0 \times h^0 \times h^{b-1})\big)$ is a multiple of a. Therefore the same is true for the cycle ξ.

As before, we set $\mu_0 = \mu - \mu'$. We have:

$$\xi = \delta^*\Big(\mu_0 \circ \big(S^{2a}(\mu_0) \cdot (h^0 \times h^0 \times h^{b-1})\big)\Big) + \delta^*\Big(\mu' \circ \big(S^{2a}(\mu') \cdot (h^0 \times h^0 \times h^{b-1})\big)\Big) + \delta^*\Big(\mu' \circ \big(S^{2a}(\mu_0) \cdot (h^0 \times h^0 \times h^{b-1})\big)\Big) + \delta^*\Big(\mu_0 \circ \big(S^{2a}(\mu') \cdot (h^0 \times h^0 \times h^{b-1})\big)\Big)\ ,$$

and we consider each of these four summands separately, one by one.

First summand. First of all we compute $S^{2a}(\mu_0)$. For every summand $h^0 \times h^{(i-1)b+a} \times l_{ib+a-1}$ in the decomposition of μ_0, and for any $r \neq a$ with $1 \leq r \leq 2a$, we have: $S^r(h^{(i-1)b+a}) = S^r(l_{ib+a-1}) = 0$ (for $r = 2a$ the relation $4a|b$, that is, $b \neq 2a$, is used) while $S^a(h^{(i-1)b+a}) = h^{(i-1)b+2a}$ and $S^a(l_{ib+a-1}) = l_{ib-1}$. Therefore

$$S^{2a}(h^0 \times h^{(i-1)b+a} \times l_{ib+a-1}) = h^0 \times h^{(i-1)b+2a} \times l_{ib-1}$$

and

$$S^{2a}(\mu_0) = \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^0 \times h^{(i-1)b+2a} \times l_{ib-1}\right)\ .$$

Now we can calculate the composition

$$\mu_0 \circ \big(S^{2a}(\mu_0) \cdot (h^0 \times h^0 \times h^{b-1})\big)$$

(see Lemma 3.8). The basis cycles which appear in the decomposition of $S^{2a}(\mu_0) \cdot (h^0 \times h^0 \times h^{b-1})$ have on the third factor place the following elements:

$$(*) \qquad h^{b-1},\ \ h^{ib+2a-1},\ \ l_{(i-1)b}\ .$$

On the other hand, the basis elements which appear in the decomposition of μ_0 itself have on the *first* factor place the following:

$$(**) \qquad h^0,\ \ h^{(i-1)b+a},\ \ l_{ib+a-1}\ .$$

It is straight-forward to see that the only pair of elements, one from $(*)$, one from $(**)$, with the product l_0 is (l_0, h^0) (look at the indices modulo $2a$). Therefore

$$\mu_0 \circ \big(S^{2a}(\mu_0) \cdot (h^0 \times h^0 \times h^{b-1})\big) = \left(h^0 \times \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1}\right)\right) \circ \big(\mathrm{Sym}(h^0 \times h^{2a}) \times l_0\big) = \mathrm{Sym}(h^0 \times h^{2a}) \times \mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1}\right)\ .$$

Applying δ^* to the cycle obtained, we get

$$\mathrm{Sym}\left(\sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib-a-1} + h^{(i-1)b+3a} \times l_{ib+a-1}\right) = h^a \times l_{b-a-1} + \dots\ .$$

It remains to show that the "remaining part" of ξ does not contain the basis cycle $h^a \times l_{b-a-1}$.

Second summand. A basis cycle of the shape $h^x \times ? \times h^y \times ?$ can be involved in the decomposition of

$$\mu' \circ \left(S^{2a}(\mu') \cdot (h^0 \times h^0 \times h^{b-1})\right)$$

only if $x, y \geq a$. In this case $\delta^*(h^x \times ? \times h^y \times ?) = h^{x+y} \times ?$ with $x + y > a$, therefore the cycle

$$\delta^*\Big(\mu' \circ \left(S^{2a}(\mu') \cdot (h^0 \times h^0 \times h^{b-1})\right)\Big)$$

does not contain the basis element $h^a \times l_{b-a-1}$.

Third summand. In order to check that the cycle

$$\delta^*\Big(\mu' \circ \left(S^{2a}(\mu_0) \cdot (h^0 \times h^0 \times h^{b-1})\right)\Big)$$

does not contain the basis element $h^a \times l_{b-a-1}$, it suffices to check that the number of basis elements in the decomposition of the composition

$$\mu_2' \circ \left(S^{2a}(\mu_1) \cdot (h^0 \times h^0 \times h^{b-1})\right)$$

is even (note that we replaced μ' by μ_2', the notation being introduced in Lemma 5.2, and we replaced μ_0 by μ_1, the notation μ_1 being introduced in the proof of Proposition 5.1 for the sum of the basis elements contained in the decomposition of μ having h^0 on the first factor place). For this, due to Lemma 5.2, it suffices to check that every of the $(c-b)/a$ compositions

$$t_{12}(\chi_j) \circ \left(S^{2a}(\mu_1) \cdot (h^0 \times h^0 \times h^{b-1})\right)\ , \quad j \in [1,\ (c-b)/a]$$

contains an even number of basis elements. We show this by a straight-forward computation. The point is that the number of summands in the decomposition of every χ_j is even and either each or none of the summands "produces" a basis element in the composition (moreover, in the first case, precisely one basis element is produced by each summand of the cycle χ_j).

Let us do the computation. The cycles $S^{2a}(\mu_1) \cdot (h^0 \times h^0 \times h^{b-1})$ and χ_j (for a fixed j) are equal respectively to $h^0 \times \alpha$ and to $h^a \times \beta$, where

$$\alpha = \sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+2a} \times l_{(i-1)b} + l_{ib-1} \times h^{ib+2a-1}\ ,$$

while

$$\beta = \sum_{i=1}^{(d-b-a+1)/c} h^{(i-1)c+b+ja} \times l_{(i-1)c+2b+(j+1)a-1} + l_{ic+b-(j-2)a-1} \times h^{ic-(j-1)a}\ ,$$

and we just need to check that the composition $\beta \circ \alpha$ is a sum of an even number of basis cycles. There are two different cases depending on the value of j. If the product ja is not 0 modulo b, then every product of every second factor of the basis cycles appearing in the decomposition of α (namely, $l_{(i-1)b}$ and $h^{ib+2a-1}$) by every first factor of the basis cycles appearing in the decomposition of β (namely, $h^{(i-1)c+b+ja}$ and $l_{ic+b-(j-2)a-1}$) is different from l_0 (to see this, look at the indices modulo b). Therefore, the composition of every basis cycle appearing in the decomposition of α with every basis cycles appearing in the decomposition of β is 0, and so, $\beta \circ \alpha = 0$ in this case.

In the contrary case — the case with $ja \equiv 0 \pmod{b}$ — for every basis cycle y in the decomposition of β there is precisely one basis cycle x in the decomposition of α such that $y \circ x \neq 0$ (note that $y \circ x$ is a basis cycle in this case). Since the number of basis cycles in the decomposition of β is even (equal to the integer $(d-b-a+1)/c$ doubled), the composition $\beta \circ \alpha$ is the sum of an even number of basis cycles.

Fourth summand. We finish the proof of Proposition 6.1 considering the cycle

$$\delta^*\Big(\mu_0 \circ \big(S^{2a}(\mu') \cdot (h^0 \times h^0 \times h^{b-1})\big)\Big) .$$

We replace μ' by μ'_1 and, furthermore, μ'_1 by χ_j with some $j \in [1,\ (c-b)/a]$. Also, we replace μ_0 by μ_2. We are going to show that the number of the basis elements of the shape $h^a \times ? \times h^0 \times ?$ appearing in the decomposition of the composition

$$\mu_2 \circ \big(S^{2a}(\chi_j) \cdot (h^0 \times h^0 \times h^{b-1})\big)$$

is even (this will finish the proof of Proposition 6.1). We have $\chi_j = h^a \times \beta$ and $\mu_2 = t_{12}(h^0 \times \alpha)$ with

$$\alpha = \sum_{i=1}^{(d-a+1)/b} h^{(i-1)b+a} \times l_{ib+a-1} + l_{ib+a-1} \times h^{(i-1)b+a} ,$$

$$\beta = \sum_{i=1}^{(d-b-a+1)/c} h^{(i-1)c+b+ja} \times l_{(i-1)c+2b+(j+1)a-1} + l_{ic+b-(j-2)a-1} \times h^{ic-(j-1)a} .$$

Therefore the number we are looking for is the number of summands in the decomposition of $\alpha \circ \big(S^{2a}(\beta) \cdot (h^0 \times h^{b-1})\big)$.

We can compute the Steenrod operation S^{2a} on the summands of the decomposition of β. The formula depends on the value of j modulo 4 because of the rules (here $S^{\leq 2a}$ stays for $\sum_{k \leq 2a} S^k$):

$$S^{\leq 2a}(h^{ia}) = \begin{cases} h^{ia} & \text{if } i \equiv 0 \pmod{4}; \\ h^{ia} + h^{(i+1)a} & \text{if } i \equiv 1 \pmod{4}; \\ h^{ia} + \qquad\quad h^{(i+2)a} & \text{if } i \equiv 2 \pmod{4}; \\ h^{ia} + h^{(i+1)a} + h^{(i+2)a} & \text{if } i \equiv 3 \pmod{4}, \end{cases}$$

while (here recall that $S(l_i) = l_i \cdot (1+h)^{2d-i+1}$, $d+1 \equiv a \pmod{b}$, and $4a$ divides b):

$$S^{\leq 2a}(l_{ia-1}) = \begin{cases} l_{ia-1} + \qquad\qquad\quad l_{(i-2)a-1} & \text{if } i \equiv 0 \pmod{4}; \\ l_{ia-1} + l_{(i-1)a-1} & \text{if } i \equiv 1 \pmod{4}; \\ l_{ia-1} & \text{if } i \equiv 2 \pmod{4}; \\ l_{ia-1} + l_{(i-1)a-1} + l_{(i-2)a-1} & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

Assume that $j \equiv 0 \pmod{4}$ or $j \equiv 1 \pmod{4}$. Then, applying the above formulae, we get that $S^{2a}(\beta) = 0$, and there is nothing more to prove in this case.

Now we assume that $j \equiv 2 \pmod{4}$ or $j \equiv 3 \pmod{4}$. Then

$$S^{2a}(\beta) \cdot (h^0 \times h^{b-1}) = \beta_1 + \beta_2 ,$$

where

$$\beta_1 = \beta \cdot (h^{2a} \times h^{b-1}) = \sum_{i=1}^{(d-b-a+1)/c} h^{(i-1)c+b+(j+2)a} \times l_{(i-1)c+b+(j+1)a} + l_{ic+b-ja-1} \times h^{ic+b-(j-1)a-1} ,$$

while

$$\beta_2 = \beta \cdot (h^0 \times h^{b+2a-1}) = \sum_{i=1}^{(d-b-a+1)/c} h^{(i-1)c+b+ja} \times l_{(i-1)c+b+(j-1)a} + l_{ic+b-(j-2)a-1} \times h^{ic+b-(j-3)a-1} .$$

If $j \equiv 3 \pmod 4$, then the compositions $\alpha \circ \beta_1$ and $\alpha \circ \beta_2$ are 0 because so are the compositions of any basis cycles included in α with any basis cycle included in β_1 or β_2 (look at the indices modulo $2a$). If $j \equiv 2 \pmod 4$, then $\alpha \circ \beta_1 = 0$ too and by the same reason (look at the indices modulo $4a$). Finally, assume that $j \equiv 2 \pmod 4$ and consider the composition $\alpha \circ \beta_2$. For every basis cycle y in the decomposition of β_2 there is precisely one basis cycle x in the decomposition of α such that $x \circ y \neq 0$ (note that $x \circ y$ is a basis cycle in this case). Since the number of basis cycles in the decomposition of β_2 is even (equal to the integer $(d-b-a+1)/c$ doubled), the composition $\alpha \circ \beta_2$ is the sum of an even number of basis cycles. □

Conjecture 1.1 is proved. The following supplement is now easy to get:

Corollary 6.2. *Let ϕ be a small quadratic form with $\dim(\phi) = 2^{n+1} - 2^m$, $m \in [1,\ n+1]$. Then the splitting pattern $\{\dim(\phi_E)_0 |\ E/F \text{ is a field extension}\}$ of the form ϕ coincides with the set $\{2^{n+1} - 2^i\}_{i=m}^{n+1}$ (in particular, the height of ϕ is equal to $n + 1 - m$).*

Proof. By Conjecture 1.1 proved above, $\dim(\phi_{F(X)})_0 = 2^{n+1} - 2^r$ for some $r \in [m+1,\ n+1]$. But by [11, conject. 0.1] (or by Item 1 of Proposition 4.28 taking in account that the highest Witt index of ϕ is 2^{n-1}), it follows that only the value $r = m + 1$ is possible. Proceeding this way (with the form $(\phi_{F(X)})_0$ and so on), we get the result. □

7. Possible dimensions

Let us recall some standard notation concerning quadratic forms: one writes $\langle a_1, \dots, a_n \rangle$, where $a_1, \dots, a_n \in F$, for the quadratic form

$$F^n \to F\ ,\ \ (x_1, \dots, x_n) \mapsto a_1 x_1^2 + \cdots + a_n x_n^2\ ;$$

$\langle\langle a_1, \dots, a_n \rangle\rangle$ for the n-fold Pfister form

$$\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle\ ,$$

and $\langle\langle a_1, \dots, a_n \rangle\rangle'$ for the pure subform of the above Pfister form (see [20, def. 1.1 of ch. 4]).

The following result provides, in particular, examples for all dimensions which are not prohibited by Conjecture 1.1.

Theorem 7.1 (A. Vishik)**.** *Take any integers $n \geq 1$ and $m \geq 2$. Let k be a field (of* $\mathrm{char}(k) \neq 2$*), t_i, t_{ij} ($i = 1, \dots, m$, $j = 1, \dots, n$) variables, and*

$$F = k(t_i, t_{ij})_{1 \leq i \leq m,\ 1 \leq j \leq n}$$

the field of rational functions in all these variables.

The splitting pattern of the quadratic form

$$\phi = t_1 \cdot \langle\langle t_{11}, \dots, t_{1n} \rangle\rangle \perp \dots \perp t_m \cdot \langle\langle t_{m1}, \dots, t_{mn} \rangle\rangle$$

over F is

$$\{2^{n+1} - 2^i |\ i = n+1, n, \dots, 1\} \cup \left(2\mathbb{Z} \cap [2^{n+1},\ m \cdot 2^n]\right) .$$

Proof. First of all, it is easy to see that all the integers $2^{n+1} - 2^i$ are in the splitting pattern of ϕ. Indeed, the anisotropic part of ϕ over the field E obtained from F by adjoining the square roots of $t_{31}, t_{41} \dots, t_{m1}$, of t_1 and of $-t_2$, is isomorphic to the (generalized Albert) form

$$\langle\langle t_{11}, \dots, t_{1n} \rangle\rangle' \perp - \langle\langle t_{21}, \dots, t_{2n} \rangle\rangle'$$

(the primes $'$ mean the pure subforms of the Pfister forms) of dimension $2^{n+1} - 2$; the splitting pattern of this form is $\{2^{n+1} - 2^i\}$ because this set is the splitting pattern of *any* anisotropic $(2^{n+1} - 2)$-dimensional quadratic form whose class lies in I^n (Corollary 6.2).

Now let us assume that some (at least one) even integers of the interval $[2^{n+1},\ m \cdot 2^n]$ are *not* in the splitting pattern of ϕ. Among all such integers we take the smallest one and call it a; let b be the biggest integer smaller that a and lying in the splitting pattern; let c be the smallest integer greater that a and lying in the splitting pattern. Let E be the field of the generic splitting tower of ϕ such that $\dim(\psi) = c$ for $\psi = (\phi_E)_0$. Let Y be the projective quadric given by the quadratic form ψ. Let $\pi \in \bar{\mathrm{Ch}}(Y^2)$ be the cycle of the set Π of Theorem 4.24 with $f(\pi) = 1$. We claim that $\pi = \mathrm{Sym}(h^0 \times l_{\mathfrak{i}_1 - 1})$ for $\mathfrak{i}_1 = \mathfrak{i}_1(Y)$. Indeed, since $\mathfrak{i}_1 = (c-b)/2 > 1$ and $\mathfrak{i}_q(Y) = 1$ for all $q \in S = S(Y) = \{1, 2, \dots, \mathfrak{h}\}$ ($\mathfrak{h}$ is the height of ψ) such that $\dim(\psi_q) \in [2^{n+1} - 2,\ b]$, the cycle π does not contain $h^{\mathfrak{i}_1 + \dots + \mathfrak{i}_{q-1}} \times l_{\mathfrak{i}_1 + \dots + \mathfrak{i}_{q-1} + \mathfrak{i}_1 - 1}$ for such q (Lemma 4.11), and also $\pi \not\ni h^i \times l_{i + \mathfrak{i}_1 - 1}$ for any $i \in \{1, 2, \dots, \mathfrak{i}_1 - 1\}$ (Lemma 4.11 as well). Finally, for the integer $q \in S$ such that $\dim(\psi_q) = 2^{n+1} - 2$, the cycle $pr^2(\pi_{E_q}) \in \bar{\mathrm{Ch}}(Y_q^2)$ (the homomorphism $pr^2 \colon \bar{\mathrm{Ch}}(Y_{E_q}^2) \to \bar{\mathrm{Ch}}(Y_q^2)$ is defined in Corollary 2.4) has the dimension

$$\dim(Y_q) + \mathfrak{i}_1 - 1 \geq \dim(Y_q) + 1 = \dim(Y_q) + \mathfrak{i}_1(Y_q)$$

and therefore is 0 by Item 3 of Proposition 4.28.

We have shown that $\pi = \mathrm{Sym}(h^0 \times l_{\mathfrak{i}_1 - 1})$. By Item 8 of Proposition 3.3, it follows that the integer $\dim(Y) - \mathfrak{i}_1 + 1$ is a power of 2, say 2^p. Since

$$\dim(Y) - \mathfrak{i}_1 + 1 = (c-2) - (c-b)/2 + 1 = (b+c)/2 - 1 ,$$

the integer 2^p sits inside of the open interval $(b,\ c)$; therefore, satisfying $2^{n+1} \leq 2^p < m \cdot 2^n$, the integer 2^p is *not* in the splitting pattern of the quadratic form ϕ. But all the integers $\leq m \cdot 2^n$ divisible by 2^n are evidently *in* the splitting pattern of ϕ. The contradiction obtained proves Theorem 7.1. $\square$

Remark 7.2. Of course, the dimensions $2^{n+1} - 2^i$ can be realized more directly by the tensor products of Pfister forms and generalized Albert forms ($u_\cdot, v_\cdot, w_\cdot$ are variables):

$$\langle\langle u_1, \dots u_{i-1}\rangle\rangle \otimes \big(\langle\langle v_1, \dots, v_{n+1-i}\rangle\rangle' \perp - \langle\langle w_1, \dots, w_{n+1-i}\rangle\rangle' \big) \, .$$

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