THE HASSE PRINCIPLE FOR SIMILARITY OF HERMITIAN FORMS

DAVID W. LEWIS, THOMAS UNGER, AND JAN VAN GEEL

ABSTRACT. The Hasse principle for similarity is established for restricted classes of skew-hermitian forms over quaternion division algebras with canonical involution and for hermitian forms over cyclic algebras with involution of the second kind. A counterexample is produced to show that the principle cannot hold for skew-hermitian forms over quaternion division algebras in general. This settles the two final cases of Hasse principles for similarity of forms that were missing in the literature.

1. INTRODUCTION

Let K be a global field, i.e. either a number field or a function field in one variable over a finite field. In the function field case we assume that $\mathrm{char}(K) \neq 2$.

The classification of symmetric bilinear forms, and more generally all kinds of skew-symmetric, hermitian and skew-hermitian forms defined over global fields (and non-commutative extensions of global fields) relies heavily on local-global principles for certain equivalence classes of the respective kinds of forms. For instance to classify forms up to isometry over K one first determines all isometry classes of forms over all completions $K_\mathfrak{p}$, $\mathfrak{p}$ a prime of K (this is simpler and it turns out to give in each case a finite list of forms of a given dimension). Then one establishes if possible a (weak) Hasse principle, i.e. the fact that forms are globally equivalent if and only if they are locally equivalent.

The standard example, to which the several other classification problems are reduced, are the "Hasse principles" for quadratic forms over global fields. These originate in seminal theorems like Hilbert's reciprocity law and the Hasse–Minkowski theorem, and can be stated as follows:

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- (Weak Hasse principle.) A quadratic form q over a global field K is hyperbolic if and only if $q \otimes K_{\mathfrak{p}}$ is hyperbolic for all (finite and infinite) primes $\mathfrak{p}$ of K.
- (Strong Hasse principle.) A quadratic form q over a global field K is isotropic if and only if $q \otimes K_{\mathfrak{p}}$ is isotropic for all (finite and infinite) primes $\mathfrak{p}$ of K.

  An element a in a global field K is represented by a quadratic form q over K if and only if a is represented by $q \otimes K_{\mathfrak{p}}$ for all (finite and infinite) primes $\mathfrak{p}$ of K.

Note that the two statements in the "strong Hasse principle" are equivalent since they are formulated for all quadratic forms. Instead of saying that $q_{\mathfrak{p}} := q \otimes K_{\mathfrak{p}}$ is hyperbolic (isotropic, ...) for all primes $\mathfrak{p}$ of K, we also say that q is *locally hyperbolic* (*locally isotropic*, ...).

In this paper we are interested in classifying forms up to similarity, i.e. isometry up to multiplying by a scalar factor. For the above mentioned forms such a classification is possible in many cases since a Hasse principle for similarity has been proven. Two cases seem to be missing in the literature: skew-hermitian forms over $(D, ^-)$, where D is a quaternion division algebra over K equipped with the canonical involution, and hermitian forms over $(D, ^-)$, where D is a cyclic algebra with $^-$ an involution of the second kind. We deal with both these remaining cases in this paper. We use the following terminology throughout the paper:

Definition 1.1. Let K be a global field and let D be a finite-dimensional K-division algebra with involution $^-$. We consider ε-hermitian forms ($\varepsilon = \pm 1$) over $(D, ^-)$, where we allow the possibility that $D = K$ and $^-$ is the identity map (e.g. symmetric bilinear, skew-symmetric bilinear, hermitian, and skew-hermitian forms). All forms considered in this paper are assumed to be nonsingular. We say that the *weak Hasse principle* holds for such forms whenever a form is globally hyperbolic if and only if it is locally hyperbolic.

Let φ be an ε-hermitian form over $(D, ^-)$. We say that the *Hasse principle for isometry (resp. similarity) holds for* φ if all forms φ' over $(D, ^-)$ that are locally isometric (resp. locally similar) to φ are globally isometric (resp. globally similar) to φ. For short, we say that **HPI** (resp. **HPS**) holds for φ.

We say that the *(strong) Hasse principle for isometry (resp. similarity)* holds for ε-hermitian forms over $(D, ^-)$ if **HPI** (resp. **HPS**) holds for all such forms φ over $(D, ^-)$.

Let us summarize the known results for all the different kinds of forms. (Reference [5] is a survey of the isometry classification problem in general.)

(a) Symmetric bilinear forms over K (or quadratic forms over K if $\mathrm{char}(K) \neq 2$): the **HPS** holds by work of Ono [7]. The **HPS** for bilinear forms in general has also been studied by Cortella [2].
(b) Skew-symmetric bilinear forms over K: the **HPS** holds since any two such forms are isometric if and only if they have the same dimension (which is necessarily even).
(c) Hermitian forms over $(K(\sqrt{\alpha}), ^-)$, where $\overline{x + y\sqrt{\alpha}} = x - y\sqrt{\alpha}$ for $x, y \in K$: the **HPS** holds by a theorem of Jacobson (which says that such a form is completely determined by its trace form, which is a quadratic form over K) and Ono's theorem. See also [2].
(d) Skew-hermitian forms over $(K(\sqrt{\alpha}), ^-)$: such forms h may be treated as hermitian forms via the equivalence $h \mapsto \sqrt{\alpha}h$. So the **HPS** holds by case (c).
(e) Hermitian forms over $(D, ^-)$, where D is a quaternion division algebra over K and $^-$ is quaternion conjugation: Jacobson's theorem applies here as well, so the **HPS** holds by Ono's theorem.
(f) Skew-hermitian forms over $(D, ^-)$: the **HPI** fails in this case; two skew-hermitian forms can be isometric at all primes without being globally isometric. On the other hand, we will show in Theorem 2.26 that the **HPS** holds for restricted classes of skew-hermitian forms. This is our main result. We also mention that Thomas [9] proved a Hasse principle for simple isometry (i.e. isometry of reduced norm 1) of skew-hermitian forms over $(D, ^-)$ which is useful in the context of Algebraic L-Theory.
(g) Hermitian forms over $(D, ^-)$, where D is a cyclic division algebra and the involution $^-$ is of the second kind: the **HPS** always holds, as we will show in Theorem 3.3. Note that as in case (d), there is no distinction here between hermitian and skew-hermitian forms. Also note that the **HPI** was proved by Landherr in 1938. This is well documented in [8, 10.6].
(h) In cases (e)–(g) when D is split the problem can be reduced by Morita theory to forms over K.

Our main reference for the theory of division algebras with involution and the local-global theory of hermitian forms is [8]. Here one can also find a detailed treatment of the Hasse principle for isometry of ε-hermitian forms ($\varepsilon = \pm 1$) over global fields. The main results in this area are due to Kneser and Springer.

2. Skew-hermitian forms over a quaternion division algebra

Throughout this section K is a global field of characteristic not 2 and D is a quaternion division algebra with center K. We endow D with the canonical involution $^-$ (quaternion conjugation), cf. [8, p.314, 11.2].

We are interested in the local-global behaviour of isometry classes and similarity classes of skew-hermitian forms over $(D, ^-)$. So we also need to consider such forms over $(D_\mathfrak{p}, ^-)$, where $D_\mathfrak{p} := D \otimes_K K_\mathfrak{p}$ and $K_\mathfrak{p}$ is some completion of K (which can be archimedean or non-archimedean in the number field case). Therefore we recall first some facts on skew-hermitian forms over a quaternion algebra $(D, ^-)$ with center a (general) field F and $^-$ the canonical involution.

Let φ be a nonsingular skew-hermitian form over $(D, ^-)$. Then φ admits a diagonalization

$$\varphi \simeq \langle \alpha_1, \dots, \alpha_n \rangle,$$

with $\alpha_i \in D_0^\times$, where D_0 is the set of pure quaternions in D (i.e. the set of skew-symmetric elements in D), and $D_0^\times$ the units in D_0. The *discriminant* of φ is defined to be

$$\operatorname{disc} \varphi := (-1)^n \det \varphi \bmod F^{\times 2},$$

where the *determinant* of φ is given by

$$\det \varphi := N(H_\varphi) \bmod F^{\times 2}$$

with H_φ any matrix representing φ and N the reduced norm from D to F.

One defines hyperbolic forms and Witt classes in the usual way, so one also has a Witt group of skew-hermitian forms, denoted by $W^{-1}(D, ^-)$. This Witt group is related to various other Witt groups by an exact sequence. We recall how this is done. Let $i \in D_0^\times$. Consider $L = F(i)$ and choose $j \in D_0^\times$ such that $ij = -ji$. Then $D = L \oplus Lj$ and D corresponds to the symbol $(a, b)_F$ with $i^2 = a, j^2 = b$, which is the quaternion algebra corresponding to the norm form $\langle 1, -a, -b, ab \rangle$. Furthermore, L carries a non-trivial automorphism determined by $i \mapsto -i$, which we also denote by $^-$.

Let V be some finite-dimensional right D-vector space. Every ε-hermitian form $h : V \times V \to D$, $\varepsilon = \pm 1$, can be written as

$$h(x, y) = f(x, y) + g(x, y)j$$

with $f := \pi_1 h$, $g := \pi_2 h$ where $\pi_1(\alpha + \beta j) = \alpha$ and $\pi_2(\alpha + \beta j) = \beta$. Since the nonsingularity of h implies the nonsingularity of f and g, isometries of h yield isometries of f and of g and since if h is hyperbolic so are f and g, we get induced maps

$$\pi_1 : W(D, {}^-) \longrightarrow W(L, {}^-)$$

and

$$\pi_2 : W^{-1}(D, {}^-) \longrightarrow W(L, {}^-).$$

(Here $W(\ \cdot\ , {}^-)$ is the Witt group of hermitian forms with respect to $^-$, and $W^{-1}(\ \cdot\ , {}^-)$ is the Witt group of skew-hermitian forms with respect to $^-$.)

This yields the following exact sequence (cf. [8, p. 359, Theorem 3.2]),

$$0 \longrightarrow W(D, {}^-) \xrightarrow{\pi_1} W(L, {}^-) \xrightarrow{\beta} W^{-1}(D, {}^-) \xrightarrow{\pi_2} W(L). \tag{1}$$

The map β is induced by $\langle u \rangle \mapsto \langle ui \rangle$, $u \in L^\times$. Note that such a sequence exists for every $L = F(\lambda)$, with $\lambda \in D_0^\times$. Also, the sequence extends to a larger exact sequence as in [6] but we do not need it for this paper.

In case D is split, i.e. $D \cong M_2(F)$, the classification of skew-hermitian forms over $(D, {}^-)$ reduces by Morita theory to the classification of quadratic forms over F. Indeed, with any skew-hermitian form h over $(D, {}^-)$ one can associate an even-dimensional quadratic form q_h (cf. [8, p. 352]) which has the following properties:

- $\dim q_h = 2 \dim h$,
- $\operatorname{disc} q_h = \operatorname{disc} h$,
- if $\lambda \in F^\times$ then $q_{\lambda h} \simeq \lambda q_h$.

Next we recall the classification of skew-hermitian forms over a quaternion algebra $(D, {}^-)$ with center a local field $K_\mathfrak{p}$, $\mathfrak{p}$ a finite or infinite prime of a global field K.

2.1. Classification over local fields: the split case.

Theorem 2.1 (cf. [8, p. 361 ff.]). *In the split case, $D \cong M_2(K_\mathfrak{p})$, we have:*

(i) For $K_\mathfrak{p} \cong \mathbb{C}$, the skew-hermitian forms are up to isometry completely classified by their dimension since the same holds for quadratic forms.

(*ii*) *For $K_{\mathfrak{p}} \cong \mathbb{R}$, the skew-hermitian forms are up to isometry completely classified by their dimension and signature of the Morita equivalent quadratic form.*
(*iii*) *For $K_{\mathfrak{p}}$, where $\mathfrak{p}$ is a finite prime ($K_{\mathfrak{p}}$ is complete with respect to a discrete valuation and has finite residue field), the skew-hermitian forms h are up to isometry completely classified by their dimension, their discriminant and by the Clifford invariant of q_h.*

Corollary 2.2. *Let φ be a skew-hermitian form over a split quaternion algebra over $K_{\mathfrak{p}}$, and let $\delta = \operatorname{disc}\varphi$.*

(*i*) *If $K_{\mathfrak{p}} = \mathbb{C}$, then $\varphi \simeq \alpha\varphi$ for all $\alpha \in \mathbb{C}$.*
(*ii*) *If $K_{\mathfrak{p}} = \mathbb{R}$, then $\varphi \simeq \alpha\varphi$ for all positive $\alpha \in \mathbb{R}^{\times}$.*
(*iii*) *If $\mathfrak{p}$ is a finite prime, then $\varphi \simeq \alpha\varphi$ for all $\alpha \in K_{\mathfrak{p}}$ such that $(\alpha, \delta)_{K_{\mathfrak{p}}}$ is trivial in ${}_2\mathrm{Br}(K_{\mathfrak{p}})$, the 2-part of the Brauer group $\mathrm{Br}(K_{\mathfrak{p}})$.*

2.2. **Classification over local fields: the nonsplit case.** In the nonsplit case, i.e. $D = (a, b)_{K_{\mathfrak{p}}}$, a quaternion division algebra over $K_{\mathfrak{p}}$, one has the following theorems:

Theorem 2.3 (**Real Case** [8, p. 364, 3.7])**.** *Let K be a real closed field and D the unique nonsplit quaternion algebra over K. Every skew-hermitian form of dimension > 1 is isotropic, and forms of equal dimension are isometric.*

Theorem 2.4 (**$\mathfrak{p}$-adic Case – Tsukamoto** [8, p. 363, 3.6], [10])**.** *Let $K_{\mathfrak{p}}$ be a local field and D the unique nonsplit quaternion algebra over K. For skew-hermitian forms over $(D, \bar{\ })$ the following statements hold:*

(*i*) *Two nonsingular forms are isometric if and only if they have the same dimension and discriminant.*
(*ii*) *Every form of dimension > 3 is isotropic.*
(*iii*) *In dimension 1 all nonsingular forms are anisotropic; there are forms of any discriminant $\neq -1$.*
(*iv*) *For any dimension > 1 there are forms of any discriminant. In dimension 2 exactly the forms of determinant 1 are isotropic. In dimension 3 exactly the forms of discriminant 1 are anisotropic.*

Remark 2.5. The theorem is stated incorrectly in [8]. There it is formulated with the determinant instead of the discriminant.

Note that for both archimedean and non-archimedean complete fields these facts yield immediately

Corollary 2.6. *Let φ be a skew-hermitian form over a quaternion division algebra over a local field $K_\mathfrak{p}$ then $\varphi \simeq \alpha\varphi$ for all $\alpha \in K_\mathfrak{p}^\times$.*

2.3. **Classification over global fields.** Finally, let K be a global field and $D = (a,b)_K$ a quaternion division algebra with center K, equipped with the canonical involution $^-$. In case K is a global function field we fix a prime $\mathfrak{p}_\infty$ such that $D \otimes_K K_{\mathfrak{p}_\infty} \neq 1$ in ${}_2\mathrm{Br}(K_{\mathfrak{p}_\infty})$. This is possible since D is a division algebra so it has to be a division algebra in at least one prime. Let A be the ring of functions regular outside $\mathfrak{p}_\infty$. In this case the primes associated to elements of $\mathrm{Spec}(A)$ are called the finite primes and $\mathfrak{p}_\infty$ is the infinite prime. In case K is a number field we take A to be the ring of integers of K, the finite primes are the ones associated to $\mathrm{Spec}(A)$ and the infinite primes correspond to the archimedean absolute values on K.

We introduce some notation:

$$\begin{aligned}
\Omega_K &= \{\text{primes of } K\},\\
\Omega_K^\infty &= \{\text{infinite primes of } K\},\\
\Omega_K^f &= \{\text{finite primes of } K\},\\
\Omega_K^{real} &= \{\mathfrak{p} \in \Omega_K^\infty \mid K_\mathfrak{p} = \mathbb{R}\},\\
\Omega_D &= \{\mathfrak{p} \in \Omega_K \mid D_\mathfrak{p} \neq 1 \text{ in } {}_2\mathrm{Br}(K_\mathfrak{p})\}.
\end{aligned}$$

The classification of skew-hermitian forms over $(D, ^-)$ up to isometry is due to Kneser and Springer.

Theorem 2.7 (Kneser–Springer, cf. [8, 10.4.1, 10.4.3])**.** *Let φ be a nonsingular skew-hermitian form over $(D, ^-)$.*

- (i) *If $\dim\varphi \geq 3$ and φ is locally isotropic, then φ is isotropic.*
- (ii) *If $\dim\varphi \geq 2$ and if $\lambda \in D_0^\times$ is represented locally by φ, then λ is represented by φ.*
- (ii') *If φ represents $\lambda \in D_0^\times$ locally, then φ represents a suitable multiple $\alpha\lambda$, $\alpha \in K^\times$.*

This result constitutes a "strong Hasse principle" for skew-hermitian forms of dimension ≥ 3. The hypotheses $\dim\varphi \geq 3$ for isotropy and $\dim\varphi \geq 2$ for the representation of elements are related to the fact that the local-global principle for hyperbolicity (the weak Hasse principle) does not hold. One has the following

Theorem 2.8 ([8, 10.4.5]). *Let s be the (even) number of primes at which D does not split, $s = |\Omega_D|$. For every $\lambda \in D_0^\times$, there exist exactly 2^{s-2} isometry classes of one-dimensional skew-hermitian forms which are locally isometric to $\langle \lambda \rangle$. All these forms can be written as $\langle \lambda\alpha \rangle$, $\alpha \in K^\times$.*

Remark 2.9. Recall from the proof of [8, 10.4.5] that the two one-dimensional skew-hermitian forms $\langle \lambda \rangle$ and $\langle \lambda\alpha \rangle$ are locally isometric if and only if $(a, \alpha)_\mathfrak{p} = 1$ or $(a, \alpha)_\mathfrak{p} = (a, b)_\mathfrak{p}$ for all $\mathfrak{p} \notin \Omega_D$.

This leads to the following local-global classification of skew-hermitian forms up to isometry.

Theorem 2.10 ([8, 10.4.6]). *For every positive even dimension there exist exactly 2^{s-2} isometry classes of locally hyperbolic forms. Every class of locally isometric forms consists of 2^{s-2} isometry classes.*

So in order to classify the isometry classes of even-dimensional forms one needs an invariant to distinguish the 2^{s-2} different isometry classes of locally hyperbolic forms. Such an invariant was introduced by Bartels in [1] using Galois cohomological methods and by Lewis in [4]. We describe a globalized version of this invariant.

2.4. **The ϑ-invariant.** The invariant takes values in a quotient of a subgroup of ${}_2\mathrm{Br}(K)$.

By taking the 2-part of the exact sequence of [8, 10.2.3(v)] we have the following reciprocity exact sequence:

$$0 \longrightarrow {}_2\mathrm{Br}(K) \xrightarrow{\iota} \bigoplus_{\mathfrak{p}\in\Omega_K} {}_2\mathrm{Br}(K_\mathfrak{p}) \xrightarrow{\sum \mathrm{inv}_\mathfrak{p}} \{1, -1\} \longrightarrow 0 \tag{2}$$

where $\mathrm{inv}_\mathfrak{p}$ is the isomorphism ${}_2\mathrm{Br}(K_\mathfrak{p}) \cong \{1, -1\}$, $\mathrm{inv}_\mathfrak{p}(H)$ is 1 if the quaternion algebra H is split and -1 otherwise. (Hilbert's reciprocity law says that the composite $\sum \mathrm{inv}_\mathfrak{p} \circ \iota$ is the zero map.)

Let T be a finite set of primes in Ω_K. Denote

$${}_2\mathrm{Br}^T(K) := \{H \in {}_2\mathrm{Br}(K) \mid \mathrm{inv}_\mathfrak{p}(H_\mathfrak{p}) = 1 \text{ for all } \mathfrak{p} \in \Omega_K \setminus T\}.$$

Sometimes it is more suitable to identify ${}_2\mathrm{Br}^T(K)$ with its image in $\bigoplus_{\mathfrak{p}\in\Omega_K} {}_2\mathrm{Br}(K_\mathfrak{p})$. Note that this image lies in the finite direct summand $\bigoplus_{\mathfrak{p}\in T} {}_2\mathrm{Br}(K_\mathfrak{p})$. It is not equal to this direct summand, it is equal to the subgroup generated by the classes of quaternion algebras over K ramified at a fixed prime $\mathfrak{p}_0$ and exactly one other prime. So it has order 2^{t-1} with $t = |T|$. Let $T = \Omega_D$, then the invariant we need takes values in ${}_2\mathrm{Br}^{\Omega_D}(K)/\langle D\rangle$, which is a group of order 2^{s-2}, $s = |\Omega_D| \in 2\mathbb{Z}$.

Let φ be a skew-hermitian form over $(D, ^-)$ which is locally hyperbolic everywhere, so the class of φ in $W^{-1}(D_\mathfrak{p}, ^-)$ is trivial for all $\mathfrak{p} \in \Omega_K$. The exact sequence (1) yields the following commutative diagram (we refer to [4] for the full details),

(3)

$$\begin{tikzcd} 0 \arrow[r] & W(D,^-) \arrow[r, "\pi_1"] \arrow[d, "p"] & W(L,^-) \arrow[r, "\beta"] \arrow[d, "q"] & W^{-1}(D,^-) \arrow[r, "\pi_2"] \arrow[d, "r"] & W(L) \arrow[d, "s"] \\ 0 \arrow[r] & \prod\limits_{\mathfrak{p}\in\Omega_K} W(D_\mathfrak{p},^-) \arrow[r, "\delta"] & \prod\limits_{\mathfrak{p}\in\Omega_K} W(L_\mathfrak{p},^-) \arrow[r, "\mu"] & \prod\limits_{\mathfrak{p}\in\Omega_K} W^{-1}(D_\mathfrak{p},^-) \arrow[r, "\nu"] & \prod\limits_{\mathfrak{p}\in\Omega_K} W(L_\mathfrak{p}) \end{tikzcd}$$

In this diagram the maps p, q and s are injective since the Hasse principle for isometry holds in these cases.

The diagram (3) yields a hermitian form ψ over $(L, ^-)$, where $L = K(i)$ with $i \in D_0^\times$, such that $\beta(\psi) = \varphi$. Let $d = \operatorname{disc}\psi$, the *discriminant* of ψ, which is defined by

$$\operatorname{disc}\psi := (-1)^{n(n-1)/2} \det\psi \bmod K^{\times 2},$$

where $n = \dim\psi$. (Note that the definition of discriminant depends on the type of form considered, cf. Section 2.) Put $i^2 = a$ and consider the symbol $(d, a)_K \in {}_2\mathrm{Br}(K)$. Define

$$\Theta(\varphi) := (d, a)_K \in {}_2\mathrm{Br}^{\Omega_D}(K).$$

We need to check that $\Theta(\varphi)$ actually is an element of ${}_2\mathrm{Br}^{\Omega_D}(K)$. Note that for $\mathfrak{p} \in \Omega_K \setminus \Omega_D$, if $K_\mathfrak{p}$ contains i, then clearly $(d, a)_K$ is trivial. If $K_\mathfrak{p}(i) = L_\mathfrak{p}$ is of degree 2, then the $\mathfrak{p}$-part of the lower sequence in (3) is

$$0 \longrightarrow W(L_\mathfrak{p}, ^-) \longrightarrow W(K_\mathfrak{p}) \longrightarrow W(L_\mathfrak{p})$$

since $W(D_\mathfrak{p}, ^-)$ is trivial in this case and $W^{-1}(D_\mathfrak{p}, ^-) = W(K_\mathfrak{p})$ by Morita theory (cf. [4, p. 234] and our earlier observations). But the form φ is locally hyperbolic so its class in $W^{-1}(D_\mathfrak{p}, ^-) = W(K_\mathfrak{p})$ is trivial. Therefore the class of ψ is also trivial in $W(L_\mathfrak{p}, ^-)$, so $d = \operatorname{disc}\psi$ must be trivial in $K_\mathfrak{p}$.

The element $\Theta(\varphi)$ only depends on the isometry class of φ (even only on the class of φ in $W^{-1}(D, ^-)$). Also, the form ψ is only determined up to elements in

$\operatorname{Ker}\beta = \operatorname{Im}\pi_1 \cong W(D, {}^-)$. In [4] it is mentioned that $\operatorname{Ker}\beta$ is the ideal generated by the two-dimensional form $\langle 1, -b\rangle$. As a consequence any form χ in $\operatorname{Ker}\beta$ will have $\det\chi = (-b)^{\dim\chi/2}$. Hence $\operatorname{disc}\psi$ is determined only up to multiples of b. Therefore $\Theta(\varphi)$ is uniquely defined only up to elements in the subgroup $\langle D\rangle$ of ${}_2\mathrm{Br}^{\Omega_D}(K)$ generated by $D = (a,b)_K$. Furthermore, in [4] it is shown that the commutative diagram yields that the locally hyperbolic skew-hermitian form φ is globally hyperbolic if and only if $\Theta(\varphi)$ is trivial in the quotient ${}_2\mathrm{Br}^{\Omega_D}(K)/\langle D\rangle$. So the invariant

$$\vartheta(\varphi) := \Theta(\varphi) \bmod \langle D\rangle$$

determines the different isometry classes over K of locally hyperbolic forms φ. Note that ϑ takes all values in an elementary 2-group of order 2^{s-2}. (In [4] this group is identified with $(\mathbb{Z}/2\mathbb{Z})^{s-1}/\sim$ using the maps $\operatorname{inv}_{\mathfrak{p}}$. Here $\sim$ is the equivalence relation defined by letting $(\varepsilon_i)_i \sim (\varepsilon_i')_i$ if and only if $\varepsilon_i = \varepsilon_i'$ for all i, or $\varepsilon_i = -\varepsilon_i'$ for all i.)

Remark 2.11. The definition of $\Theta(\varphi)$ and $\vartheta(\varphi)$ required a choice of the quadratic subfield L of D, i.e. a choice of the pure quaternion i. However, one can show that $\Theta(\varphi)$ and $\vartheta(\varphi)$ do not depend on this choice (cf. [4]).

Example 2.12. Let $K = \mathbb{Q}$ and D the quaternion division algebra $(-5,-13)_{\mathbb{Q}}$. Then $\Omega_D = \{2, 5, 13, \infty\}$.

Consider the skew-hermitian form $\langle\beta i, -i\rangle$ with $\beta \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ and $i^2 = -5$. We have $\langle\beta i\rangle \simeq_{\mathbb{Q}_p} \langle i\rangle$ for all $p \in \Omega_D$ by Corollary 2.6. Assume $\beta \in \mathbb{Z}$ is such that $(-5,\beta)_{\mathbb{Q}_p}$ is trivial for all $p \notin \Omega_D$, then we also have for all $p \notin \Omega_D$ that $\langle\beta i\rangle \simeq_{\mathbb{Q}_p} \langle i\rangle$. This follows from Corollary 2.2(*iii*) since $(-5,-13)_{\mathbb{Q}_p}$ is split in this case and $\operatorname{disc}\langle i\rangle = -5$.

So the form $\langle\beta i, -i\rangle$ is locally hyperbolic. We obtain

$$\Theta(\langle\beta i, -i\rangle) = (\beta, -5)_{\mathbb{Q}}.$$

We consider the symbols $(-5,-5)_{\mathbb{Q}_p}$ (which is ramified at $p = \infty$ and $p = 2$), $(-2,-5)_{\mathbb{Q}_p}$ (which is ramified at $p = \infty$ and $p = 5$) and $(-39,-5)_{\mathbb{Q}_p}$ (which is ramified at $p = \infty$ and $p = 13$). Now we see that all possible values of the ϑ-invariant can occur since we find a suitable β by just taking products of the above three symbols. They generate the group of invariants as we mentioned already at the beginning of Subsection 2.4.

Remark 2.13. In [1] Bartels defines, for a pair of skew-hermitian forms φ_1, φ_2 of the same dimension and discriminant over $(D,\bar{\ })$, a relative invariant

$$c(\varphi_1,\varphi_2) \in H^2(K,\mathbb{Z}/2\mathbb{Z})/\langle (a,b)\rangle,$$

where (a,b) is the symbol correspondig to the quaternion algebra D under the identification of $H^2(K,\mathbb{Z}/2\mathbb{Z})$ with ${}_2\mathrm{Br}(K)$.

Satz 4 of [1] provides some properties of this relative invariant:

Theorem 2.14.

(*i*) $c(\varphi_1,\varphi_2)^2 = 1$,
(*ii*) $c(\varphi_1,\varphi_2) = c(\varphi_2,\varphi_1)$,
(*iii*) $c(\varphi_1,\varphi_2)c(\varphi_2,\varphi_3) = c(\varphi_1,\varphi_3)$,
(*iv*) $c(\varphi_1 \perp \varphi_3, \varphi_2 \perp \varphi_3) = c(\varphi_1,\varphi_2)$,
(*v*) $c(\varphi,\alpha\varphi) = (\alpha,d) \bmod (a,b)$ *with* $\alpha \in K^\times$ *and* $d = \operatorname{disc}\varphi$.

In [4] the first author shows that for a locally hyperbolic skew-hermitian form ψ over $(D,\bar{\ })$ one has (identifying the 2-component of the Brauer group with the second Galois cohomology group, taking coefficients in $\mathbb{Z}/2\mathbb{Z}$),

$$\vartheta(\psi) = c(\psi,\mathbb{H}_{2n}),$$

where $\mathbb{H}_{2n}$ is a global hyperbolic form of dimension $2n = \dim\psi$ (both forms have trivial invariant). The proof is based on decomposing the form ψ in an orthogonal sum of 2-dimensional skew-hermitian forms of discriminant 1, and applying Theorem 2.14.

2.5. **The Hasse principle for similarity.** Let K be a global field and let $D = (a,b)_K$ be a quaternion division algebra over K.

Lemma 2.15. *Let φ and φ' be skew-hermitian forms over $(D,\bar{\ })$ which are locally isometric. Let $\delta = \operatorname{disc}\varphi$. Then*

(*i*) $\delta = \operatorname{disc}\varphi = \operatorname{disc}\varphi'$.
(*ii*) *For $(\alpha_\mathfrak{p})_{\mathfrak{p}\in\Omega_K}$ the following statements are equivalent:*
  (*a*) *for all $\mathfrak{p} \in \Omega_K^{real}\backslash\Omega_D$, $\alpha_\mathfrak{p} > 0$ and for all $\mathfrak{p} \in \Omega_K^f\backslash\Omega_D$ such that $\sqrt{\delta} \notin K_\mathfrak{p}$, $\alpha_\mathfrak{p} \in N_{K_\mathfrak{p}(\sqrt{\delta})/K}\big(K_\mathfrak{p}(\sqrt{\delta})\big) \subset K_\mathfrak{p}^\times$,*
  (*b*) $\varphi \simeq_{K_\mathfrak{p}} \alpha_\mathfrak{p}\varphi'$, $\forall \mathfrak{p} \in \Omega_K$.
(*iii*) *For $\alpha \in K$ the following statements are equivalent:*

(a) $\alpha \in \left(\bigcap_{\mathfrak{p}\in\Omega_K^{real}\setminus\Omega_D} K_{\mathfrak{p},+}\right) \cap \left(\bigcap_{\substack{\mathfrak{p}\in\Omega_K^f\setminus\Omega_D \\ \sqrt{\delta}\notin K_{\mathfrak{p}}}} N_{K_{\mathfrak{p}}(\sqrt{\delta})/K}\big(K_{\mathfrak{p}}(\sqrt{\delta})\big)\right) \subset K^\times$,

(b) $\varphi \simeq_{K_{\mathfrak{p}}} \alpha\varphi'$, $\forall \mathfrak{p} \in \Omega_K$,

(c) *for all* $\mathfrak{p} \in \Omega_K \setminus \Omega_D$, $(\alpha, \delta)_{K_{\mathfrak{p}}} = 1$ *in* ${}_2\mathrm{Br}(K_{\mathfrak{p}})$.

($K_{\mathfrak{p},+}$ denotes the positive elements of $K_{\mathfrak{p}}$ when $\mathfrak{p}$ is a real prime.)

Proof. (i) Since the forms are locally isometric their discriminants (as field elements) differ locally by a square, and therefore also globally by the global squares theorem (cf. [8]).

(ii) $\varphi \simeq_{K_{\mathfrak{p}}} \alpha_{\mathfrak{p}}\varphi'$ is by Morita theory equivalent to $q_\varphi \simeq_{K_{\mathfrak{p}}} q_{\alpha_{\mathfrak{p}}\varphi'}$ for all $\mathfrak{p} \in \Omega_K \setminus \Omega_D$ (here q_ψ denotes the quadratic form which is Morita equivalent to ψ). Since $q_{\alpha_{\mathfrak{p}}\varphi'} = \alpha_{\mathfrak{p}} q_{\varphi'}$, this is equivalent to

$$\alpha_{\mathfrak{p}} > 0 \text{ for all } \mathfrak{p} \in \Omega_K^{real} \setminus \Omega_D,$$

and

$$c(q_\varphi) \simeq_{K_{\mathfrak{p}}} c(q_{\alpha_{\mathfrak{p}}\varphi'}) =_{K_{\mathfrak{p}}} c(q_{\varphi'})(\alpha_{\mathfrak{p}}, \delta)_{K_{\mathfrak{p}}} \text{ for all } \mathfrak{p} \in \Omega_K^f \setminus \Omega_D$$

(where $c(q)$ is the Clifford invariant of q). Since φ and φ' are locally isometric, $c(q_\varphi) \simeq_{K_{\mathfrak{p}}} c(q_{\varphi'})$. So the latter conditions are equivalent to $(\alpha_{\mathfrak{p}}, \delta)_{K_{\mathfrak{p}}}$ being trivial in ${}_2\mathrm{Br}(K_{\mathfrak{p}})$ for all $\mathfrak{p} \in \Omega_K^f \setminus \Omega_D$. It remains to show that these last local conditions are equivalent to $\alpha_{\mathfrak{p}}$ being in the appropriate norm groups.

Since $(\alpha_{\mathfrak{p}}, \delta)_{K_{\mathfrak{p}}}$ is trivial in ${}_2\mathrm{Br}(K_{\mathfrak{p}})$ for all $\mathfrak{p}$ such that $\sqrt{\delta} \in K_{\mathfrak{p}}$, statement (ii) follows from the general fact that $(x, y)_F$, with $y \notin F^{\times 2}$, is trivial if and only if $x \in N_{F(\sqrt{y})/F}\big(F(\sqrt{y})\big)$.

The first two equivalences in (iii) follow from (ii) viewing α as an element of all the completions of K. The third equivalence follows from the proof of (ii). ∎

Lemma 2.16. *Let φ, φ' and δ be as in Lemma 2.15. Assume $\alpha \in K_{\mathfrak{p},+}$ for all $\mathfrak{p} \in \Omega_K^{real} \setminus \Omega_D$ and $\alpha \in \bigcap_{\substack{\mathfrak{p}\in\Omega_K\setminus\Omega_D \\ \sqrt{\delta}\notin K_{\mathfrak{p}}}} N_{K_{\mathfrak{p}}(\sqrt{\delta})/K}\big(K_{\mathfrak{p}}(\sqrt{\delta})\big)$. Then*

$$\vartheta(\varphi \perp -\alpha\varphi') = \vartheta(\varphi \perp -\varphi')\overline{(\alpha, \delta)}_K \text{ in } {}_2\mathrm{Br}(K)/\langle D \rangle.$$

(Here $\overline{(\alpha, \delta)}_K$ denotes the image of $(\alpha, \delta)_K$ in ${}_2\mathrm{Br}(K)/\langle D \rangle$.)

Proof. We have (cf. Remark 2.13) $\vartheta(\varphi \perp -\varphi') = c(\varphi \perp -\varphi', \mathbb{H}_{2n})$, c being the Bartels invariant in $H^2(K, \mathbb{Z}/2\mathbb{Z})$. So, using Theorem 2.14,

$$\begin{aligned}\vartheta(\varphi \perp -\varphi')\vartheta(\varphi \perp -\alpha\varphi') &= c(\varphi \perp -\varphi', \mathbb{H}_{2n})c(\varphi \perp -\alpha\varphi', \mathbb{H}_{2n}) \\ &= c(\varphi \perp -\varphi', \varphi \perp -\alpha\varphi') \\ &= \overline{(-\varphi', \alpha\varphi')} \\ &= \overline{(\alpha, \delta)}.\end{aligned}$$

Interpreting the last symbol as an element of ${}_2\mathrm{Br}(K)/\langle D\rangle$ proves the lemma. ∎

Remark 2.17. It follows from [4, Prop. 5] that two skew-hermitian forms φ and φ' over $(D, ^-)$ are isometric if and only if
(1) φ and φ' are locally isometric, and
(2) $\vartheta(\varphi \perp -\varphi')_{\mathfrak{p}}$ is trivial for all $\mathfrak{p} \in \Omega_D$.

Counterexample 2.18. This is adapted from the example in [3, p. 174], and indeed is a counterexample to Hijikata's own "result" in which he claims that the **HPS** is true in dimension three.

Just as in Example 2.12, we let $K = \mathbb{Q}$ and consider the quaternion division algebra $(-5, -13)_{\mathbb{Q}}$, generated by i and j with $i^2 = -5$, $j^2 = -13$ and $ij = k$. We have $\Omega_D = \{2, 5, 13, \infty\}$.

The forms $\langle -3i\rangle$ and $\langle -i\rangle$ are locally isometric since $(-3, -5)_{\mathbb{Q}_p} = 1$ for all $p \notin \Omega_D$ (this follows from Corollary 2.2(*iii*)). These forms are not globally isometric however, since $\vartheta(\langle -3i, -i\rangle) = \Theta(\langle -3i, -i\rangle) \bmod \langle D\rangle = \overline{(-3, -5)_{\mathbb{Q}}}$ is not trivial: upon identifying $\Theta(\langle -3i, -i\rangle)$ with its image in $\bigoplus_{p \in \Omega_D} {}_2\mathrm{Br}(\mathbb{Q}_p)$ we see that $(-3, -5)_{\mathbb{Q}_5} = -1$, but $(-3, -5)_{\mathbb{Q}_{13}} = 1$ (note that -1 is a square in $\mathbb{Q}_p$ for both $p = 5$ and 13).

Now let $\varphi_1 = \langle -3i, j, k\rangle$ and $\varphi_2 = \langle i, j, k\rangle$, then φ_1 and φ_2 are locally isometric. Since ϑ is only determined up to Witt equivalence we have

$$\vartheta(\varphi_1 \perp -\varphi_2) = \vartheta(\langle -3i, -i\rangle) = \overline{(-3, -5)_{\mathbb{Q}}},$$

which is not trivial by our earlier observation. Since $\mathrm{disc}(\varphi_1)_p = \mathrm{disc}(\varphi_2)_p = 1$ for both $p = 5$ and 13, there is no way we can choose an α to make $\vartheta(\varphi_1 \perp -\alpha\varphi_2)$ trivial. In other words, φ_1 and φ_2 are not globally similar.

Definition 2.19. For $\delta \in K^\times$ we define

$$G_\delta := \mathrm{Br}\big(K(\sqrt{\delta})/K\big) \cap {}_2\mathrm{Br}^{\Omega_D}(K).$$

($\mathrm{Br}\big(K(\sqrt{\delta})/K\big)$ is the kernel of the natural map $\mathrm{Br}(K) \to \mathrm{Br}\big(K(\sqrt{\delta})\big)$). Since $K(\sqrt{\delta})$ is equal to K or to a quadratic extension of K, it follows that $\mathrm{Br}\big(K(\sqrt{\delta})/K\big)$ is a subgroup of ${}_2\mathrm{Br}(K)$.)

We denote the image of G_δ in the quotient ${}_2\mathrm{Br}^{\Omega_D}(K)/\langle D\rangle$ by $\overline{G}_\delta$.

Lemma 2.20. *Let φ and ψ be two locally isometric skew-hermitian form over $(D, {}^-)$. Let $\delta = \operatorname{disc}\varphi$ $(= \operatorname{disc}\psi)$. Then there exists an $\alpha \in K^\times$ such that $\varphi \simeq_K \alpha\psi$ (i.e. φ and ψ are globally similar) if and only if $\vartheta(\varphi \perp -\psi) \in \overline{G}_\delta$.*

Proof. If $\varphi \simeq_K \alpha\psi$, then $\vartheta(\varphi \perp -\alpha\psi)$ is defined and trivial. Lemma 2.16 implies that

$$1 = \vartheta(\varphi \perp -\alpha\psi) = \vartheta(\varphi \perp -\psi)\overline{(\alpha,\delta)_K}.$$

Since $\overline{(\alpha,\delta)_K} \in \overline{G}_\delta$, so does $\vartheta(\varphi \perp -\psi)$.

Conversely, assume that $\vartheta(\varphi \perp -\psi) \in \overline{G}_\delta$. Let $\vartheta(\varphi \perp -\psi) =: E \bmod \langle D\rangle$, then either $E \otimes_K K(\sqrt{\delta})$ or $(E \otimes_K D) \otimes_K K(\sqrt{\delta})$ is trivial in $\mathrm{Br}\big(K(\sqrt{\delta})\big)$. So either $E = (\alpha,\delta)_K$, or $E \otimes_K D = (\alpha,\delta)_K$, with $\alpha \in K^\times$. In both cases it follows that $(\alpha,\delta)_K \otimes K_\mathfrak{p}$ is trivial in $\mathrm{Br}(K_\mathfrak{p})$ for all $\mathfrak{p} \notin \Omega_D$ (E and D are both in ${}_2\mathrm{Br}^{\Omega_D}(K)$). Then, using Lemma 2.15, we obtain $\varphi \simeq_{K_\mathfrak{p}} \alpha\psi$ for all $\mathfrak{p} \in \Omega_K$. This implies that $\varphi \perp -\alpha\psi$ is locally hyperbolic and so $\vartheta(\varphi \perp -\alpha\psi)$ is defined. Lemma 2.16 then gives

$$\vartheta(\varphi \perp -\alpha\psi) = \vartheta(\varphi \perp -\psi)\overline{(\alpha,\delta)_K} = \overline{(\alpha,\delta)_K}^2 = 1.$$

This yields $\varphi \simeq_K \alpha\psi$. ∎

Corollary 2.21. *The number of global similarity classes in a local isometry class of a form φ is equal to $\dfrac{2^{s-2}}{|\overline{G}_\delta|}$, with $\delta = \operatorname{disc}\varphi$.*

Proof. Follows directly from the lemma. ∎

Proposition 2.22. *Let $(D, {}^-)$ be a quaternion division algebra with canonical involution over K. Let φ and ψ be n-dimensional skew-hermitian forms over $(D, {}^-)$. Assume that φ and ψ are locally similar, i.e. for all $\mathfrak{p} \in \Omega_K$ there is an $\alpha_\mathfrak{p} \in K_\mathfrak{p}^\times$ such that $\varphi \simeq_{K_\mathfrak{p}} \alpha_\mathfrak{p}\psi$. Then there exists an $\alpha \in K^\times$ such that for all $\mathfrak{p} \in \Omega_K$, $\varphi \simeq_{K_\mathfrak{p}} \alpha\psi$.*

Proof. Let $\delta = \operatorname{disc}\varphi$. Since φ and ψ are locally similar, it is clear that also $\delta = \operatorname{disc}(\gamma\psi)$ for any scalar γ. We are looking for an $\alpha \in K^\times$ such that the forms φ and $\alpha\psi$ are locally isometric. By the classification results stated at the beginning of this section this means that we are looking for an $\alpha \in K^\times$ such that φ and $\alpha\psi$ have the same local invariants for the various primes of K.

In the nonsplit case there is no problem: all forms of equal dimension are isometric at the real primes (cf. Theorem 2.3) and all forms of equal dimension and equal discriminant are isometric at the finite primes and at the prime $\mathfrak{p}_\infty$ in the function field case (cf. Theorem 2.4).

In the split case (cf. Theorem 2.1) we need conditions on the sign of α at the real primes and on the Clifford invariants of the forms q_φ and $q_{\alpha\psi}$, Morita equivalent to φ and $\alpha\psi$ respectively, at the finite primes. By our assumption we have

$$c(q_\varphi) = c(q_{\alpha_\mathfrak{p}\psi}) = c(q_\psi)(\alpha_\mathfrak{p}, \delta)_{K_\mathfrak{p}} \text{ in } {}_2\mathrm{Br}(K_\mathfrak{p})$$

(c denotes the Clifford invariant) in this case, and thus α also has to satisfy $(\alpha, \delta)_{K_\mathfrak{p}} = (\alpha_\mathfrak{p}, \delta)_{K_\mathfrak{p}}$ at these primes.

Upon closer inspection we see that everything will work, except at the following sets of primes:

$$S_0 := \{\mathfrak{p} \in \Omega_K^f \setminus \Omega_D \,|\, \mathfrak{p} \text{ non-dyadic and } (\alpha_\mathfrak{p}, \delta)_{K_\mathfrak{p}} \neq 1 \text{ in } {}_2\mathrm{Br}(K_\mathfrak{p})\},$$

and

$$S_1 := \{\mathfrak{p} \in \Omega_K \setminus \Omega_D \,|\, \mathfrak{p} \text{ dyadic or infinite and } (\alpha_\mathfrak{p}, \delta)_{K_\mathfrak{p}} \neq 1 \text{ in } {}_2\mathrm{Br}(K_\mathfrak{p})\}.$$

Let $S := S_0 \cup S_1$. We claim that S is a finite set of primes. Clearly it suffices to show that S_0 is a finite set. To see this, let Λ be a maximal A-order in D, where A is the ring of integers in K in case K is a number field, and the ring of functions that are regular outside $\mathfrak{p}_\infty$ in case K is a global function field. Then φ and ψ have diagonalizations $\langle u_1, \dots, u_n\rangle$, respectively $\langle v_1, \dots, v_n\rangle$ with $u_i, v_i \in D_0^\times \cap \Lambda$. (Since Λ contains a K-basis for D, D is the skewfield of fractions of Λ and every element of D is a fraction λ/d with $\lambda \in \Lambda$ and $d \in K^\times$. If λ/d is a pure quaternion, then so is λ. So up to multiplying by squares in K we may assume that every entry of the diagonalization of φ and ψ is in Λ.)

Note that for almost all $\mathfrak{p} \in \Omega_K^f$ the entries $u_1, \dots, u_n, v_1, \dots, v_n$ are in $\Lambda_\mathfrak{p} = \Lambda \otimes A_\mathfrak{p}$, where $A_\mathfrak{p}$ is the complete discrete valuation ring in $K_\mathfrak{p}$. Let $\mathfrak{p} \in S_0$, then $D \otimes K_\mathfrak{p} \cong M_2(K_\mathfrak{p})$ and this isomorphism can be chosen so that the image of $\Lambda_\mathfrak{p}$ is equal to $M_2(A_\mathfrak{p})$.

For the forms q_φ and q_ψ, Morita equivalent to $\varphi \otimes K_\mathfrak{p}$ and $\psi \otimes K_\mathfrak{p}$ respectively, we can consider first and second residue forms (since $\mathfrak{p}$ is non-dyadic). By the above observation the forms q_φ and q_ψ have entries which are units for almost all $\mathfrak{p} \in S_0$ (for $\langle a \rangle$ with $a = \left(\begin{smallmatrix} u & v \\ w & -u \end{smallmatrix}\right)$, $u, v, w \in M_2(A_\mathfrak{p})$ the form $q_{\langle a \rangle}$ is represented by the matrix $\left(\begin{smallmatrix} -w & -u \\ -u & v \end{smallmatrix}\right)$, cf. [8, pp. 361–362].) It follows that for all such $\mathfrak{p}$ the second residue form of q_φ and of q_ψ is trivial. Assume that $\alpha_\mathfrak{p}$ is in the square class of $\pi_\mathfrak{p}$ (a uniformizing element in $A_\mathfrak{p}$). Then the second residue form of $\alpha_\mathfrak{p} q_\psi$ is not trivial, contradicting $\alpha_\mathfrak{p} q_\psi \simeq_{K_\mathfrak{p}} q_\varphi$. This implies that for almost all $\mathfrak{p} \in S_0$ the element $\alpha_\mathfrak{p}$ is a unit times a square in almost all $A_\mathfrak{p}$. Since δ is a unit for almost all $\mathfrak{p}$, it follows that for almost all $\mathfrak{p} \in S_0$ the quaternion algebra $(\alpha_\mathfrak{p}, \delta)_{K_\mathfrak{p}}$ is trivial in ${}_2\mathrm{Br}(K_\mathfrak{p})$. By definition of S_0 this can only be if $|S_0|$ is finite, thereby proving the claim.

Fix a prime $\mathfrak{q} \in \Omega_D$ (note that $\mathfrak{q} \notin S$). Consider the following element $(\omega_\mathfrak{p}) \in \bigoplus_{\mathfrak{p} \in \Omega_K} {}_2\mathrm{Br}(K_\mathfrak{p})$,

$$\begin{cases} \omega_\mathfrak{p} := (\alpha_\mathfrak{p}, \delta)_{K_\mathfrak{p}} & \text{if } \mathfrak{p} \in S, \\ \omega_\mathfrak{q} := D \otimes K_\mathfrak{q} & \text{if } |S| \text{ is odd}, \\ \omega_\mathfrak{q} := 1 & \text{if } |S| \text{ is even}, \\ \omega_\mathfrak{r} := 1 & \text{if } \mathfrak{r} \notin S \cup \{\mathfrak{q}\}. \end{cases}$$

Note that $\sum_{\mathfrak{p} \in \Omega_K} \mathrm{inv}_\mathfrak{p}(\omega_\mathfrak{p}) = 1$. The reciprocity exact sequence (2) in 2.4 implies the existence of an element $\omega \in {}_2\mathrm{Br}(K)$ such that $\omega \otimes K_\mathfrak{p} = \omega_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega_K$. Using weak approximation we can find an element $\beta \in K$ such that $\beta \in K_\mathfrak{p}^{\times 2}$ for all $\mathfrak{p} \in S$, $\beta\delta \notin K_\mathfrak{q}^{\times 2}$, and—in case K is a number field—$\beta\delta$ is positive for all real primes. With this choice of β we have that $\omega \otimes_K K\big(\sqrt{\beta\delta}\big) \otimes_{K(\sqrt{\beta\delta})} K\big(\sqrt{\beta\delta}\big)_\mathfrak{P}$ is trivial in ${}_2\mathrm{Br}\Big(K\big(\sqrt{\beta\delta}\big)_\mathfrak{P}\Big)$ for all $\mathfrak{P} \in \Omega_{K(\sqrt{\beta\delta})}$ (to see this note that if $\omega_\mathfrak{p}$ is non-trivial in ${}_2\mathrm{Br}(K_\mathfrak{p})$, $K_\mathfrak{p}\big(\sqrt{\beta\delta}\big)$ is a quadratic extension of $K_\mathfrak{p}$, and therefore there is a unique prime $\mathfrak{P}$ in $K\big(\sqrt{\beta\delta}\big)$ lying over $\mathfrak{p}$ in K and $K_\mathfrak{p}\big(\sqrt{\beta\delta}\big) = K\big(\sqrt{\beta\delta}\big)_\mathfrak{P}$). The first part of the reciprocity sequence then implies that $\omega \otimes_K K\big(\sqrt{\beta\delta}\big)$ is trivial in ${}_2\mathrm{Br}\Big(K\big(\sqrt{\beta\delta}\big)\Big)$. So $\omega = (\alpha, \beta\delta)_K$ for some $\alpha \in K^\times$.

Since in the case where K is a number field, $\beta\delta$ is positive for all real primes of K, we can, using [7, Lemma 5], multiply α by a norm ν of $K\big(\sqrt{\beta\delta}\big)$ such that $\mathrm{sign}_\mathfrak{p} \nu\alpha = \mathrm{sign}_\mathfrak{p} \alpha_\mathfrak{p}$ for all real primes $\mathfrak{p}$ of K. Then $(\nu\alpha, \beta\delta)_K = (\alpha, \beta\delta)_K$. This means that without loss of generality we may assume that $\mathrm{sign}_\mathfrak{p} \alpha = \mathrm{sign}_\mathfrak{p} \alpha_\mathfrak{p}$ for all real primes $\mathfrak{p}$ of K.

Thus we have found an $\alpha \in K^\times$ that satisfies the conditions outlined at the start of the proof. We conclude that $\varphi \simeq_{K_\mathfrak{p}} \alpha\psi$ for all $\mathfrak{p} \in \Omega_K$. ∎

Corollary 2.23. *The Hasse principle for similarity holds for all skew-hermitian forms of discriminant δ over $(D, ^-)$ if and only if $\overline{G}_\delta = {}_2\mathrm{Br}^{\Omega_D}(K)/\langle D\rangle$.*

Proof. The Hasse principle for similarity holds for a form φ over K if and only if all forms locally similar to φ are globally similar to φ. Proposition 2.22 implies that the forms locally similar to φ are locally isometric to the form $\alpha\varphi$ with $\alpha \in K$. So the Hasse principle for similarity holds for φ if and only if there is only one global similarity class in the local isometry class of φ. The statement then follows from Corollary 2.21. ∎

Corollary 2.24. *Let $\mu \in K^\times$ be such that $D \otimes_K K(\sqrt{\mu})$ is trivial in $\mathrm{Br}(K(\sqrt{\mu}))$. Then the Hasse principle for similarity holds for all skew-hermitian forms of discriminant μ.*

Proof. The hypothesis implies that $\overline{G}_\delta = {}_2\mathrm{Br}^{\Omega_D}(K)/\langle D\rangle$. ∎

Corollary 2.25. *If $|\Omega_D| = 2$ then the Hasse principle for similarity holds for all skew-hermitian forms over $(D, ^-)$.*

Proof. The hypothesis implies ${}_2\mathrm{Br}^{\Omega_D}(K)/\langle D\rangle = 1$. ∎

Finally we obtain the following theorem:

Theorem 2.26.

(*i*) *Let $\delta \in K^\times$ such that $\delta \notin K_\mathfrak{p}^{\times 2}$ for all $\mathfrak{p} \in \Omega_D$. Then the Hasse principle for similarity holds for skew-hermitian forms φ over $(D, ^-)$ with $\operatorname{disc}\varphi = \delta \bmod K^{\times 2}$.*

(*ii*) *Let φ be a skew-hermitian form over $(D, ^-)$ of odd dimension and such that $\varphi \otimes K_\mathfrak{p}$ has maximal Witt index for all $\mathfrak{p} \in \Omega_D$. Then the Hasse principle for similarity holds for φ.*

Proof. The first statement follows directly from Corollary 2.24.

For the second statement it is enough to show that the hypotheses on φ imply that $\operatorname{disc}\varphi \notin K_\mathfrak{p}^{\times 2}$ for all $\mathfrak{p} \in \Omega_D$.

To do this we examine two cases.

Case 1. If $\mathfrak{p} \in \Omega_D \cap \Omega_K^{real}$ (in case K is a global function field this case is empty), then $D \otimes_K K_\mathfrak{p} = (-1,-1)_{K_\mathfrak{p}}$. Since $K_\mathfrak{p}$ is real closed it is well-known that all one-dimensional skew-hermitian forms over $D \otimes_K K_\mathfrak{p}$ are isometric to $\langle i\rangle$, so $\varphi \otimes K_\mathfrak{p} = \langle i, i, \ldots, i\rangle$ and $\operatorname{disc}\varphi = (-1)^n N(i)^n = -N(i) = i^2 = -1$. (Here N is the reduced norm so that $N(\lambda) = -\lambda^2$ for $\lambda \in D_0^\times$.) It follows that $\operatorname{disc}\varphi \notin K_\mathfrak{p}^{\times 2}$.

Case 2. For $\mathfrak{p}$ the prime at infinity if K is a function field, and for $\mathfrak{p} \in \Omega_D \cap \Omega_K^f$ if K is a number field, $D \otimes_K K_\mathfrak{p}$ is the unique quaternion division algebra over $K_\mathfrak{p}$ (up to isomorphism). By Tsukamoto's results (cf. [10]) either $\varphi \otimes K_\mathfrak{p} \cong \langle \lambda \rangle \perp \mathbb{H}$ where λ is a pure quaternion and $\mathbb{H}$ a hyperbolic form or else $\varphi \otimes K_\mathfrak{p} \cong \langle \mu \rangle \perp \mathbb{H}$, with μ the unique 3-dimensional anisotropic form over $D \otimes_K K_\mathfrak{p}$ (up to isometry) and $\mathbb{H}$ hyperbolic. If $\varphi \otimes K_\mathfrak{p} \cong \langle \lambda \rangle \perp \mathbb{H}$ we see that $\operatorname{disc} \varphi = (-1)^n N(\lambda) = \lambda^2$ which can take any value except 1 in $K_\mathfrak{p}^\times / K_\mathfrak{p}^{\times 2}$. The second possibility is ruled out by the hypothesis on the Witt index.

The result now follows. ■

3. Hermitian forms over a cyclic algebra

Let k be a global field of characteristic not two, $K = k(\sqrt{a})$ a quadratic extension field with nontrivial automorphism $^-$ and D a finite-dimensional division algebra with center K, equipped with a K/k-involution, also denoted by $^-$. It is well-known that D has to be a cyclic algebra.

If $\mathfrak{p}$ is a prime of k, we denote the completion of k at $\mathfrak{p}$ by $k_\mathfrak{p}$. Then $K_\mathfrak{p} := K \otimes_k k_\mathfrak{p}$ will either be a field or a double-field and $D_\mathfrak{p} := D \otimes_k k_\mathfrak{p}$ will be an algebra with center $K_\mathfrak{p}$. The involution of D extends in a unique way to an involution of $D_\mathfrak{p}$, again denoted by $^-$.

Let $\varphi : V \times V \longrightarrow D$ be a nonsingular hermitian form over $(D, ^-)$, where V is a finite-dimensional right D-vector space. Then $V_\mathfrak{p} := V \otimes_k k_\mathfrak{p}$ is a free right $D_\mathfrak{p}$-module of rank $\dim_D V$ and φ extends to a hermitian form $\varphi_\mathfrak{p} : V_\mathfrak{p} \times V_\mathfrak{p} \longrightarrow D_\mathfrak{p}$. We use the notation of [8, 10.6].

Recall [8, p. 375] that the *determinant* of φ is defined by

$$\det \varphi := N(H) \cdot N_{K/k}(K^\times) \in k^\times / N_{K/k}(K^\times),$$

where H is a matrix of φ with respect to a basis of V and $N : D \longrightarrow K$ denotes the reduced norm (we have in fact $N(H) \in k$).

If $\mathfrak{p}$ is a real nondecomposed prime of k, the extension $K_\mathfrak{p}/k_\mathfrak{p}$ can be identified with $\mathbb{C}/\mathbb{R}$, the algebra $D_\mathfrak{p}$ splits and the hermitian form $\varphi_\mathfrak{p}$ is completely determined by a quadratic form $q_{\varphi_\mathfrak{p}}$ via Morita theory. In this case, the *signature* of φ at $\mathfrak{p}$ is defined by

$$\operatorname{sgn}_\mathfrak{p}(\varphi) = \operatorname{sgn}(\varphi_\mathfrak{p}) := \operatorname{sgn}(q_{\varphi_\mathfrak{p}}),$$

see [8, p. 376 ff.].

In 1938, Landherr proved the following Hasse principle for isometry of hermitian forms:

Theorem 3.1. *Two hermitian forms φ and ψ over $(D, ^-)$ are isometric if and only if $\varphi_{\mathfrak{p}}$ and $\psi_{\mathfrak{p}}$ are isometric for all primes $\mathfrak{p}$ of k.*

Corollary 3.2. *In the situation of Theorem 3.1, we have $\varphi \simeq \psi$ if and only if $\dim \varphi = \dim \psi$, $\det \varphi = \det \psi$ and $\operatorname{sgn}_{\mathfrak{p}} \varphi = \operatorname{sgn}_{\mathfrak{p}} \psi$ for all real nondecomposed primes $\mathfrak{p}$ of k.*

The following theorem is an easy consequence of Landherr's Theorem:

Theorem 3.3. *Two hermitian forms φ and ψ over $(D, ^-)$ are similar if and only if $\varphi_{\mathfrak{p}}$ and $\psi_{\mathfrak{p}}$ are similar for all primes $\mathfrak{p}$ of k.*

Proof. One direction is trivial. For the other direction we assume that there exist $\alpha_{\mathfrak{p}} \in k_{\mathfrak{p}}$ such that $\varphi_{\mathfrak{p}} \simeq \alpha_{\mathfrak{p}}\psi_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of k. We immediately have that $\dim \varphi = \dim \psi =: n$. We also let $m = \deg D$. Two cases need to be considered:

Case 1: nm odd. Let $\alpha = \det \varphi \cdot \det \psi$ (modulo norms). Then $\alpha \equiv \alpha_{\mathfrak{p}}$ (modulo norms). Thus,

$$\varphi_{\mathfrak{p}} \simeq \alpha_{\mathfrak{p}}\psi_{\mathfrak{p}} \simeq \alpha\psi_{\mathfrak{p}} = (\alpha\psi)_{\mathfrak{p}},$$

for all primes $\mathfrak{p}$ of k. Hence $\varphi \simeq \alpha\psi$, by Theorem 3.1. (Note that $\det(\alpha_{\mathfrak{p}}\psi_{\mathfrak{p}}) = N(\alpha_{\mathfrak{p}})^n \det \psi_{\mathfrak{p}} = \alpha_{\mathfrak{p}}^{nm} \det \psi_{\mathfrak{p}} = \alpha_{\mathfrak{p}} \det \psi_{\mathfrak{p}}$.)

Case 2: nm even. Let $\varepsilon_{\mathfrak{p}} \in \{+1, -1\}$ be the sign of $\alpha_{\mathfrak{p}}$ for all real nondecomposed primes $\mathfrak{p}$ of k. By weak approximation we may choose an element $\alpha \in k$ such that α is of sign $\varepsilon_{\mathfrak{p}}$ for all real nondecomposed $\mathfrak{p}$. Then

$$\operatorname{sgn}_{\mathfrak{p}} \varphi = \operatorname{sgn} \varphi_{\mathfrak{p}} = \varepsilon_{\mathfrak{p}} \operatorname{sgn} \psi_{\mathfrak{p}} = \operatorname{sgn}(\alpha\psi_{\mathfrak{p}}) = \operatorname{sgn}_{\mathfrak{p}}(\alpha\psi)$$

for all real nondecomposed primes $\mathfrak{p}$ of k. Furthermore, $\det \varphi = \det(\alpha\psi) = \det \psi$ (since nm is even) follows directly from the fact that $\det \varphi_{\mathfrak{p}} = \det \psi_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of k. We conclude that $\varphi \simeq \alpha\psi$. ∎

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Department of Mathematics, University College Dublin, Belfield, Dublin 4, IRELAND

E-mail address: `david.lewis@ucd.ie`

Department of Mathematics, University College Dublin, Belfield, Dublin 4, IRELAND

E-mail address: `thomas.unger@ucd.ie`

Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, 9000 Gent, BELGIUM

E-mail address: `jvg@cage.ugent.be`