Identities on Maximal Subgroups of $GL_n(D)$

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Abstract

Let D be a division ring with centre F. Assume that M is a maximal subgroup of $GL_n(D)$, $n \geq 1$ such that $Z(M)$ is algebraic over F. Group identities on M and polynomial identities on the F-linear hull $F[M]$ are investigated. It is shown that if $F[M]$ is a PI-algebra, then $[D : F] < \infty$. When D is noncommutative and F is infinite, it is also proved that if M satisfies a group identity and $F[M]$ is algebraic over F, then we have either $M = K^*$, where K is a field and $[D : F] < \infty$ or M is absolutely irreducible. For a finite dimensional division algebra D, assume that N is a subnormal subgroup of $GL_n(D)$ and M is a maximal subgroup of N. If M satisfies a group identity, it is shown that M is abelian-by-finite.

1 Introduction

Let D be a division ring with centre F, and let n be a positive integer. Denote by $A := M_n(D)$ the full $n \times n$ matrix ring over D and by $A^* := GL_n(D)$, the units of A. Given a subgroup M of A^*, we shall say that M is *maximal* in A^* if for any subgroup L of A^* with $M \subset L$, one concludes that $L = A^*$.

The study of maximal subgroups of A^* begins in [1] and [9] in relation with an investigation of the structure of finitely generated normal subgroups of $GL_n(D)$, where D is of finite dimension over its centre F. In those papers we essentially show that maximal subgroups arise naturally in A^*, and finitely generated subnormal subgroups of A^*, are central. This result is used to prove that a maximal subgroup of A^* can not be finitely generated. The reader may consult [7], and the references thereof for more recent results on multiplicative subgroups of A^*. The object of this note is to investigate the algebraic structure of D when the F-linear hull $F[M]$ satisfies a polynomial identity. We also study the structure of M whenever M satisfies a group identity. To be more precise, let M be a maximal subgroup of A^* such that the centre of M, $Z(M)$ is algebraic over F. It is shown that if $F[M]$ is a PI-algebra, then $[D : F] < \infty$. As consequences of this, some results of [1] that are proved there for the case $n = 1$ may be generalized for $n > 1$. For examples, it is shown if $[A : F] = \infty$, then A^* contains no abelian maximal subgroup which is algebraic over F. Also, if M satisfies a multilinear polynomial identity, then $[D : F] < \infty$. In this direction, it is proved that if D is noncommutative and M/F^* is locally finite, then we have either M is absolutely irreducible and D is locally finite dimensional or $[D : F] < \infty$ and $M = K^*$, where K is a subfield of A. We then turn to the case where M satisfies a group identity. Let D be a noncommutative division ring with infinite centre F and $n \geq 1$. Assume that M is a maximal subgroup of A^* such that $F[M]$ is algebraic over F. It is shown that if M satisfies a group identity, then we have either $M = K^*$, where K is a field and $[D : F] < \infty$ or M is absolutely irreducible. In particular, if M is a noncommutative soluble maximal subgroup of A^* such that $F[M]$ is algebraic over F, then M is abelian-by-locally finite. For a noncommutative finite dimensional F-central division algebra D, assume that N is a subnormal subgroup of A^* and M is a maximal subgroup of N. It is proved that if M satisfies a group identity, then M is abelian-by-finite.

2 Notations and conventions

Let D be a division ring with centre F and G be a subgroup of $A^* = GL_n(D)$. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated by elements of G over F. We also denote by D^n the space of row n-vectors over D. Then D^n is a $D - G$ bimodule in the obvious manner. G is said to be an *irreducible* (reducible) subgroup of $GL_n(D)$ whenever D^n is irreducible (reducible) as $D - G$ bimodule. Considering the elements of D^n as column vectors, we may regard D^n as a $G - D$ bimodule. It is easily shown that D^n is irreducible (reducible) as a $G - D$ bimodule precisely when it has the property as $D - G$ bimodule. We shall say that G is *absolutely irreducible* if $M_n(D) = F[G]$. For any group G we denote its centre by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the *normalizer* of H in G, and $< H, K >$ the group generated by H and K, where K is a subgroup of G. We shall say that H is *abelian-by-finite* if there is an abelian normal subgroup K of H such that H/K is finite. Let S be a subset of $M_n(D)$, then the *centralizer* of S in $M_n(D)$ is denoted by $C_{M_n(D)}(S)$. We shall identify the centre FI of $M_n(D)$ with F. By a dilatation matrix $D_{ii}(d)$, $d \in D^*$ we understand a diagonal $n \times n$ matrix whose diagonal entries are all 1 except the (i, i)-th entry which is d. Some notations and conventions for linear groups and skew linear groups from [11] and [12] are frequently used throughout.

3 Polynomial identities on $F[M]$

Given a maximal subgroup of A^*, this section essentially deals with conditions on M that imply either the commutativity of M or $[D : F] < \infty$. The main result is Theorem 5 which asserts that if $F[M]$ is a PI-algebra and $Z(M)$ is algebraic over F, then $[D : F] < \infty$. Using this, it is shown that if M satisfies a multilinear polynomial identity and $Z(M)$ is algebraic over F, then $[D : F] < \infty$. Furthermore, it is proved that if either $n = 1$ and D is noncommutative or $n > 1$ and D is infinite, then there exists no maximal subgroup M of A^* containing F^* such that $[M : F^*] < \infty$. For a noncommutative division ring with centre F, it is also shown that if M/F^* is locally finite, then we have either M is absolutely irreducible and D is locally finite dimensional or $[D:F]<\infty$ and $M=K^*$, where K is a subfield of A. We begin our material with

Proposition 1. *Let D be a division ring with centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$. Then M is either irreducible or it contains an isomorphic copy of D^*.*

Proof. The case $n=1$ follows from Proposition 1 of [1]. So, we may assume that $n \geq 2$. Now, consider the F-algebra $F[M]$. Since M is maximal we conclude that either $GL_n(D) = F[M]^*$ or $F[M]^* = M$. The first case implies that $M_n(D) = F[M]$, i.e., M is absolutely irreducible and so it is irreducible. Thus, we may assume that $F[M]^* = M$. If M is not irreducible, then D^n has a nontrivial submodule as $D - F[M]$ bimodule. Thus, by 1.1.1 of [12], there exists an invertible $n \times n$ matrix P over D such that $PMP^{-1} \subset \Sigma$, where $\Sigma = \{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL_s(D), C \in GL_{n-s}(D), B \in M_{s\times(n-s)}(D)\}$. It is clear that PMP^{-1} is also a maximal subgroup of $GL_n(D)$ and we have $PMP^{-1} \subseteq \Sigma \subset GL_n(D)$. Therefore, $PMP^{-1} = \Sigma$ and since Σ contains a copy of D^* we obtain the result.

We shall need the following lemmas to prove our main theorem.

Lemma 2. *Let D be a division ring of infinite dimension over its centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$. If $F[M]$ is a PI-algebra, then $F[M]$ is a prime ring.*

Proof. By Proposition 1, we know that either M contains a copy of D^* or M is irreducible. If the first case happens, then D is a PI-algebra. This implies, by Kaplansky's Theorem (cf. [11]), that $[D:F]<\infty$ which is a contradiction. So we may assume that M is irreducible. Now, using 1.1.14 of [12, p. 9], we conclude that $F[M]$ is a prime ring.

Lemma 3. *Let D be a division ring of infinite dimension over its centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$. If $F[M]$ is a PI-algebra, then $Z(F[M])$ is a field.*

Proof. Set $A = F[M]$. Assume that $X \in Z(A)$. We consider two cases:

Case 1. If $X \in GL_n(D)$ and $X^{-1} \notin A$, then we have $< M, X > \subseteq C_{M_n(D)}(Z(A))$ and by maximality of M we conclude that $C_{M_n(D)}(Z(A)) = M_n(D)$. This means that $Z(A) = F$ which is a contradiction. Thus $X^{-1} \in Z(A)$.

Case 2. Assume that $X \notin GL_n(D)$. Then there exists an $n \times n$ matrix P in $GL_n(D)$ such that the first row of PXP^{-1} is zero. We note that PMP^{-1} is a maximal subgroup of $GL_n(D)$, $F[PMP^{-1}]$ is a PI-algebra, and $PXP^{-1} \in Z(F[PMP^{-1}]) = PZ(F[M])P^{-1}$. Set $J := XF[M]$. Then J is an ideal in $F[M]$ and $PXP^{-1}(PF[M]P^{-1}) = PJP^{-1} = PXP^{-1}F[PMP^{-1}]$ is also an ideal in $F[PMP^{-1}]$. Put $J' = PJP^{-1}$ and $B := \{Y \in M_n(D) | YJ' \subset J'\}$. It is clear that B is a ring and $PAP^{-1} \subseteq B$. On the other hand PXP^{-1} is a matrix whose first row is zero. Therefore, $D_{11}(d) \in B$ for all $d \in D$. It is clear that $PMP^{-1} \subseteq B^*$. Thus, $B^* = GL_n(D)$ or $B^* = PMP^{-1}$. In the first case $B = M_n(D)$ and so J' is a right ideal and clearly J is a right ideal in $M_n(D)$. In the second case $D_{11}(d) \in PMP^{-1}$ for all $d \in D$. Therefore, $F[PMP^{-1}]$ contains a copy of D^* and so D is a PI-algebra which implies $[D : F] < \infty$ that is a contradiction. Similarly, there exists a matrix $Q \in GL_n(D)$ such that QXQ^{-1} is a matrix whose first column is zero. Set $C = \{Y \in M_n(D) | (QJQ^{-1})Y \subset QJQ^{-1}\}$. As above, one may show that J is a left ideal in $M_n(D)$. Consequently, J is an ideal in $M_n(D)$. Since $J \neq 0$ we obtain $J = M_n(D)$. Therfore, $F[M] = M_n(D)$ and so $[D : F] < \infty$ which is a contradiction. Thus, $Z(A)$ is a field and the proof is complete.

Lemma 4. *Let D be a division ring of infinite dimension over its centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$. If $F[M]$ is a PI-algebra, then $F[M]$ is simple and we have $[F[M] : Z(F[M])] < \infty$.*

Proof. By Lemma 2-3, we conclude that $F[M]$ is a prime ring whose centre is a field. Since $F[M]$ is also a PI-algebra, by Theorem 7.5 of [3], we conclude that $F[M]$ is simple. Finally, the rest of the result follows from Kaplansky's Theorem.

Theorem 5. *Let D be a division ring with centre $Z(D) = F$ and $n \geq 1$.*

Assume that M is a maximal subgroup of $GL_n(D)$ such that $Z(M)$ is algebraic over F. If $F[M]$ is a PI-algebra, then $[D : F] < \infty$.

PROOF. We have $M \subseteq F[M]^*$. By maximality of M we conclude that either $F[M]^* = GL_n(D)$ or $F[M]^* = M$. The first case gives us $F[M] = M_n(D)$. Now, use Kaplansky's Theorem to obtain $[D : F] < \infty$. Therefore, we may assume that $F[M]^* = M$. To complete the proof, we show that the assumption $[D : F] = \infty$ leads to a contradiction. Thus, suppose $[D : F] = \infty$. Then, by Lemma 4 and Artin-Wedderburn's Theorem, we have $F[M] \cong M_{n_1}(D_1)$ for some positive integer n_1 and division ring D_1, and so $M \cong GL_{n_1}(D_1)$. We claim that $Z(M) = F^*$. For otherwise, since $Z(M)$ is algebraic over F there exists $a \in Z(M) \setminus F^*$ such that $[F(a) : F] < \infty$. Now, we have $F[M] \subseteq C_{M_n(D)}(F(a)) := A$. If $F[M] \neq A$, then, by the Centralizer Theorem, we conclude that A is a simple Artinian ring. Therefore, there exists a positive integer n_2 and a division ring D_2 such that $A \cong M_{n_2}(D_2)$ and so $A^* \cong GL_{n_2}(D_2)$. We know that $M \subseteq A^*$. If $M = A^*$, then we clearly have $F[M] = A$ which is a contradiction to our assumption. Thus, by maximality of M, one concludes that $A^* = GL_n(D)$, i.e., $C_{M_n(D)}(F(a)) = A = M_n(D)$ which contradicts the fact that $a \in Z(M) \setminus F$. Therefore, we must have $A = F[M]$ and so $Z(F[M]) = Z(A) = F(a)$ by the Centralizer Theorem. Thus, by Lemma 4, this means that $[F[M] : F] < \infty$. Now, apply the Centralizer Theorem again to obtain $M_n(D) \otimes F(a)^{0p} \cong M_s(F) \otimes F[M]$ for some positive integer s. The last isomorphism implies that $[D : F] < \infty$ which contradicts our assumption. Therefore, we must have $Z(M) = F^*$ and the claim is established. Now, since $F[M]$ is simple and $F[M]^* = M$ we obtain $Z(F[M]) = F$. Finally, consider the simple Artinian ring $B := C_{M_n(D)}(F[M])$. There exists a positive integer n_3 and a division ring D_3 such that $B \cong M_{n_3}(D_3)$. If $F \neq C_{M_n(D)}(F[M])$, then $F^* \subset B^* \cong GL_{n_3}(D_3)$. Therefore. there is an $X \in B^* \setminus F^*$ such that $< M, X > \subseteq C_{M_n(D)}(F(X))$ and so $M_n(D) = C_{M_n(D)}(F(X))$ which is a contradiction to the fact that $X \notin F^*$. Thus, $F = C_{M_n(D)}(F[M])$. Now, using the Centralizer Theorem as above, one concludes that $[D : F] < \infty$ which is a contradiction and this completes the proof.

The next result generalizes Theorem 4.1 of [1].

Corollary 6. *Let D be a division ring with centre $Z(D) = F$ and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$ such that $Z(M)$ is algebraic over F. If M satisfies a multi-linear polynomial identity, then $[D : F] < \infty$.*

The following result is also a generalization of Corollary 4.2 of [1].

Corollary 7. *Let D be a division ring of infinite dimension over its centre $Z(D) = F$ and $n \geq 1$. Then $GL_n(D)$ contains no abelian maximal subgroup which is algebraic over F.*

Corollary 8. *Let D be a division ring with centre F. If either $n = 1$ and D is noncommutative or $n > 1$ and D is infinite, then there exists no maximal subgroup M of $GL_n(D)$, $n \geq 1$, containing F^* such that $[M : F^*] < \infty$.*

Proof. Assume that there is a maximal subgroup M such that $[M : F^*] < \infty$. Then, we have $[F[M] : F] < \infty$, i.e., $F[M]$ is a PI-algebra and M is algebraic over F. Thus, by Theorem 5, we obtain $[D : F] < \infty$. Let $x_1, \ldots, x_t$ be the representatives for cosets of F^* in M, i.e., $M = F^*x_1 \cup \ldots \cup F^*x_t$. Then, we have $M =< x_1, \ldots, x_t > F^*$, where $< x_1, \ldots, x_t >$ is the group generated by $x_1, \ldots, x_t$. Take $x \in GL_n(D) \backslash M$. By maximality of M, we obtain $GL_n(D) =< x_1, \ldots, x_t, x > F^*$. Put $H =< x_1, \ldots, x_t, x >$. Thus, $GL_n(D) = HF^*$ and consequently we have $SL_n(D) = H' \subset H$, i. e., H is normal in $GL_n(D)$. Now, by Theorem 5 of [2], we conclude that $H \subset F^*$, i. e., $GL_n(D) = F^*$ which means that $n = 1$ and $D = F$ that is a contradiction and so the result follows.

Corollary 9. *Let R be a semisimple Artinian F-algebra with $[Z(R) : F] < \infty$. Assume that M is a maximal subgroup of R^* such that $Z(M)$ is algebraic over F. If $F[M]$ satisfies a polynomial identity, then we have $[R : F] < \infty$.*

Now, assume that M is a maximal subgroup of $GL_n(D)$ containing F^* such that M/F^* is locally finite. The next result gives some information about the algebraic structure of M.

Theorem 10. *Let D be a noncommutative division ring with centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$ containing F^* such that M/F^* is locally finite. Then we have either M is absolutely irreducible and D is locally finite dimensional or $[D : F] < \infty$ and $M = K^*$, where K is a subfield of $M_n(D)$.*

Proof. Consider the F-algebra $F[M]$. If $F[M]^* = GL_n(D)$, then we have $F[M] = M_n(D)$. Since M/F^* is locally finite we conclude that D is locally finite dimenisional and so the result follows. Thus, we may assume throughout that $F[M]^* \neq GL_n(D)$. By maximality of M we conclude that $F[M]^* = M$. We observe that since M/F^* is locally finite then $F[M]$ is locally finite dimensional. Therefore, for each finite set of elemnts $m_1 \cdots m_r \in M$ we have $[F[m_1 \cdots m_r] : F] < \infty$. We next claim that $F[M]$ is a PI-algebra satisfying $P(X, Y) = (XY - YX)^n$. To do this, let $x, y \in F[M]$. Then, there exist elements $m_i, n_j \in M$ with $1 \leq i \leq t$, $1 \leq j \leq s$ such that $x = m_1 + \cdots m_t$ and $y = n_1 + \cdots n_s$, and we have $[F[m_1, \cdots m_t, n_1, \cdots n_s] : F] < \infty$. Therefore, $A = F[m_1, \cdots m_t, n_1, \cdots n_s]$ is an Artinian PI -ring and so the Jacobson radical $J = J(A)$ of A is nilpotent. Therefore, by 1.3.9 of [12], we conclude that $J^n = 0$. Now, by Wedderburn-Artin Theorem , there exist positive integers $n_1, \cdots, n_k$ such that $B := \frac{A}{J(A)} \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ as F-algebras , for some division rings D_i, where $1 \leq i \leq k$. Now, we have $B^* = \{m + J | m \in M \cap A\} \cong GL_{n_1}(D_1) \times \cdots GL_{n_k}(D_k)$. Since B^* is torsion over F we conclude that each D_i is torsion over F where $F \subseteq Z(D_i)$. Thus D_i is torsion over $Z(D_i)$ and by a result of Kaplansky (cf. [5]), we obtain $D_i = Z(D_i) := F_i$. Therefore, we obtain $B \cong M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_k)$. If there exists i such that $n_i \neq 1$, then the dilatation $n_i \times n_i$ matrix $D_{11}(f)$, where $f \in F_i$, is torsion over F. In particular, this implies that F^* is a torsion group. Now, F^* and M/F^* are locally finite. Therefore, by a well-known result of group theory, M is locally finite. By Proposition 1, either M contains an isomorphic copy of D^* or M is irreducible. The first case says that D^* is locally finite. This implies, by a result of Jacobson (cf. [5]), that $D = F$ which is a contradiction.

Thus, we may assume that M is irreducible and so it is completely reducible. Therefore, by 1.1.14 of [12], we conclude that $F[M]$ is semisimple Artinian. Thus, as in the above case, there exist positive integers $m_1, \cdots, m_k$ such that $F[M] \cong M_{m_1}(D_1) \times \cdots \times M_{m_k}(D_k)$, for some division rings D_i, where $1 \leq i \leq k$. Now, we have $F[M]^* = M \cong GL_{m_1}(D_1) \times \cdots GL_{m_k}(D_k)$. Since M is locally finite, as above, we conclude that $D_i = F_i = Z(D_i)$. Therefore, $F[M] \cong M_{m_1}(F_1) \times \cdots \times M_{m_k}(F_k)$. This means that $F[M]$ satisfies a polynomial identity and so by Theorem 5 we conclude that $[D : F] < \infty$. Since F^* is torsion we obtain $D = F$ which is a contradiction. Therefore, in the decomposition of B for all i we have $n_i = 1$, i.e., $B \cong F_1 \times \cdots \times F_k$. This implies that for each $x, y \in A$ we have $xy - yx \in J$ and therefore $(xy - yx)^n = 0$ since $J^n = 0$. Thus, $F[M]$ satisfies $P(X,Y) = (XY - YX)^n$. Therefore, by Theorem 5, we obtain $[D : F] < \infty$. Since M is irreducible and $[D : F] < \infty$, by 1.1.12 of [12], we conclude that $F[M]$ is simple Artinian, i.e., $F[M] \cong M_t(D_1)$ for some t and division ring D_1. Now, we may use a similar argument as above to deduce that $D_1 = F_1 = Z(D_1)$. If $t \neq 1$, we may conclude that F is torsion and since $[D : F] < \infty$ we obtain $D = F$ which is a contradiction. Therefore, $t = 1$ and we have $F[M]^* = M \cong F_1^*$ which completes the proof.

Corollary 11. *Let D be a division ring with centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$. If M is locally finite, then we have $D = F$.*

Proof. If $F^* \not\subseteq M$, then $SL_n(D) \subset M$. Since M is torsion, we conclude that $SL_n(D)$ is torsion. Thus, by Corollary 2 of [8], we conclude that $D = F$. So, we may assume that $F^* \subseteq M$. This implies that F^* is torsion and so M/F^* is torsion. Now, Theorem 10 implies that D is algebraic over F. Therefore, D is algebraic over its prime subfield and so by a result of Jacobson we conclude that $D = F$.

4 Group identities on M

Let D be a noncommutative division ring with infinite centre F and n be a positive integer. Given a maximal subgroup M of A^* such that $F[M]$ is algebraic over F, the key result of this section is to show that if M satisfies a group identity, then we have either $M = K^*$, where K is a field and $[D : F] < \infty$ or M is absolutely irreducible. Particularly, if M is a noncommutative soluble maximal subgroup of A^* such that $F[M]$ is algebraic over F, it is proved that M is abelian-by-locally finite. When D is of finite dimension over F, assume that N is a subnormal subgroup of A^* and M is a maximal subgroup of N. It is proved that if M satisfies a group identity, then M is abelian-by-finite.

THEOREM 1. *Let D be a noncommutative division ring with infinite centre F and $n \geq 1$. Assume that M is a maximal subgroup of $GL_n(D)$ such that $F[M]$ is algebraic over F. If M satisfies a group identity, then we have either $M = K^*$, where K is a field and $[D : F] < \infty$ or M is absolutely irreducible.*

PROOF. If $F^* \nsubseteq M$, then M is normal in $GL_n(D)$. This means that $F[M]$ is normal in $M_n(D)$. Thus, by a result of [10], we conclude that either $F[M] \subset F$ or $F[M] = M_n(D)$. If the first case occurs, then we have $M \subseteq F^*$ which is a contradiction to the fact that M is a maximal subgroup of $GL_n(D)$. The second case says that M is absolutely irreducible. So, let $F^* \subset M$, we have two cases to consider. If $F[M]^* = GL_n(D)$, then M is absolutely irreducible. So, assume that $F[M]^* = M$. By Proposition 3.1, we know that either M contains an isomorphic copy of D^* or M is irreducible. If the first case occurs, then D^* satisfies a group identity. But since F is infinite this is impossible by a result of Amitsur (cf. [11]). Therefore, M must be irreducible. Thus, by 1.1.14 of [12, p. 9], we conclude that $F[M]$ is a prime ring. Now, by Theorem 5.5 of [6], we have $F[M] \cong M_r(K)$ for some positive integer r, where K is an extension field of F. This shows that $F[M]$ satisfies a polynomial identity. Now, by Theorem 3.5, we conclude that $[D : F] < \infty$. Finally, we have $F[M]^* = M \cong GL_r(K)$. If $r \neq 1$, then M contains a copy of $SL_r(K)$, i.e., M contains a free subgroup (cf. [12]). This contradicts the fact that M satisfies a group identity. Therefore, we have $r = 1$ and this completes the proof.

COROLLARY 2. *Let D be a division ring with infinite centre F and $n \geq 1$. Assume that M is a noncommutative maximal subgroup of $GL_n(D)$ such that $F[M]$ is algebraic over F. If M satisfies a group identity, then $M_n(D)$ is algebraic over F.*

COROLLARY 3. *Let D be a division ring with infinite centre F and $n \geq 1$. Assume that M is a noncommutative nilpotent maximal subgroup of $GL_n(D)$ such that $F[M]$ is algebraic over F. Then M is centre-by-locally finite. Therefore, M/F^* is locally finite and so D is locally finite dimensional.*

PROOF. M satisfies a group identity since M is nilpotent. Thus, by Theorem 1, we conclude that M is absolutely irreducible. Now, by a result of [12, p. 213], this implies that M is centre-by-locally finite and so the proof is complete.

COROLLARY 4. *Let D be a division ring with infinite centre F and $n \geq 1$. Assume that M is a noncommutative soluble maximal subgroup of $GL_n(D)$ such that $F[M]$ is algebraic over F. Then M is abelian-by-locally finite.*

PROOF. M satisfies a group identity since M is soluble. By Theorem 1, we conclude that M is absolutely irreducible. Now, by a result of [13], this implies that M is abelian-by-locally finite and so the proof is complete.

To prove our final result, we need the following

LEMMA 5. *Let D be a division algebra of finite dimension over its centre F. Assume that G is a subgroup of D^*. If G satisfies a group identity, then G is abelian-by-finite.*

PROOF. Assume that G satisfies a group identity. Since $[D : F] < \infty$ we conclude that G is a linear group. By a theorem of Platonov (cf. [14. p. 149]), G is soluble-by-finite, i.e., there exists a soluble normal subgroup N of G such that G/N is finite. Since $[D : F] < \infty$ we conclude that $F[N]$ is a division ring, and therefore $F[N]$ is semisimple. Thus, N as a linear group over F is completely reducible. Therefore, N is a completely reducible soluble linear group. So, by a theorem of [4, p. 111], N is abelian-by-finite and consequently, G is abelian-by-finite which completes the proof.

THEOREM 6. *Let D be a noncommutative division algebra of finite dimension over its centre F and $n \geq 1$. Assume that N is a subnormal subgroup of $GL_n(D)$ and M is a maximal subgroup of N. If M satisfies a group identity, then M is abelian-by-finite.*

PROOF. The case $n = 1$ follows from Lemma 4. So, we may assume that $n \geq 2$ and N is a subnormal subgroup of $GL_n(D)$. By Theorem 11 of [8], we have either $N \subset F^*$ or $SL_n(D) \subset N$, i.e., N is normal in $GL_n(D)$. We now claim that M is irreducible. For otherwise assume that M is reducible. Therefore, D^n has a nontrivial submodule as $D-F[M]$ bimodule. Thus, there exists an invertible $n \times n$ matrix P over D such that $PMP^{-1} \subset \Sigma$, where $\Sigma = \{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL_s(D), C \in GL_{n-s}(D), B \in M_{s\times(n-s)}(D)\} \cap N$. It is clear that PMP^{-1} is also a maximal subgroup of N. Set

$$T = \begin{pmatrix} SL_s(D) & o \\ o & SL_{n-s}(D) \end{pmatrix}.$$

Then, we have $T \subset SL_n(D)$ and so $T \subset \Sigma$. Now, we clearly have $PMP^{-1} \subseteq \Sigma \subseteq N$. By maximality of PMP^{-1} we conclude that either $PMP^{-1} = \Sigma$ or $N = \Sigma$. If the second case occurs, then $I + e_{n1} \in SL_n(D)$ whereas $I + e_{n1} \notin \Sigma = N$ and this contradicts the fact that $SL_n(D) \subseteq N$. Therefore, $PMP^{-1} = \Sigma$ which implies that Σ satisfies a group identity, and so $T \subset \Sigma$ satisfies a group identity. Now, by 4.5.1 of [12], we conclude that $D = F$ which is a contradiction. Thus, M is irreducible as claimed. Therefore, by 1.1.12 of [12], $F[M]$ is simple Artinian. So, there exists a positive integer t and a division ring D_1 such that $F[M] \cong M_t(D_1)$ as F-algebras. Thus, $F[M]^* \cong GL_t(D_1)$ and $(F[M]^*)' \cong SL_t(D_1)$. Since N is normal in $GL_n(D)$ we have $(F[M]^*)' \subset N$. If $F[M]^* \cap N = M$, then we have $SL_t(D_1) \cong (F[M]^*)' \subseteq F[M]^* \cap N = M$. Since M satisfies a group identity, by 4.5.1 of [12] again, we conclude that either D_1 is a locally finite field or $t = 1$ and $D_1 = Z(D_1)$. In the first case F is also a locally finite field and since $[D : F] < \infty$ we conclude that $D = F$ which is a contradiction. The second case implies that M is abelian and the result follows. Finally, if $N \cap F[M]^* = N$, then $N \subseteq F[M]^*$. Since N is normal in $GL_n(D)$ we conclude that $F[M] = M_n(D)$, i.e., M is absolutely irreducible. By a result of [14, p. 149], M is soluble-by-finite, i.e., there is a soluble normal subgroup N_1 such that M/N_1 is finite. Now, by 1.2.12 of [12], $F[N_1]$ is semisimple Artinian. Therefore, N_1 is a soluble completely reducible linear group. By a theorem of [4, p. 111], N_1 is abelian-by-finite and therefore M is abelian-by-finite which completes the proof.

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