Degenerations of Rank 3 Quadratic Bundles and Rank 4 Azumaya Bundles over Schemes

T. E. Venkata Balaji
Mathematisches Institut, Georg-August-Universität
Bunsenstrasse 3-5, D-37073 Göttingen, Germany
tevbal@uni-math.gwdg.de

Dedicated to Professor Martin Kneser

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Abstract

This paper uses scheme-theoretic methods to address algebraic problems about rank 3 quadratic forms and rank 4 Azumaya algebras. It extends well-known results on semiregular rank 3 quadratic bundles and rank 4 Azumaya algebra bundles to their degenerations by studying the scheme of specialisations of Azumaya algebra bundle structures introduced in Part A of [3]. The Witt-invariant of a rank 3 quadratic bundle (V, q), which by definition is the isomorphism class (as algebra bundle) of its even-Clifford algebra $C_0(V, q)$, is shown to determine the pair (V, q) upto tensoring by a discriminant line bundle. The special, usual and the general orthogonal groups of (V, q) are computed and canonically determined in terms of $\mathrm{Aut}(C_0(V, q))$, and it is shown that the general orthogonal group is always a semidirect product. Any element of $\mathrm{Aut}(C_0(V, q))$ can be lifted to a self-similarity, and in fact to an element of the orthogonal group provided the determinant of the automorphism is a square. The special orthogonal group and the group of determinant 1 automorphisms of $C_0(V, q)$ are naturally isomorphic. If the base scheme X is integral and q is semiregular at some point of X, then every automorphism of $C_0(V, q)$ has determinant 1 and is thus induced from a self-isometry; the orthogonal group is also seen to be a semidirect product in this case. If X is affine with coordinate ring a UFD, then every specialised algebra structure on a rank 4 vector bundle over it arises as the even-Clifford algebra of a global rank 3 quadratic bundle (V, q), so that the set of rank 3 quadratic bundles upto tensoring by a discriminant line bundle naturally corresponds to the set of isomorphism classes of specialised rank 4 algebra bundles. These results are seen as limiting versions of the natural bijection $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mathsf{O}_3)/\mathrm{Disc}(X) \cong \check{\mathrm{H}}^1_{\text{étale}}(X, \mathsf{PGL}_2)$. The multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit that is part of a global basis is written down explicitly. For a connected proper scheme X of finite type over an algebraically closed field, the hypothesis of self-duality on a unital associative algebra bundle of square rank over X forces the algebra to be either globally Azumaya or to be nowhere-Azumaya. This implies that the existence of a non-Azumaya specialisation which is Azumaya at some point excludes the possibility of the existence of global Azumaya algebra structures on that bundle. The use of the nice technical notion of semiregularity introduced by Kneser in [1] allows all of the above results to be valid over an arbitrary base scheme X, some (or even all) of whose points may have residue fields of characteristic two.

Keywords: semiregular, quadratic bundle, Azumaya bundle, Witt-invariant, Clifford algebra, discriminant bundle, orthogonal group, similarity, similitude.

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Contents

1 Overview of the Main Results

The smoothness of the schematic closure of Azumaya algebra structures on a fixed vector bundle of rank 4 over any scheme was obtained in Part A of [3]. Part B of that work had applied this result to obtain desingularisations (with good specialisation properties) of certain moduli spaces over fairly general base schemes. The present work is concerned with applications to the study of degenerations of rank 3 quadratic bundles over a scheme.

For a rank 3 vector bundle V over a scheme X with a quadratic form q on V (taking values in $\mathcal{O}_X$), consider the natural functorial association of the pair (V, q) to its even Clifford algebra bundle $C_0(V, q)$ of rank 4 over X. If q is a semiregular quadratic form, then $C_0(V, q)$ is an Azumaya algebra bundle over X i.e., a twisted form for the the (2×2)-matrix algebra over $\mathcal{O}_X$ for the étale topology on X. (A generic quadratic form on a vector bundle of odd rank is semiregular and such forms are technically the "good" ones—they were introduced by Kneser in [1] and are studied in detail by Knus in his book [2]—coinciding with the usual regular quadratic forms when the residue field of every point of X has characteristic different from 2). Therefore the association is one that "generically" takes a rank 3 semiregular quadratic bundle to a rank 4 Azumaya algebra bundle. This may also be described as the natural bijection

$$\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mathsf{O}_3)/\mathrm{Disc}(X) \cong \check{\mathrm{H}}^1_{\text{étale}}(X, \mathsf{PGL}_2)$$

where the left side classifies semiregular rank 3 quadratic bundles upto tensoring by a discriminant line bundle on X, and the right side classifies Azumaya algebra bundles of rank 4 on X. If none of the residue fields of the points of X is of characteristic two, then the subscript 'fppf' on the left side may be replaced with 'étale'. What follows is an attempt to extend this bijection 'to the limit' i.e., with degenerate (possibly non-semiregular) quadratic bundles on the left and specialised (possibly non-Azumaya) bundles on the right. We extend the left side by replacing it with the set of quadratic bundles of rank 3 on X upto tensoring by a discriminant line bundle and extend the right side by replacing it with algebra-isomorphism classes of specialisations, of rank 4 Azumaya algebras, in the sense of Part A, [3]. We find that the resulting map is always an injection; we are able to prove that it is a surjection in certain cases, for example when X is affine with coordinate ring a UFD, and hope that it will always be a bijection.

A detailed analysis of the origins behind the smoothness result in Part A, [3], allows us to extend many of the known results for semiregular rank 3 quadratic bundles (and rank 4 Azumaya bundles), as in Chap.V, §3 of the book of Knus [2], to degenerate rank 3 quadratic bundles and specialisations of rank 4 Azumaya algebra bundles. For example, it turns out that the Witt-invariant of (V, q), which by definition is the isomorphism class (as algebra bundle) of $C_0(V, q)$, determines the pair (V, q) upto tensoring by a discriminant line bundle. For another example, the orthogonal groups related to (V, q) viz. the special, usual and the general orthogonal groups (contained in that order inside $\mathrm{Aut}(V) \times \Gamma(X, \mathcal{O}_X^*) = \mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$) may be computed and canonically determined in terms of $\mathrm{Aut}(C_0(V, q))$.

Let us fix some notations and terminology before proceeding further. In the following we switch between the language of (geometric) vector bundles and the (equivalent) language of locally-free sheaves of finite constant positive rank; as a convention, sheaves corresponding to vector bundles $V, W, L \ldots$ are respectively denoted $\mathcal{V}, \mathcal{W}, \mathcal{L}\ldots$ Relative to any fixed base scheme B, consider the natural transformation between two natural (covariant) functors

$$\mathsf{Witt} : \mathscr{Q}_3 \longrightarrow \mathscr{A}_4, \qquad \mathscr{Q}_3, \mathscr{A}_4 : (B - \mathfrak{Schemes})^{op} \longrightarrow \mathfrak{Sets}$$

defined as follows. If $X \longrightarrow B$ is a B-scheme, recall that a pair (L, h) is said to be a discriminant line bundle on X if L is a line bundle and $h : L \otimes_X L \cong \mathbb{A}^1_X$ is an isomorphism of line bundles (i.e., coming from an isomorphism of the associated locally-free sheaves of $\mathcal{O}_X$-modules of rank 1). In other words h is a non-singular bilinear form on L with values in the trivial line bundle (hence necessarily symmetric). Let $\mathsf{Disc}(X)$ be the set of isometry classes of such discriminant bundles. It is naturally an abelian group of exponent 2 under the tensor product operation. It acts on the set of isometry classes of quadratic bundles (V, q) on X, since given (L, h) there is the natural quadratic bundle $(V, q) \otimes (L, h)$ viz. the quadratic form $q \otimes h$ on $V \otimes L$. Define $\mathscr{Q}_3(X)$ to be the set of equivalence classes under this action i.e., the set of rank 3 quadratic bundles on X (isometric) upto tensoring by a discriminant bundle. Also define $\mathscr{A}_4(X)$ to be the set of isomorphism classes of associative unital algebra structures on rank 4 vector bundles over X (here unital means that there exists a global nowhere-vanishing section which serves as the unit for the algebra multiplication). Associating to (V, q) its Witt-invariant i.e., the isomorphism class of its even Clifford algebra $C_0(V, q)$ gives rise to a functor in X:

$$\mathsf{Witt}(X) : \mathscr{Q}_3(X) \longrightarrow \mathscr{A}_4(X) : [(V, q)] \mapsto [C_0(V, q)]$$

since tensoring a quadratic bundle with a discriminant bundle (which includes replacing it with a similar—in particular with an isometric—quadratic bundle) does not affect the isomorphism class of its even Clifford algebra (see Prop. 2.5). Observe that by considering only semiregular quadratic bundles we obtain a subfunctor $\mathscr{Q}_3^{sr} \hookrightarrow \mathscr{Q}_3$ (since tensoring by a discriminant bundle preserves semiregularity, see Prop.2.2). Also by considering only Azumaya algebra bundle structures we obtain a subfunctor $\mathscr{A}_4^{Azu} \hookrightarrow \mathscr{A}_4$ so that the transformation Witt takes $\mathscr{Q}_3^{sr}$ into $\mathscr{A}_4^{Azu}$. In fact, $\mathsf{Witt}(X) : \mathscr{Q}_3^{sr}(X) \longrightarrow \mathscr{A}_4^{Azu}(X)$ is bijective for each scheme X, and this may be deduced from the case of an affine X which is proved in §3, Chap.V, [2] (see Theorem 1.14). It follows from the main result of Part A of [3] that there exists a subfunctor $\mathscr{A}_4^{Sp-Azu} \hookrightarrow \mathscr{A}_4$ that contains $\mathscr{A}_4^{Azu}$ as well as the image of Witt. This subfunctor may be described as those algebra bundle structures that are scheme-theoretic specialisations of Azumaya algebra structures, or also as those that are locally (in the Zariski topology) the even Clifford algebras of quadratic bundles. It turns out that these specialisations are quaternion algebra bundles i.e., those that have a standard involution (in the sense of Knus, para.1.3, Chap.I, [2]) but of course even over an algebraically closed field there are quaternion algebras that are not even-Clifford algebras of quadratic forms.

THEOREM 1.1 *For each X, the map $\mathsf{Witt}(X) : \mathscr{Q}_3(X) \longrightarrow \mathscr{A}_4^{Sp-Azu}(X)$ is injective.*

For two quadratic bundles (V, q) and (V', q') denote by $\mathrm{Sim}[(V, q), (V', q')]$ the set of similarities (also called similitudes) from (V, q) to (V', q') (with multipliers being global sections of $\mathcal{O}_X^*$) and by $\mathrm{Iso}[(V, q), (V', q')]$ the subset of isometries (i.e., similarities with trivial multipliers). When $V = V'$, the subset of isometries with trivial determinant is denoted $\text{S-Iso}[(V, q), (V, q')]$. On taking $q = q'$ these sets naturally become subgroups of $\mathrm{Aut}(V) \times \Gamma(X, \mathcal{O}_X^*) = \mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$ and we get

$$\mathrm{Sim}[(V, q), (V, q)] = \mathrm{GO}(V, q) \supset \mathrm{Iso}[(V, q), (V, q)] = \mathrm{O}(V, q) \supset \text{S-Iso}[(V, q), (V, q)] = \mathrm{SO}(V, q).$$

Of course, $\mathrm{O}(V, q)$ and $\mathrm{SO}(V, q)$ may as usual be considered as subgroups of $\mathrm{GL}(V) \equiv \mathrm{GL}(V) \times \{1\}$. Since in general a quadratic bundle (V, q) on a non-affine scheme X may not be induced from a global bilinear form, one is unable to identify the $(\mathbb{Z}/2.\mathbb{Z})$– graded vector bundle underlying its Clifford algebra bundle with that underlying the exterior algebra bundle of V (which is the same as the Clifford algebra bundle of the zero quadratic form).

PROPOSITION 1.2 *Every isomorphism of algebra-bundles $\phi : C_0(V, q) \cong C_0(V', q')$ is naturally associated to an isomorphism of bundles $\phi_{\Lambda^2} : \Lambda^2(V) \cong \Lambda^2(V')$ which induces a map*

$$\zeta_{\Lambda^2} : \mathrm{Iso}[C_0(V, q), C_0(V', q')] \longrightarrow \mathrm{Iso}[\Lambda^2(V), \Lambda^2(V')] : \phi \mapsto \phi_{\Lambda^2}$$

where $\mathrm{Iso}[C_0(V, q), C_0(V', q')]$ is the set of algebra bundle isomorphisms. When $V = V'$, we may thus denote the subset of those ϕ for which $\det(\phi_{\Lambda^2}) \in \mathrm{Aut}[\det(\Lambda^2(V))] \cong \Gamma(X, \mathcal{O}_X^*)$ is a square (respectively $= 1$) by $\mathrm{Iso}'[C_0(V, q), C_0(V, q')]$ (respectively by the smaller subset $\text{S-Iso}[C_0(V, q), C_0(V, q')]$). Taking $q = q'$ in these sets and replacing "Iso" by "Aut" in their notations respectively defines the groups $\mathrm{Aut}(C_0(V, q)) \supset \mathrm{Aut}'(C_0(V, q)) \supset \text{S-Aut}(C_0(V, q))$.

THEOREM 1.3 *For rank 3 quadratic bundles (V,q) and (V,q') with the same underlying bundle V on a scheme X, one has the following commuting diagram of natural maps of sets with the downward arrows being the canonical inclusions, the horizontal arrows being surjective and the top horizontal arrow being bijective:*

$$\begin{array}{ccc} \text{S-Iso}[(V,q),(V,q')] & \xrightarrow{\cong} & \text{S-Iso}[C_0(V,q),C_0(V,q')] \\ \text{inj}\big\downarrow & & \big\downarrow\text{inj} \\ \text{Iso}[(V,q),(V,q')] & \xrightarrow{\text{onto}} & \text{Iso}'[C_0(V,q),C_0(V,q')] \\ \text{inj}\big\downarrow & & \big\downarrow\text{inj} \\ \text{Sim}[(V,q),(V,q')] & \xrightarrow{\text{onto}} & \text{Iso}[C_0(V,q),C_0(V,q')] \end{array}$$

With respect to the surjections of the horizontal arrows in the diagram above, we further have the following (where l is the function that associates to any similitude its multiplier, $\det(g,l) := \det(g)$ for a similitude with multiplier l and ζ_{Λ^2} is the map of Prop.1.2 above):

(a) *there is a family of sections $s_{2k+1} : \text{Iso}[C_0(V,q),C_0(V,q')] \longrightarrow \text{Sim}[(V,q),(V,q')]$ indexed by the integers such that $l \circ s_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2}$ and such that $(\det^2 \circ s_{2k+1}) \times (l^{-3} \circ s_{2k+1}) = \det \circ \zeta_{\Lambda^2}$;*

(b) *there is also a section $s' : \text{Iso}'[C_0(V,q),C_0(V,q')] \longrightarrow \text{Iso}[(V,q),(V,q')]$ such that $\det^2 \circ s' = \det \circ \zeta_{\Lambda^2}$;*

(c) *there is a family of sections $s^+_{2k+1} : \text{Iso}[C_0(V,q),C_0(V,q')] \longrightarrow \text{Sim}[(V,q),(V,q')]$ indexed by the integers which is multiplicative when followed by the natural inclusions into $\text{GL}(V) \times \Gamma(X,\mathcal{O}_X^*)$, i.e., if $\phi_i \in \text{Iso}[C_0(V,q_i),C_0(V,q_{i+1})]$ then $s^+_{2k+1}(\phi_2 \circ \phi_1) = s^+_{2k+1}(\phi_2) \circ s^+_{2k+1}(\phi_1) \in \text{GL}(V) \times \Gamma(X,\mathcal{O}_X^*)$. Further, $l \circ s^+_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2}$ and $(\det^2 \circ s^+_{2k+1}) \times (l^{-3} \circ s^+_{2k+1}) = \det \circ \zeta_{\Lambda^2}$.*

(d) *The maps s_{2k+1} and s' above may not be multiplicative but are mutliplicative upto $\mu_2(\Gamma(X,\mathcal{O}_X))$ i.e., these followed by the quotient map, on taking the quotient of $\text{GL}(V) \times \Gamma(X,\mathcal{O}_X^*)$ by $\mu_2(\Gamma(X,\mathcal{O}_X)) \times \{1\}$, become multiplicative.*

THEOREM 1.4 *For a rank 3 quadratic bundle (V,q) on a scheme X, one has the following natural commutative diagram of groups with exact rows, where the downward arrows are the canonical inclusions and where l is the function that associates to any orthogonal similitude its multiplier:*

$$\begin{array}{ccccccccc} & & & & \text{SO}(V,q) & \xrightarrow{\cong} & \text{S-Aut}(C_0(V,q)) & & \\ & & & & \text{inj}\big\downarrow & & \big\downarrow\text{inj} & & \\ 1 & \longrightarrow & \mu_2(\Gamma(X,\mathcal{O}_X)) & \longrightarrow & \text{O}(V,q) & \longrightarrow & \text{Aut}'(C_0(V,q)) & \longrightarrow & 1 \\ & & \text{inj}\big\downarrow & & \text{inj}\big\downarrow & & \big\downarrow\text{inj} & & \\ 1 & \longrightarrow & \Gamma(X,\mathcal{O}_X^*) & \longrightarrow & \text{GO}(V,q) & \longrightarrow & \text{Aut}(C_0(V,q)) & \longrightarrow & 1 \\ & & & & \det^2 \times l^{-3}\big\downarrow & & \big\downarrow\det & & \\ & & & & \Gamma(X,\mathcal{O}_X^*) & = & \Gamma(X,\mathcal{O}_X^*) & & \end{array}$$

Further, we have

(a) *There are splitting homomorphisms $s^+_{2k+1} : \text{Aut}(C_0(V,q)) \longrightarrow \text{GO}(V,q)$ such that $l \circ s^+_{2k+1} = \det^{2k+1}$ and $(\det^2 \circ s^+_{2k+1}) \times (l^{-3} \circ s^+_{2k+1}) = \det$. The restriction of s^+_{2k+1} to $\text{Aut}'(C_0(V,q))$ does not necessarily take values in $\text{O}(V,q)$, but the further restriction to $\text{S-Aut}(C_0(V,q))$ does take values in $\text{SO}(V,q)$. In particular, $\text{GO}(V,q)$ is a semidirect product. The maps s_{2k+1} and s' of Theorem 1.3 above (under the current hypotheses) may not be homomorphisms but are homomorphisms upto $\mu_2(\Gamma(X,\mathcal{O}_X))$.*

(b) *Suppose X is integral and $q \otimes \kappa(x)$ is semiregular at some point x of X with residue field $\kappa(x)$. Then any automorphism of $C_0(V,q)$ has determinant 1 so that $\text{Aut}(C_0(V,q)) = \text{Aut}'(C_0(V,q)) = \text{S-Aut}(C_0(V,q))$ and in particular, $\text{O}(V,q)$ is the semidirect product of $\mu_2(\Gamma(X,\mathcal{O}_X))$ and $\text{SO}(V,q)$.*

We next turn to the question whether the map $\mathsf{Witt}(X) : \mathscr{Q}_3(X) \longrightarrow \mathscr{A}_4^{Sp-Azu}(X)$ is always surjective. This is the same as asking if, given a specialisation A of rank 4 Azumaya bundles on X, there exists a rank 3 quadratic bundle (V,q) such that $C_0(V,q) \cong A$ (as algebra bundles)—of course Theorem 1.1 will guarantee that (V,q) is unique upto tensoring by a discriminant bundle. It is this question of surjectivity that seems to involve the geometry of X.

THEOREM 1.5 *Let X be a scheme and A a specialisation of rank 4 Azumaya algebra bundles on X. Let $\mathcal{A}$ denote the locally-free $\mathcal{O}_X$-algebra corresponding to A. Let $\mathcal{O}_X.1_{\mathcal{A}} \hookrightarrow \mathcal{A}$ denote the image of the canonical morphism $\mathcal{O}_X \longrightarrow \mathcal{A}$ defined by the nowhere-vanishing global section of $\mathcal{A}$ corresponding to the unit for algebra multiplication.*

(a) *There exists a rank 3 vector bundle V on X (with corresponding $\mathcal{O}_X$-module $\mathcal{V}$) such that the following hold:*

- **(1)** $\det(\mathcal{A}) \otimes \Lambda^2(\mathcal{V}) \cong \mathcal{A}/\mathcal{O}_X.1_{\mathcal{A}}$, *from which the following may be deduced:*
- **(2)** $\det(\Lambda^2(\mathcal{V})) \cong (\det(\mathcal{A}))^{\otimes -2}$;
- **(3)** $\mathcal{V} \cong (\mathcal{A}/\mathcal{O}_X.1_{\mathcal{A}})^{\vee} \otimes \det(\mathcal{V}) \otimes \det(\mathcal{A})$;
- **(4)** $\det(\mathcal{A}^{\vee}) \cong (\det(\mathcal{A}))^{\otimes -3} \otimes (\det(\mathcal{V}))^{\otimes -2}$ *which implies that* $\det(\mathcal{A}) \otimes \det(\mathcal{A}^{\vee}) \in 2.\mathrm{Pic}(X)$.

(b) *If there exists a quadratic bundle (V, q) on X such that $A \cong C_0(V, q)$ then $\mathcal{A}/\mathcal{O}_X.1_{\mathcal{A}} \cong \Lambda^2(V)$ which implies that $\det(\mathcal{A}^{\vee}) \in 2.\mathrm{Pic}(X)$, and therefore using (4) of (a) above, that $\det(\mathcal{A}) \in 2.\mathrm{Pic}(X)$.*

A bilinear form b (with values in $\mathcal{O}_X$) on a vector bundle V on X induces a quadratic form q_b given on sections by $x \mapsto b(x, x)$. Further, b also defines a $(\mathbb{Z}/2.\mathbb{Z})$-graded linear isomorphism $\psi_b : C(V, q_b) \cong \Lambda(V)$ which is unique with respect to certain properties (see (2d), Theorem 2.1). When X is affine, every (global) quadratic form on a vector bundle is induced from a (global) bilinear form. Therefore the following theorem is optimal for affine X.

THEOREM 1.6 *With X and A as in Theorem 1.5 above, there exists a rank 3 quadratic bundle (V, q) (on X) induced from a rank 3 bilinear-form bundle (V, b) such that $C_0(V, q) \cong A$ (isomorphism as algebra bundles) iff the following two conditions hold:*

(1) *the determinant bundle of A is an element of* $2.\mathrm{Pic}(X)$*;*

(2) *$\mathcal{O}_X.1_A$ is an $\mathcal{O}_X$-direct summand of A.*

The hypothesis **(2)** *is satisfied in the following cases:*

(i) *X is an affine scheme;*

(ii) *the residue field of every point of X has characteristic $\neq 2$.*

The hypothesis **(1)** *is satisfied when A is itself an Azumaya algebra bundle of rank 4 over X. It is also satisfied whenever* $\det(\mathcal{A}) \cong \mathcal{O}_X$ *(for example, when A is (globally) free) and whenever* $\mathrm{Pic}(X) = 2.\mathrm{Pic}(X)$*—a condition which is trivially satisfied when* $\mathrm{Pic}(X) = 0$*, for example when* $X = \mathrm{Spec}(R)$ *and R is a UFD or a local ring or a field. It follows therefore that if* $X = \mathrm{Spec}(R)$ *is an affine scheme such that* $\mathrm{Pic}(X) = 2.\mathrm{Pic}(X)$*, then the map* $\mathsf{Witt}(X) : \mathscr{Q}_3(X) \longrightarrow \mathscr{A}_4^{Sp-Azu}(X)$ *is surjective, and hence a bijection.*

Before proceeding further, we need to recall a few concepts from Part A of [3]. For a rank n^2 vector bundle W on a scheme X and $w \in \Gamma(X, W)$ a nowhere-vanishing global section, recall that if Id-w-Azu$_W$ is the open X–subscheme of Azumaya algebra structures on W with identity w then its schematic image (or the scheme of specialisations or the limiting scheme) in the bigger X–scheme Id-w-Assoc$_W$ of associative w-unital algebra structures on W is the X–scheme Id-w-Sp-Azu$_W$. By definition, the set of distinct specialised w-unital algebra structures on W corresponds precisely to the set of global sections of this last scheme over X.

THEOREM 1.7 *Let X be a connected proper scheme of finite type over an algebraically closed field and let W be a vector bundle on X of rank n^2 for some $n \geq 2$.*

(a) *If W is self-dual ($\Leftrightarrow W \cong W^{\vee}$) and A is an associative unital algebra structure on W such that $\mathcal{A} \otimes_X \mathcal{O}_{X,x}$ is Azumaya even for a single point $x \in X$, then A is Azumaya at every point of X. This may also be stated as follows: if a section to* Id-w-Assoc$_W$ *over X topologically meets* Id-w-Azu$_W$*, then it factors as a morphism through the open subscheme* Id-w-Azu$_W$*, where $w := 1_A$ and A corresponds to the given section.*

(b) *Let the rank of W be 4. If there exists an associative unital algebra structure on W which is not globally Azumaya but is Azumaya atleast at one point of X, then there does not exist any global Azumaya algebra structure on W. Thus, if there is a section over X of* Id-w-Assoc$_W$ *that topologically meets both the open subscheme* Id-w-Azu$_W$ *and its complement (with $w = 1_A$ where A corresponds to the given section), then the X-schemes* Id-w'-Assoc$_W$ *(with w' global nowhere-vanishing) cannot have sections that land topologically inside* Id-w'-Azu$_W$ *and hence in particular the X-schemes* Id-w'-Azu$_W$ *have no sections over X.*

Recall that an integral separated Noetherian scheme is said to be locally-factorial if each of its local rings is a unique factorisation domain (=UFD=factorial ring).

THEOREM 1.8 *Let X be a scheme and W a rank 4 vector bundle on X with a global nowhere-vanishing section w. Let D_X denote the closed subset* $\text{Id-}w\text{-Sp-Azu}_W \setminus \text{Id-}w\text{-Azu}_W$.

(a) *X is irreducible iff $\text{Id-}w\text{-Sp-Azu}_W$ is irreducible iff $\text{Id-}w\text{-Azu}_W$ is irreducible iff D_X is irreducible. The set of irreducible components of X is locally finite—for example this happens when X is locally noetherian—iff the same is true of the corresponding set for $\text{Id-}w\text{-Sp-Azu}_W$ or for $\text{Id-}w\text{-Azu}_W$. If X is noetherian and finite-dimensional then the same are true for $\text{Id-}w\text{-Sp-Azu}_W$ and $\text{Id-}w\text{-Azu}_W$.*

(b) *If $X' \longrightarrow X$ is a morphism of schemes, and if (W', w') denotes the pullback of (W, w), then we have a canonical isomorphism*

$$\text{Id-}w'\text{-Azu}_{W'} \cong \text{Id-}w\text{-Azu}_W \times_{\text{Id-}w\text{-Sp-Azu}_W} \text{Id-}w'\text{-Sp-Azu}_{W'}$$

In particular, the topological image of $D_{X'}$ is D_X. Moreover, when $X' \longrightarrow X$ is a homeomorphism onto its topological image—which is for example the case when it is a closed or an open immersion, then $D_X \cap \text{Id-}w'\text{-Sp-Azu}_{W'}$ can be identified with $D_{X'}$.

(c) *If X is a scheme which is finite dimensional and whose set of irreducible components is locally finite, then the closed subset D_X is a divisor i.e., it has codimension 1 in $\text{Id-}w\text{-Sp-Azu}_W$.*

(d) *X is affine iff $\text{Id-}w\text{-Sp-Azu}_W$ is affine iff $\text{Id-}w\text{-Azu}_W$ is affine. If X is regular in codimension 1 (respectively locally-factorial) then so are $\text{Id-}w\text{-Sp-Azu}_W$ and $\text{Id-}w\text{-Azu}_W$.*

(e) *Assume that X is locally-factorial and W is self-dual (i.e., $W \cong W^\vee$). Then the (Weil) divisor $n.(D_X)$ is principal for some positive integer n, so that the natural homomorphism given by restriction of line bundles $\text{Pic}(\text{Id-}w\text{-Sp-Azu}_W) \longrightarrow \text{Pic}(\text{Id-}w\text{-Azu}_W)$ is an isomorphism iff $n = 1$. The integer n is divisible by 2 iff $\det(W \otimes_X \text{Id-}w\text{-Sp-Azu}_W) \in 2.\text{Pic}(\text{Id-}w\text{-Sp-Azu}_W)$ so that in this case if there exists a specialised algebra structure on W with unit w, then $\det(W) \in 2.\text{Pic}(X)$; if in addition, $\mathcal{O}_X.w$ is an $\mathcal{O}_X$-direct summand of W, then W is of the form $\Lambda^{even}(V)$ for a rank 3 vector bundle V on X, and for every X-scheme T and for every specialisation A_T on $W_T := (W \otimes_X T)$ with unit $w_T := (w \otimes_X T)$, there exists a rank 3 quadratic bundle (V_T, q_T) induced from a rank 3 bilinear form bundle (V_T, b_T) on T such that $C_0(V_T, q_T) \cong A_T$ as algebra bundles. (this last conclusion holds for any affine scheme T over X, even if $\mathcal{O}_X.w$ were not an $\mathcal{O}_X$-direct summand of W).*

Notice that under the hypotheses (1) and (2) of Theorem 1.6, there exists by (1), Theorem 1.5, a rank 3 bundle V on X such that $\Lambda^{even}(\mathcal{V}) = \mathcal{O}_X \oplus \Lambda^2(\mathcal{V}) \cong \mathcal{A}$ with $\mathcal{O}_X \cong \mathcal{O}_X.1_{\mathcal{A}}$. In this situation the 'if' part of Theorem 1.6 as well as the latter part of (e) of Theorem 1.8 are consequences of the next one which describes the specialisations as bilinear forms. We continue with the notations introduced before Theorem 1.7. If $\text{Stab}_w \subset \text{GL}_W$ is the stabiliser subgroupscheme of w, recall from Theorems 3.4 and 3.8, Part A, [3], that there exists a canonical action of Stab_w on $\text{Id-}w\text{-Sp-Azu}_W$ such that the natural inclusions

$$(\clubsuit) \qquad \text{Id-}w\text{-Azu}_W \hookrightarrow \text{Id-}w\text{-Sp-Azu}_W \hookrightarrow \text{Id-}w\text{-Assoc-Alg}_W$$

are all Stab_w-equivariant. Now let V be a rank 3 vector bundle on the scheme X and Bil_V be the associated rank 9 vector bundle of bilinear forms on V with values in $\mathcal{O}_X$. Let $\text{Bil}_V^{sr} \hookrightarrow \text{Bil}_V$ correspond to the open subscheme of semiregular bilinear forms—we say that a bilinear form is semiregular if the quadratic form it induces is semiregular (though it may turn out that a semiregular bilinear form may be degenerate). Let $W := \Lambda^{even}(V)$ and let $w \in \Gamma(X, W)$ be the nowhere-vanishing global section corresponding to the unit for the natural multiplication in the even-exterior algebra bundle. There is an obvious natural action of GL_V on Bil_V. There is also a natural morphism of groupschemes $\text{GL}_V \longrightarrow \text{Stab}_w$ given on valued points by $g \mapsto \Lambda^{even}(g)$ and therefore the natural inclusions marked by $(\clubsuit)$ above are GL_V-equivariant. Finally, note that there is an obvious involution Σ on $\text{Id-}w\text{-Assoc-Alg}_W$ given by $A \mapsto \text{opposite}(A)$ which leaves the open subscheme $\text{Id-}w\text{-Azu}_W$ invariant.

THEOREM 1.9

(1) *Let V be a rank 3 vector bundle on the scheme X, $W := \Lambda^{even}(V)$ and $w \in \Gamma(X, W)$ correspond to 1 in the even-exterior algebra bundle. There is a natural GL_V-equivariant morphism of X–schemes $\Upsilon' = \Upsilon'_X : \text{Bil}_V \longrightarrow \text{Id-}w\text{-Assoc}_W$ whose schematic image is precisely the scheme of specialisations $\text{Id-}w\text{-Sp-Azu}_W$. Further if Υ' factors canonically through $\Upsilon = \Upsilon_X : \text{Bil}_V \longrightarrow \text{Id-}w\text{-Sp-Azu}_W$, then Υ is a GL_V-equivariant isomorphism and it maps the GL_V-stable open subscheme Bil_V^{sr} isomorphically onto the GL_V-stable open subscheme $\text{Id-}w\text{-Azu}_W$.*

(2) *The involution* Σ *of* Id-w-Assoc-Alg$_W$ *defines a unique involution (also denoted by* Σ*) on the scheme of specialisations* Id-w-Sp-Azu$_W$ *leaving the open subscheme* Id-w-Azu$_W$ *invariant, and therefore via the isomorphism* Υ*, it defines an involution on* Bil_V*. This involution is none other than the one on valued points given by* $B \mapsto \mathrm{transpose}(-B)$.

(3) *For an* X*-scheme* T*, let* V_T *(resp.* W_T*) denote the pullback of* V *(resp.* W*) to* T*, and let* w_T *be the global section of* W_T *induced by* w*. Then the base-changes of* Υ'_X *and* Υ_X *to* T*, namely* $\Upsilon'_X \times_X T : \mathrm{Bil}_V \times_X T \longrightarrow$ Id-w-Assoc$_W \times_X T$ *and* $\Upsilon_X \times_X T : \mathrm{Bil}_V \times_X T \cong$ Id-w-Sp-Azu$_W \times_X T$ *may be canonically identified with the corresponding ones over* T *namely* $\Upsilon'_T : \mathrm{Bil}_{V_T} \longrightarrow$ Id-w_T-Assoc$_{W_T}$ *and* $\Upsilon_T : \mathrm{Bil}_{V_T} \cong$ Id-w_T-Sp-Azu$_{W_T}$.

The explicit computation of the morphism Υ locally over X is an important step in proving the above theorem. To describe this, suppose that V is free of rank 3 over X, so that we may fix a basis $\{e_1, e_2, e_3\}$ for V, which naturally gives rise to a basis of Bil_V. For any X-scheme T, a T-valued point B of Bil_V is just a global bilinear form with values in $\mathcal{O}_T$ on the pull-back $V \otimes_X T$ of V to T. Such a B is given uniquely by a (3×3)-matrix (b_{ij}) with the b_{ij} being global sections of the trivial line bundle $\mathbb{A}^1_T$ (or equivalently, elements of $\Gamma(T, \mathcal{O}_T)$). The chosen basis for V also gives rise to the basis $\{\epsilon_0 := w = 1\ ;\ \epsilon_1 := e_1 \wedge e_2\ ,\ \epsilon_2 := e_2 \wedge e_3\ ,\ \epsilon_3 := e_3 \wedge e_1\}$ of $W = \Lambda^{even}(V)$. A T-valued point A of Id-w-Assoc-Alg$_W$ is just a $w_T := (w \otimes_X T)$-unital associative algebra structure on the bundle $W_T := W \otimes_X T$. Let $\cdot_A$ denote the multiplication in the algebra bundle A, and for ease of notation, let s° denote the section $s \otimes_X T$ induced from a section s (for example, $(w \otimes_X T) = w^\circ, \epsilon_i \otimes_X T = \epsilon_i^\circ$ etc).

THEOREM 1.10 *In addition to the hypothesis of Theorem 1.9, assume that* V *is free of rank 3. Then fixing a basis for* V *and adopting the notations above, the map* $\Upsilon(T)$ *takes* $B = (b_{ij})$ *to* $(A, 1_A, \cdot) = (W_T, w_T = w^\circ, \cdot_A)$ *with multiplication given as follows, where* $M_{ij}(B)$ *is the determinant of the minor of the element* b_{ij} *in* B :

- $\epsilon_1^\circ \cdot_A \epsilon_1^\circ = -M_{33}(B)w^\circ + (b_{21} - b_{12})\epsilon_1^\circ$
- $\epsilon_2^\circ \cdot_A \epsilon_2^\circ = -M_{11}(B)w^\circ + (b_{32} - b_{23})\epsilon_2^\circ$
- $\epsilon_3^\circ \cdot_A \epsilon_3^\circ = -M_{22}(B)w^\circ + (b_{13} - b_{31})\epsilon_3^\circ$
- $\epsilon_1^\circ \cdot_A \epsilon_2^\circ = -M_{31}(B)w^\circ - b_{23}\epsilon_1^\circ - b_{12}\epsilon_2^\circ - b_{22}\epsilon_3^\circ$
- $\epsilon_2^\circ \cdot_A \epsilon_3^\circ = +M_{12}(B)w^\circ - b_{33}\epsilon_1^\circ - b_{31}\epsilon_2^\circ - b_{23}\epsilon_3^\circ$
- $\epsilon_3^\circ \cdot_A \epsilon_1^\circ = +M_{23}(B)w^\circ - b_{31}\epsilon_1^\circ - b_{11}\epsilon_2^\circ - b_{12}\epsilon_3^\circ$
- $\epsilon_1^\circ \cdot_A \epsilon_3^\circ = +M_{32}(B)w^\circ + b_{13}\epsilon_1^\circ + b_{11}\epsilon_2^\circ + b_{21}\epsilon_3^\circ$
- $\epsilon_2^\circ \cdot_A \epsilon_1^\circ = -M_{13}(B)w^\circ + b_{32}\epsilon_1^\circ + b_{21}\epsilon_2^\circ + b_{22}\epsilon_3^\circ$
- $\epsilon_3^\circ \cdot_A \epsilon_2^\circ = -M_{21}(B)w^\circ + b_{33}\epsilon_1^\circ + b_{13}\epsilon_2^\circ + b_{32}\epsilon_3^\circ$

The key to the proofs of Theorems 1.1, 1.3 and 1.4 lies in a deeper study of a different identification of the scheme of specialisations, namely one related to the scheme of quadratic forms on a trivial rank 3 bundle in the special situation when W is free and w part of a global basis. Without loss of generality we may in this situation therefore take V to be a free rank 3 vector bundle on X and $(W, w) = (\Lambda^{even}(V), 1)$, so that we are in the situation of Theorem 1.10 above. This relationship with quadratic forms was shown in Theorem 5.3, Part A, [3], which we briefly recall next. Let Quad_V denote the bundle of quadratic forms on V (with values in $\mathcal{O}_X$) and Quad_V^{sr} the open subscheme of semiregular quadratic forms. Let A_0 denote the algebra bundle structure (with unit $w = 1$) on $W = \Lambda^{even}(V)$ given by $\Lambda^{even}(V)$ itself. Fix a basis for V and adopt the notations preceding Theorem 1.10 above. Then Stab_w is the semidirect product of a commutative 3-dimensional subgroupscheme $\mathsf{L}_w \cong (\mathbb{A}^3_X, +)$ with the stabiliser subgroupscheme Stab_{A_0} of A_0 in Stab_w (Lemma 5.1, Part A, [3]).

THEOREM 1.11 (Definition 5.2 & Theorem 5.3, Part A, [3]) *There is a natural isomorphism* $\Theta : \mathrm{Quad}_V \times_X \mathsf{L}_w \cong$ Id-w-Sp-Azu$_W$ *which maps the open subscheme* $\mathrm{Quad}_V^{sr} \times_X \mathsf{L}_w$ *isomorphically onto the open subscheme* Id-w-Azu$_W$.

Section 3 is essentially devoted to studying Θ. There we compute Θ explicitly and in Theorem 3.1 we write out the multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit that is part of a global basis. It turns out that Θ is not equivariant with respect to GL_V, but nevertheless satisfies a 'twisted' form of equivariance (Theorem 3.4). A T-valued point q of $\mathrm{Quad}_V \cong \mathbb{A}^6_X$ may be identified uniquely with a 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23})$ corresponding to the quadratic form $(x_1, x_2, x_3) \mapsto \Sigma_i \lambda_i x_i^2 + \Sigma_{i<j} \lambda_{ij} x_i x_j$. A T-valued point $\underline{t}$ of $\mathsf{L}_w \cong (\mathbb{A}^3_X, +)$ may be

identified uniquely with a 3-tuple (t_1, t_2, t_3) which corresponds to the valued point of Stab_w given by the (4×4)-matrix

$$\left(\begin{array}{c|ccc} 1 & t_1 & t_2 & t_3 \\ \hline 0 & & I_3 & \end{array} \right)$$

where I_3 is the (3×3)-identity matrix. With these notations, the identification of Theorems 1.9 and 1.10 may be compared with that of the above Theorem 1.11 as follows.

THEOREM 1.12 *The isomorphism* $\Upsilon^{-1} \circ \Theta : \mathrm{Quad}_V \times_X \mathsf{L}_w \cong \mathrm{Bil}_V$ *takes the valued point* $(\ q,\ \underline{t}\ ) = (\ (\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23}),\ \ (t_1, t_2, t_3)\ \ )$ *to the valued point* $B = (b_{ij})$ *given by*

$$B = \begin{pmatrix} \lambda_1 & t_1 & \lambda_{13} - t_3 \\ \lambda_{12} - t_1 & \lambda_2 & t_2 \\ t_3 & \lambda_{23} - t_2 & \lambda_3 \end{pmatrix}$$

Moreover, under this identification, the involution $B \mapsto (-B)^t$ *on* Bil_V *(induced from the isomorphism* Υ *of Theorem 1.9) translates into the involution on* $\mathrm{Quad}_V \times_X \mathsf{L}_w$ *given by*

$$(\ q,\ \ (t_1, t_2, t_3)\ \ ) \mapsto (\ -q,\ \ (t_1 - \lambda_{12}, t_2 - \lambda_{23}, t_3 - \lambda_{13})\ \ ).$$

We next make some comments relevant to the question of the surjectivity of $\mathsf{Witt}(X) : \mathscr{Q}_3(X) \longrightarrow \mathscr{A}_4^{Sp-Azu}(X)$. Given a specialised algebra structure A (on a rank 4 vector bundle on X) which is itself an Azumaya algebra, we first indicate briefly how to naturally retrieve a rank 3 quadratic bundle (V, q) such that $C_0(V, q) \cong A$. Let R be a unital commutative ring and A a unital associative R-algebra. Given any involution σ on A, we may define the associated trace $tr_\sigma : x \mapsto x + \sigma(x)$ and the associated norm $n_\sigma : x \mapsto x.\sigma(x)$; in para.1.3, Chap.I, [2], Knus calls σ *standard* if σ fixes $R.1_A$ and both tr_σ and n_σ take values in $R.1_A$. In Prop.1.3.4 of the same chapter, he proves that a standard involution is unique if it exists, provided the R-module underlying A is finitely generated projective and faithful. Thereafter, in para.1.3.7, Knus defines A to be a *quaternion* algebra if A is a projective R-module of rank 4 and A has a standard involution. Thus we may define a rank 4 algebra bundle on a scheme X to be a quaternion algebra bundle if it is locally (in the Zariski topology) a quaternion algebra in Knus' sense, and it would follow that the local standard involutions glue to define a unique global standard involution on the bundle.

PROPOSITION 1.13 *Any specialised algebra bundle is a quaternion algebra bundle.*

This result can be deduced from the following two facts:

(1) Any specialised algebra is locally (in the Zariski topology) the even Clifford algebra of a rank 3 quadratic bundle (Theorems 3.8 and 5.3, Part A, [3]).

(2) The even Clifford algebra of a quadratic module of rank 3 over a commutative ring has a standard involution which is none other than the restriction of the 'standard' involution on the full Clifford algebra (Prop.3.1.1, Chap.V, [2]). **Q.E.D.**

THEOREM 1.14 *For each scheme* X*, the map* $\mathsf{Witt}(X) : \mathscr{Q}_3^{sr}(X) \longrightarrow \mathscr{A}_4^{Azu}(X)$ *is bijective.*

The proof of the above Theorem follows from Prop.3.2.3 and Prop.3.2.4, Chap.V, [2] generalised to the scheme-theoretic setting. We recall how the surjectivity is established. Let A be a specialised algebra bundle on the scheme X. By the results just quoted if A is Azumaya, or more generally by Prop.1.13, we have the existence of a unique standard involution σ_A on A, to which are associated the norm $n_{\sigma_A} : A \longrightarrow \mathbb{A}^1_X$ given on sections by $x \mapsto x.\sigma_A(x)$ and the trace $tr_{\sigma_A} : A \longrightarrow \mathbb{A}^1_X$ given on sections by $x \mapsto x + \sigma_A(x)$. Let $A' := \mathrm{kernel}(tr_{\sigma_A}) \hookrightarrow A$ be the subsheaf of trace zero elements. As the calculations in para.3.2, Chap.V, [2] show, the trace map is surjective if A is itself an Azumaya algebra; if this is the case, then it is further shown there that the rank 3 quadratic bundle $(V, q) := (A', n_{\sigma_A}|A')$ is semiregular and its even Clifford algebra $C_0(V, q) \cong A$.

However the above method of retrieving a canonical rank 3 quadratic bundle fails badly for specialised non-Azumaya algebras. Consider even the case of $X = \mathrm{Spec}(k)$ where k is a field of characteristic two and the Clifford algebra $A = C_0(V, q)$ of a quadratic form q on $V = k^{\oplus 3}$ which is a perfect square (i.e., a square of a linear form or equivalently a sum of squares). In this case an easy computation shows that the subspace A' of trace zero elements is the full space A. However, the underlying module of A trivially satisfies the hypotheses of Theorem 1.6, and so the existence of (V, q) follows from that Theorem.

The good algebraic properties of Azumaya algebras are reflected as good geometric properties of the scheme of Azumaya algebra structures on a fixed vector bundle: this scheme is separated, of finite type and smooth relative to the base scheme (over which the vector bundle is fixed) and also base-changes well relative to the base scheme (Theorem 3.4, Part A, [3]). When the vector bundle is of rank 4, the nice thing that happens is that all these good properties also pass over to the limit i.e., to the scheme of specialisations, defined to be the schematic image of the scheme of Azumaya algebra structures (Theorem 3.8, Part A, [3]). If A is Azumaya over X, then as seen above, $A \cong C_0(V,q)$ with (V,q) semiregular. So by (b), Theorem 1.5, we have that $\det(A) \in 2.\mathrm{Pic}(X)$. On the other hand, by the same Theorem, this is a necessary condition for a specialisation to arise as $C_0(V,q)$ for a global q. Therefore the author expects this condition to hold in general (the assertions in (e), Theorem 1.8, arose after an attempt to investigate this viewpoint). Of course, if this holds and also if $\mathcal{O}_X.1_A$ is an $\mathcal{O}_X$-direct summand of A, then by Theorem 1.6 we get something more, i.e., even a bilinear form bundle (V,b) such that $C_0(V,q_b) \cong A$. But the latter hypothesis is seemingly strong for non-affine X. The author hopes that the surjectivity of $\mathsf{Witt}(X)$ is valid for general X, and believes that this may follow from a deeper understanding of the geometry of the scheme of specialisations. We next state some results on rank 3 quadratic forms and specialised algebras in certain particular cases.

PROPOSITION 1.15 *Let S be a commutative semilocal ring that is 2-perfect i.e., such that the square map $S \longrightarrow S : s \mapsto s^2$ is surjective, and V a free rank 3 S-module. Then the set of semiregular quadratic S-forms on V forms a single $\mathrm{GL}(V)$–orbit; in other words, upto isometry, $\exists$ only one semiregular quadratic S-module structure on V.*

Corollary 1.16 *Let S be a commutative local ring that is 2-perfect. Then any two rank 4 Azumaya S-algebras are isomorphic. If S were only semilocal, the conclusion still holds provided the identity elements for multiplication for each of the two Azumaya S-algebras can be completed to an S-basis.*

Since $\mathsf{Witt}(X) : \mathscr{Q}_3^{sr}(X) \cong \mathscr{A}_4^{Azu}(X)$ is bijective, taking $X = \mathrm{Spec}(S)$ with S as in the Prop.1.15 proves the first assertion of the above corollary. The second may be deduced by an application of Theorem 1.11 alongwith Prop.1.15.

Let W be a rank 4 vector bundle on a scheme X, $w \in \Gamma(X,W)$ a nowhere-vanishing global section and $\mathrm{Stab}_w \subset \mathrm{GL}_W$ the stabiliser subgroupscheme of w. Recall that the natural inclusion Id-w-Azu$_W \hookrightarrow$ Id-w-Sp-Azu$_W$ is Stab_w-equivariant (see (♣), page 6). When $X = \mathrm{Spec}(k)$ where k is an algebraically closed field, there is a canonical Stab_w-stratification of the k-variety underlying Id-w-Sp-Azu$_W$ as follows.

THEOREM 1.17

(1) *Let k be a quadratically closed field and $X = \mathrm{Spec}(k)$. Then $\mathscr{Q}_3(X)$ has 4 elements which correspond to (a) semiregular quadratic modules; (b) rank 2 quadratic modules i.e., those that are not semiregular but which are regular on a two-dimensional subspace; (c) nonzero perfect squares and (d) the zero form. If V is a 3-dimensional vector space over k and $\{e_1,e_2,e_3\}$ a k-basis for V, then representatives for these 4 GL_V-orbits in the space Quad_V of quadratic forms on V can respectively be taken to be: (a) $q^{(1)}(\Sigma_{i=1}^3 x_ie_i) = x_1x_2 + x_3^2$; (b) $q^{(2)}(\Sigma_{i=1}^3 x_ie_i) = x_1x_2$; (c) $q^{(3)}(\Sigma_{i=1}^3 x_ie_i) = x_3^2$; (d) $q^{(4)} = 0$.*

(2) *In addition to the hypotheses and notations of (1) above, assume that k is an algebraically closed field. Then the four orbits $\mathrm{Quad}_V^{(i)} := \mathrm{GL}_V \cdot q^{(i)}$ for $1 \le i \le 4$ form a stratification of the k-variety Quad_V in the sense that we have*

$$\overline{\mathrm{Quad}_V^{(1)}} = \mathrm{Quad}_V \quad \textit{and} \quad \overline{\mathrm{Quad}_V^{(i+1)}} = \overline{\mathrm{Quad}_V^{(i)}} \ \setminus \ \mathrm{Quad}_V^{(i)} \quad \textit{for} \quad 1 \le i \le 3$$

and further we also have

$$\mathrm{Sing}(\overline{\mathrm{Quad}_V^{(i+1)}}) = \overline{\mathrm{Quad}_V^{(i+1)}} \ \setminus \ \mathrm{Quad}_V^{(i+1)} \quad \textit{for} \quad 1 \le i \le 2$$

unless the characteristic of k is 2 in which case $\overline{\mathrm{Quad}_V^{(3)}}$ is itself smooth (the notation $\overline{T}$ denotes the orbit closure and $\mathrm{Sing}(T)$ denotes the subset of singular (non-smooth) points of T, each given the canonical reduced induced closed subscheme structure).

(3) *Continuing with the notations and hypotheses of (2) above, set $(W,w) := (\Lambda^{even}(V), 1)$. For ease of notation denote Id-w-Sp-Azu$_W$ by SpAzu and Stab_w by H. Then the four orbits $\mathrm{SpAzu}^{(i)} := H \cdot \Theta(q^{(i)}, I_4)$ for $1 \le i \le 4$ form a stratification of the k-variety SpAzu in the sense that we have*

$$\overline{\mathrm{SpAzu}^{(1)}} = \mathrm{SpAzu} \quad \textit{and} \quad \overline{\mathrm{SpAzu}^{(i+1)}} = \overline{\mathrm{SpAzu}^{(i)}} \ \setminus \ \mathrm{SpAzu}^{(i)} \quad \textit{for} \quad 1 \le i \le 3$$

and further we also have

$$\mathrm{Sing}(\overline{\mathrm{SpAzu}^{(i+1)}}) = \overline{\mathrm{SpAzu}^{(i+1)}} \ \setminus \ \mathrm{SpAzu}^{(i+1)} \quad \textit{for} \quad 1 \le i \le 2$$

unless the characteristic of k is 2 in which case $\overline{\mathrm{SpAzu}^{(3)}}$ is itself smooth.

2 Reduction of Theorems 1.1 & 1.3 to the Free Case

In this section, we prove Prop.1.2 and reduce the proof of Theorem 1.1 to Theorem 1.3. Thereafter we reduce the proof of Theorem 1.3 to the case when V is free. Our means towards this end are several standard results which we recall below. Bourbaki's tensor operations are described first, followed by basic facts on tensoring quadratic bundles with discriminant bundles, and thereafter by some facts about similitudes between quadratic bundles. On the way we justify the definitions of the functors $\mathscr{Q}_3^{sr} \hookrightarrow \mathscr{Q}_3$, and the natural transformation Witt introduced in §1 above.

THEOREM 2.1 (Bourbaki's Tensor Operations, §9, Chap.9, [4]; para.1.7, Chap.IV, [2]) *In the following, let R be a commutative ring (with 1) and V an R-module.*

(1) *Let $q : V \longrightarrow R$ be a quadratic form on V and $f \in V^\vee := \mathrm{Hom}_R(V, R)$ be a functional on V. Then there exists an R-linear endomorphism t_f of the tensor algebra TV which is unique with respect to the first three of the following properties it satisfies:*

- **(a)** $t_f(1) = 0$;
- **(b)** $t_f(x \otimes y) = f(x).y - x \otimes t_f(y)$ *for every $x \in V$ and every $y \in TV$;*
- **(c)** *If $J(q)$ is the two-sided ideal of TV generated by the set $\{x \otimes x - q(x).1 \mid x \in V\}$, then $t_f(J(q)) \subset J(q)$;*
- **(d)** *t_f is homogeneous of degree -1 (for elements it does not annihilate);*
- **(e)** *(Recall that the Clifford algebra of q is $C(V, q) := TV/J(q)$). By (c) above, t_f induces a $(\mathbb{Z}/2.\mathbb{Z})$-antigraded endomorphism $d_f^q : C(V, q) \longrightarrow C(V, q)$;*
- **(f)** $t_f \circ t_f = 0$;
- **(g)** *if $g \in V^\vee$ is also a functional, then $t_f \circ t_g + t_g \circ t_f = 0$;*
- **(h)** *if $\alpha \in \mathrm{End}_R(V)$, then $t_f \circ T(\alpha) = T(\alpha) \circ t_{\alpha^* f}$ where $\alpha^* f \in \mathrm{End}_R(V)$ is defined by $x \mapsto f(\alpha(x))$;*
- **(i)** *$t_f \equiv 0$ on the subalgebra of TV generated by* $\mathrm{kernel}(f)$.

(2) *Let $q, q' : V \longrightarrow R$ be two quadratic forms whose difference is the quadratic form q_b induced by a bilinear form $b \in \mathrm{Bil}_R(V) := \mathrm{Hom}_R(V \otimes_R V, R)$ i.e., $q'(x) - q(x) = q_b(x) := b(x, x)\, \forall x \in V$. Further, for any $x \in V$ denote by b_x the functional on V given by $y \mapsto b(x, y)$. Then there exists an R-linear automorphism Ψ_b of TV which is unique with respect to the first three of the following properties it satisfies:*

- **(a)** $\Psi_b(1) = 1$;
- **(b)** $\Psi_b(x \otimes y) = x \otimes \Psi_b(y) + t_{b_x}(\Psi_b(y))$ *for any $x \in V$ and any $y \in TV$;*
- **(c)** $\Psi_b(J(q')) \subset J(q)$;
- **(d)** *by the previous property, Ψ_b induces an isomorphism of $(\mathbb{Z}/2.\mathbb{Z})$-graded R-modules*

$$\psi_b : C(V, q') \cong C(V, q);$$

  in particular, given a quadratic form $q_1 : V \longrightarrow R$, we may take a bilinear form b_1 that induces q_1 (i.e., such that $q_1(x) = q_{b_1}(x) := b_1(x, x)\, \forall x \in V$), and get a $(\mathbb{Z}/2.\mathbb{Z})$-graded linear isomorphism $\psi_{b_1} : C(V, q_1 = q_{b_1}) \cong C(V, 0) = \Lambda(V)$;
- **(e)** $\Psi_b(T^{2n}V) \subset \oplus_{(i \le n)} T^{2i}V$ *and* $\Psi_b(T^{2n+1}V) \subset \oplus_{(\text{odd } i \le 2n+1)} T^i V$;
- **(f)** *in particular, for $x, x' \in V$, $\Psi_b(x \otimes x') = x \otimes x' + b(x, x').1_{TV}$ so that for $\psi_b : C_0(V, q_b) \cong C_0(V, 0) = \Lambda^{even}(V)$ we have $\psi_b(x.x') = x \wedge x' + b(x, x').1$ where $x.x'$ denotes the product in $C(V, q_b)$.*
- **(g)** *if $f \in V^\vee$ and t_f is given by (1) above, then $\Psi_b \circ t_f = t_f \circ \Psi_b$;*
- **(h)** *if b_i are bilinear forms on V, then $\Psi_{b_1 + b_2} = \Psi_{b_1} \circ \Psi_{b_2}$ and $\Psi_0 =$ Identity on TV;*
- **(i)** *for any $\alpha \in \mathrm{End}_R(V)$, $\Psi_b \circ T(\alpha) = T(\alpha) \circ \Psi_{(b.\alpha)}$ where $(b.\alpha)(x, x') := b(\alpha(x), \alpha(x'))\, \forall x, x' \in V$;*
- **(j)** *by property (h), one has a homomorphism of groups $(\mathrm{Bil}_R(V), +) \longrightarrow (\mathrm{Aut}_R(TV), \circ) : b \mapsto \Psi_b$; the associative unital monoid $(\mathrm{End}_R(V), \circ)$ acts on $\mathrm{Bil}_R(V)$ on the right by $b' \rightsquigarrow b'.\alpha$ and acts on the left (resp. on the right) of $\mathrm{End}_R(TV)$ by $\alpha.\Phi := T(\alpha) \circ \Phi$ (resp. by $\Phi.\alpha := \Phi \circ T(\alpha)$), and the homomorphism $b \mapsto \Psi_b$ satisfies $\alpha.\Psi_{(b.\alpha)} = \Psi_b.\alpha$; the group $\mathrm{Aut}_R(V) = \mathrm{GL}_R(V)$ acts on the left of $\mathrm{Bil}_R(V)$ by $g.b : (x, x') \mapsto b(g^{-1}(x), g^{-1}(x'))$ and on the left of $\mathrm{Aut}_R(TV)$ by conjugation via the natural group homomorphism $\mathrm{GL}_R(V) \longrightarrow \mathrm{Aut}_R(TV) : g \mapsto T(g)$ i.e., $g.\Phi := T(g) \circ \Phi \circ T(g^{-1})$, and the homomorphism $b \mapsto \Psi_b$ is $\mathrm{GL}_R(V)$-equivariant: $\Psi_{g.b} = g.\Psi_b$.*

(3) *For a commutative R-algebra S (with 1), let $(q \otimes_R S), (q' \otimes_R S) : (V \otimes_R S =: V_S) \longrightarrow S$ be the quadratic S-forms induced from the quadratic forms q, q' of (2) above and $(b \otimes_R S) \in \mathrm{Bil}_S(V_S)$ the bilinear S-form induced from the bilinear form b of (2) above. Then as a result of the uniqueness properties (2a)–(2c) satisfied by Ψ_b and $\Psi_{(b\otimes_R S)}$, the S-linear automorphisms $(\Psi_b \otimes_R S)$ and $\Psi_{(b\otimes_R S)}$ may be canonically identified. In particular, the $(\mathbb{Z}/2.\mathbb{Z})$-graded S-linear isomorphism $(\psi_b \otimes_R S) : C(V_S, (q' \otimes_R S)) \cong C(V_S, (q \otimes_R S))$ induced from ψ_b of (2d) above may be canonically identified with $\psi_{(b\otimes_R S)}$.*

Tensoring by Discriminant Bundles. We recall that a bilinear form (resp. alternating form, resp. quadratic form) with values in $\mathcal{O}_X$ on a vector bundle V over an open set $U \hookrightarrow X$ of the scheme X is by definition a section over U of the vector bundle Bil_V (resp. of Alt^2_V, resp. of Quad_V), or equivalently, an element of $\Gamma\left(U, \mathrm{Bil}_{\mathcal{V}} := (T^2_{\mathcal{O}_X}(\mathcal{V}))^\vee\right)$ (resp. of $\Gamma\left(U, \mathrm{Alt}^2_{\mathcal{V}} := (\Lambda^2_{\mathcal{O}_X}(\mathcal{V}))^\vee\right)$, resp. of $\Gamma\left(U, \mathrm{Quad}_{\mathcal{V}}\right)$), where the sheaf $\mathrm{Quad}_{\mathcal{V}}$—the (coherent locally-free) sheaf of $\mathcal{O}_X$-modules corresponding to the bundle Quad_V of quadratic forms on V—is defined by the exactness of the following sequence:

$$0 \longrightarrow \mathrm{Alt}^2_{\mathcal{V}} \longrightarrow \mathrm{Bil}_{\mathcal{V}} \longrightarrow \mathrm{Quad}_{\mathcal{V}} \longrightarrow 0.$$

In terms of the corresponding (geometric) vector bundles over X, the above translates into the following sequence of morphisms of vector bundles, with the first one a closed immersion and the second one a Zariski locally-trivial principal Alt^2_V-bundle:

$$\mathrm{Alt}^2_V \hookrightarrow \mathrm{Bil}_V \twoheadrightarrow \mathrm{Quad}_V.$$

Given a quadratic form $q \in \Gamma(U, \mathrm{Quad}_{\mathcal{V}})$, recall that the usual 'associated' bilinear form $b_q \in \Gamma(U, \mathrm{Bil}_{\mathcal{V}})$ is given on sections (over open subsets of U) by $v \otimes v' \mapsto q(v + v') - q(v) - q(v')$. This association in general does not lead to a bijective correspondence between quadratic forms and symmetric bilinear forms (which is nevertheless correct when $2 \in \Gamma(X, \mathcal{O}^*_X)$ or equivalently when the residue field of each point of X is of characteristic $\neq 2$). Given a (not-necessarily symmetric!) bilinear form b, we also have the induced quadratic form q_b given on sections by $v \mapsto b(v \otimes v)$. Since a surjection of sheaves does not necessarily imply a surjection on global sections, a global quadratic form may not be induced from a global bilinear form (unless we assume something more, for e.g., that the scheme is affine, or more generally that the sheaf cohomology group $\mathrm{H}^1(X, \mathrm{Alt}^2_{\mathcal{V}}) = 0$).

PROPOSITION 2.2 *Let (V, q) be a quadratic bundle on X and (M, b) be a symmetric bilinear form bundle on X (i.e., M is a vector bundle on X and b is a global symmetric bilinear form on M with values in $\mathcal{O}_X$).*

(1) *Then we can tensor (V, q) with (M, b) to get a unique quadratic bundle $(V \otimes M, q \otimes b)$. The quadratic form on $V \otimes M$ is given on sections by $v \otimes m \mapsto q(v).b(m \otimes m)$ and has associated bilinear form $b_{q\otimes b} = b_q \otimes b$.*

(2) *When the rank of M is 1 i.e., M is a line bundle, (M, b) is regular iff (M, q_b) is semiregular iff $b : \mathcal{M} \otimes \mathcal{M} \cong \mathcal{O}_X$ is an isomorphism (i.e., iff (M, b) is a discriminant bundle).*

(3) *Let V be of odd rank and (M, b) a discriminant bundle. Then (V, q) is semiregular iff $(V, q) \otimes (M, b) = (V \otimes M, q \otimes b)$ is semiregular.*

By taking an affine open cover for X (which always exists by definition since X is a scheme), the proof of the assertions in the above proposition may be reduced to the case when X is itself an affine scheme. In this case we may further assume that V and M are themselves free. Then (1) follows from para.8.4, Chap.I, [2], while (2) and (3) are easy consequences of the definition and basic properties of semiregularity (§3, Chap.IV, [2]). Note that statments (1) and (3) above justify respectively the definition of the functor $\mathscr{Q}_3$ and its subfunctor $\mathscr{Q}^{sr}_3$ introduced in §1.

PROPOSITION 2.3 *Let V and V' be vector bundles of the same rank on the scheme X, with associated locally-free sheaves $\mathcal{V}$ and $\mathcal{V}'$ respectively. Let $\alpha : V' \cong V \otimes L$ be an isomorphism of bundles where (L, h) is a discriminant bundle on X.*

(1) *Over any open subset $U \hookrightarrow X$, given a bilinear form $b \in \Gamma(U, \mathrm{Bil}_{\mathcal{V}})$, we can define a bilinear form $b' \in \Gamma(U, \mathrm{Bil}_{\mathcal{V}'})$ using α and h as follows: we let $b' := b \circ \zeta_{(\alpha,h)}$ where $\zeta_{(\alpha,h)} : \mathcal{V}' \otimes \mathcal{V}' \cong \mathcal{V} \otimes \mathcal{V}$ is the $\mathcal{O}_X$-module isomorphism given by the composition of the following natural morphisms:*

$$\mathcal{V}' \otimes \mathcal{V}' \overset{\alpha\otimes\alpha\,(\cong)}{\longrightarrow} \mathcal{V} \otimes \mathcal{L} \otimes \mathcal{V} \otimes \mathcal{L} \overset{\mathrm{SWAP}(2,3)\,(\equiv)}{\longrightarrow} \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{L} \otimes \mathcal{L} \overset{\mathrm{Id}\otimes h\,(\cong)}{\longrightarrow} \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{O}_X \overset{\mathrm{CANON}\,(\equiv)}{\longrightarrow} \mathcal{V} \otimes \mathcal{V}.$$

Then the association $b \mapsto b'$ induces $\mathcal{O}_X$-linear isomorphisms shown by vertical downward arrows in the following diagram (with exact rows) making it commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Alt}^2_{\mathcal{V}} & \longrightarrow & \mathrm{Bil}_{\mathcal{V}} & \longrightarrow & \mathrm{Quad}_{\mathcal{V}} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & \mathrm{Alt}^2_{\mathcal{V}'} & \longrightarrow & \mathrm{Bil}_{\mathcal{V}'} & \longrightarrow & \mathrm{Quad}_{\mathcal{V}'} & \longrightarrow & 0
\end{array}$$

Therefore one also has the following commutative diagram of vector bundle morphisms with the vertical downward arrows being isomorphisms:

$$\begin{array}{ccccc}
\mathrm{Alt}^2_V & \xrightarrow[\text{immersion}]{\text{closed}} & \mathrm{Bil}_V & \xrightarrow[\text{trivial}]{\text{locally}} & \mathrm{Quad}_V \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathrm{Alt}^2_{V'} & \xrightarrow[\text{immersion}]{\text{closed}} & \mathrm{Bil}_{V'} & \xrightarrow[\text{trivial}]{\text{locally}} & \mathrm{Quad}_{V'}
\end{array}$$

(2) *Let $b \in \Gamma(X, \mathrm{Bil}_{\mathcal{V}})$ be a global bilinear form and let it induce $b' \in \Gamma(X, \mathrm{Bil}_{\mathcal{V}'})$ via α and h as defined in (1) above. Let $\Psi_b \in \mathrm{Aut}_{\mathcal{O}_X}(T\mathcal{V})$ (resp. $\Psi_{b'} \in \mathrm{Aut}_{\mathcal{O}_X}(T\mathcal{V}')$) be the $(\mathbb{Z}/2.\mathbb{Z})$-graded linear automorphism of the tensor algebra (with even elements given degree 0 and odd elements given degree 1) induced by b (resp. by b') defined locally (and hence globally) as in (2e) of Theorem 2.1 above. Let $Z_{(\alpha,h)} : T^{even}_{\mathcal{O}_X}(\mathcal{V}') \cong T^{even}_{\mathcal{O}_X}(\mathcal{V})$ be the $\mathcal{O}_X$-algebra isomorphism induced via the isomorphism $\zeta_{(\alpha,h)} : T^2_{\mathcal{O}_X}(\mathcal{V}') \cong T^2_{\mathcal{O}_X}(\mathcal{V})$ defined in (1) above. Then the following diagram commutes:*

$$\begin{array}{ccc}
T^{even}_{\mathcal{O}_X}(\mathcal{V}') & \xrightarrow[\cong]{Z_{(\alpha,h)}} & T^{even}_{\mathcal{O}_X}(\mathcal{V}) \\
\Psi_{b'}\downarrow\cong & & \cong\downarrow\Psi_b \\
T^{even}_{\mathcal{O}_X}(\mathcal{V}') & \xrightarrow[Z_{(\alpha,h)}]{\cong} & T^{even}_{\mathcal{O}_X}(\mathcal{V})
\end{array}$$

thereby inducing (see (2d), Theorem 2.1) the following commutative diagram of $\mathcal{O}_X$-linear isomorphisms

$$\begin{array}{ccc}
C_0(\mathcal{V}', q_{b'}) & \xrightarrow[\cong]{\text{via } Z_{(\alpha,h)}} & C_0(\mathcal{V}, q_b) \\
\psi_{b'}\downarrow\cong & & \cong\downarrow\psi_b \\
\Lambda^{even}_{\mathcal{O}_X}(\mathcal{V}') & \xrightarrow[\text{via } Z_{(\alpha,h)}]{\cong} & \Lambda^{even}_{\mathcal{O}_X}(\mathcal{V})
\end{array}$$

(3) *Let b and b' be as in (2) above. Then $\alpha : V' \cong V \otimes L$ induces an isometry of bilinear form bundles $\alpha : (V', b') \cong (V, b) \otimes (L, h)$ and also an isometry of the induced quadratic bundles $\alpha : (V', q_{b'}) \cong (V, q_b) \otimes (L, h)$. Moreover, if we are just given a global quadratic form q on V (resp. q' on V'), then we may define the global quadratic form q' on V' (resp. q on V) via $q' := (q \otimes h) \circ \alpha$ (resp. via $q := (q' \otimes (h^{\vee})^{-1}) \circ (\alpha \otimes L^{-1})^{-1}$) and again $\alpha : (V', q') \cong (V, q) \otimes (L, h)$ becomes an isometry of quadratic bundles.*

Similitudes. Let (V, q) and (V', q') be quadratic bundles on the scheme X. We recall that a morphism of vector bundles $g : V \longrightarrow V'$ is called a similitude (or similarity) with multiplier $l \in \Gamma(X, \mathcal{O}_X^*)$, and we write $g : (V, q) \cong_l (V', q')$, if g is an isomorphism of vector bundles and the following diagram involving their associated locally-free sheaves commutes (where q and q' are considered as morphisms of sheaves of *sets*):

$$\begin{array}{ccc}
\mathcal{V} & \xrightarrow[\cong]{g} & \mathcal{V}' \\
q\downarrow & & \downarrow q' \\
\mathcal{O}_X & \xrightarrow[\text{MULT. BY } l]{\cong} & \mathcal{O}_X
\end{array}$$

Moreover, when the multiplier $l = 1$, we write $g : (V, q) \cong (V', q')$ and call g an isometry.

PROPOSITION 2.4 *Let $g : (V, q) \cong_l (V', q')$ be a similarity with multiplier $l \in \Gamma(X, \mathcal{O}_X^*)$.*

(1) *There exists a unique isomorphism of $\mathcal{O}_X$-algebra bundles $C_0(g,l) : C_0(V,q) \cong C_0(V',q')$ such that $C_0(g,l)(v.v') = l^{-1}g(v).g(v')$ on sections.*

(2) *There exists a unique vector bundle isomorphism $C_1(g,l) : C_1(V,q) \cong C_1(V',q')$ such that*

(a) $C_1(g,l)(v.c) = g(v).C_0(g,l)(c)$ *and*

(b) $C_1(g,l)(c.v) = C_0(g,l)(c).g(v)$

for any section v of V and any section c of $C_0(V,q)$. Thus $C_1(g,l)$ is $C_0(g,l)$-semilinear.

(3) *If $g_1 : (V',q') \cong_{l_1} (V'',q'')$ is another similarity with multiplier l_1, then the composition $g_1 \circ g : (V,q) \cong_{ll_1} (V'',q'')$ is also a similarity with multiplier given by the product of the multipliers. Further $C_i(g_1 \circ g, ll_1) = C_i(g_1,l_1) \circ C_i(g,l)$ for $i = 0,1$.*

The proof of the above follows from the case of an affine scheme which is established in Prop.7.1.1, Chap.IV, [2]. A local computation shows that tensoring by a discriminant bundle amounts to (locally) applying a similarity. In this case also one gets a global isomorphism of even Clifford algebras:

PROPOSITION 2.5 *Let (V,q) be a quadratic bundle on a scheme X and (L,h) be a discriminant bundle. There exists a unique isomorphism of algebra bundles $\gamma_{(L,h)} : C_0((V,q) \otimes (L,h)) \cong C_0(V,q)$ given by $\gamma_{(L,h)}((v \otimes \lambda).(v' \otimes \lambda')) = h(\lambda \otimes \lambda')v.v'$ for any sections v, v' of V and λ, λ' of L.*

The proof of the above may be reduced to the case of an affine scheme, in which case, we may further assume that L is free, so that tensoring by (L,h) is the same as applying a similarity. In this case, the result follows from Prop.2.4. The above result justifies the definition of the natural transformation Witt introduced in §1.

Proof of Prop.1.2: Start with an isomorphism of algebra-bundles $\phi : C_0(V,q) \cong C_0(V',q')$. Let $\{U_i\}_{i\in I}$ be an affine open covering of X (which may also be chosen so as to trivialise the rank 3 vector bundles V and V' if needed). Choose bilinear forms $b_i \in \Gamma(U_i, \mathrm{Bil}_V)$ and $b'_i \in \Gamma(U_i, \mathrm{Bil}_{V'})$ such that $q|U_i = q_{b_i}$ and $q'|U_i = q_{b'_i}$ for each $i \in I$. By (2d), Theorem 2.1, we have isomorphisms of vector bundles ψ_{b_i} and $\psi_{b'_i}$, which preserve 1 by (2a) of the same Theorem, and we define the isomorphism of vector bundles $\phi_{\Lambda_i^{ev}}$ so as to make the following diagram commute:

$$\begin{tikzcd} C_0(V|U_i, q|U_i) \arrow[r, "\phi|U_i", "\cong"'] \arrow[d, "\psi_{b_i}"', "\cong"] & C_0(V'|U_i, q'|U_i) \arrow[d, "\cong"', "\psi_{b'_i}"] \\ \Lambda^{even}(V|U_i) \arrow[r, "\cong", "\phi_{\Lambda_i^{ev}}"'] & \Lambda^{even}(V'|U_i) \end{tikzcd}$$

The linear isomorphism $\phi_{\Lambda_i^{ev}}$ preserves 1 and therefore it induces a linear isomorphism from $\Lambda^2(V|U_i)$ to $\Lambda^2(V'|U_i)$, which we denote by $(\phi_{\Lambda^2})_i$. Observe that $(\phi_{\Lambda^2})_i$ is independent of the choice of the bilinear forms b_i and b'_i. For, replacing these respectively by $\widehat{b_i}$ and $\widehat{b'_i}$, it follows from (2f), Theorem 2.1, that $\psi_{b_i} \circ (\psi_{\widehat{b_i}})^{-1}$ (resp. $\psi_{b'_i} \circ (\psi_{\widehat{b'_i}})^{-1}$) followed by the canonical projection onto $\Lambda^2(V|U_i)$ (resp. onto $\Lambda^2(V'|U_i)$) is the same as the projection itself. By this observation, it is also clear that the isomorphisms $\{(\phi_{\Lambda^2})_i\}_{i\in I}$ agree on (any open affine subscheme of, and hence on all of) any intersection $U_i \cap U_j$. Therefore they glue to give a global isomorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \cong \Lambda^2(V')$. **Q.E.D, Prop.1.2.**

Reduction of Proof of Theorem 1.1 to Theorem 1.3. We start with an isomorphism of algebra-bundles $\phi : C_0(V,q) \cong C_0(V',q')$, construct the isomorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \cong \Lambda^2(V')$ and keep the notations introduced in the proof of Prop.1.2. Firstly we deduce a linear isomorphism $\det((\phi_{\Lambda^2})^\vee)^{-1} : \det((\Lambda^2(V))^\vee) \cong \det((\Lambda^2(V'))^\vee)$. Since V and V' are of rank 3, there are canonical isomorphisms $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ and $\eta' : \Lambda^2(V') \equiv (V')^\vee \otimes \det(V')$. It follows therefore that if we set $L := \det(V') \otimes (\det(V))^{-1}$ then we get a discriminant line bundle (L,h) and a vector bundle isomorphism $\alpha : V' \cong V \otimes L$. Now for each $i \in I$, the bilinear form $b_i \in \Gamma(U_i, \mathrm{Bil}_V)$ induces, via $\alpha|U_i$ and $(L|U_i, h|U_i)$ and (1), Prop.2.3, a bilinear form $b''_i \in \Gamma(U_i, \mathrm{Bil}_{V'})$. By (3) of the same Proposition, over each U_i we get an isometry of bilinear form bundles $\alpha|U_i : (V'|U_i, b''_i) \cong (V|U_i, b_i) \otimes (L|U_i, h|U_i)$ and also an isometry of quadratic bundles $\alpha|U_i : (V'|U_i, q_{b''_i}) \cong (V|U_i, q_{b_i} = q|U_i) \otimes (L|U_i, h|U_i)$. On the other hand, by an assertion in (3), Prop.2.3, we could also define the global quadratic bundle (V', q'') using (V,q), α and (L,h), so that we have an isometry of quadratic bundles $\alpha : (V', q'') \cong (V,q) \otimes (L,h)$. It follows therefore that the $q_{b''_i}$ glue to give q''. Notice that in general the b''_i (resp. the b_i) need not glue to give a global bilinear form b'' (resp. b) such that $q_{b''} = q''$ (resp. $q_b = q$). By (1), Prop.2.4, there exists a unique isomorphism of $\mathcal{O}_X$-algebra bundles $C_0(\alpha, 1) : C_0(V', q'') \cong C_0((V, q) \otimes (L, h))$ and by Prop.2.5 we have a unique isomorphism of algebra bundles $\gamma_{(L,h)} : C_0((V, q) \otimes (L, h)) \cong C_0(V, q)$. Therefore the composition of the following sequence of isomorphisms of algebra bundles on X

$$C_0(V', q'') \overset{C_0(\alpha,1)\,(\cong)}{\longrightarrow} C_0((V, q) \otimes (L, h)) \overset{\gamma_{(L,h)}\,(\cong)}{\longrightarrow} C_0(V, q) \overset{\phi\,(\cong)}{\longrightarrow} C_0(V', q')$$

is an element of $\mathrm{Iso}[C_0(V', q''), C_0(V', q')]$, which, granting Theorem 1.3, is induced by a similarity in $\mathrm{Sim}[(V', q''), (V', q')]$. Therefore, we would have that (V', q'') and (V', q') are globally similar (which means that they differ by a discriminant bundle with underlying line bundle being trivial), and this combined with the fact that (V, q) and (V', q'') are isometric upto the discriminant bundle (L, h) (by the construction above), we would have that (V, q) and (V', q') also differ by a discriminant bundle. Therefore the proof of Theorem 1.1 reduces to the proof of Theorem 1.3.

Reduction of Theorem 1.3 to the Free Case: For a similarity g with multiplier l, we have $C_0(g, l)$ given by (1), Prop.2.4, so that the map $\mathrm{Sim}[(V, q), (V, q')] \longrightarrow \mathrm{Iso}[C_0(V, q), C_0(V, q')]$ mentioned in the statement is the natural $g \mapsto C_0(g, l)$. We shall show (locally in the Zariski topology and hence globally) the equality $\det\left((C_0(g, l))_{\Lambda^2}\right) = l^{-3}\det^2(g)$ so that $\mathrm{Iso}[(V, q), (V, q')]$ and $\text{S-Iso}[(V, q), (V, q')]$ are respectively mapped into $\mathrm{Iso}'[C_0(V, q), C_0(V, q')]$ and $\text{S-Iso}[C_0(V, q), C_0(V, q')]$ as claimed. This equality will be verified in Lemma 3.11 following the proof of the surjectivity of $\mathrm{Sim}[(V, q), (V, q')] \longrightarrow \mathrm{Iso}[C_0(V, q), C_0(V, q')]$. We start with an isomorphism of algebra-bundles $\phi : C_0(V, q) \cong C_0(V, q')$, which by the above discussion leads to the automorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \cong \Lambda^2(V)$. Firstly, define the global bundle automorphism $g' \in \mathrm{GL}\left(V \otimes (\det(V))^{-1}\right)$ so that the following diagram commutes:

$$\begin{array}{ccc}
(\Lambda^2(V))^\vee & \xrightarrow[\cong]{((\phi_{\Lambda^2})^\vee)^{-1}} & (\Lambda^2(V))^\vee \\
{\scriptstyle (\eta^\vee)^{-1}}\downarrow \equiv & & \equiv \downarrow {\scriptstyle (\eta^\vee)^{-1}} \\
V \otimes (\det(V))^{-1} & \xrightarrow[g']{\cong} & V \otimes (\det(V))^{-1}
\end{array}$$

where $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ is the canonical isomorphism (since V is of rank 3). Now let $g \in \mathrm{GL}(V) \overset{\cong}{\longleftarrow} \mathrm{GL}(V \otimes (\det(V))^{-1})$ be the image of g' i.e., the image of $g' \otimes \det(V)$ under the canonical identification $\mathrm{GL}(V \otimes (\det(V))^{-1} \otimes \det(V)) \equiv \mathrm{GL}(V)$. Next, let $l \in \Gamma(X, \mathcal{O}_X^*)$ be a global section such that $\gamma(l) := (l^3).\det(\phi_{\Lambda^2})$ has a square root in $\Gamma(X, \mathcal{O}_X^*)$. For example, we have the following special cases when this is true:

Case 1. If $\det(\phi_{\Lambda^2})$ is itself a square, set $l := 1$. If further $\det(\phi_{\Lambda^2}) = 1$, set $\sqrt{\gamma(l)} = 1$, otherwise let $\sqrt{\gamma(l)}$ denote any fixed square root of $\det(\phi_{\Lambda^2})$.

Case 2. If $\det(\phi_{\Lambda^2})$ is not a square, given an integer k, take $l = (\det(\phi_{\Lambda^2}))^{2k+1}$ and let $\sqrt{\gamma(l)}$ denote any fixed square root of $(\det(\phi_{\Lambda^2}))^{6k+4}$.

For each integer k, we now associate to ϕ the element $g_l^\phi := (l^{-1}\sqrt{\gamma(l)}\,)g$ with g as defined above. We shall show the following locally for the Zariski toplogy on X (more precisely, for each open subscheme of X over which V is free):

(1) that g_l^ϕ is a similitude from q to q' with multiplier l (Lemma 3.9);

(2) that g_l^ϕ induces ϕ i.e., with the notations of (1), Prop.2.4, that $C_0(g_l^\phi, l) = \phi$ (Lemma 3.10);

(3) that $\det(g_l^\phi) = \sqrt{\gamma(l)}$ so that $\det^2(g_l^\phi) = \det(\phi_{\Lambda^2})$ when $\det(\phi_{\Lambda^2})$ is itself a square (Lemma 3.8) and

(4) that the map $\text{S-Iso}[(V, q), (V, q')] \longrightarrow \text{S-Iso}[C_0(V, q), C_0(V, q')]$ is injective (Lemma 3.12).

It would follow then that these statements are also true globally. The maps $s_{2k+1} : \phi \mapsto g_l^\phi$ with l as in Case 2 and $s' : \phi \mapsto g_l^\phi$ with l as in Case 1 will then give the sections to the maps (which would imply their surjectivities) as mentioned in Theorem 1.3. But these maps are not necessarily multiplicative since a computation reveals that if $\phi_i \in \mathrm{Iso}[C_0(V, q_i), C_0(V, q_{i+1})]$ is associated to $g_{l_i}^{\phi_i} \in \mathrm{Sim}[(V, q_i), (V, q_{i+1})]$, and $\phi_2 \circ \phi_1$ to $g_{l_{21}}^{\phi_2\circ\phi_1}$, then $g_{l_{21}}^{\phi_2\circ\phi_1} = \delta g_{l_2}^{\phi_2} \circ g_{l_1}^{\phi_1}$ for $\delta \in \mu_2(\Gamma(X, \mathcal{O}_X^*))$ because of the ambiguity in the initial global choices of square roots for $\gamma(l_i)$ and $\gamma(l_{21})$. However this can be remedied as follows. For any given $\phi \in \mathrm{Iso}[C_0(V, q), C_0(V, q')]$, irrespective of whether or not $\det(\phi_{\Lambda^2})$ is a square, take

$$l = (\det(\phi_{\Lambda^2}))^{2k+1}, \gamma(l) = l^3\det(\phi_{\Lambda^2}), \sqrt{\gamma(l)} := (\det(\phi_{\Lambda^2}))^{3k+2} \text{ and } s^+_{2k+1}(\phi) := g_l^\phi = \left(l^{-1}\sqrt{\gamma(l)}\right) g.$$

Then it is clear that each s^{+}_{2k+1} is multiplicative with the properties as claimed in the statement. We thus reduce the proof of Theorem 1.3 to the case when the rank 3 vector bundle V is free. This will be taken up next in §3.

3 Investigation of Θ ; Proof of Theorem 1.4

In this section, we conclude the proofs of Theorems 1.1 and 1.3 which were begun in §2 and also prove Theorem 1.4. As means to these ends, we carry out two explicit computations. Firstly we compute the isomorphism Θ of Theorem 1.11. This provides us with the multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit which is part of a global basis (Theorem 3.1 below). This result will also be used in §4 in the proof of Theorem 1.12. It turns out that Θ is not equivariant with respect to GL_V, but nevertheless satisfies a 'twisted' form of equivariance (Theorem 3.4). Secondly, we explicitly compute the algebra bundle isomorphism $C_0(g,l) : C_0(V,q) \cong C_0(V,q')$ of (1), Prop.2.4 induced by a similarity $g : (V,q) \cong_l (V,q')$ with multiplier $l \in \Gamma(X, \mathcal{O}_X^*)$ in the case when V is free of rank 3 (Theorem 3.5).

The Action of GL_V. Let V be a vector bundle over a scheme X with associated locally-free sheaf $\mathcal{V}$. The X-smooth X-groupscheme GL_V acts naturally on the left on the sheaves $\mathrm{Alt}^2_{\mathcal{V}}$, $\mathrm{Bil}_{\mathcal{V}}$ and $\mathrm{Quad}_{\mathcal{V}}$ of alternating, bilinear and quadratic forms on $\mathcal{V}$. Namely, for $U \hookrightarrow X$ an open subscheme, and for $b \in \Gamma(U, \mathrm{Bil}_{\mathcal{V}})$ (resp. $a \in \Gamma(U, \mathrm{Alt}^2_{\mathcal{V}})$, resp. $q \in \Gamma(U, \mathrm{Quad}_{\mathcal{V}})$), and for $g \in \Gamma(U, \mathrm{GL}_V) = \mathrm{GL}(V|U)$, the corresponding form of the same type $g.b$ (resp. $g.a$, resp. $g.q$) is defined on sections (over open subsets of U) by $(g.b)(v,v') := b(g^{-1}(v), g^{-1}(v'))$ (resp.$(g.a)(v,v') := a(g^{-1}(v), g^{-1}(v'))$, resp. $(g.q)(v) := q(g^{-1}(v))$). It is immediate that the following short-exact-sequence of sheaves, indicated in the discussion before Prop.2.2, is equivariant with respect to this action:

$$(\spadesuit) \qquad 0 \longrightarrow \mathrm{Alt}^2_{\mathcal{V}} \longrightarrow \mathrm{Bil}_{\mathcal{V}} \longrightarrow \mathrm{Quad}_{\mathcal{V}} \longrightarrow 0.$$

Equivalently, the X-groupscheme GL_V acts on the corresponding geometric vector bundles such that both of the X-morphisms of X-vector bundles in the following sequence are GL_V-equivariant:

$$\mathrm{Alt}^2_V \hookrightarrow \mathrm{Bil}_V \twoheadrightarrow \mathrm{Quad}_V.$$

Notice that it is one and the same thing to require that $\mathrm{GL}(V|U) \ni g : (V|U,q) \cong_l (V|U,q')$ be a similitude with multiplier $l \in \Gamma(U, \mathcal{O}_X)$, and to require that $g.q = l^{-1}q'$.

Definition of the isomorphism Θ. We briefly recall the definition of Θ from Part A of [3]. We keep the notations introduced just before Theorem 1.11; for ease of notation, the pullback of a section s (of a vector bundle or its associated sheaf) is denoted by s°. Since V is free of rank 3 on X, we choose an identification $\mathcal{V} \equiv \mathcal{O}_X.e_1 \oplus \mathcal{O}_X.e_2 \oplus \mathcal{O}_X.e_3$. This gives the identification of the dual bundle as $\mathcal{V}^\vee \equiv \mathcal{O}_X.f_1 \oplus \mathcal{O}_X.f_2 \oplus \mathcal{O}_X.f_3$ (defined uniquely by $f_i(e_j) = \delta_{ij}$, the Kronecker delta). Therefore the dual of the sheaf of quadratic forms on V, which is $(\mathrm{Quad}_{\mathcal{V}})^\vee := (\mathrm{Bil}_{\mathcal{V}}/\mathrm{Alt}^2_{\mathcal{V}})^\vee = ((T^2\mathcal{V})^\vee/(\Lambda^2\mathcal{V})^\vee)^\vee$ has global $\mathcal{O}_X$-basis given by $\{e_i \otimes e_i; (e_i \otimes e_j + e_j \otimes e_i)\}$. This leads to an identification of the associated sheaf of symmetric algebras $\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Quad}^\vee_{\mathcal{V}}) \equiv \mathcal{O}_X[Y_1,Y_2,Y_3,Y_{12},Y_{13},Y_{23}]$, where $e_i \otimes e_i \equiv Y_i$ and $e_i \otimes e_j + e_j \otimes e_i \equiv Y_{ij}$, and therefore $\mathrm{Quad}_V := \mathbf{Spec}\left(\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Quad}^\vee_{\mathcal{V}})\right) \equiv \mathbf{Spec}(\mathcal{O}_X[Y_1,Y_2,Y_3,Y_{12},Y_{13},Y_{23}]) = \mathbb{A}^6_X$. Consider the universal quadratic bundle $(\mathbf{V},\mathbf{q})$ where $\mathbf{V}$ is the pullback of V by $\mathrm{Quad}_V \longrightarrow X$. The universal quadratic form $\mathbf{q}$ is given by $(x_1,x_2,x_3) \mapsto \Sigma_i Y_i.(x_i)^2 + \Sigma_{i<j} Y_{ij}.x_i.x_j$ and moreover the global bilinear form on $\mathbf{V}$ given by

$$\mathbf{b}(\mathbf{q}) : \left((x_1,x_2,x_3),(x'_1,x'_2,x'_3)\right) \mapsto \Sigma_i Y_i.x_i.x'_i + Y_{12}.x_2.x'_1 + Y_{23}.x_3.x'_2 + Y_{13}.x_1.x'_3$$

induces $\mathbf{q}$ (the bilinear form 'associated in the usual sense' to $\mathbf{q}$, viz. $b_{\mathbf{q}}$ is not $\mathbf{b}(\mathbf{q})$ but in fact its symmetrisation). Therefore, by (2d), Theorem 2.1, we get an isomorphism of vector bundles $\psi_{\mathbf{b}(\mathbf{q})}$: $C_0(\mathbf{V},\mathbf{q} = q_{\mathbf{b}(\mathbf{q})}) \cong \Lambda^{even}(\mathbf{V}) =: \mathbf{W}$ which, according to (2a) and (2f) of the same Theorem, carries the ordered Poincaré-Birkhoff-Witt basis $\{1; e_1^\circ.e_2^\circ, e_2^\circ.e_3^\circ, e_3^\circ.e_1^\circ\}$ onto the corresponding ordered basis of the even exterior algebra (=even Clifford algebra of the zero quadratic form on $\mathbf{V}$) given by $\{w^\circ = 1^\circ = 1; e_1^\circ \wedge e_2^\circ, e_2^\circ \wedge e_3^\circ, e_3^\circ \wedge e_1^\circ\}$. The choices $e_3^\circ.e_1^\circ$ and $e_3^\circ \wedge e_1^\circ$ instead of the usual $e_1^\circ.e_3^\circ$ and $e_1^\circ \wedge e_3^\circ$ are deliberate—for example, $\psi_{\mathbf{b}(\mathbf{q})}$ would carry $\{1; e_1^\circ.e_2^\circ, e_2^\circ.e_3^\circ, e_1^\circ.e_3^\circ\}$ onto $\{w^\circ = 1^\circ = 1; e_1^\circ \wedge e_2^\circ, e_2^\circ \wedge e_3^\circ, e_1^\circ \wedge e_3^\circ + Y_{13}.w^\circ\}$ which depends on Y_{13}. Thus the even Clifford algebra bundle $C_0(\mathbf{V},\mathbf{q} = q_{\mathbf{b}(\mathbf{q})})$ induces via $\psi_{\mathbf{b}(\mathbf{q})}$ a w°-unital algebra structure on the pullback bundle $\mathbf{W}$ of $W := \Lambda^{even}(V)$ (where w corresponds to

1 in $\Lambda^{even}(V)$). But by definition, this algebra structure corresponds precisely to an X-morphism θ : $\mathrm{Quad}_V \longrightarrow$ Id-w-Sp-Azu_W. The isomorphism Θ is now given by the composition of the following X-morphisms (cf. Def.5.2, Part A, [3]):

$$\mathrm{Quad}_V \times_X \mathsf{L}_w \xrightarrow{\theta\times\mathrm{ID}} \text{Id-}w\text{-Sp-Azu}_W \times_X \mathsf{L}_w \xrightarrow{\mathrm{SWAP}\,(\cong)} \mathsf{L}_w \times_X \text{Id-}w\text{-Sp-Azu}_W \xrightarrow{\mathrm{ACTION}} \text{Id-}w\text{-Sp-Azu}_W.$$

The association of $\mathbf{q}$ with $\mathbf{b(q)}$ also defines a splitting of the exact sequence (♠) above, so that more generally, given a valued point $q \in (\mathrm{Quad}_V)(T)$, we may associate uniquely a valued point $b(q) \in (\mathrm{Bil}_V)(T)$ which induces it. That this association is not GL_V-equivariant is reflected in the lack of equivariance of the isomorphism Θ (Theorem 3.4).

THEOREM 3.1 *Let T be an X-scheme. Let q be a T-valued point of $\mathrm{Quad}_V \equiv \mathbb{A}^6_X$ which is identified uniquely with a 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23})$ corresponding to the quadratic form $(x_1, x_2, x_3) \mapsto \Sigma_i \lambda_i x_i^2 + \Sigma_{i<j} \lambda_{ij} x_i x_j$. Let $\underline{t}$ be a T-valued point of $\mathsf{L}_w \equiv (\mathbb{A}^3_X, +)$ which is identified uniquely with a 3-tuple (t_1, t_2, t_3) that corresponds to the T-valued point of Stab_w given by the (4×4)-matrix*

$$\left(\begin{array}{c|ccc} 1 & t_1 & t_2 & t_3 \\ \hline 0 & & I_3 & \end{array}\right)$$

where I_3 is the (3×3)-identity matrix. Then the multiplication table for the specialised algebra structure $\Theta(q, \underline{t}) = \underline{t}.\theta(q)$ on the pullback bundle W_T with unit $w^\circ = w_T$ is given as follows (in terms of the global basis $\{w^\circ = 1^\circ = 1\ ;\ \epsilon_1^\circ := e_1^\circ \wedge e_2^\circ\ ,\ \epsilon_2^\circ := e_2^\circ \wedge e_3^\circ\ ,\ \epsilon_3^\circ := e_3^\circ \wedge e_1^\circ\}$ induced from that of $W = \Lambda^{even}(V)$:)

- $\epsilon_1^\circ.\epsilon_1^\circ = (t_1\lambda_{12} - \lambda_1\lambda_2 - t_1^2).w^\circ + (\lambda_{12} - 2t_1).\epsilon_1^\circ\,;$
- $\epsilon_2^\circ.\epsilon_2^\circ = (t_2\lambda_{23} - \lambda_2\lambda_3 - t_2^2).w^\circ + (\lambda_{23} - 2t_2).\epsilon_2^\circ\,;$
- $\epsilon_3^\circ.\epsilon_3^\circ = (t_3\lambda_{13} - \lambda_1\lambda_3 - t_3^2).w^\circ + (\lambda_{13} - 2t_3).\epsilon_3^\circ\,;$
- $\epsilon_1^\circ.\epsilon_2^\circ = (\lambda_2\lambda_{13} - \lambda_2 t_3 - t_1t_2).w^\circ - t_2\epsilon_1^\circ - t_1\epsilon_2^\circ - \lambda_2\epsilon_3^\circ\,;$
- $\epsilon_2^\circ.\epsilon_3^\circ = (\lambda_3\lambda_{12} - \lambda_3 t_1 - t_2t_3).w^\circ - \lambda_3\epsilon_1^\circ - t_3\epsilon_2^\circ - t_2\epsilon_3^\circ\,;$
- $\epsilon_3^\circ.\epsilon_1^\circ = (\lambda_1\lambda_{23} - \lambda_1 t_2 - t_1t_3).w^\circ - t_3\epsilon_1^\circ - \lambda_1\epsilon_2^\circ - t_1\epsilon_3^\circ\,;$
- $\epsilon_2^\circ.\epsilon_1^\circ = (\lambda_2 t_3 - (\lambda_{12} - t_1)(\lambda_{23} - t_2)).w^\circ + (\lambda_{23} - t_2)\epsilon_1^\circ + (\lambda_{12} - t_1)\epsilon_2^\circ + \lambda_2\epsilon_3^\circ\,;$
- $\epsilon_3^\circ.\epsilon_2^\circ = (\lambda_3 t_1 - (\lambda_{13} - t_3)(\lambda_{23} - t_2)).w^\circ + \lambda_3\epsilon_1^\circ + (\lambda_{13} - t_3)\epsilon_2^\circ + (\lambda_{23} - t_2)\epsilon_3^\circ\,;$
- $\epsilon_1^\circ.\epsilon_3^\circ = (\lambda_1 t_2 - (\lambda_{12} - t_1)(\lambda_{13} - t_3)).w^\circ + (\lambda_{13} - t_3)\epsilon_1^\circ + \lambda_1\epsilon_2^\circ + (\lambda_{12} - t_1)\epsilon_3^\circ.$

Proof of Theorem 3.1: For clarity, let $*_q$ denote the multiplication in $C_0(V_T, q)$, and for uniformity, let $\epsilon_0 := w$. Since $q = q_{b(q)}$, we have by (2d), Theorem 2.1, the isomorphism $\psi_{b(q)} : C_0(V_T, q) \cong \Lambda^{even}(V_T) = W_T$. Let $*_{b(q)}$ denote the product in the algebra structure $\theta(q)$ thus induced on W_T. Since the ϵ_i° are a basis for W_T, it is enough to compute the products $\epsilon_i^\circ *_{b(q)} \epsilon_j^\circ$ for $1 \le i, j \le 3$. For example, consider the product $\epsilon_2^\circ *_{b(q)} \epsilon_1^\circ$. Using the properties of the multiplication in $C(V_T, q)$, and the properties of the isomorphism $\psi_{b(q)}$ from (2), Theorem 2.1, we get the following:

$$\begin{aligned}
\epsilon_2^\circ *_{b(q)} \epsilon_1^\circ &= \psi_{b(q)}\left(\{\psi_{b(q)}^{-1}(e_2^\circ \wedge e_3^\circ)\} *_q \{\psi_{b(q)}^{-1}(e_1^\circ \wedge e_2^\circ)\}\right)\\
&= \psi_{b(q)}\left((e_2^\circ *_q e_3^\circ) *_q (e_1^\circ *_q e_2^\circ)\right)\\
&= \psi_{b(q)}\left((\lambda_{23}(1^\circ) - e_3^\circ *_q e_2^\circ) *_q (\lambda_{12}(1^\circ) - e_2^\circ *_q e_1^\circ)\right)\\
&= \psi_{b(q)}\left(\lambda_{23}\lambda_{12}(1^\circ) - \lambda_{23}e_2^\circ *_q e_1^\circ - \lambda_{12}e_3^\circ *_q e_2^\circ + (e_3^\circ *_q e_2^\circ) *_q (e_2^\circ *_q e_1^\circ)\right)\\
&= \psi_{b(q)}\left(\lambda_{23}\lambda_{12}(1^\circ) - \lambda_{23}(\lambda_{12}(1^\circ) - e_1^\circ *_q e_2^\circ) - \lambda_{12}(\lambda_{23}(1^\circ) - e_2^\circ *_q e_3^\circ) + e_3^\circ *_q (e_2^\circ *_q e_2^\circ) *_q e_1^\circ\right)\\
&= (-\lambda_{12}\lambda_{23})w^\circ + \lambda_{23}\epsilon_1^\circ + \lambda_{12}\epsilon_2^\circ + \lambda_2\epsilon_3^\circ.
\end{aligned}$$

In a similar fashion, the other products may be computed; this amounts to computing θ on T-valued points. The following result is needed to compute Θ from θ.

Lemma 3.2 *Let $*_{(b(q),\underline{t})}$ denote the multiplication in the algebra $\Theta(q, \underline{t}) = \underline{t}.\theta(q)$ and as before, $*_{b(q)}$ denote the multiplication in $\theta(q)$. Then we have*

1. $\underline{t}(\epsilon_i^\circ) = t_i w^\circ + \epsilon_i^\circ$ *for* $1 \le i \le 3$;

2. $(\underline{t})^{-1}(\epsilon_i^\circ) = -t_i w^\circ + \epsilon_i^\circ$ *for* $1 \le i \le 3$;

3. $\epsilon_i^\circ *_{(b(q),\underline{t})} \epsilon_j^\circ = \underline{t}(\epsilon_i^\circ *_{b(q)} \epsilon_j^\circ) - t_j\epsilon_i^\circ - t_i\epsilon_j^\circ - t_it_jw^\circ.$

While the first two of the above formulae follow easily by direct computation, the third follows by using the first two alongwith the following:

$$\epsilon_i^\circ *_{(b(q),\underline{t})} \epsilon_j^\circ = \underline{t}\left((\underline{t}^{-1}(\epsilon_i^\circ)) *_{b(q)} (\underline{t}^{-1}(\epsilon_j^\circ)) \right).$$

We may now compute the multiplication in the algebra $\Theta(q,\underline{t}) = \underline{t}.\theta(q)$ by making use of the formulas listed in the above lemma and the expressions for the products of the form $\epsilon_i^\circ *_{b(q)} \epsilon_j^\circ$ whose computation had already been illustrated before the lemma. **Q. E. D., Theorem 3.1.**

Computation of $C_0(g,l)$ in the free case. We continue with the notations introduced above. In the following we study the lack of equivariance of the isomorphism Θ relative to GL_V and show that it satisfies a curious 'twisted' version of equivariance. Firstly we consider the morphism of X-groupschemes $\Lambda^{even} : \mathrm{GL}_V \longrightarrow \mathrm{Stab}_w$ given on valued points by $g \mapsto \Lambda^{even}(g)$. Recall that $\Lambda^{even}(V) =: A_0 \in$ Id-w-Sp-Azu$_W(X)$ is the even graded part of the Clifford algebra of the zero quadratic form on V. A simple computation reveals the following result.

Lemma 3.3 *For each X-scheme T, define the map*

$$\mathrm{GL}(V_T) \longrightarrow \mathrm{Stab}((A_0)_T) : g \mapsto \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & B(g) \end{pmatrix} \in \mathrm{Stab}(w_T)$$

where $B(g) := \det(g)\left(E_{12}E_{23}(g^{-1})^t E_{23}E_{12}\right)$ *with*

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the above maps define a morphism of X-groupschemes which in fact is none other than Λ^{even} : $\mathrm{GL}_V \longrightarrow \mathrm{Stab}_w$; in other words: $B(g) = \Lambda^2(g)$.

Recall from Lemma 5.1, Part A, [3], that Stab_w is the semidirect product of Stab_{A_0} and L_w, so that Stab_{A_0} naturally acts on L_w by "outer conjugation". Let GL_V act on L_w via the homomorphism Λ^{even} i.e., for $g \in \mathrm{GL}(V_T)$ and $\underline{t} \in \mathsf{L}_w(T)$,

$$g.\underline{t} := \Lambda^{even}(g).\underline{t} := \Lambda^{even}(g)\,\underline{t}\,\Lambda^{even}(g^{-1}).$$

Any element $h \in \mathrm{Stab}(w_T)$ can be uniquely written as $h = h_s h_l = h_l' h_s$ where $h_s \in \mathrm{Stab}((A_0)_T)$ and $h_l,\ h_l' \in \mathsf{L}_w(T)$. Then the relation between h_l and h_l' can be written as $h_l' = h_s.h_l$ or $h_l = h_s^{-1}.h_l'$ where "." stands for the action of Stab_{A_0} on L_w. Thus one has a GL_V-action on $\mathrm{Quad}_V \times_X \mathsf{L}_w$ induced by the diagonal embedding $\mathrm{GL}_V \overset{\Delta}{\hookrightarrow} \mathrm{GL}_V \times_X \mathrm{GL}_V$. Since Id-$w$-Sp-Azu$_W$ comes with a canonical action of Stab_w on it, we let GL_V act on Id-w-Sp-Azu$_W$ via Λ^{even}. The following result describes the lack of GL_V-equivariance of the isomorphism Θ.

THEOREM 3.4 *Let T be an X-scheme. For T-valued points $g, q, \underline{t}$ respectively of GL_V, Quad_V, and L_w, there exists a unique T-valued point of L_w given by an isomorphism $h_l'(g,q)$ of $\mathcal{O}_T$-algebra bundles*

$$h_l'(g,q) : g.\Theta(q,\underline{t}) \xrightarrow{\cong} \Theta(g.q, g.\underline{t}).$$

Further, $h_l'(g,q)$ satisfies the formula

$$h_l'(gg',q) = h_l'(g,g'.q)(g.h_l'(g',q)).$$

Therefore Θ satisfies a 'twisted' version of GL_V-equivariance. The next theorem, which was originally motivated by the proof of this 'twisted equivariance', will be of central importance to us for the rest of this section.

THEOREM 3.5 *Given a similarity $g : (V_T, q) \cong_l (V_T, q')$ with multiplier $l \in \Gamma(T, \mathcal{O}_T^*)$, let $h(g,l,q,q')$ be the automorphism of (W_T, w_T) given by the composition of the following isomorphisms:*

$$W_T \overset{(\psi_{b(q)})^{-1}\,(\cong)}{\longrightarrow} C_0(V_T,q) \overset{C_0(g,l)\,(\cong)}{\longrightarrow} C_0(V_T,q') \overset{\psi_{b(q')}\,(\cong)}{\longrightarrow} W_T$$

where the algebra bundle isomorphism $C_0(g,l)$ comes from (1), Prop.2.4 and the linear isomorphisms $\psi_{b(q)}$ and $\psi_{b(q')}$ come from (2d), Theorem 2.1. In terms of actions, this means that $h(g,l,q,q').\theta(q) = \theta(q')$. Write $h(g,l,q,q') \in \mathrm{Stab}(w_T)$ uniquely as a product

$$h(g,l,q,q') = h_s(g,l,q,q')h_l(g,l,q,q')$$

with the first factor in $\mathrm{Stab}_{A_0}(T)$ *and the second in* $\mathsf{L}_w(T)$ *as explained earlier. Then* $h_s(g,l,q,q')$ *depends only on* g *and* l *and not on* q *or* q'*. In fact, one has*

$$h_s(g,l,q_1,q_2) = h_s(g,l) := \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & l^{-1}\Lambda^2(g) \end{pmatrix} \forall q_1, q_2 \in \mathrm{Quad}(V_T).$$

PROOF: We directly compute the $\mathcal{O}_T$-linear automorphism $h(g,l,q,q')$ of W_T as follows. Of course, this automorphism fixes $w^\circ = w_T$. So we need to only compute the images of the three remaining basis elements $\epsilon_1^\circ = e_1^\circ \wedge e_2^\circ$, $\epsilon_2^\circ = e_2^\circ \wedge e_3^\circ$ and $\epsilon_3^\circ = e_3^\circ \wedge e_1^\circ$ in terms of the basis elements w° and ϵ_i°. Let q and $l(g.q) = q'$ respectively correspond to the 6-tuples $(\mu_1,\mu_2,\mu_3,\mu_{12},\mu_{13},\mu_{23})$ and $(\mu_1',\mu_2',\mu_3',\mu_{12}',\mu_{13}',\mu_{23}') \in \Gamma(T,\mathcal{O}_T{}^{\oplus 6})$. (We caution the reader that $l(g.q) \neq (lg).q = l^{-2}(g.q)$!) Let $g \in \mathrm{GL}(V_T) \equiv \mathrm{GL}_3(\Gamma(T,\mathcal{O}_T))$ be given by the matrix (g_{ij}). Observe that the μ' are polynomials in the μ and g_{ij}. In the following computation, for the sake of clarity, we denote the product in $C(V_T,q)$ by $*_q$. For example, we have

$$\begin{aligned}
h(g,l,q,q')\epsilon_1 &= \psi_{b(q')} \circ C_0(g,l)\left((\psi_{b(q)})^{-1}(e_1^\circ \wedge e_2^\circ)\right) = \psi_{b(q')}\left(C_0(g,l)\left(e_1^\circ *_q e_2^\circ\right)\right) \text{(by (2f), Theorem 2.1)} \\
&= \psi_{b(q')}\left(l^{-1}(g(e_1^\circ) *_{q'} g(e_2^\circ))\right) \qquad \text{(by (1), Prop.2.4)} \\
&= l^{-1}\psi_{b(q')}\left((g_{11}e_1^\circ + g_{21}e_2^\circ + g_{31}e_3^\circ) *_{q'} (g_{12}e_1^\circ + g_{22}e_2^\circ + g_{32}e_3^\circ)\right) \\
&= l^{-1}\psi_{b(q')}\big(\, (g_{11}g_{12}\mu_1' + g_{21}g_{22}\mu_2' + g_{31}g_{32}\mu_3' + g_{21}g_{12}\mu_{12}' + g_{31}g_{22}\mu_{23}' + g_{11}g_{32}\mu_{13}')w^\circ + \\
&\quad + (g_{11}g_{22} - g_{21}g_{12})e_1^\circ *_{q'} e_2^\circ + (g_{21}g_{32} - g_{31}g_{22})e_2^\circ *_{q'} e_3^\circ + (g_{31}g_{12} - g_{11}g_{32})e_3^\circ *_{q'} e_1^\circ \,\big) \\
&= l^{-1}\left(P_1(g,l,q,q')w^\circ + C_{33}(g)\epsilon_1^\circ + C_{13}(g)\epsilon_2^\circ + C_{23}(g)\epsilon_3^\circ\right) \qquad \text{(by (2f), Theorem 2.1)}
\end{aligned}$$

where $P_1(g,l,q,q')$ is the polynomial in the μ' and g_{ij} (as computed in the previous step) and where $C_{ij}(g)$ represents the cofactor determinant of the element g_{ij} of the matrix $g = (g_{ij})$. Similarly one computes the values of $h(g,l,q,q')\epsilon_2$ and $h(g,l,q,q')\epsilon_3$. Then the matrix of $h(g,l,q,q')$ is given by

$$h(g,l,q,q') = \begin{bmatrix} 1 & l^{-1}P_1(g,l,q,q') & l^{-1}P_2(g,l,q,q') & l^{-1}P_3(g,l,q,q') \\ 0 & l^{-1}C_{33}(g) & l^{-1}C_{31}(g) & l^{-1}C_{32}(g) \\ 0 & l^{-1}C_{13}(g) & l^{-1}C_{11}(g) & l^{-1}C_{12}(g) \\ 0 & l^{-1}C_{23}(g) & l^{-1}C_{21}(g) & l^{-1}C_{22}(g) \end{bmatrix}$$

which implies that

$$h_s(g,l,q,q') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & l^{-1}C_{33}(g) & l^{-1}C_{31}(g) & l^{-1}C_{32}(g) \\ 0 & l^{-1}C_{13}(g) & l^{-1}C_{11}(g) & l^{-1}C_{12}(g) \\ 0 & l^{-1}C_{23}(g) & l^{-1}C_{21}(g) & l^{-1}C_{22}(g) \end{bmatrix} \text{ depends only on } g \text{ and } l.$$

Next define the matrix

$$\widehat{g} = \begin{bmatrix} g_{33} & g_{13} & g_{23} \\ g_{31} & g_{11} & g_{21} \\ g_{32} & g_{12} & g_{22} \end{bmatrix} \text{ so that } h_s(g,l,q,q') = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & l^{-1}C(\widehat{g})^t \end{bmatrix}$$

where $C(\widehat{g})$ is the cofactor matrix of $\widehat{g}$. Now if E_{12} and E_{23} are the matrices defined in Lemma 3.3 above, premultiplying by E_{ij} has the effect of interchanging the ith and jth rows, while postmultiplying has a similar effect on the columns. Thus we get $\widehat{g} = E_{12}E_{23}(g^t)E_{23}E_{12}$ from which it follows that

$$C(\widehat{g})^t = \text{ Adjoint } (\widehat{g}) = \text{ det } (\widehat{g}).(\widehat{g})^{-1} = \text{ det } (g).\left(E_{12}E_{23}(g^{-1})^t E_{23}E_{12}\right),$$

showing that $C(\widehat{g})^t = \Lambda^2(g)$ by Lemma 3.3. **Q.E.D., Theorem 3.5.**

Proof of Theorem 3.4. Note that g is an isometry from (V_T,q) to $(V_T,g.q)$ and hence according to (1), Prop.2.4, induces the algebra isomorphism $C_0(g,l=1) : C_0(V_T,q) \cong C_0(V_T,g.q)$. Let $h(g,q) := h(g,1,q,g.q)$ where $h(g,l,q,q')$ was defined in Theorem 3.5 above. As explained in page 17, there are two canonical decompositions of $\mathrm{Stab}(w_T)$, one leading to the unique (ordered) decomposition of $h(g,q)$ as

$h_l'(g,q)h_s(g,q)$, and the other leading to the unique ordered decomposition $h_s(g,q)h_l(g,q)$. By Theorem 3.5 above, $h_s(g,q_1) = h_s(g,q_2) = h_s(g,1) = \Lambda^{even}(g) =: h_s(g)\ \forall q_1,q_2 \in \mathrm{Quad}(V_T)$ and hence we get:

$$\begin{aligned}
\Theta(g.q, g.\underline{t}) &:= (g.\underline{t}).\theta(g.q)\\
&= (\Lambda^{even}(g)\underline{t}\Lambda^{even}(g^{-1})).(h(g,1,q,g.q).\theta(q))\\
&= (h_s(g)\underline{t}h_s^{-1}(g)).(h(g,q).\theta(q))\\
&= ((h_s(g)\underline{t}h_s^{-1}(g))(h_s(g,q)h_l(g,q))).\theta(q)\\
&= (h_s(g)\underline{t}h_l(g,q)).\theta(q)\\
&= (h_s(g)h_l(g,q)).(\underline{t}.\theta(q))\\
&= (h_s(g,q)h_l(g,q)).(\Theta(q,\underline{t}))\\
&= (h_l'(g,q)h_s(g,q)).(\Theta(q,\underline{t}))\\
&= h_l'(g,q).(h_s(g,q).\Theta(q,\underline{t}))\\
&= h_l'(g,q).(g.\Theta(q,\underline{t})).
\end{aligned}$$

Note that $h_l(g,q)$ was explicitly computed in the proof of Theorem 3.5 above to be

$$h_l(g,q) = \begin{bmatrix} 1 & P_1(g,1,q,g.q) \quad P_2(g,1,q,g.q) \quad P_3(g,1,q,g.q) \\ \mathbf{0} & I_3 \end{bmatrix} \in \mathsf{L}_w(T).$$

The formula for $h_l'(g_1g_2,q)$ stated in the theorem is gotten thus:

$$\begin{aligned}
h_l'(g_1g_2,q) &= (h_l'(g_1g_2,q)h_s(g_1g_2))h_s^{-1}(g_1g_2)\\
&= h(g_1g_2,q)h_s^{-1}(g_1g_2).
\end{aligned}$$

Now by (3) of Prop.2.4 it follows that

$$\begin{aligned}
h_l'(g_1g_2,q) &= (h(g_1,g_2.q)h(g_2,q))h_s^{-1}(g_1g_2)\\
&= h_l'(g_1,g_2.q)h_s(g_1,g_2.q)h_s(g_2,q)h_l(g_2,q)h_s^{-1}(g_1g_2)\\
&= h_l'(g_1,g_2.q)h_s(g_1)h_s(g_2)h_l(g_2,q)h_s^{-1}(g_1g_2)\\
&= h_l'(g_1,g_2.q)h_s(g_1g_2)h_l(g_2,q)h_s^{-1}(g_1g_2)\\
&= h_l'(g_1,g_2.q)((g_1g_2).h_l(g_2,q)).
\end{aligned}$$

But $g_2^{-1}.h_l'(g_2,q) = h_s^{-1}(g_2)(h_l'(g_2,q)h_s(g_2)) = h_s^{-1}(g_2)(h_s(g_2)h_l(g_2,q)) = h_l(g_2,q)$ and therefore

$$\begin{aligned}
h_l'(g_1g_2,q) &= h_l'(g_1,g_2.q).((g_1g_2).(g_2^{-1}.h_l'(g_2,q)))\\
&= h_l'(g_1,g_2.q).(g_1.h_l'(g_2,q)).
\end{aligned}$$

Finally, one has to show the uniqueness of $h_l'(g,q) \in \mathsf{L}_w(T)$. Suppose $h_l \in \mathsf{L}_w(T)$ is also an algebra isomorphism $h_l : g.\Theta(q,\underline{t}) \xrightarrow{\cong} \Theta(g.q,g.\underline{t})$, i.e., $h_l.(g.\Theta(q,\underline{t})) = \Theta(g.q,g.\underline{t})$. Notice that while showing the 'twisted' equivariance of Θ above, we have also proved that $\Theta(g.q,g.\underline{t}) = h(g,q)\Theta(q,\underline{t})$. Therefore we get

$$\begin{aligned}
&(h_lh_s(g)).(\underline{t}.\theta(q)) = h(g,q).\Theta(q,\underline{t})\\
&\Rightarrow (h_lh_s(g)).(\underline{t}.\theta(q)) = (h_s(g)h_l(g,q)).\Theta(q,\underline{t})\\
&\Rightarrow (h_s^{-1}(g)h_lh_s(g)).(\underline{t}.\theta(q)) = (h_l(g,q)\underline{t}).\theta(q)\\
&\Rightarrow \Theta(q,(g^{-1}.h_l)\underline{t}) = \Theta(q,h_l(g,q)\underline{t}).
\end{aligned}$$

But since Θ is an isomorphism by Theorem 1.11, this implies that $(g^{-1}.h_l)\underline{t} = h_l(g,q)\underline{t}$ which gives

$$h_l = h_s(g)h_l(g,q)h_s^{-1}(g) = h_l'(g,q).$$

Q.E.D., Theorem 3.4.

Proofs of Theorems 1.3 and 1.1: We remind the reader that towards the end of §2, we had reduced the proof of Theorem 1.1 to that of Theorem 1.3, and had indicated in page 14 that it would be enough to prove the latter in the case when V is free—which has been the case in this section so far. Starting with an isomorphism of algebra bundles $\phi : C_0(V,q) \cong C_0(V,q')$ we arrive at the element $g_l^{\phi} \in \mathrm{GL}(V)$ as defined in page 14; to briefly recall this, firstly $g \in \mathrm{GL}(V)$ was defined by the following commuting diagram:

$$\begin{tikzcd}
(C_0(V,q))^\vee \arrow[r, "((\psi_{b(q)})^\vee)^{-1}", "\cong"'] & (\Lambda^{even}(V))^\vee \arrow[r, "\text{surjection}"] & (\Lambda^2(V))^\vee \arrow[r, "(\eta^\vee)^{-1}", "\equiv"'] & V \otimes (\det(V))^{-1} \arrow[r, "\otimes\det(V)", "\equiv"'] & V \\
(C_0(V,q'))^\vee \arrow[u, "\phi^\vee", "\cong"'] \arrow[r, "((\psi_{b(q')})^\vee)^{-1}", "\cong"'] & (\Lambda^{even}(V))^\vee \arrow[u, "(\phi_{\Lambda^{ev}})^\vee", "\cong"'] & (\Lambda^2(V))^\vee \arrow[l, "\text{inclusion}"'] \arrow[u, "(\phi_{\Lambda^2})^\vee", "\cong"'] \arrow[r, "(\eta^\vee)^{-1}", "\equiv"'] & V \otimes (\det(V))^{-1} \arrow[u, "(g')^{-1}", "\cong"'] \arrow[r, "\otimes\det(V)", "\equiv"'] & V \arrow[u, "g^{-1}", "\cong"']
\end{tikzcd}$$

Secondly, we had defined $g_l^{\phi} := (l^{-1}\sqrt{\gamma(l)}\,)g$. Our current special choices of bilinear forms $b(q)$ and $b(q')$ that induce q and q' respectively do not affect the generality, as was observed in the proof of Prop.1.2. We shall now show that g_l^{ϕ} is a similitude from (V,q) to (V,q') with multiplier l and that this similitude induces ϕ i.e., with the notations of (1), Prop. 2.4, that $C_0(g_l^{\phi}, l) = \phi$. We shall also check that $\det^2(g_l^{\phi}) = \det(\phi_{\Lambda^2})$ when $\det(\phi_{\Lambda^2})$ is itself a square. We proceed with the proof which will follow from several lemmas.

Lemma 3.6 *Write the element* $h := \phi_{\Lambda^{ev}} \in (\mathrm{Stab}_w)(X)$ *uniquely as an ordered product* $h = h_s h_l = h_l' h_s$ *where* $h_l, h_l' \in (\mathsf{L}_w)(X)$ *and* $h_s \in (\mathrm{Stab}_{A_0})(X)$ *as explained in page 17; let* B *be the matrix corresponding to* ϕ_{Λ^2}*, and let the matrices* E_{ij} *be as defined in Lemma 3.3. Then we have matrix representations:*

$$h_s = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \text{ and } g_l^{\phi} = \left(l^{-1}\sqrt{\gamma(l)}\right) E_{23}E_{12}((B)^t)^{-1}E_{12}E_{23}$$

The proof of the above lemma follows from the fact that the matrix of the canonical isomorphism $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ is given by $E_{23}E_{12}$, which can be verified by a simple computation.

Lemma 3.7 $h_s(\mathrm{Identity}, l^{-1}, q', l^{-1}q') = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & l \times I_3 \end{pmatrix}$ and $h_s(\mathrm{Identity}, l^{-1}, q', l^{-1}q').\theta(q') = \theta(l^{-1}q')$.

The identity map on V is obviously a similarity with multiplier l^{-1} from (V,q') to $(V, l^{-1}q')$. Hence the above lemma follows by taking $T = X$, $g = \mathrm{Identity}$, and the l^{-1} and the q' at hand for the l and the q of Theorem 3.5 (caution: the q' there would have to be replaced by $l^{-1}q'$!). This can also be seen directly from the multiplication tables for $\theta(l^{-1}q') = \Theta(l^{-1}q', I_4)$ and $\theta(q') = \Theta(q', I_4)$ written out in Theorem 3.1, where we must take $T = X$ and $\underline{t} = I_4$ i.e., $t_i = 0\,\forall\, i$. We observe from the multiplication table that each of the coefficients of ϵ_i for $1 \leq i \leq 3$ is a single λ, whereas each coefficient of $w = 1 = \epsilon_0$ is a product of two λs, and this observation implies the lemma above. As the reader might have noticed, there are two crucial facts about the identifications in this section; namely, firstly, for any X-scheme T, each of the maps $\psi_{b(q)}$ (for different q) identify $(C_0(V_T,q),1)$ with the same (W_T, w_T) and secondly, relative to the bases chosen, all these identifying maps have trivial determinant. The latter is also true of the identification η, since it is given by the matrix $E_{23}E_{12}$ (as was noted after Lemma 3.6). It therefore follows that $\det(\phi) = \det(\phi_{\Lambda^{ev}}) = \det(\phi_{\Lambda^2}) = \det(g') = \det(g) = \det(B^{-1})$. But we had chosen $l \in \Gamma(X, \mathcal{O}_X^*)$ such that $\gamma(l) := (l^3).\det(\phi_{\Lambda^2}) = l^3\det(B)$. Using these facts alongwith Lemma 3.6 above, a straightforward computation gives the following.

Lemma 3.8 $\det(g_l^{\phi}) = \sqrt{\gamma(l)}$ *from which it follows that* $B(g_l^{\phi}) = l \times B$ *where* $B(g_l^{\phi})$ *and* B *are as defined in Lemmas 3.3 and 3.6 respectively. In particular,* $\det^2(g_l^{\phi}) = \det(\phi_{\Lambda^2})$ *when* $\det(\phi_{\Lambda^2})$ *is itself a square, since in this case we had chosen* $l := 1$.

Lemma 3.9 g_l^{ϕ} *is a similitude from* (V,q) *to* (V,q') *with multiplier* l.

The hypothesis $\phi : C_0(V,q) \cong C_0(V,q')$ is an algebra isomorphism translates in terms of actions into $h.\theta(q) = \theta(q')$ where $h = \phi_{\Lambda^{ev}} \in (\mathrm{Stab}_w)(X)$. Let $h(g_l^{\phi}, q) := h(g_l^{\phi}, 1, q, g_l^{\phi}.q)$ where $h(g, l, q, q')$ was defined in Theorem 3.5 above. Then we have

$$\begin{aligned}
\Theta(g_l^\phi.q, I_4) &:= \theta(g_l^\phi.q) = h(g_l^\phi,q).\theta(q) = h(g_l^\phi,q).(h^{-1}.\theta(q')) = \left(h_l'(g_l^\phi,q)h_s(g_l^\phi,q){h_s}^{-1}(h_l')^{-1}\right).\theta(q')\\
&= \left(h_l'(g_l^\phi,q)\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & B(g_l^\phi)\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & B^{-1}\end{pmatrix}(h_l')^{-1}\right).\theta(q') \quad \text{(by Theorem 3.5; Lemmas 3.3 \& 3.6)}\\
&= \left(h_l'(g_l^\phi,q)\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & l\times I_3\end{pmatrix}(h_l')^{-1}\right).\theta(q') \quad \text{(by Lemma 3.8)}\\
&= \left(h_l'(g_l^\phi,q)\, h_s(\text{Identity}, l^{-1}, q', l^{-1}q')\,(h_l')^{-1}\right).\theta(q') \quad \text{(by Lemma 3.7)}\\
&= (h_l'(g_l^\phi,q)h_l'').\left(h_s(\text{Identity}, l^{-1}, q', l^{-1}q').\theta(q')\right) \quad \text{(since }\mathrm{Stab}_w\text{ is a semidirect product)}\\
&= (h_l'(g_l^\phi,q)h_l'').\theta(l^{-1}q')) \quad \text{(by Lemma 3.7)}\\
&=: \Theta(l^{-1}q', (h_l'(g_l^\phi,q)h_l''))
\end{aligned}$$

But since Θ is an isomorphism, this implies the claim of the above Lemma i.e., that $g_l^\phi.q = l^{-1}q'$ (and further that $h_l'(g_l^\phi,q) = (h_l'')^{-1}$).

Lemma 3.10 *$g_l^\phi : (V,q)\cong_l(V,q')$ induces ϕ i.e., with the notations of (1), Prop.2.4, $C_0(g_l^\phi,l) = \phi$.*

We have $C_0(g_l^\phi,l)=\phi \iff h(g_l^\phi,l,q,q') := \psi_{b(q')}\circ C_0(g_l^\phi,l)\circ\psi_{b(q)}^{-1} = \psi_{b(q')}\circ\phi\circ\psi_{b(q)}^{-1} =: \phi_{\Lambda^{ev}} =: h$. Now using successively Theorem 3.5, Lemma 3.3, Lemma 3.8 and Lemma 3.6, we get the following sequence of equalities:

$$h_s(g_l^\phi,l,q,q') = \begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & l^{-1}\times\Lambda^2(g_l^\phi)\end{pmatrix} = \begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & l^{-1}\times B(g_l^\phi)\end{pmatrix} = \begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & B\end{pmatrix} = h_s$$

Therefore the present hypotheses translated in terms of actions give

$$\begin{aligned}
h(g_l^\phi,l,q,q').\theta(q) &= \theta(q') = h.\theta(q)\\
\Rightarrow h_s(g_l^\phi,l,q,q').(h_l(g_l^\phi,l,q,q').\theta(q)) &= h_s.(h_l.\theta(q))\\
\Rightarrow \Theta(q, h_l(g_l^\phi,l,q,q')) &= \Theta(q,h_l).
\end{aligned}$$

But Θ being an isomorphism, the last equality implies that $h_l(g_l^\phi,l,q,q') = h_l$ which gives $h(g_l^\phi,l,q,q') = h$ as wanted.

Lemma 3.11

(1) *For a similarity $g\in\mathrm{Sim}[(V,q),(V,q')]$ with multiplier l and the induced isomorphism $C_0(g,l)\in\mathrm{Iso}[C_0(V,q),C_0(V,q')]$ given by (1), Prop.2.4, we have the equality $\det((C_0(g,l))_{\Lambda^2}) = l^{-3}\det^2(g)$. Therefore the map*

$$\mathrm{Sim}[(V,q),(V,q')] \longrightarrow \mathrm{Iso}[C_0(V,q),C_0(V,q')] : g\mapsto C_0(g,l)$$

maps the subset $\mathrm{Iso}[(V,q),(V,q')]$ into the subset $\mathrm{Iso}'[C_0(V,q),C_0(V,q')]$ and $\text{S-Iso}[(V,q),(V,q')]$ into $\text{S-Iso}[C_0(V,q),C_0(V,q')]$.

(2) *In the case $q'=q$, if $C_0(g,l)$ is the identity on $C_0(V,q)$, then $g = l^{-1}\det(g)\times\mathrm{Id}_V$, and further if $g\in\mathrm{O}(V,q)$ then $g=\det(g)\times\mathrm{Id}_V$ with $\det^2(g)=1$.*

By definition, $(C_0(g,l))_{\Lambda^{ev}} = \psi_{b(q')}\circ C_0(g,l)\circ\psi_{b(q)}^{-1}$, and the latter isomorphism is $h(g,l,q,q')$ from Theorem 3.5 which further gives a formula for $h_s(g,l,q,q')$. Now using the facts that $\psi_{b(q)}$ and $\psi_{b(q')}$ have trivial determinant (as noted before Lemma 3.8) we get assertion (1):

$$\det\left((C_0(g,l))_{\Lambda^2}\right) = \det\left((C_0(g,l))_{\Lambda^{ev}}\right) = \det(h(g,l,q,q')) = \det(h_s(g,l,q,q')) = l^{-3}\det^2(g).$$

If $q=q'$ and $C_0(g,l)$ is the identity, then the same argument in fact shows that $l^{-1}\Lambda^2(g) = I_3$ and by using the formula in Lemma 3.3 for $B(g)=\Lambda^2(g)$, we get $g = l^{-1}\det(g)I_3$; taking determinants in the last equality gives $\det^2(g) = l^3$, so that when $g\in\mathrm{O}(V,q)$ i.e., $l=1$, $\det^2(g)\in\mu_2(\Gamma(X,\mathcal{O}_X))$ and assertion (2) follows.

Lemma 3.12 *The map $\text{S-Iso}[(V,q),(V,q')] \longrightarrow \text{S-Iso}[C_0(V,q),C_0(V,q')] : g\mapsto C_0(g,1,q,q')$ is a bijection.*

Given $\phi \in$ S-Iso$[C_0(V,q), C_0(V,q')]$, by the definition in Prop.1.2 we have $\det(\phi_{\Lambda^2}) = 1$, so by Lemma 3.8 $\det(g_l^\phi) = \sqrt{\gamma(l)} := 1$ by our earlier choice (see Case 1 on page 14). Therefore $g_l^\phi \in$ S-Iso$[(V,q),(V,q')]$ and is, according to Lemma 3.10, such that $C_0(g_l^\phi, l = 1, q, q') = \phi$ which gives the surjectivity. As for the injectivity, if $g_1, g_2 : (V,q)\cong_1(V,q')$ are isometries with determinant 1 such that $C_0(g_1, 1, q, q') = C_0(g_2, 1, q, q')$, then we have $h(g_1, 1, q, q') = h(g_2, 1, q, q')$ so that $h_s(g_1, 1, q, q') = h_s(g_2, 1, q, q')$ whence by Theorem 3.5 and Lemma 3.3 $B(g_1) = B(g_2) \Rightarrow g_1 = g_2$.

To sum up the above discussion: Let now V be a not-necessarily globally-trivial rank 3 vector bundle on the scheme X, and given ϕ, we define $l, \gamma(l), \sqrt{\gamma(l)}$ and g_l^ϕ globally as before. Then the statements of Lemmas 3.9, 3.10, 3.11 and 3.12 are again valid, for the objects involved are defined globally, but the assertions are of a local nature. Given an isomorphism of algebra bundles $\phi \in$ Iso$[C_0(V,q), C_0(V,q')]$, when $\det(\phi_{\Lambda^2}) \notin (\Gamma(X, \mathcal{O}_X^*))^2$, we have thus lifted ϕ to a family of similarities $s_{2k+1}(\phi) := g_l^\phi \in$ Sim$[(V,q),(V,q')]$ with multipliers $l := (\det(\phi_{\Lambda^2}))^{2k+1}$ parametrised by integers k; when $1 \neq \det(\phi_{\Lambda^2}) \in (\Gamma(X, \mathcal{O}_X^*))^2$, we have lifted ϕ to a similarity with multiplier $l = 1$ i.e., to an isometry $s'(\phi) := g_l^\phi \in$ Iso$[(V,q),(V,q')]$ such that $\det(\phi_{\Lambda^2}) = \det^2(g_l^\phi)$, and moreover, when ϕ_{Λ^2} has determinant 1, we have lifted it to a unique isometry g_l^ϕ with determinant 1. Of course, s_{2k+1} and s' are multiplicative only upto an element of $\mu_2(\Gamma(X, \mathcal{O}_X^*))$, but this can be remedied by considering s_{2k+1}^+ as noted at the end of section 2. **Q.E.D., Theorems 1.3 & 1.1.**

Proof of Theorem 1.4. Taking $q' = q$ in Theorem 1.3 gives the commutative diagram of groups and homomorphisms as asserted in the statement of the theorem. We continue with the notations used in the proof of Theorem 1.3. For $g \in$ GO(V,q) with multiplier l, the equality $\det(C_0(g,l)) = \det((C_0(g,l))_{\Lambda^2}) = l^{-3}\det^2(g)$ was proved in (1), Lemma 3.11. Assertion (2) of the same Lemma gives exactness at GO(V,q) and at O(V,q).

We proceed to prove assertion (b). Let $\phi \in$ Aut$(C_0(V,q))$, and consider the self-similarity $s_{2k+1}^+(\phi) = g_l^\phi$ with multiplier $l = \det(\phi)^{2k+1}$. For the moment, assume that V is trivial over X with global basis $\{e_1, e_2, e_3\}$, and set $e_i' = g_l^\phi(e_i)$. It follows from Kneser's definition of the half-discriminant d_0—see formula (3.1.4), Chap.IV, [2]—that $d_0(q, \{e_i\}) = d_0(q, \{e_i'\}) \det^2(g_l^\phi)$. Since we have $g_l^\phi.q = l^{-1}q$, a simple computation shows that $d_0(q, \{e_i'\}) = l^3 d_0(q, \{e_i\})$. The hypothesis that $q \otimes \kappa(x)$ is semiregular means that the image of the element $d_0(q, \{e_i\}) \in \Gamma(X, \mathcal{O}_X)$ in $\kappa(x)$ is nonzero. Since X is integral, we therefore deduce that $\det^2(g_l^\phi) = l^{-3}$. On the other hand, we know that $\det^2(g_l^\phi)l^{-3} = \det(\phi)$. It follows that $\det^{12k+7}(\phi) = 1 \,\forall\, k \in \mathbb{Z}$, which forces $\det(\phi) = 1$ as claimed. **Q.E.D., Theorem 1.4.**

4 Specialisations as Bilinear Forms: Theorems 1.9–1.12

In this section we reduce the proof of Theorem 1.9 to Theorem 1.10. We prove the latter and using it alongwith Theorem 3.1, deduce Theorem 1.12.

Preliminaries on the notion of Schematic Image.

Definition 4.1 (Defs.6.10.1-2, Chap.I, EGA I [6]) Let $f : X \longrightarrow Y$ be a morphism of schemes. If there exists a smallest closed subscheme $Y' \hookrightarrow Y$ such that the inverse image scheme $f^{-1}(Y') := Y' \times_Y ({}_f X)$ is equal to X, one calls Y' the *schematic image* of f (or of X in Y under f). If X were a subscheme of Y and f the canonical immersion, and if f has a schematic image Y', then Y' is called the *schematic limit* or the *limiting scheme* of the subscheme $X \overset{f}{\hookrightarrow} Y$.

PROPOSITION 4.2 (Prop.6.10.5, Chap.I, EGA I) *The schematic image Y' of X by a morphism $f : X \longrightarrow Y$ exists in the following two cases: (1) $f_*(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_Y$-module, which is for example the case when f is quasi-compact and quasi-separated; (2) X is reduced.*

PROPOSITION 4.3 *In each of the following statements whenever a schematic image is mentioned, we assume that one of the two hypotheses of the above Prop. is true so that the schematic image does exist.*

1. *Let Y' be the schematic image of X under a morphism $f : X \longrightarrow Y$ and let f factor as $X \xrightarrow{g} Y' \overset{j}{\hookrightarrow} Y$. Then Y' is topologically the closure of $f(X)$ in Y, the morphism g is schematically dominant (i.e., $g^\# : \mathcal{O}_{Y'} \longrightarrow g_*(\mathcal{O}_X)$ is injective) and the schematic image of X in Y' (under g) is Y' itself. If X is reduced (respectively integral) then the same is true of Y'.*

2. *The schematic image of a closed subscheme under its canonical closed immersion is itself.*

3. *(Transitivity of Schematic Image) Let there be given morphisms* $X \xrightarrow{f} Y \xrightarrow{g} Z$*, such that the schematic image* Y' *of* X *under* f *exists, and further such that if* g' *is the restriction of* g *to* Y'*, the schematic image* Z' *of* Y' *by* g' *exists. Then the schematic image of* X *under* $g \circ f$ *exists and equals* Z'*.*

4. *Let* $f : X \longrightarrow Y$ *be a morphism which factors through a closed subscheme* Y_1 *of* Y *by a morphism* $f_1 : X \longrightarrow Y_1$*. Then the scheme-theoretic image* Y' *of* X *in* Y *is the same as the scheme-theoretic image* Y_1' *of* X *in* Y_1 *considered canonically as closed subscheme of* Y*.*

5. *If* $f : X \longrightarrow Y$ *has a schematic image* Y' *then* f *is schematically dominant iff* $Y' = Y$*.*

6. *The formation of schematic image commutes with flat base change: i.e., if* $f : X \longrightarrow Y$ *is a morphism of* S*-schemes which has a schematic image* Y' *then for a flat morphism* $S' \longrightarrow S$*, one has that the induced* S'*-morphism* $f \times_S S' : X \times_S S' \longrightarrow Y \times_S S'$ *has a schematic image and it may be canonically identified with* $Y' \times_S S'$*. In particular this means that the formation of schematic image is local over the base.*

Assertions (1) and (3) are respectively Prop.6.10.5 and Prop.6.10.3 in EGA I. The defining property of schematic image gives (2), while (4) can be deduced from the first three. As for (5), from (1) it follows that $Y' = Y$ implies $f = g$ is schematically dominant. For the other way around, one uses the following characterisation of a schematically dominant morphism in Prop.5.4.1 of EGA I: if $f : X \longrightarrow Y$ is a morphism of schemes, then f is schematically dominant iff for every open subscheme U of Y and every closed subscheme Y_1 of U such that there exists a factorisation $f^{-1}(U) \xrightarrow{g_1} Y_1 \overset{j_1}{\hookrightarrow} U$, of the restriction $f^{-1}(U) \longrightarrow U$ of f (where j_1 is the canonical closed immersion), one has $Y_1 = U$—given f is schematically dominant, one just has to take $U = Y$, $Y_1 = Y'$ and $g_1 = g$. Assertion (6) follows from statement (ii) (a) of Theorem 11.10.5 of EGA IV [7].

Preliminaries from Part A of [3]. Until further notice we assume that W is a vector bundle of fixed positive rank on the scheme X. Given any X-scheme T, by a T-algebra structure on $W_T := W \times_X T$ (also referred to as T-algebra bundle), we mean a morphism $W_T \times_T W_T \longrightarrow W_T$ of vector bundles on T arising from a morphism of the associated locally-free sheaves. Given such a T-algebra structure and $T' \longrightarrow T$ an X-morphism, it is clear that one gets by pullback (i.e., by base-change) a canonical T'-algebra structure on $W_{T'}$. Thus one has a contravariant "functor of algebra structures on W" from $\{X - Schemes\}$ to $\{Sets\}$ denoted Alg_W whose set of T-valued points is the set of T-algebra structures on W_T viz. $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{W}_T \otimes \mathcal{W}_T, \mathcal{W}_T)$. By Prop.9.6.1, Chap.I of EGA I [6], it follows that the functor Alg_W is represented by the X-scheme

$$\mathrm{Alg}_W := \mathrm{Spec}\left(\mathsf{Sym}_X\left[\left(\mathcal{W}_X{}^\vee \otimes_X \mathcal{W}_X{}^\vee \otimes_X \mathcal{W}_X\right)^\vee\right]\right).$$

Hence Alg_W is affine (hence separated), of finite presentation over X and in fact smooth of relative dimension $\mathrm{rank}_X(W)^3$. If $X' \longrightarrow X$ is an extension of base, then the construction Alg_W base-changes well i.e., one may canonically identify $\mathrm{Alg}_W \times_X X'$ with $\mathrm{Alg}_{W'}$ where $W' = W \times_X X'$ (cf. Prop.9.4.11, Chap.I, EGA I [6]). We remark that an algebra structure may fail to be associative and may fail to have a (two-sided) identity element for multiplication. However, a multiplicative identity for an associative algebra structure must be a nowhere vanishing section (Lemma 2.3, and (2)⇒(4) of Lemma 2.4, Part A, [3]).

The general linear groupscheme associated to W viz GL_W naturally acts on Alg_W on the left, so that for each X-scheme T, $\mathrm{Alg}_W(T)$ mod $\mathrm{GL}_W(T)$ is the set of isomorphism classes of T-algebra structures on $W \times_X T$.

Let $w \in \Gamma(X, \mathcal{W})$ be a nowhere vanishing section. For any X-scheme T, let $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_W(T)$ denote the subset of $\mathrm{Alg}_W(T)$ consisting of associative algebra structures with multiplicative identity the nowhere vanishing section w_T over T induced from w. Thus we obtain a contravariant subfunctor $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_W$ of Alg_W. Let $\mathrm{Stab}_w(T) \subset \mathrm{GL}_W(T)$ denote the stabiliser subgroup of w_T, so that one gets a subfunctor in subgroups $\mathrm{Stab}_w \subset \mathrm{GL}_W$. It is in fact represented by a closed subgroupscheme (also denoted by) Stab_w and further behaves well under base change relative to X i.e., $\mathrm{Stab}_w \times_X T$ can be canonically identified with Stab_{w_T} for any X-scheme T. These follow from para 9.6.6 of Chap.I, EGA I [6]. It is clear that the natural action of GL_W on Alg_W induces one of Stab_w on $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_W$. It is easy to check (p.489, Part A, [3]) that the functor $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_W$ is a sheaf in the big Zariski site over X and further that this functor is represented by a natural closed subscheme $\mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_W \hookrightarrow \mathrm{Alg}_W$ which is Stab_w-invariant. Further the construction Id-w-Assoc$_W$ behaves well with respect to base-change (relative to X). Consider the subfunctor Id-w-Azu$_W$ $\hookrightarrow$ Id-w-Assoc$_W$ corresponding to Azumaya algebras.

THEOREM 4.4 (Theorem 3.4, Part A, [3])

(1) Id-w-Azu$_W$ *is represented by a* Stab$_w$*-stable open subscheme* Id-w-Azu$_W$ $\hookrightarrow$ Id-w-Assoc$_W$ *and the canonical open immersion is an affine morphism.*

(2) Id-w-Azu$_W$ *is affine (hence separated) and of finite presentation over X, and the construction* Id-w-Azu$_W$ *behaves well with respect to base-change (relative to X).*

(3) *Further,* Id-w-Azu$_W$ *is smooth of relative dimension $(m^2-1)^2$ and geometrically irreducible over X, where $m^2 := rank_X(\mathcal{W})$.*

THEOREM 4.5 (Theorem 3.8, Part A, [3])

(1) *The open immersion* Id-w-Azu$_W$ $\hookrightarrow$ Id-w-Assoc$_W$ *has a schematic image denoted* Id-w-Sp-Azu$_W$ *which is affine (hence separated) and of finite type over X and is naturally a* Stab$_w$*-stable closed subscheme of* Id-w-Assoc$_W$*, the action extending the natural one on the open subscheme* Id-w-Azu$_W$*.*

(2) *When the rank of W over X is 4,* Id-w-Sp-Azu$_W$ *is locally (over X) isomorphic to relative 9-dimensional affine space; in fact over every open affine subscheme U of X where W becomes trivial and w becomes part of a global basis,* Id-w-Sp-Azu$_W|_U \cong \mathbb{A}^9_U$*. Thus* Id-$w$-Sp-Azu$_W$ *is smooth of relative dimension 9 and geometrically irreducible over X. In particular, it is of finite presentation over X.*

(3) *When $rank_X(W) = 4$, the construction* Id-w-Sp-Azu$_W \longrightarrow X$ *base changes well.*

Definition of the Morphism Υ'. We adopt the notations of Theorem 1.9. Let T be an X-scheme. Given a bilinear form $b \in \mathrm{Bil}_V(T) = \mathrm{Bil}(V_T)$, consider the linear isomorphism $\psi_b : C_0(V_T, q_b) \cong \Lambda^{even}(V_T) = W_T$ of (2d), Theorem 2.1. Let A_b denote the algebra bundle structure on W_T with unit $w_T = 1$ induced via ψ_b from the even Clifford algebra $C_0(V_T, q_b)$. By definition, $A_b \in$ Id-w-Assoc$_W(T)$ and we get a map of T-valued points

$$\Upsilon'(T) : \mathrm{Bil}_V(T) \longrightarrow \text{Id-}w\text{-Assoc}_W(T) : b \mapsto A_b.$$

This is functorial in T because of (3), Theorem 2.1, and hence defines an X-morphism $\Upsilon' : \mathrm{Bil}_V \longrightarrow$ Id-w-Assoc$_W$. The morphism Υ' is GL_V-equivariant due to (2j), Theorem 2.1.

Semiregular Bilinear forms. Fundamental problems in dealing with quadratic forms over arbitrary base schemes arise essentially from two abnormalities in characteristic two: firstly, the mapping that associates a quadratic form to its symmetric bilinear form is not bijective and secondly, there do not exist regular quadratic forms on any odd-rank bundle. The remedy for this is to consider semiregular quadratic forms, a concept due to M.Kneser [1] and elaborated upon by Knus in [2], which in fact works over an arbitrary base scheme (and hence in a characteristic-free way) and further reduces to the usual notion of regular form in characteristics $\neq 2$. Let $\mathrm{Spec}(R) = U \hookrightarrow X$ be an open affine subscheme of X such that $V|U$ is trivial. Consider a quadratic form $q \in \Gamma(U, \mathrm{Quad}_V)$ on $V|U$ and its associated symmetric bilinear form b_q. The matrix of this bilinear form relative to any fixed basis is a symmetric matrix of odd rank and in particular, if U is of pure characteristic two (i.e., the residue field of any point of U is of characteristic two), then this matrix is also alternating and is hence singular, immediately implying that q cannot be regular. However, computing the the determinant of such a matrix in *formal variables* $\{\zeta_i, \zeta_{ij}\}$ shows that it is twice the following polynomial

$$P_3(\zeta_i, \zeta_{ij}) = 4\zeta_1\zeta_2\zeta_3 + \zeta_{12}\zeta_{13}\zeta_{23} - (\zeta_1\zeta_{23}^2 + \zeta_2\zeta_{13}^2 + \zeta_3\zeta_{12}^2).$$

The value $P_3(q(e_i), b_q(e_i, e_j))$ corresponding to a basis $\{e_1, e_2, e_3\}$ is called the *half-discriminant* of q relative to that basis, and q is said to be semiregular if its half-discriminant is a unit. It turns out that this definition is independent of the basis chosen (§3, Chap.IV, [2]). Even if $V|U$ were only projective (i.e., locally-free but not free), the semiregularity of q may be defined as the semiregularity of $q \otimes_R R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$, and it turns out that with this definition, the notion of a quadratic form being semiregular is local and is well-behaved under base-change (Prop.3.1.5, Chap.IV, [2]). We may thus define the subfunctor of Quad_V of semiregular quadratic forms. This subfunctor is represented by a GL_V-invariant open subscheme $i : \mathrm{Quad}_V^{sr} \hookrightarrow \mathrm{Quad}_V$ because, over each affine open subscheme $U \hookrightarrow X$ which trivialises V, it corresponds to localisation by the non-zerodivisor P_3. Note that this canonical open immersion is affine and schematically dominant as well. We next turn to semiregular bilinear forms. Recall that we had defined a bilinear form b to be semiregular iff its induced quadratic form q_b is semiregular. Thus by definition, Bil_V^{sr} is the fiber product:

$$\begin{array}{ccc} \mathrm{Bil}_V & \xrightarrow{p} & \mathrm{Quad}_V \\ {\scriptstyle i'}\uparrow & & \uparrow{\scriptstyle i} \\ \mathrm{Bil}_V^{sr} & \xrightarrow[p']{} & \mathrm{Quad}_V^{sr} \end{array}$$

Since p is a Zariski-locally-trivial principal Alt_V^2-bundle, it is smooth and surjective (in particular faithfully flat). It therefore follows that the affineness and schematic dominance of i imply those of i'. We record these facts below.

PROPOSITION 4.6 *The open immersion $\mathrm{Bil}_V^{sr} \hookrightarrow \mathrm{Bil}_V$ is a GL_V-equivariant schematically dominant affine morphism. Further this open immersion behaves well under base-change (relative to X).*

Reduction of Proof of Theorem 1.9 to Theorem 1.10. We first recall the following crucial fact (see (1), Prop.3.2.4, Chap.IV [2]): *The even Clifford algebra of a semiregular quadratic form is an Azumaya algebra.* Using this fact and the definition of Υ', we see that the morphism Υ' restricted to Bil_V^{sr} factors through Id-w-Azu_W by a morphism Υ^{sr} such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Bil}_V & \xrightarrow{\Upsilon'} & \text{Id-}w\text{-}\mathrm{Assoc}_W \\ \uparrow & & \uparrow \\ \mathrm{Bil}_V^{sr} & \xrightarrow[\Upsilon^{sr}]{} & \text{Id-}w\text{-}\mathrm{Azu}_W \end{array}$$

where the vertical arrows are the canonical open immersions. The above diagram base changes well in view of (2), Theorem 4.4, Prop.4.6 and (3), Theorem 2.1. Notice that since the structure morphism $\mathrm{Bil}_V \longrightarrow X$ is an affine morphism, and since the same is true of Id-w-$\mathrm{Assoc}_W \longrightarrow X$, it is also true of Υ'. In particular, Υ' is quasi-compact and separated, and therefore has a schematic image by case (1) of Prop.4.2. The same is true of each of the two vertical arrows and of Υ^{sr} in view of Prop.4.6 and (1) of Theorem 4.4. Further, as noted in Prop.4.6, $\mathrm{Bil}_V^{sr} \hookrightarrow \mathrm{Bil}_V$ is schematically dominant and therefore by (5), Prop.4.3, the limiting scheme of the former in the latter is the latter itself. So using the commutative diagram above, the transitivity of the schematic image (assertion (3), Prop.4.3), and the definition of Id-w-Sp-Azu_W (assertion (1), Theorem 4.5), we see that in order to prove (1), Theorem 1.9, it is enough to show that

(*) Υ^{sr} is schematically dominant and surjective, and Υ' is a closed immersion.

We now claim that the above properties are equivalent to

(**) Υ^{sr} is proper and Υ' is a closed immersion.

Suppose **(**)** holds. To show **(*)**, we only need to show that Υ^{sr} is surjective and schematically dominant. From **(**)** it follows that $\Upsilon_K^{sr} := \Upsilon^{sr} \otimes_X K$ is functorially injective and proper for each algebraically closed field K with an X-morphism $\mathrm{Spec}(K) \longrightarrow X$.

That both the K-schemes $\mathrm{Bil}_V^{sr} \otimes_X K$ and Id-w-$\mathrm{Azu}_W \otimes_X K$ are integral and smooth of the same dimension follows from the smoothness of relative dimension 9 and geometric irreducibility $/X$ of Bil_V^{sr} (which is obvious), and of Id-w-Azu_W from (3), Theorem 4.4. Since Υ_K^{sr} is differentially injective at each closed point, it has to be a smooth morphism by Theorem 17.11.1 of EGA IV [7] and thus has to be an open map. But by **(**)** it is also proper and hence a closed map. Thus Υ_K^{sr} is bijective etale, and hence an isomorphism. This also gives that Υ^{sr} is surjective. Now from Cor.11.3.11 of EGA IV and from the flatness of Bil_V^{sr} over X, it follows that Υ^{sr} is itself flat, and hence schematically dominant since it is faithfully flat (being already surjective). Therefore **(**)** $\Longrightarrow$ **(*)**.

The property of a morphism being proper is local on the target (see for e.g., (f), Cor.4.8, Chap.IV, [9]) and the same is true of the property of being a closed immersion. Therefore, in verifying (**), we may assume that V is free over X (so that $W = \Lambda^{even}(V)$ is also free over X and and w is part of a global basis). We are now in the situation of Theorem 1.10. Granting it, we see immediately from the multiplication table that **(**)** holds. For the table shows that the composition of the following X-morphisms

$$\mathrm{Bil}_V \xrightarrow{\Upsilon'} \text{Id-}w\text{-}\mathrm{Assoc}_W \overset{\text{CLOSED}}{\hookrightarrow} \mathrm{Alg}_W$$

is a closed immersion, which implies that Υ' is also a closed immersion. Further, the multiplication table also shows that both Υ' and Υ^{sr} satisfy the valuative criterion for properness, and are therefore proper. Thus the conditions **(**)** are verified. So we have reduced the proof of (1), Theorem 1.9 to Theorem 1.10. As for statement (2) of Theorem 1.9, firstly, the involution Σ of Id-w-Assoc-Alg$_W$ defines a unique involution (also denoted by Σ) on the scheme of specialisations Id-w-Sp-Azu$_W$ (leaving the open subscheme Id-w-Azu$_W$ invariant) because of the defining property of the schematic image involved; for we may verify that an automorphism of a scheme T which leaves an open subscheme U stable will also leave stable the limiting scheme of U in T (of course we assume here that the canonical open immersion $U \hookrightarrow T$ is a quasi-compact open immersion, which ensures the existence of the limiting scheme). Secondly, a glance at the multiplication table of Theorem 1.10 keeping in view the definition of opposite algebra shows that indeed the induced $\Sigma \in \mathrm{Aut}_X(\mathrm{Bil}_V)$ takes the T-valued point $B = (b_{ij})$ to transpose$(-B) = (-b_{ji})$. Finally, assertion (3) of Theorem 1.9 is a consequence of (1) taking into account (3), Theorem 4.5.

Proof of Theorem 1.10. Given $B = (b_{ij}) \in \mathrm{Bil}_V(T)$, by our definition above, $(\Upsilon'(T))(B) = A_B$ is the algebra structure induced from the linear isomorphism $\psi_B : C_0(V_T, q_B) \cong \Lambda^{even}(V_T)$ of (2d), Theorem 2.1. The stated multiplication table for $A = A_B$ is a consequence of straightforward calculation, keeping in mind (2f), Theorem 2.1 and the standard properties of the multiplication in the even Clifford algebra $C_0(V_T, q_B)$. **Q.E.D., Theorems 1.10 & 1.9.**

Proof of Theorem 1.12. The proof follows by comparing the multiplication table relative to Θ as computed in Theorem 3.1 with the multiplication table relative to Υ of Theorem 1.10 computed above. **Q.E.D., Theorem 1.12.**

5 Properties of Specialisations/Self-duality: 1.5–1.8

In this section, we prove Theorem 1.5 and use it alongwith Theorem 1.9 to prove Theorem 1.6. Thereafter we investigate the specialised algebras when the underlying bundle is self-dual and prove Theorems 1.7 and 1.8.

Proof of Theorem 1.5. Let W be the rank 4 vector bundle underlying the specialised algebra A and $w \in \Gamma(X, W)$ be the global section corresponding to 1_A. We choose an affine open covering $\{U_i\}_{i\in I}$ of X such that $W|U_i$ is trivial and $w|U_i$ is part of a global basis $\forall\, i$. Therefore on the one hand, for each $i \in I$, we can find a linear isomorphism $\zeta_i : \Lambda^{even}\left(\mathcal{O}_X^{\oplus 3}|U_i\right) \cong W|U_i$ taking $1_{\Lambda^{even}}$ onto $w|U_i$. The $(w|U_i)$-unital algebra structure $A|U_i$ induces via ζ_i an algebra structure A_i on $\Lambda^{even}\left(\mathcal{O}_X^{\oplus 3}|U_i\right)$ (so that ζ_i becomes an algebra isomorphism). Since A_i is also a specialised algebra structure ((3), Theorem 4.5), by Theorem 1.9 we can on the other hand also find a quadratic form q_i on $\mathcal{O}_X^{\oplus 3}|U_i$ induced from a bilinear form b_i so that the algebra structure A_i is precisely the one induced by the linear isomorphism $\psi_{b_i} : C_0\left(\mathcal{O}_X^{\oplus 3}|U_i, q_i\right) \cong \Lambda^{even}\left(\mathcal{O}_X^{\oplus 3}|U_i\right)$ given by (2d) of Theorem 2.1. For each pair of indices $(i,j) \in I \times I$, let ζ_{ij} and ϕ_{ij} be defined so that the following diagram commutes:

$$\begin{array}{ccccc}
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) & \xrightarrow[\cong]{\psi_{b_i}|U_{ij}} & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) & \xrightarrow[\cong]{\zeta_i|U_{ij}} & A|U_{ij} \\
{\scriptstyle \phi_{ij}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \zeta_{ij}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle =}\big\downarrow \\
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij}) & \xrightarrow[\psi_{b_j}|U_{ij}]{\cong} & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) & \xrightarrow[\zeta_j|U_{ij}]{\cong} & A|U_{ij}
\end{array}$$

The above diagram means that the algebras A_i glue along $U_{ij} := U_i \cap U_j$ via ζ_{ij} to give (an algebra bundle isomorphic to) A, and in the same vein, the even Clifford algebras $C_0(\mathcal{O}_X^{\oplus 3}|U_i, q_i)$ glue along the U_{ij} via ϕ_{ij} to give A as well. Now consider the similarity $g_{l_{ij}}^{\phi_{ij}} = s_{-1}^{+}(\phi_{ij}) : (\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) \cong_{l_{ij}} (\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij})$ with multiplier $l_{ij} := \det(\phi_{ij})^{-1}$ given by (c), Theorem 1.3. Since s_{-1}^{+} is multiplicative, and since ϕ_{ij} satisfy the cocycle condition, it follows that $s_{-1}^{+}(\phi_{ij})$ also satisfy the cocycle condition and therefore glue the $\mathcal{O}_X^{\oplus 3}|U_i$ along the U_{ij} to give a rank 3 vector bundle V on X. (Observe that unfortunately the q_i do not glue!) We shall now revert to the notations of Section 3. By Theorem 3.5, we have

$$h_s(g_{l_{ij}}^{\phi_{ij}}, l_{ij}, q_i|U_{ij}, q_j|U_{ij}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & l_{ij}^{-1}\Lambda^2(g_{l_{ij}}^{\phi_{ij}}) \end{pmatrix}$$

which means that $(\phi_{ij})_{\Lambda^2} = \det(\phi_{ij}) \times \Lambda^2(g_{l_{ij}}^{\phi_{ij}})$. This immediately implies part (1) of assertion (a) of Theorem 1.5, from which parts (2)—(4) can be deduced using the standard properties of the determinant and the perfect pairings between suitable exterior powers of a bundle. To prove (b), let $A = C_0(V, q)$. Using a trivialisation $\{U_i\}_{i\in I}$ for the quadratic bundle (V, q) with transition functions g_{ij}, and remembering that by definition the g_{ij} are isometries, we have the following commutative diagram on the intersections U_{ij}:

$$\begin{array}{ccc} C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) & \xrightarrow[\cong]{\psi_{b(q_i)}|U_{ij}} & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) \\ {\scriptstyle C_0(g_{ij},1,q_i|U_{ij},q_j|U_{ij})}\Big\downarrow\cong & & \cong\Big\downarrow{\scriptstyle h(g_{ij},1,q_i|U_{ij},q_j|U_{ij})} \\ C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij}) & \xrightarrow[\psi_{b(q_j)}|U_{ij}]{\cong} & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) \end{array}$$

But by the formula in Theorem 3.5, we have

$$h_s(g_{ij}, 1, q_i|U_{ij}, q_j|U_{ij}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \Lambda^2(g_{ij}) \end{pmatrix}$$

which immediately implies that $C_0(V, q)/\mathcal{O}_X.1_{C_0} \cong \Lambda^2(V)$. **Q.E.D., Theorem 1.5.**

Proof of 'if' part of Theorem 1.6. Suppose that the hypotheses hold; by (1) $\exists L \in \mathrm{Pic}(X)$ such that $\det(A) \cong L^{\otimes 2}$ and by (2), we may choose a splitting $A \cong \mathcal{O}_X.1_A \oplus (A/\mathcal{O}_X.1_A)$. Using assertion (1) of (a), Theorem 1.5, we see that there exists a rank 3 vector bundle V' on X such that

$$A \cong \mathcal{O}_X.1_A \oplus (A/\mathcal{O}_X.1_A) \cong \mathcal{O}_X.1_A \oplus (L^{\otimes 2} \otimes \Lambda^2(V')) \cong \mathcal{O}_X.1_A \oplus \Lambda^2(V' \otimes L) \cong \Lambda^{even}(V)$$

where $V := V' \otimes L$ and the last isomorphism is chosen so as to map $\mathcal{O}_X.1_A$ isomorphically onto $\mathcal{O}_X.1_{\Lambda^{ev}(V)}$. Therefore if $(W, w) := (\Lambda^{even}(V), 1)$, then by the above identification A induces an element of Id-w-Sp-Azu$_W(X)$, and since $\Upsilon : \mathrm{Bil}_V \cong$ Id-w-Sp-Azu$_W$ is an X-isomorphism by (1), Theorem 1.9, it follows that there exists a global quadratic form $q = q_b$ induced from a bilinear form b such that the algebra structure $\Upsilon(b) \cong A$. To finish the proof, we only have to remember that $\Upsilon(b)$ is the algebra structure induced from the linear isomorphism $\psi_b : C_0(V, q = q_b) \cong \Lambda^{even}(V) =: W$ of (2d), Theorem 2.1, which preserves 1 by (2a) of the same Theorem.

Proof of 'only if' part of Theorem 1.6. Suppose $\exists\,(V, q)$ such that $C_0(V, q) \cong A$, and that q is induced from a bilinear form b. Then by (2d), Theorem 2.1, we have a linear isomorphism $\psi_b : C_0(V, q = q_b) \cong \Lambda^{even}(V)$. Since ψ_b preserves 1 by (2a) of the same Theorem, we get (2). Further (1) follows from assertion (b) of Theorem 1.5.

Proofs of assertions on validity of hypotheses (1) & (2). If (i) holds, the validity of (2) is a simple exercise in commutative algebra. Since A is a quaternion algebra bundle by Prop.1.13, we have the associated trace map $tr_{\sigma_A} : A \longrightarrow \mathbb{A}^1_X$. Since (ii) is equivalent to $2 \in \Gamma(X, \mathcal{O}_X^*)$, when (ii) holds, the map $2^{-1}tr_{\sigma_A}$ splits the canonical inclusion $\mathcal{O}_X.1_A \hookrightarrow A$, showing that (2) holds. If A is Azumaya, then as we saw in the discussion following Theorem 1.14, $A \cong C_0(V, q)$ with (V, q) semiregular. So (1) follows from (b), Theorem 1.5. **Q.E.D., Theorem 1.6.**

Relations with Self-Duality. In the following we let X be a scheme and W a rank 4 vector bundle over X. We recall the following result:

PROPOSITION 5.1 (Prop.3.3, Part A, [3])

1. *Let T be an X-scheme and A an associative unital algebra structure on $W_T := W \otimes_X T$. Then the subset*

$$U(T, A) := \{t \in T \mid \mathcal{A}_t \textit{ is an Azumaya } \mathcal{O}_{T,t} - \textit{algebra}\}$$

is an open (possibly empty) subset. When $U(T, A)$ is nonempty, denote by the same symbol the canonical open subscheme structure. Then if $f : T' \longrightarrow T$ is an X-morphism such that the topological image intersects $U(T, A)$, then $U(T', f^*(A) = A \otimes_T T') \cong U(T, A) \times_T T'$ as open subschemes of T'. Further $U(T, A) \hookrightarrow T$ is an affine morphism.

2. *$U(T, A)$ is the maximal open subset restricted to which A is Azumaya.*
3. *Further let $f : T' \longrightarrow T$ be a morphism of X-schemes such that $f^*(A)$ is Azumaya. Then f factors through the open subscheme $U(T, A)$ defined above.*

Proof of Theorem 1.7. We first make certain observations when X is any reduced scheme and W is a rank n^2 vector bundle over X with nowhere-vanishing global section w. It is not hard to see that the structure morphism Id-w-Azu$_W \longrightarrow X$ is in fact a morphism of finite presentation. Hence, in view of Prop.17.5.7, EGA IV [7], assertion (3) of Theorem 4.4 implies that Id-w-Azu$_W$ is reduced. Since Id-w-Sp-Azu$_W$ is the schematic image of Id-w-Azu$_W$, it follows from (1), Prop.4.3 that Id-w-Sp-Azu$_W$ is reduced as well. When X is integral, it is easy to see that Id-w-Azu$_W$ is integral as well. Consider any affine open subscheme $U = \mathrm{Spec}(R) \hookrightarrow X$ such that $W|U$ is trivial and $w|U$ is part of a global basis. There are $(w|U)$-unital Azumaya algebra structures on $W|U$ which are isomorphic to the $(n \times n)$-matrix algebra over R. Consider the orbit morphism $\mathrm{Stab}_{w|U} \longrightarrow$ Id-$(w|U)$-Azu$_{W|U}$ corresponding to one such algebra structure. Assertions (2) and (3) of Theorem 4.4 show that the topological image of this morphism is dense. Further, $\mathrm{Stab}_{w|U}$ is integral since it is $\cong \mathbb{A}^{12}_U$ and since U is integral. Thus Id-$(w|U)$-Azu$_{W|U}$ is integral. Recall (e.g., Prop.2.1.6 & Cor.2.1.7, Chap.0.2, EGA I [6]) that a nonempty topological space—whose set of irreducible components is locally finite—is locally irreducible iff each irreducible component is open; further it is irreducible iff it is locally irreducible and connected. Now since Id-w-Azu$_W$ can be covered by irreducible open subschemes which pairwise intersect (as X is irreducible), it follows that Id-w-Azu$_W$ is integral as well. Therefore (1), Prop.4.3 implies that Id-w-Sp-Azu$_W$ is also integral in the case when X is integral. Using again the facts that Id-w-Azu$_W$ behaves well under base change and that the schematic image of an irreducible scheme is irreducible (by (1), Prop.4.3), we may infer from the foregoing that X is irreducible iff Id-w-Azu$_W$ is irreducible iff Id-w-Sp-Azu$_W$ is irreducible.

We next observe that assertion (b) follows from (a) and from the following result: *The vector bundle underlying a rank 4 Azumaya algebra bundle A is self-dual.* To see this, first remember from Prop.1.13 that A comes with a standard involution $\sigma = \sigma_A$. The associated norm $n_\sigma : x \mapsto x.\sigma(x)$ is a (global) quadratic form with values in $\mathcal{O}$ and its associated bilinear form b_{n_σ} may be given in terms of the associated trace tr_σ as $b_{n_\sigma}(x, x') = tr_\sigma(x\sigma(x')) = x\sigma(x') + x'\sigma(x)$. Now n_σ is nonsingular (=regular), as may be verified locally over affine open subschemes—for details see para 7.3.5, Chap.I, [2].

As for assertion (a), we begin with a general remark: if $X' \longrightarrow X$ is a surjective k-morphism, k being the algebraically closed base field, and if (a) holds for the pair $(X', A' := A \otimes_X X')$, then it also holds for the pair (X, A) where A is an associative unital algebra structure on W. For, by (3), Prop.5.1, $U(X', A') = X' \Longrightarrow U(X, A) = X$. Since the canonical morphism $X_{RED} \hookrightarrow X$ is a surjective closed immersion, we may therefore assume that X is reduced. Let $x \in U(X, A) \cap X_\alpha$ where X_α is an irreducible component of X given the canonical reduced induced closed subscheme structure. If (a) holds for $(X_\alpha, A \otimes_X X_\alpha)$, then by (1), Prop.5.1, we have that $A \otimes_X X_\alpha$ is Azumaya, and hence by (3) of the same Proposition, the canonical closed immersion $X_\alpha \hookrightarrow X$ factors through $U(X, A)$. Now as X is connected, we observe that (a) holds for X if it holds for each irreducible component given the reduced induced closed subscheme structure (which is also a proper scheme of finite type over k); thus we may as well assume that X is irreducible. By Chow's Lemma (Theorem 5.6.1, EGA II [8]) there exists an integral projective scheme X' of finite type over k alongwith a surjective projective k-morphism $X' \longrightarrow X$. Therefore we may further assume that X is projective. Finally, we may also assume that X is normal, for if $\widetilde{X}$ is the normalisation of X, then the canonical morphism $\widetilde{X} \longrightarrow X$ is finite and surjective, hence also projective. So we have reduced to the case when X is an integral normal projective scheme over k. However, the following proof works with the hypothesis of X being regular in codimension 1, which is satisfied when X is normal.

Now given an associative w-unital algebra structure A on W such that the open subset $U(X, A)$ of Prop.5.1 is nonempty, the restriction to the dense open subset $U(X, A)$ of the section over X corresponding to A factors through Id-w-Azu$_W$ as a morphism (and not just topologically); further, since both X and Id-w-Sp-Azu$_W$ are reduced, this section factors through Id-w-Sp-Azu$_W$ itself. Thus any A such that $U(X, A) \neq \emptyset$ corresponds to a specialised algebra structure. Let there be given an isomorphism of vector bundles $\phi : W \cong W^\vee$ and an A such that $U(X, A) \neq \emptyset$. To prove (a), we must show that $U(X, A) = X$. We proceed by contradiction: suppose $D(X, A) := X \backslash U(X, A)$ is nonempty. Consider the composition of the following morphisms of vector bundles:

$$W \otimes_X W \equiv A \otimes_X A^{op} \overset{(a\otimes b)\mapsto(x\mapsto axb)}{\longrightarrow} \mathrm{End}_{Linear}(A) \equiv W^\vee \otimes_X W \overset{\phi\times Id}{\longrightarrow} W \otimes_X W$$

The above composite gives an endomorphism of the vector bundle $W \otimes W$ which is an isomorphism precisely at the local rings of the points of $U(X, A)$. Therefore, the induced element $s(\phi) \in \Gamma(X, \mathcal{O}_X) \equiv \mathrm{H}^0(X, \mathrm{End}(\det(W \otimes W)))$ goes into the maximal ideal of the local ring at each point of $D(X, A)$ and to a unit of the local ring at each point of $U(X, A)$. It follows that if $D(X, A)$ is the irredundant union of divisors $\{D_i | 1 \leq i \leq n\}$, then the divisor defined by $(s(\phi), \mathcal{O}_X)$ namely $\Sigma_i n_i D_i$ is principal. Under the present hypotheses on X, every principal divisor has degree zero (see for e.g., (d), Ex.6.2, Chap.II, [9]), so that we get $0 = \Sigma_i n_i.(\text{degree}(D_i))$. This is a contradiction, since $n_i \geq 1$ as it is the order of vanishing of $s(\phi)$ on D_i for each i, and since the degree of each D_i is positive (see for e.g., (a), Prop.7.6, Chap.I, [9]). **Q.E.D., Theorem 1.7.**

Proof of assertions (a)—(d) of Theorem 1.8. The proofs follow essentially from the properties of Id-w-Sp-Azu$_W \longrightarrow X$ and Id-w-Azu$_W \longrightarrow X$ as mentioned in Theorems 4.5 and 4.4. In this regard see also the first paragraph of the proof of Theorem 1.7. We indicate proofs for the not-so-obvious assertions, especially (c) and for the implication (X irreducible $\Rightarrow D_X$ irreducible) of (a). The converse implication would follow from the fact that $D_X \longrightarrow X$ is topologically surjective. Since the structure morphisms Id-w-Sp-Azu$_W \longrightarrow X$ and Id-w-Azu$_W \longrightarrow X$ are smooth, they are faithfully flat (hence surjective). Therefore in view of (b)—which is actually the result of good base-change properties of Id-w-Sp-Azu$_W$ and Id-w-Azu$_W$ relative to X, the irreducible components of D_X and of Id-w-Sp-Azu$_W$ are induced from those of X by pullback, provided we check the particular case when X is irreducible and reduced.

So let X be integral and first assume that W is trivial and w is part of a global basis. Without loss of generality, we may take $(W, w) = (\Lambda^{even}(V), 1)$ for V a rank 3 trivial vector bundle over X. After fixing a suitable basis for V, we may define the morphism Θ, which by Theorem 1.11 is an isomorphism that maps the closed subset $(\text{Quad}_V \times_X \mathsf{L}_w) \backslash (\text{Quad}_V^{sr} \times_X \mathsf{L}_w)$ onto $D_0 :=$ Id-w-Sp-Azu$_W \backslash$Id-w-Azu$_W$. Therefore the irreducibility of D_0 is equivalent to that of the closed subset $\text{Quad}_V \backslash \text{Quad}_V^{sr}$. Recall from the discussion on semiregular forms (page 24, Section 4) that the open subscheme Quad_V^{sr} corresponds to localisation by the polynomial P_3. This polynomial is irreducible as an element of $R[\zeta_i, \zeta_{ij}] \cong R[\text{Quad}_V]$ when $X = \text{Spec}(R)$ is affine, though it is not clear if it is a prime element (unless we assume something more e.g., R a UFD). The closed subset $\text{Quad}_V \backslash \text{Quad}_V^{sr}$ may be given the canonical closed subscheme structure $Z(P_3)$ corresponding to the vanishing of P_3. Let $q^{(2)}$ be the global quadratic form on V given by $(x_1, x_2, x_3) \mapsto x_1 x_2$. It can be checked that $q^{(2)}$ is not semiregular, but that its restriction to the rank two (direct summand) vector subbundle generated by $\{e_1, e_2\}$ is regular. Therefore the X-valued point corresponding to $q^{(2)}$ lands topologically inside the closed subset underlying $Z(P_3)$. Consider the orbit morphism $O(q^{(2)}) : \text{GL}_V \longrightarrow \text{Quad}_V$ corresponding to this X-valued point which also lands topologically inside $Z(P_3)$. It will follow from assertion (2), Theorem 1.17, that the topological image of $O(q^{(2)})$ is dense in $Z(P_3)$. On the other hand this topological image is irreducible, since $\text{GL}_V \cong \mathbb{A}^9_X$. It follows that $\text{Quad}_V \backslash \text{Quad}_V^{sr}$, and hence D_0, is irreducible in the case when W is trivial and w is part of a global basis over X. Since the reduced closed subscheme structure on $Z(P_3)$ is given by the radical of the ideal (P_3), it follows that $\sqrt{(P_3)}$ is the minimal prime divisor of (P_3) and by Krull's *Hauptidealsatz* this prime has height 1. Therefore, the codimension of D_0 is also 1 in the present case.

Now consider the case of a general (W, w), choose an affine open covering $\{U_i = \text{Spec}(R_i)\}_{i \in I}$ of X such that $W_i := W|U_i$ is trivial and $w_i := w|U_i$ is part of a global basis $\forall\, i$. The subset $D_i :=$ Id-w_i-Sp-Azu$_{W_i} \backslash$Id-w_i-Azu$_{W_i}$ is irreducible for each i by the preceding paragraph. On the other hand, by Theorems 4.4 and 4.5, the subsets D_i form an open cover of $D :=$ Id-w-Sp-Azu$_W \backslash$Id-w-Azu$_W$. Since X is irreducible, we have $D_i \cap D_j \neq \emptyset$ when $i \neq j$. Thus D is locally irreducible and connected, and hence irreducible (for e.g., by Cor.2.1.7, Chap.0, EGA I [6]). Since X is integral and noetherian, assertion (2) of Theorem 4.5 implies that Id-w-Sp-Azu$_W$ is integral and noetherian as well. Therefore the codimension of D in Id-w-Sp-Azu$_W$ is atleast 1. On the other hand (for e.g., by Prop.14.2.3, Chap.0, EGA IV, [7]) this codimension is bounded above by $1 = \text{Codim}\,(D_i, \text{Id-}w_i\text{-Sp-Azu}_{W_i})$ for any i, since Id-w_i-Sp-Azu$_{W_i} \hookrightarrow$ Id-w-Sp-Azu$_W$ is an open subset whose intersection with D is precisely D_i.

Proof of assertion (e) of Theorem 1.8. Let us remind the reader that the pullbacks $W \otimes_X$ Id-w-Sp-Azu$_W$ and $W \otimes_X$ Id-w-Azu$_W$ are naturally endowed with ($w \otimes_X$ Id-w-Sp-Azu$_W$)-unital associative $\mathcal{O}_{\text{Id-}w\text{-Sp-Azu}_W}$-algebra structures with which they become respectively the universal specialisation and universal Azumaya algebra relative to the pair (W, w)—for details see the proof of Theorem 3.4, Part A, [3]. By (c) and (d), the natural map Pic(Id-w-Sp-Azu$_W$) $\longrightarrow$ Pic(Id-w-Azu$_W$) is surjective and its kernel is generated by the image of $\mathbb{Z}.(D_X)$. Given an isomorphism $\phi : W \cong W^\vee$, we consider its pullback to Id-w-Sp-Azu$_W$, and proceeding along the lines of the proof of Theorem 1.7, with the triple (Id-w-Sp-Azu$_W$, $W \otimes_X$ Id-w-Sp-Azu$_W$, $\phi \otimes_X$ Id-w-Sp-Azu$_W$) replacing the triple (X, A, ϕ) there, we infer that $n.(D_X)$ is principal for some $n \geq 1$. Now $W \otimes_X$ Id-w-Azu$_W$ is an Azumaya algebra bundle, and as seen in the discussion following Theorem 1.14, it is isomorphic to the even Clifford algebra of a canonically obtained rank 3 quadratic bundle on Id-w-Azu$_W$. Therefore it follows from (b), Theorem 1.5 that det($W \otimes_X$ Id-w-Sp-Azu$_W$) maps to an element of 2.Pic(Id-w-Azu$_W$). The remaining assertions in (e) are now consequences of Theorem 1.6. **Q.E.D., Theorem 1.8.**

6 Stratification of the Variety of Specialisations

In this section, we prove Prop.1.15 and Theorem 1.17.

Proof of Prop.1.15. Fix an S-basis $\{e_1, e_2, e_3\}$ for V, and with respect to this basis, let q^1 denote the quadratic form given by $(x_1e_1 + x_2e_2 + x_3e_3) \longmapsto x_1x_2 + x_3^2$. It is easy to see that this quadratic form is semiregular. We show that any semiregular quadratic form q can be moved to q^1 i.e., that $\exists\, g \in \mathrm{GL}(V)$ such that $g \cdot q = q^1$.

By Prop.3.17, Chap.IV, [2], there exists a basis $\{e_1', e_2', e_3'\}$ for V such that q restricted to the submodule generated by e_1' and e_2' is *regular* and further such that $q(e_3') \in S^*$, $b_q(e_1', e_2') = 1$ and $b_q(e_1', e_3') = 0 = b_q(e_2', e_3')$. Let $g' \in \mathrm{GL}(V)$ be the automorphism that maps e_i' onto the e_i for each i and consider the quadratic form $q' := g' \cdot q$. Then by definition of the $\mathrm{GL}(V)$-action on the set $\mathrm{Quad}(V)$ of quadratic S-forms on V we have $q'(e_3) \in S^*$, $b_{q'}(e_1, e_2) = 1$ and $b_{q'}(e_1, e_3) = 0 = b_{q'}(e_2, e_3)$. So if we assume that $q'(e_i) = \lambda_i$ ($\Leftrightarrow q(e_i') = \lambda_i$), then $q'(x_1e_1 + x_2e_2 + x_3e_3) = \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2 + x_1x_2\ \forall\ x_i \in S$. Thus it is enough to show that q' can be moved to q^1. We look for an invertible matrix $g'' = (u_{ij}) \in \mathrm{GL}(V)$ such that $g'' \cdot q' = q^1$. Writing this condition equivalently as $q' = (g'')^{-1} \cdot q^1$ and comparing the polynomials in the x_i gives 6 equations in terms of the u_{ij} and the λ_i which are to be satisfied. We choose the u_{ij} as follows. First set $u_{11} = u_{22} = 0$ and let $u_{12} \in S^*$ be a free parameter. Since every element of S has square roots in S, it makes sense to choose $u_{31} = \pm\sqrt{\lambda_1}$ and $u_{32} = \pm\sqrt{\lambda_2}$. We let $\alpha = 1 + 2u_{31}u_{32}$, $\beta = 1 - 2u_{31}u_{32}$ and $u_{21} = \beta/u_{12}$. Since $q' = g \cdot q$ and since q is semiregular, q' is also semiregular. Its half-discriminant relative to the present basis of V is (remembering that $\lambda_3 \in S^*$) $d_{q'}(e_1, e_2, e_3) = \lambda_3.(4\lambda_1\lambda_2 - 1) \in S^*$. This implies that $\alpha\beta = 1 - 4\lambda_1\lambda_2 \in S^* \implies \alpha, \beta \in S^*$. Therefore it makes sense to define $u_{33} = \pm\sqrt{(\beta\lambda_3)/\alpha}, u_{13} = -2u_{31}u_{33}u_{12}/\beta$ and $u_{23} = -2u_{32}u_{33}/u_{12}$. Note that $u_{33} \in S^*$. A computation shows that the determinant of the matrix $g'' = (u_{ij})$ defined above is $-u_{33}\alpha \in S^*$ and hence g'' is invertible. It is also easily checked that $g'' \cdot q' = q^1$. **Q.E.D., Prop.1.15.**

Proof of assertion (2) of Theorem 1.17. (We shall not prove assertion (1) since it is well-known). Recall from the discussion on semiregular forms (page 24, Section 4) that the open subscheme Quad_V^{sr} corresponds to localisation by the polynomial P_3 and that this polynomial is prime as an element of $k[\zeta_i, \zeta_{ij}] \cong k[\mathrm{Quad}_V]$. Here a quadratic form q corresponding to the point $(\lambda_i, \lambda_{ij}) \in \mathbb{A}_k^6$ is given by $(x_1, x_2, x_3) \mapsto \Sigma_i\lambda_ix_i^2 + \Sigma_{i<j}\lambda_{ij}x_ix_j$. For ease of readability (and typesetting !) let us denote the closure $\overline{T}$ of a subset T (given the reduced closed subscheme structure) by $\langle T\rangle$ in what follows. Since $\mathrm{Quad}_V^{(1)}$ is the same as the variety underlying the open subscheme Quad_V^{sr} of semiregular quadratic forms, that $\langle\mathrm{Quad}_V^{(1)}\rangle = \mathrm{Quad}_V$ follows from the fact that Quad_V is irreducible.

By assertion (1) of the present Theorem, Quad_V is the disjoint union of the $\mathrm{Quad}_V^{(i)}$; therefore $\langle\mathrm{Quad}_V^{(1)}\rangle\backslash\mathrm{Quad}_V^{(1)}$ is the disjoint union of $\{\mathrm{Quad}_V^{(i)} | 2 \leq i \leq 4\}$ and also equals the closed subset $Z(P_3)$ defined by the vanishing of P_3. An explicit computation shows that the dimension of the stabilizer of $q^{(2)}$ in $\mathrm{GL}(V)$ is 4. Since $\mathrm{Quad}_V^{(2)}$ is an open dense subvariety of $\langle\mathrm{Quad}_V^{(2)}\rangle \subset V(P_3)$, its closure is thus 5-dimensional. But since P_3 is an irreducible polynomial, $Z(P_3)$ is also an irreducible 5-dimensional subvariety. It follows that $\langle\mathrm{Quad}_V^{(2)}\rangle = \langle\mathrm{Quad}_V^{(1)}\rangle\backslash\mathrm{Quad}_V^{(1)}$.

Since $\mathrm{Quad}_V^{(2)}$ is smooth in its closure ($= Z(P_3)$ as seen above), the singularities of its closure are contained in $\mathrm{Quad}_V^{(3)} \cup \mathrm{Quad}_V^{(4)}$ which consists of quadratic forms that are perfect squares i.e., squares of linear forms. These singularities may be identified with points $(\lambda_i, \lambda_{ij}) \in \mathbb{A}_k^6 \cong \mathrm{Quad}_V$ at which all the partial derivatives of P_3 vanish. A simple computation shows that this set is the symmetric determinantal variety given by the vanishing of the (2×2)-minors of the matrix of the symmetric bilinear form associated to the generic quadratic form given by $(x_1, x_2, x_3) \mapsto \Sigma_i\zeta_ix_i^2 + \Sigma_{i<j}\zeta_{ij}x_ix_j$; but it can also be shown that this set precisely corresponds to the perfect squares. Therefore $\mathrm{Sing}\ (\langle\mathrm{Quad}_V^{(2)}\rangle) = \langle\mathrm{Quad}_V^{(2)}\rangle\backslash\mathrm{Quad}_V^{(2)}$.

That $\langle\mathrm{Quad}_V^{(i+1)}\rangle = \langle\mathrm{Quad}_V^{(i)}\rangle\backslash\mathrm{Quad}_V^{(i)}$ for $i = 2$ follows from the above and the obvious fact that any quadratic form can be specialised to the zero quadratic form. The case $i = 3$ is trivial.

To see that $\langle\mathrm{Quad}_V^{(3)}\rangle$ is smooth if Char(k)=2, we first note from the above and assertion (1) of the present Theorem that the closure of $\mathrm{Quad}_V^{(3)}$ consists of perfect squares. Since char(k)=2, under the identification $\mathrm{Quad}_V \cong \mathbb{A}_k^6$, the perfect squares are seen to correspond to the copy of $\mathbb{A}_k^3$ in $\mathbb{A}_k^6$ given by the vanishing of the last three coordinates λ_{ij}, $1 \leq i < j \leq 3$.

To see that the zero quadratic form is a singularity of $\langle\mathrm{Quad}_V^{(3)}\rangle$ if char(k) $\neq 2$, we note from the above that $\langle\mathrm{Quad}_V^{(3)}\rangle$ is defined by the same equations that define the singularities of $Z(P_3)$ and is a certain symmetric determinantal variety. It is a well-known fact—an application of Standard Monomial

Theory (for e.g., see [10] or [11])—that the ideal defining these equations is itself *reduced*, i.e., it is the ideal of the variety. Checking the Jacobian criterion now shows that the zero quadratic form is indeed a singular point.

Proof of assertion (3) of Theorem 1.17. The proof will follow from a series of observations. We remind the reader of the X-morphism $\theta : \mathrm{Quad}_V \longrightarrow \text{Id-}w\text{-Sp-Azu}_W$ which was used to define the isomorphism Θ (page 15, Section 3).

***Claim 1: There are exactly 4 H-orbits in* SpAzu.**

By Theorem 1.11, any point of SpAzu is of the form $\underline{t}\cdot\theta(q)$. Its H-orbit is $H\cdot\theta(q)$. There exists a unique i, $(1 \leq i \leq 4)$, and some $g \in \mathrm{GL}(V)$ such that $q = g\cdot q^{(i)}$. Consider the algebra isomorphism $C_0(g,1)$ of (1), Prop.2.4 and the induced isomorphism $h(g,1,q^{(i)},q)$ of Prop.3.5. By definition, $\theta(q) = \theta(g\cdot q^{(i)}) = h(g,1,q^{(i)},q)\cdot\theta(q^{(i)}) \Rightarrow H\cdot\theta(q) = H\cdot\theta(q^{(i)})$ so that there are atmost 4 orbits. To see that there are atleast 4, we use the following result.

***Claim 2. For* $q \in \mathrm{Quad}_V$*, we have* $\mathsf{L}_w\cdot\Theta((GL(V)\cdot q)\times\{I_4\}) = \Theta((GL(V)\cdot q)\times\mathsf{L}_w) = H\cdot\theta(q)$.**

On the one hand, we have

$$\begin{aligned}\Theta((GL(V)\cdot q)\times\mathsf{L}_w) &= \{\underline{t}\cdot\theta(g\cdot q) \mid \underline{t}\in\mathsf{L}_w \text{ and } g\in GL(V)\}\\ &= \{\underline{t}\cdot(h(g,1,q,g\cdot q)\cdot\theta(q)) \mid \underline{t}\in\mathsf{L}_w \text{ and } g\in GL(V)\}\\ &\subset H\cdot\theta(q).\end{aligned}$$

Conversely take any $h\cdot\theta(q)\in H\cdot\theta(q)$. By Theorem 1.11, there exists a unique $\underline{t'}\in\mathsf{L}_w$ and a unique $q'\in\mathrm{Quad}_V$ such that $h\cdot\theta(q) = \underline{t'}\cdot\theta(q')$. Therefore $((\underline{t'})^{-1}.h)\cdot\theta(q) = \theta(q')$. Since k is algebraically- and hence quadratically closed, by (b), Theorem 1.3, there exists $g\in GL(V)$ such that $q' = g\cdot q$ and $(\underline{t'})^{-1}.h = h(g,1,q,q')$. Hence $h\cdot\theta(q) = \underline{t'}\cdot\theta(g\cdot q) = \Theta(g\cdot q,\underline{t'}) \Rightarrow H\cdot\theta(q)\subset\Theta((GL(V)\cdot q)\times\mathsf{L}_w)$. This settles Claim 2. As for Claim 1, if $q,q'\in\mathrm{Quad}_V$ are such that their $GL(V)$-orbits are distinct, then because Θ is an isomorphism, Claim 2 shows that $H\cdot\theta(q)$ is distinct from $H\cdot\theta(q')$.

***Claim 3. For each i, with* $1\leq i\leq 4$, $\langle\mathrm{Quad}_V^{(i)}\times\mathsf{L}_w\rangle = \langle\mathrm{Quad}_V^{(i)}\rangle\times\mathsf{L}_w$.**

If $f: X\longrightarrow Y$ is a *smooth morphism* and $U\hookrightarrow Y$ is an open subset, then $f^{-1}(\langle U\rangle) = \langle f^{-1}(U)\rangle$. Since $\mathsf{L}_w\longrightarrow\mathrm{Spec}(k)$ is smooth, so is the induced morphism $\langle\mathrm{Quad}_V^{(i)}\rangle\times\mathsf{L}_w\longrightarrow\langle\mathrm{Quad}_V^{(i)}\rangle$. Taking f to be this morphism and $U = \mathrm{Quad}_V^{(i)}$ gives Claim 3.

***Claim 4. The $GL(V)$-stratification of* Quad_V *induces a $GL(V)$-stratification of* $\mathrm{Quad}_V\times\mathsf{L}_w$ *(the $GL(V)$-action on* L_w *being taken to be trivial) with strata given by* $(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i)} := \mathrm{Quad}_V^{(i)}\times\mathsf{L}_w,\ (1\leq i\leq 4)$.**

To prove Claim 4, the only thing that needs to be checked is that

$$\langle(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i+1)}\rangle = \langle(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i)}\rangle\backslash(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i)}.$$

This follows by applying Claim 3 twice:

$$\begin{aligned}\langle(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i+1)}\rangle &= \langle\mathrm{Quad}_V^{(i+1)}\times\mathsf{L}_w\rangle = \langle\mathrm{Quad}_V^{(i+1)}\rangle\times\mathsf{L}_w = (\langle\mathrm{Quad}_V^{(i)}\rangle\backslash\mathrm{Quad}_V^{(i)})\times\mathsf{L}_w\\ &= \langle\mathrm{Quad}_V^{(i)}\rangle\times\mathsf{L}_w\backslash\mathrm{Quad}_V^{(i)}\times\mathsf{L}_w = \langle(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i)}\rangle\backslash(\mathrm{Quad}_V\times\mathsf{L}_w)^{(i)}.\end{aligned}$$

Now according to Claim 2, we have $\mathrm{SpAzu}^{(i)} = \Theta(GL(V)\cdot q^{(i)}\times\mathsf{L}_w) = \Theta((\mathrm{Quad}_V\times\mathsf{L}_w)^{(i)})$. This combined with Claim 4 and the fact that Θ is an isomorphism completes the proof of assertion (3) of Theorem 1.17. **Q.E.D., Theorem 1.17.**

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