The normalization constant of a certain invariant measure on $GL_n(D_{\mathbf{A}})$

Yoshihide Nakamura and Takao Watanabe *

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Abstract

The ratio of the Tamagawa measure and a certain invariant measure on the group $GL_n(D_{\mathbf{A}})$ is computed, where $D_{\mathbf{A}}$ is the adèle of a division algebra D over a global field. An explicit formula of the ratio is described in terms of the special values of the zeta function of D. This formula yields (i) an explicit lower bound of the Hermite–Rankin constant $\gamma_{n,m}(D)$ of D and (ii) an explicit asymptotic behavior of the distribution of rational points on Brauer–Severi variety.

Introduction

Let G be a connected reductive algebraic group defined over a global field k and $G(\mathbf{A})$ the adèle group of G. Since $G(\mathbf{A})$ is a locally compact unimodular group, it has a non-trivial invariant measure. The invariant measure $\omega_{\mathbf{A}}^G$ on $G(\mathbf{A})$ induced from the invariant gauge form ω^G on G defined over k is called the Tamagawa measure, which is a canonical invariant measure on $G(\mathbf{A})$ in a sense. There is another useful invariant measure on $G(\mathbf{A})$ defined as follows: We fix a parabolic subgroup R of G defined over k and a maximal compact subgroup K of $G(\mathbf{A})$ which possesses an Iwasawa decomposition $G(\mathbf{A}) = R(\mathbf{A})K$. Let $\omega_{\mathbf{A}}^R$ denote the Tamagawa measure of $R(\mathbf{A})$ and ω_K the invariant measure on K normalized so that $\omega_K(K) = 1$. Then the product $\omega_{\mathbf{A}}^R \cdot \omega_K$ defines an invariant measure, say $\omega_{(G(\mathbf{A}),R(\mathbf{A}))}$, on $G(\mathbf{A})$. Since an invariant measure is unique up to constant, there is the positive constant $C_{G,R,K}$ such that $\omega_{\mathbf{A}}^G = C_{G,R,K} \cdot \omega_{(G(\mathbf{A}),R(\mathbf{A}))}$. We call $C_{G,R,K}$ the normalization constant of $\omega_{(G(\mathbf{A}),R(\mathbf{A}))}$.

In general, the constant $C_{G,R,K}$ has a description by an Euler product such as

$$C_{G,R,K} = \prod_v \epsilon_v J_v \,,$$

where v runs over all places of k and ϵ_v are elementary constants determined by G and R. Every J_v is an integral of the form

$$J_v = \int_{U_R^-(k_v)} \eta_v(u_v) d\omega_v^{U_R^-}(u_v) \,,$$

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where U_R^- denotes the unipotent radical of the opposite k-parabolic subgroup of R and η_v the function on $G(k_v)$ induced by the modular character of $R(k_v)$. In §2.1, we will show this formula in detail. In principle, the constant $C_{G,R,K}$ can be explicitly computed by using this formula and the reduction of J_v to the cases of semisimple rank one groups due to Gindikin–Karpelevič formula (see §2.2 and §2.3). Indeed, an explicit formula of $C_{G,R,K}$ is known in the case where G is a k-quasisplit group ([L]), an orthogonal group ([Ik]) and a unitary group ([Ic]). However, except for the case that G is a k-quasisplit group, its actual computation is not easy.

In this paper, we give an explicit formula of $C_{G,R,K}$ in the case that G is an inner k-from of general linear groups, *i.e.*, G is the algebraic group determined by $G(k) = M_n(D)^\times = GL_n(D)$, where D is a division k-algebra. We fix a minimal k-parabolic subgroup P of G and a certain maximal compact subgroup K of $G(\mathbf{A})$ such that $G(\mathbf{A}) = P(\mathbf{A})K$. Since $C_{G,R,K} = C_{G,P,K}/C_{M_R,M_R\cap P,M_R(\mathbf{A})\cap K}$ holds for any standard k-parabolic subgroup R of G with a Levi subgroup M_R, it is sufficient to compute $C_{G,P,K}$. Then the integral J_v occurring in the Euler product of $C_{G,P,K}$ is decomposed into a product of integrals over a division k_v-algebra $D(v)$ which is equivalent to $D \otimes_k k_v$ in the Brauer group of k_v. By computing the integrals over $D(v)$, we obtain the value of J_v, and as a consequence, the explicit formula of $C_{G,P,K}$ is described in terms of special values of the zeta function $Z_D(s)$ of D (see §3.6).

Our motivation of computing $C_{G,R,K}$ is the following. In [Wa], the second author introduced the fundamental Hermite constant $\gamma(G,Q;k)$ of the pair of a connected reductive k-group G and a maximal k-parabolic subgroup Q of G. Then the constant $C_{G,Q,K}$ appeared in the Minkowski–Hlawka type lower bound of $\gamma(G,Q;k)$. Thus an explicit formula of $C_{G,R,K}$ yields an explicit lower bound of $\gamma(G,Q,k)$. In the case of $G(k) = GL_n(D)$, we will take up this application in §4.2. Moreover, we will apply the formula of $C_{G,R,K}$ to give an explicit asymptotic behavior of the distribution of rational points on Brauer–Severi variety in §4.3.

Notations

Let k be a global field, *i.e.*, an algebraic number field of finite degree over $\mathbf{Q}$ or an algebraic function field of one variable over a finite field. In the latter case, we identify the constant field of k with the finite field $\mathbf{F}_q$ with q elements. Let $\mathfrak{V}$ be the set of all places of k. We write $\mathfrak{V}_\infty$, $\mathfrak{V}_\mathbf{R}$, $\mathfrak{V}_\mathbf{C}$ and $\mathfrak{V}_f$ for the sets of all infinite places, all real places, all imaginary places and all finite places of k, respectively. For $v \in \mathfrak{V}$, k_v denotes the completion of k at v. If $v \in \mathfrak{V}_f$, $\mathfrak{o}_v$ denotes the maximal compact subring of k_v and q_v the cardinality of the residual field of k_v. We fix, once and for all, a Haar measure μ_v on k_v normalized so that $\mu_v(\mathfrak{o}_v) = 1$ if $v \in \mathfrak{V}_f$, $\mu_v([0,1]) = 1$ if $v \in \mathfrak{V}_\mathbf{R}$ and $\mu_v(\{a \in k_v : a\overline{a} \le 1\}) = 2\pi$ if $v \in \mathfrak{V}_\mathbf{C}$. Then the absolute value $|\cdot|_v$ on k_v is defined as $|a|_v = \mu_v(aC)/\mu_v(C)$, where C is an arbitrary compact subset of k_v with nonzero measure. Let $\mathbf{A}$ be the adèle ring of k, $|\cdot|_\mathbf{A} = \prod_{v\in\mathfrak{V}} |\cdot|_v$ the idele norm on the idele group $\mathbf{A}^\times$ and $\mu_\mathbf{A} = \prod_{v\in\mathfrak{V}} \mu_v$ an invariant measure on $\mathbf{A}$. The measure $\mu_\mathbf{A}$ is characterized by

$$\mu_\mathbf{A}(\mathbf{A}/k) = \begin{cases} |D_k|^{1/2} & \text{(if } k \text{ is an algebraic number field of discriminant } D_k\text{).} \\ q^{g(k)-1} & \text{(if } k \text{ is a function field of genus } g(k)\text{).} \end{cases}$$

The zeta function $\zeta_k(s)$ of k is defined to be

$$\zeta_k(s) = \prod_{v\in\mathfrak{V}_f} (1-q_v^{-s})^{-1} .$$

The residue of $\zeta_k(s)$ at $s=1$ is denoted by ρ_k.

Let k_1 be an arbitrary field. If $\mathfrak{A}_1$ is a central simple k_1-algebra, then $\mathrm{Nr}_{\mathfrak{A}_1/k_1}$ and $\tau_{\mathfrak{A}_1/k_1}$ stand for the reduced norm and the reduced trace of $\mathfrak{A}_1$, respectively. The unit group of $\mathfrak{A}_1$ is denoted by $\mathfrak{A}_1^\times$.

1 Normalization constant of an invariant measure

1.1 Tamagawa measure

Let G be a connected affine algebraic group defined over k. For any k-algebra A, $G(A)$ stands for the set of A-rational points of G. Let $\mathbf{X}^*(G)$ and $\mathbf{X}^*_k(G)$ be the free $\mathbf{Z}$-modules consisting of all rational characters and all k-rational characters of G, respectively. The absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on $\mathbf{X}^*(G)$. The representation of $\mathrm{Gal}(\overline{k}/k)$ in the space $\mathbf{X}^*(G)\otimes_{\mathbf{Z}}\mathbf{Q}$ is denoted by σ_G and the corresponding Artin L-function is denoted by $L(s,\sigma_G)=\prod_{v\in\mathfrak{V}_f} L_v(s,\sigma_G)$. We set $\sigma_k(G)=\lim_{s\to 1}(s-1)^n L(s,\sigma_G)$, where $n=\mathrm{rank}\,\mathbf{X}^*_k(G)$. Let ω^G be a nonzero right invariant gauge form on G defined over k. From ω^G and the fixed Haar measure μ_v on k_v, one can construct a right invariant Haar measure ω^G_v on $G(k_v)$. Then, the Tamagawa measure on $G(\mathbf{A})$ is well defined by

$$\omega^G_{\mathbf{A}} = \mu_{\mathbf{A}}(\mathbf{A}/k)^{-\dim G}\omega^G_\infty\omega^G_f ,$$

where

$$\omega^G_\infty = \prod_{v\in\mathfrak{V}_\infty}\omega^G_v \quad\text{and}\quad \omega^G_f = \sigma_k(G)^{-1}\prod_{v\in\mathfrak{V}_f} L_v(1,\sigma_G)\omega^G_v .$$

For each $g\in G(\mathbf{A})$, we define the homomorphism $\vartheta_G(g)\,:\,\mathbf{X}^*_k(G)\longrightarrow\mathbf{R}_+$ by $\vartheta_G(g)(\chi)=|\chi(g)|_{\mathbf{A}}$ for $\chi\in\mathbf{X}^*_k(G)$. Then ϑ_G is a homomorphism from $G(\mathbf{A})$ into $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*_k(G),\mathbf{R}_+)$. We write $G(\mathbf{A})^1$ for the kernel of ϑ_G. The Tamagawa measure $\omega_{G(\mathbf{A})^1}$ on $G(\mathbf{A})^1$ is defined as follows:

- The case of $\mathrm{ch}(k)=0$. If a $\mathbf{Z}$-basis $\chi_1,\cdots,\chi_n$ of $\mathbf{X}^*_k(G)$ is fixed, then $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*_k(G),\mathbf{R}_+)$ is identified with $(\mathbf{R}_+)^n$ and ϑ_G gives rise to an isomorphism from $G(\mathbf{A})^1\backslash G(\mathbf{A})$ onto $(\mathbf{R}_+)^n$. Put the Lebesgue measure dt on $\mathbf{R}$ and the invariant measure dt/t on $\mathbf{R}_+$. Then $\omega_{G(\mathbf{A})^1}$ is the measure on $G(\mathbf{A})^1$ such that the quotient measure $\omega_{G(\mathbf{A})^1}\backslash\omega^G_{\mathbf{A}}$ is the pullback of the measure $\prod_{i=1}^n dt_i/t_i$ on $(\mathbf{R}_+)^n$ by ϑ_G. The measure $\omega_{G(\mathbf{A})^1}$ is independent of the choice of the $\mathbf{Z}$-basis $\chi_1,\cdots,\chi_n$.

- The case of $\mathrm{ch}(k)>0$. The value group of the idele norm $|\cdot|_{\mathbf{A}}$ is the cyclic group $q^{\mathbf{Z}}$ generated by q. Thus the image $\mathrm{Im}\vartheta_G$ of ϑ_G is contained in $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*_k(G),q^{\mathbf{Z}})$ and $G(\mathbf{A})^1$ is an open normal subgroup of $G(\mathbf{A})$. Since the index of $\mathrm{Im}\vartheta_G$ in $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*_k(G),q^{\mathbf{Z}})$ is finite ([O, I, Proposition 5.6]),

  $$d^*_G = (\log q)^{\mathrm{rank}\mathbf{X}^*_k(G)}[\mathrm{Hom}_{\mathbf{Z}}(\mathbf{X}^*_k(G),q^{\mathbf{Z}})\,:\,\mathrm{Im}\vartheta_G]$$

  is well defined. The measure $\omega_{G(\mathbf{A})^1}$ is defined to be the restriction of the measure $(d^*_G)^{-1}\omega^G_{\mathbf{A}}$ to $G(\mathbf{A})^1$.

In both cases, we put the counting measure $\omega_{G(k)}$ on $G(k)$. The volume of $G(k)\backslash G(\mathbf{A})^1$ with respect to the measure $\omega_{G(k)}\backslash\omega_{G(\mathbf{A})^1}$ is called the Tamagawa number of G and denoted by $\tau(G)$.

1.2 Another Haar measure on $G(\mathbf{A})$ and its normalization constant

In the following, let G be a connected reductive group defined over k. We fix a maximal k-split torus S in G and a minimal k-parabolic subgroup P of G which contains S. The centralizer of S in G gives a Levi subgroup M_P of P. Thus P has a Levi decomposition: $P = M_P U_P$, where U_P denotes the unipotent radical of P. Let R be a k-parabolic subgroup of G such that $P \subset R$. Such R is called a standard k-parabolic subgroup. There exists a unique Levi subgroup M_R of R such that $M_P \subset M_R$. The unipotent radical of R is denoted by U_R. We fix a maximal compact subgroup K of $G(\mathbf{A})$ satisfying the following property; For every standard k-parabolic subgroup R of G, $K \cap M_R(\mathbf{A})$ is a maximal compact subgroup of $M_R(\mathbf{A})$, and furthermore $M_R(\mathbf{A})$ possesses an Iwasawa decomposition $(M_R(\mathbf{A}) \cap U_P(\mathbf{A}))M_P(\mathbf{A})(K \cap M_R(\mathbf{A}))$.

If a standard k-parabolic subgroup R of G is given, then one can define another Haar measure $\omega_{(G(\mathbf{A}),R(\mathbf{A}))}$ of $G(\mathbf{A})$ as follows. Let $\omega_{\mathbf{A}}^{M_R}$ and $\omega_{\mathbf{A}}^{U_R}$ be the Tamagawa measures of $M_R(\mathbf{A})$ and $U_R(\mathbf{A})$, respectively. The modular character δ_R^{-1} of $R(\mathbf{A})$ is a function on $M_R(\mathbf{A})$ which satisfies the integration formula

$$\int_{U_R(\mathbf{A})} f(mum^{-1}) d\omega_{\mathbf{A}}^{U_R}(u) = \delta_R(m)^{-1} \int_{U_R(\mathbf{A})} f(u) d\omega_{\mathbf{A}}^{U_R}(u).$$

Let ω_K be the Haar measure on K normalized so that the total volume equals one. Then the mapping

$$f \mapsto \int_{U_R(\mathbf{A}) \times M_R(\mathbf{A}) \times K} f(umh)\delta_R(m)^{-1} d\omega_{\mathbf{A}}^{U_R}(u) d\omega_{\mathbf{A}}^{M_R}(m) d\omega_K(h)\,, \quad (f \in C_0(G(\mathbf{A})))$$

defines an invariant measure on $G(\mathbf{A})$ and is denoted by $\omega_{(G(\mathbf{A}),R(\mathbf{A}))}$.

Since a non-trivial invariant measure on $G(\mathbf{A})$ is unique up to constant, there exists a positive constant $C_{G,R,K}$ such that

$$\omega_{\mathbf{A}}^G = C_{G,R,K} \cdot \omega_{(G(\mathbf{A}),R(\mathbf{A}))}\,.$$

We call $C_{G,R,K}$ the normalization constant of $\omega_{(G(\mathbf{A}),R(\mathbf{A}))}$. For simplicity, we often write $C_{G,R}$ for $C_{G,R,K}$. It is easy to show the following compatibility of three constants $C_{G,R,K}$, $C_{G,P,K}$ and $C_{M_R,M_R\cap P,M_R(\mathbf{A})\cap K}$:

$$C_{G,R,K} = \frac{C_{G,P,K}}{C_{M_R,M_R\cap P,M_R(\mathbf{A})\cap K}}.$$

2 A formula of $C_{G,R}$

2.1 An expression of $C_{G,R}$ by a product of integrals

Let G, R and K be the same as in §1.2. We consider the right G-homogeneous space $\mathfrak{X}_R = U_R\backslash G$. Since U_R is a split unipotent subgroup, one has $\mathfrak{X}_R(\mathbf{A}) = U_R(\mathbf{A})\backslash G(\mathbf{A})$.

Since both U_R and G are unimodular, $\omega^{U_R}\backslash\omega^G$ gives a unique (up to constant) G-invariant gauge form on $\mathfrak{X}_R$ defined over k. The $G(\mathbf{A})$-invariant measure on $\mathfrak{X}_R(\mathbf{A})$ defined from $\omega^{U_R}\backslash\omega^G$ is equal to

$$\omega_{\mathbf{A}}^{U_R}\backslash\omega_{\mathbf{A}}^G = C_{G,R}\delta_R^{-1}\omega_{\mathbf{A}}^{M_R}\omega_K\,. \tag{1}$$

We take the opposite parabolic subgroup R^- of R. We denote by U_R^- the unipotent radical of R^-, *i.e.*, $U_R^- = U_{R^-}$. Then one has the Levi decomposition $R^- = U_R^- M_R$ and $R\cap R^- = M_R$. By [B-T, Proposition 4.10 d)], the product morphism $U_R \times R^- \longrightarrow G$ is injective and gives an isomorphism of variety from $U_R\times R^-$ onto a Zariski open set in G. Thus R^- is regarded as a Zariski open subset of $\mathfrak{X}_R$. Since $(\omega^{U_R}\backslash\omega^G)|_{R^-}$ yields a right invariant gauge form on R^- defined over k, there exists a constant $\lambda\in k^\times$ such that

$$(\omega^{U_R}\backslash\omega^G)|_{R^-} = \lambda\omega^{U_R^-}\omega^{M_R}\,. \tag{2}$$

For each $v\in\mathfrak{V}$, define the function $\eta_v : G(k_v)\longrightarrow \mathbf{R}_+$ by $\eta_v(umh)=\delta_R(m)$ for $u\in U_R(k_v), m\in M_R(k_v)$ and $h\in K_v$. We take a right K-invariant $\Phi\in C_0(\mathfrak{X}_R(\mathbf{A}))$ of the form $\Phi=\prod_{v\in\mathfrak{V}}\Phi_v$, $\Phi_v\in C_0(\mathfrak{X}_R(k_v))$. On the one hand, by (1), we have

$$\begin{aligned}\int_{\mathfrak{X}_R(\mathbf{A})}\Phi(x)d(\omega_{\mathbf{A}}^{U_R}\backslash\omega_{\mathbf{A}}^G)(x) &= C_{G,R}\int_{M_R(\mathbf{A})\times K}\Phi(mh)\delta_R(m)^{-1}d\omega_{\mathbf{A}}^{M_R}(m)d\omega_K(h)\\ &= C_{G,R}\int_{M_R(\mathbf{A})}\Phi(m)\delta_R(m)^{-1}d\omega_{\mathbf{A}}^{M_R}(m).\end{aligned} \tag{3}$$

On the other hand, by (2),

$$\begin{aligned}&\int_{\mathfrak{X}_R(\mathbf{A})}\Phi(x)d(\omega_{\mathbf{A}}^{U_R}\backslash\omega_{\mathbf{A}}^G)(x)\\
=\ &\frac{\mu_{\mathbf{A}}(\mathbf{A}/k)^{\dim U_R-\dim G}}{\sigma_k(G)}\prod_{v\in\mathfrak{V}_\infty}\int_{\mathfrak{X}_R(k_v)}\Phi_v(x_v)d(\omega_v^{U_R}\backslash\omega_v^G)(x_v)\\
&\quad\times\prod_{v\in\mathfrak{V}_f}L_v(1,\sigma_G)\int_{\mathfrak{X}_R(k_v)}\Phi_v(x_v)d(\omega_v^{U_R}\backslash\omega_v^G)(x)\\
=\ &\frac{\mu_{\mathbf{A}}(\mathbf{A}/k)^{\dim U_R-\dim G}}{\sigma_k(G)}\prod_{v\in\mathfrak{V}_\infty}\int_{M_R(k_v)\times U_R^-(k_v)}\Phi_v(m_vu_v)\delta_R(m_v)^{-1}|\lambda|_v d\omega_v^{M_R}(m_v)d\omega_v^{U_R^-}(u_v)\\
&\quad\times\prod_{v\in\mathfrak{V}_f}L_v(1,\sigma_G)\int_{M_R(k_v)\times U_R^-(k_v)}\Phi_v(m_vu_v)\delta_R(m_v)^{-1}|\lambda|_v d\omega_v^{M_R}(m_v)d\omega_v^{U_R^-}(u_v).\\
=\ &\frac{\mu_{\mathbf{A}}(\mathbf{A}/k)^{\dim U_R-\dim G}}{\sigma_k(G)}\prod_{v\in\mathfrak{V}_\infty}\int_{M_R(k_v)\times U_R^-(k_v)}\Phi_v(m_vu_v)\delta_R(m_v)^{-1}d\omega_v^{M_R}(m_v)d\omega_v^{U_R^-}(u_v)\\
&\quad\times\prod_{v\in\mathfrak{V}_f}L_v(1,\sigma_G)\int_{M_R(k_v)\times U_R^-(k_v)}\Phi_v(m_vu_v)\delta_R(m_v)^{-1}d\omega_v^{M_R}(m_v)d\omega_v^{U_R^-}(u_v)\end{aligned}$$

since $|\lambda|_{\mathbf{A}}=1$. We decompose $u_v\in U_R^-(k_v)$ into $u'_vm'_vh'_v$, $u'_v\in U_R(k_v), m'_v\in M_R(k_v)$ and

$h'_v \in K_v$. Then one has

$$\begin{aligned}
&\int_{M_R(k_v)\times U_R^-(k_v)} \Phi_v(m_v u_v)\delta_R(m_v)^{-1} d\omega_v^{M_R}(m_v) d\omega_v^{U_R^-}(u_v) \\
&= \int_{M_R(k_v)\times U_R^-(k_v)} \Phi_v((m_v u'_v m_v^{-1})(m_v m'_v) h'_v)\delta_R(m_v)^{-1} d\omega_v^{M_R}(m_v) d\omega_v^{U_R^-}(u_v) \\
&= \int_{M_R(k_v)\times U_R^-(k_v)} \Phi_v(m_v m'_v)\delta_R(m_v)^{-1} d\omega_v^{M_R}(m_v) d\omega_v^{U_R^-}(u_v) \\
&= \int_{M_R(k_v)\times U_R^-(k_v)} \Phi_v(m_v)\delta_R(m_v(m'_v)^{-1})^{-1} d\omega_v^{M_R}(m_v) d\omega_v^{U_R^-}(u_v) \\
&= \left(\int_{M_R(k_v)} \Phi_v(m_v)\delta_R(m_v)^{-1} d\omega_v^{M_R}(m_v)\right)\left(\int_{U_R^-(k_v)} \eta_v(u_v) d\omega_v^{U_R^-}(u_v)\right).
\end{aligned}$$

By the definition of Tamagawa measures,

$$\begin{aligned}
\int_{M_R(\mathbf{A})} \Phi(m)\delta_R(m)^{-1} d\omega_{\mathbf{A}}^{M_R}(m) &= \frac{\mu_{\mathbf{A}}(\mathbf{A}/k)^{-\dim M_R}}{\sigma_k(M_R)} \prod_{v\in\mathfrak{V}_\infty} \int_{M_R(k_v)} \Phi_v(m_v)\delta_R(m_v)^{-1} d\omega_v^{M_R}(m_v) \\
&\quad \times \prod_{v\in\mathfrak{V}_f} L_v(1,\sigma_{M_R}) \int_{M_R(k_v)} \Phi_v(m_v)\delta_R(m_v)^{-1} d\omega_v^{M_R}(m_v).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathfrak{X}_R(\mathbf{A})} \Phi(x) d(\omega_{\mathbf{A}}^{U_R}\backslash\omega_{\mathbf{A}}^G)(x) &= \frac{\mu_{\mathbf{A}}(\mathbf{A}/k)^{\dim R-\dim G}\sigma_k(M_R)}{\sigma_k(G)} \int_{M_R(\mathbf{A})} \Phi(m)\delta_R(m)^{-1} d\omega_{\mathbf{A}}^{M_R}(m) \\
&\quad \times \prod_{v\in\mathfrak{V}_\infty} J_v \prod_{v\in\mathfrak{V}_f} \frac{L_v(1,\sigma_G)}{L_v(1,\sigma_{M_R})} J_v\,, \qquad (4)
\end{aligned}$$

where

$$J_v = \int_{U_R^-(k_v)} \eta_v(u_v) d\omega_v^{U_R^-}(u_v)\,.$$

From (3), (4) and $\dim R - \dim G = -\dim U_R$, we obtain the following.

Theorem 1 *Notations being as above, we have*

$$C_{G,R} = \frac{\mu_{\mathbf{A}}(\mathbf{A}/k)^{-\dim U_R}\sigma_k(M_R)}{\sigma_k(G)} \prod_{v\in\mathfrak{V}_\infty} J_v \prod_{v\in\mathfrak{V}_f} \frac{L_v(1,\sigma_G)}{L_v(1,\sigma_{M_R})} J_v\,.$$

2.2 Reduction of J_v to the case of minimal k_v-parabolic subgroups

We explain how to compute the local integral J_v. Let $P^{(v)}$ be a minimal parabolic subgroup of G defined over k_v such that $P^{(v)}(k_v) \subset R(k_v)$. Then $P^{(v)}$ has a Levi subgroup $M^{(v)}$ such that $M^{(v)}(k_v) \subset M_R(k_v)$. Let $U^{(v)}$ be the unipotent radical of $P^{(v)}$ and $U^{(v)-}$ be the unipotent radical of the opposite parabolic subgroup of $P^{(v)}$. We set $P_{M_R}^{(v)} = P^{(v)} \cap M_R$, $U_{M_R}^{(v)} = U^{(v)} \cap M_R$ and $U_{M_R}^{(v)-} = U^{(v)-} \cap M_R$. Then $P_{M_R}^{(v)}$ is a minimal parabolic subgroup of M_R defined over k_v with the unipotent radical $U^{(v)}_{M_R}$ and a Levi subgroup $M^{(v)}$. The unipotent group $U_R(k_v)$ is a normal subgroup of $U^{(v)}(k_v)$, and $U^{(v)}(k_v)$ has a semidirect product decomposition $U_R(k_v)U^{(v)}_{M_R}(k_v)$. Let $\delta^{-1}_{P^{(v)}} : M^{(v)}(k_v) \longrightarrow \mathbf{R}_+$ and $\delta^{-1}_{P^{(v)}_{M_R}} : M^{(v)}(k_v) \longrightarrow \mathbf{R}_+$ be the modular characters of $P^{(v)}(k_v)$ and $P^{(v)}_{M_R}(k_v)$, respectively. One has a relation

$$\delta_R^{-1}\Big|_{M^{(v)}(k_v)} = \delta^{-1}_{P^{(v)}} \cdot \delta_{P^{(v)}_{M_R}} \,. \tag{5}$$

Define the function $\eta^G_{P^{(v)}} : G(k_v) \longrightarrow \mathbf{R}_+$ by $\eta^G_{P^{(v)}}(umh) = \delta_{P^{(v)}}(m)$ for $u \in U^{(v)}(k_v)$, $m \in M^{(v)}(k_v)$ and $h \in K_v$. In a similar fashion, the function $\eta^{M_R}_{P^{(v)}_{M_R}} : M_R(k_v) \longrightarrow \mathbf{R}_+$ is defined by $\eta^{M_R}_{P^{(v)}_{M_R}}(umh) = \delta_{P^{(v)}_{M_R}}(m)$ for $u \in U^{(v)}_{M_R}(k_v)$, $m \in M^{(v)}(k_v)$ and $h \in K_v \cap M_R(k_v)$. We set

$$J_v^G = \int_{U^{(v)-}(k_v)} \eta^G_{P^{(v)}}(u) d\omega_{U^{(v)-}(k_v)}(u), \qquad J_v^{M_R} = \int_{U^{(v)-}_{M_R}(k_v)} \eta^{M_R}_{P^{(v)}_{M_R}}(u) d\omega_{U^{(v)-}_{M_R}(k_v)}(u) \,.$$

Here, we fix invariant measures $\omega_{U^{(v)-}(k_v)}$ on $U^{(v)-}(k_v)$ and $\omega_{U^{(v)-}_{M_R}(k_v)}$ on $U^{(v)-}_{M_R}(k_v)$ such that

$$\omega_{U^{(v)-}(k_v)} = \omega_v^{U_R^-} \cdot \omega_{U^{(v)-}_{M_R}(k_v)} \,.$$

Let us compute J_v^G following the decomposition $U^{(v)-} = U_R^-(k_v) U^{(v)-}_{M_R}(k_v)$:

$$J_v^G = \int_{U^{(v)-}(k_v)} \eta^G_{P^{(v)}}(u) d\omega_{U^{(v)-}(k_v)}(u) = \int_{U^{(v)-}_{M_R}(k_v)} d\omega_{U^{(v)-}_{M_R}(k_v)}(u_2) \int_{U_R^-(k_v)} \eta^G_{P^{(v)}}(u_1 u_2) d\omega_v^{U_R^-}(u_1) \,.$$

Let $u_i = \alpha_i \beta_i \gamma_i$, $\alpha_i \in U^{(v)}(k_v)$, $\beta_i \in M^{(v)}(k_v)$ and $\gamma_i \in K_v$ for $i = 1, 2$. Since

$$\eta^G_{P^{(v)}}(u_1 u_2) = \eta^G_{P^{(v)}}(\alpha_2\beta_2(\alpha_2\beta_2)^{-1} u_1 (\alpha_2\beta_2)) = \delta_{P^{(v)}}(\beta_2) \eta^G_{P^{(v)}}((\alpha_2\beta_2)^{-1} u_1 (\alpha_2\beta_2)),$$

one has

$$\begin{aligned} J_v^G &= \int_{U^{(v)-}_{M_R}(k_v)} d\omega_{U^{(v)-}_{M_R}(k_v)}(u_2) \int_{U_R^-(k_v)} \delta_{P^{(v)}}(\beta_2) \delta_{R^-}(\beta_2) \eta^G_{P^{(v)}}(u_1) d\omega_v^{U_R^-}(u_1) \\ &= \int_{U^{(v)-}_{M_R}(k_v)} \delta_{P^{(v)}}(\beta_2) \delta_{R^-}(\beta_2) d\omega_{U^{(v)-}_{M_R}(k_v)}(u_2) \int_{U_R^-(k_v)} \delta_{P^{(v)}}(\beta_1) d\omega_v^{U_R^-}(u_1) \,. \end{aligned}$$

By $\delta_{R^-}(\beta_2) = \delta_R(\beta_2)^{-1}$, $\delta_R(\beta_1) = \delta_{P^{(v)}}(\beta_1)$ and (5), we obtain

$$\begin{aligned} J_v^G &= \int_{U^{(v)-}_{M_R}(k_v)} \delta_{P^{(v)}_{M_R}}(\beta_2) d\omega_{U^{(v)-}_{M_R}(k_v)}(u_2) \cdot \int_{U_R^-(k_v)} \delta_R(\beta_1) d\omega_v^{U_R^-}(u_1) \\ &= J_v^{M_R} \cdot J_v \,. \end{aligned}$$

Therefore, one has

$$J_v = \frac{J_v^G}{J_v^{M_R}} \,. \tag{6}$$

2.4 Gindikin-Karpelevič formula of J_v^G

We set

$$J_v^G(s) = \int_{U^{(v)-}(k_v)} \eta^G_{P^{(v)}}(u)^{s+1/2} d\omega_{U^{(v)-}(k_v)}(u),$$

where s is a complex number with $\Re(s) > 0$. We recall the Gindikin–Karpelevič formula of $J_v^G(s)$ (cf. [K, Chap.VII, §5, Corollary 7.5]). Let $S^{(v)}$ be a maximal k_v-split torus of $M^{(v)}$, $\Sigma_v(G)$ the relative root system of G with respect to $S^{(v)}$ and $\Sigma_v^+(G)$ the set of positive roots of $\Sigma_v(G)$ corresponding to the minimal k_v-parabolic subgroup $P^{(v)}$. We set

$$\mathfrak{a}_v = X^*_{k_v}(S^{(v)}/Z_G^{(v)}) \otimes_{\mathbf{Z}} \mathbf{R},$$

where $Z_G^{(v)}$ denotes the maximal central k_v-split torus of G. Note that the real vector space $\mathfrak{a}_v$ is identified with $X^*_{k_v}(M^{(v)}/Z_G^{(v)}) \otimes_{\mathbf{Z}} \mathbf{R}$ since $M^{(v)}/S^{(v)}$ is anisotropic over k_v. The set of simple roots of $\Sigma_v^+(G)$ gives a basis of $\mathfrak{a}_v$, and hence $\Sigma_v(G)$ is regarded as a subset of $\mathfrak{a}_v$. Thus, for each $\beta \in \Sigma_v(G)$, the function $\xi_\beta^G : G(\mathbf{A}) \longrightarrow \mathbf{R}_+$ is well defined by $\xi_\beta^G(umh) = |\beta(m)|_v$ for $u \in U^{(v)}(k_v)$, $m \in M^{(v)}(k_v)$ and $h \in K_v$. We fix an admissible inner product $(\cdot,\cdot)$ on $\mathfrak{a}_v$ and define the coroot $\beta^\vee$ of $\beta \in \Sigma_v(G)$ by

$$\beta^\vee = \frac{2}{(\beta,\beta)}\beta.$$

For $\beta \in \Sigma_v^+(G)$, the connected component $(\mathrm{Ker}\beta)^0$ of the kernel of β is a subtorus of $S^{(v)}$. We denote by $G_{(\beta)}$ the centralizer of $(\mathrm{Ker}\beta)^0$ in G. Then $G_{(\beta)}$ is a reductive k_v-subgroup of G with semisimple k_v-rank one. We set $P_{(\beta)} = G_{(\beta)} \cap P^{(v)}$, $M_{(\beta)} = G_{(\beta)} \cap M^{(v)}$, $U_{(\beta)} = G_{(\beta)} \cap U^{(v)}$, $U^-_{(\beta)} = G_{(\beta)} \cap U^{(v)-}$ and $K_{(\beta)} = G_{(\beta)}(k_v) \cap K_v$. We assume that $G_{(\beta)}(k_v) = P_{(\beta)}(k_v)K_{(\beta)}$ holds for all $\beta \in \Sigma_v^+(G)$. Then we define the function $\eta_\beta : G_{(\beta)}(k_v) \longrightarrow \mathbf{R}_+$ by $\eta_\beta(umh) = \delta_{P_{(\beta)}}(m)$ for $u \in U_{(\beta)}(k_v)$, $m \in M_{(\beta)}(k_v)$ and $h \in K_{(\beta)}$, where $\delta^{-1}_{P_{(\beta)}} : M_{(\beta)}(k_v) \longrightarrow \mathbf{R}_+$ denote the modular character of $P_{(\beta)}(k_v)$. Moreover, we write ρ_v^G for the half-sum of positive roots and $\xi_{\rho_v^G} : G(\mathbf{A}) \longrightarrow \mathbf{R}_+$ for the function corresponding to ρ_v^G, *i.e.*,

$$\rho_v^G = \frac{1}{2}\sum_{\beta\in\Sigma_v^+} (\dim U_{(\beta)})\beta, \qquad \xi_{\rho_v^G} = \prod_{\beta\in\Sigma_v^+} (\xi_\beta^G)^{\dim U_{(\beta)}/2}.$$

There is a relation $\xi^2_{\rho_v^G} = \eta^G_{P^{(v)}}$. With these notations, the Gindikin–Karpelevič formula of $J_v^G(s)$ is stated as follows:

$$J_v^G(s) = \prod_{\substack{\beta \in \Sigma_v^+(G) \\ \beta/2 \notin \Sigma_v^+(G)}} \int_{U^-_{(\beta)}(k_v)} \xi_\beta^G(u)^{(\rho_v^G,\beta^\vee)s} \eta_\beta(u)^{1/2} d\omega_{U^-_{(\beta)}(k_v)}(u). \tag{7}$$

Here, we fix a family of invariant measures $\omega_{U^-_{(\beta)}(k_v)}$, $\beta \in \Sigma_v^+(G)$ such that

$$\omega_{U^{(v)-}(k_v)} = \prod_{\substack{\beta \in \Sigma_v^+ \\ \beta/2 \notin \Sigma_v^+}} \omega_{U^-_{(\beta)}(k_v)}$$

holds. In principle, $C_{G,R}$ can be computed by Theorem 1 and formulas (6), (7).

3 An explicit formula of $C_{G,P}$ in the case of $G(k) = GL_n(D)$

3.1 Central simple algebras

Let D be a central division k-algebra of degree d^2. Let $D_v = D \otimes_k k_v$ for $v \in \mathfrak{V}$ and $D_{\mathbf{A}} = D \otimes_k \mathbf{A}$. Since D_v is a central simple k_v-algebra, it is isomorphic with an algebra $M_{d/d_v}(D(v))$, where $D(v)$ is a division k_v-algebra of degree d_v^2. The set $\mathfrak{V}$ is divided into two subsets $\mathfrak{V}_1 = \{v \in \mathfrak{V} \,:\, d_v = 1\}$ and $\mathfrak{V}_2 = \{v \in \mathfrak{V} \,:\, d_v > 1\}$. We write $\mathfrak{V}_{\mathbf{R},1}$, $\mathfrak{V}_{\mathbf{R},2}$, $\mathfrak{V}_{f,1}$ and $\mathfrak{V}_{f,2}$ for $\mathfrak{V}_{\mathbf{R}} \cap \mathfrak{V}_1$, $\mathfrak{V}_{\mathbf{R}} \cap \mathfrak{V}_2$, $\mathfrak{V}_f \cap \mathfrak{V}_1$ and $\mathfrak{V}_f \cap \mathfrak{V}_2$, respectively. We fix a maximal order $\mathfrak{O}_D$ of D. For $v \in \mathfrak{V}_f$, the completion of $\mathfrak{O}_D$ in D_v is denoted by $\mathfrak{O}_{D_v}$, which is a maximal order of D_v. Since any maximal order of D_v is conjugate to $\mathfrak{O}_{D_v}$, there is an isomorphism from D_v onto $M_{d/d_v}(D(v))$ such that the image of $\mathfrak{O}_{D_v}$ equals $M_{d/d_v}(\mathfrak{O}_{D(v)})$, where $\mathfrak{O}_{D(v)}$ denotes a unique maximal order of $D(v)$.

For every $v \in \mathfrak{V}_f$, we denote by $\mathfrak{d}_v$ the different of $\mathfrak{O}_{D_v}/\mathfrak{o}_v$, *i.e.*,

$$\mathfrak{d}_v^{-1} = \{a \in D_v \,:\, \tau_{D_v/k_v}(a\mathfrak{O}_{D_v}) \subset \mathfrak{o}_v\}.$$

Then the different $\mathfrak{d}_{\mathfrak{O}_D}$ of $\mathfrak{O}_D$ is given by $\prod_{v \in \mathfrak{V}_f} \mathfrak{d}_v$. The absolute norm $\mathrm{N}\mathfrak{d}_{D/k}$ of $\mathfrak{d}_{\mathfrak{O}_D}$ is defined to be

$$\mathrm{N}\mathfrak{d}_{D/k} = \prod_{v \in \mathfrak{V}_f} |\mathfrak{O}_{D_v}/\mathfrak{d}_v|\,,$$

which is independent of the choice of the maximal order $\mathfrak{O}_D$ (cf. [R, Theorems (25.3) and (25.7)])

Now we consider the central simple k-algebra $\mathfrak{A} = M_n(D)$ and its maximal order $\mathfrak{O}_{\mathfrak{A}} = M_n(\mathfrak{O}_D)$. We identify $\mathfrak{A}_v = \mathfrak{A} \otimes_k k_v$ with $M_n(D_v)$ for $v \in \mathfrak{V}$ and $\mathfrak{A}_{\mathbf{A}} = \mathfrak{A} \otimes_k \mathbf{A}$ with $M_n(D_{\mathbf{A}})$. For $v \in \mathfrak{V}_f$, set $\mathfrak{O}_{\mathfrak{A}_v} = M_n(\mathfrak{O}_{D_v})$, which is a maximal order of $\mathfrak{A}_v$. Hereafter, G denotes an affine algebraic k-group defined by $G(k) = \mathfrak{A}^\times = GL_n(D)$. The adèle group $G(\mathbf{A})$ of G is the unit group of $\mathfrak{A}_{\mathbf{A}}$. If $v \in \mathfrak{V}_\infty$, we define an involution $a \mapsto a^*$ of $\mathfrak{A}_v$ as follows. We fix an algebra isomorphism $\mathfrak{A}_v \cong M_{nd/d_v}(D(v))$. Then, for $a = (a_{ij}) \in \mathfrak{A}_v$ ($a_{ij} \in D(v)$), the involution a^* is defined to be $a^* = (\overline{a}_{ij})^t$, where the superscript t means the transpose of a matrix and $a_{ij} \mapsto \overline{a}_{ij}$ denotes the canonical involution of the division algebra $D(v)$, *i.e.*, it is the identity map, the complex conjugate or the quaternion conjugate according as $v \in \mathfrak{V}_{\mathbf{R},1}$, $v \in \mathfrak{V}_{\mathbf{C}}$ or $v \in \mathfrak{V}_{\mathbf{R},2}$. By using this involution, we define the subgroup K_v of $G(k_v) = \mathfrak{A}_v^\times$ by $K_v = \{a \in \mathfrak{A}_v^\times \,:\, a^{-1} = a^*\}$. If $v \in \mathfrak{V}_f$, set $K_v = \mathfrak{O}_{\mathfrak{A}_v}^\times$. Then $K = \prod_{v \in \mathfrak{V}} K_v$ gives a maximal compact subgroup of $G(\mathbf{A})$. Let P be the minimal k-parabolic subgroup of G which consists of upper triangular matrices in G. We will compute the constant $C_{G,P} = C_{G,P,K}$.

3.2 Self-dual measures

It is convenient to use a self-dual measure on $D_{\mathbf{A}}$ in order to compute $C_{G,P}$. We recall its construction. We fix a non-trivial character $\psi \,:\, \mathbf{A}/k \longrightarrow \mathbf{C}^1$ as follows. If $\mathrm{ch}(k) > 0$, we arbitrarily choose a non-trivial ψ. If $\mathrm{ch}(k) = 0$, we define the character ψ_0 on the adèle group $\mathbf{A}_{\mathbf{Q}}$ of $\mathbf{Q}$ by

$$\psi_0(x) = e^{-2\pi\sqrt{-1}x_\infty} \prod_{p\,:\,\mathrm{prime}} e^{2\pi\sqrt{-1}(x_p \bmod \mathbf{Z}_p)}$$

for $x = (x_\infty, x_2, x_3, \cdots) \in \mathbf{A}_{\mathbf{Q}}$, and then set $\psi = \psi_0 \circ \mathrm{Tr}_{k/\mathbf{Q}}$. For every $v \in \mathfrak{V}$, ψ induces a character $\psi_v \,:\, k_v \longrightarrow \mathbf{C}^1$. Let $\mathfrak{C}$ be an arbitrary central simple k-algebra and $\mathfrak{C}_v = \mathfrak{C} \otimes_k k_v$

for $v \in \mathfrak{V}$ and $\mathfrak{C}_{\mathbf{A}} = \mathfrak{C} \otimes_k \mathbf{A}$. An invariant measure $\nu_{\mathfrak{C}_v}$ on the locally compact additive group $\mathfrak{C}_v$ is called the self-dual measure with respect to ψ_v if

$$\Phi(x) = \int_{\mathfrak{C}_v} \left\{ \int_{\mathfrak{C}_v} \Phi(z)\psi_v(\tau_{\mathfrak{C}_v/k_v}(yz))d\nu_{\mathfrak{C}_v}(z) \right\} \psi_v(-\tau_{\mathfrak{C}_v/k_v}(xy))d\nu_{\mathfrak{C}_v}(y)$$

holds for any Schwartz–Bruhat function Φ on $\mathfrak{C}_v$. The product measure $\nu_{\mathfrak{C}_{\mathbf{A}}} = \prod_{v\in\mathfrak{V}} \nu_{\mathfrak{C}_v}$ on $\mathfrak{C}_{\mathbf{A}}$ satisfies

$$\Phi(x) = \int_{\mathfrak{C}_{\mathbf{A}}} \left\{ \int_{\mathfrak{C}_{\mathbf{A}}} \Phi(z)\psi_v(\tau_{\mathfrak{C}/k}(yz))d\nu_{\mathfrak{C}_{\mathbf{A}}}(z) \right\} \psi_v(-\tau_{\mathfrak{C}/k}(xy))d\nu_{\mathfrak{C}_{\mathbf{A}}}(y)$$

for any Schwartz–Bruhat function Φ on $\mathfrak{C}_{\mathbf{A}}$. The invariant measure $\nu_{\mathfrak{C}_{\mathbf{A}}}$ is called the self-dual measure of $\mathfrak{C}_{\mathbf{A}}$ with respect to ψ.

For $v \in \mathfrak{V}$, let $\nu_{D(v)}$ be the self-dual measure on $D(v)$ with respect to ψ_v. It is known by [T, Propositions 5, 6, 7 and 8] that the product measure $\nu_{D(v)}^{d^2/d_v^2}$ coincides with the self-dual measure on $M_{d/d_v}(D(v))$ with respect to ψ_v. Hence one can identify ν_{D_v} with $\nu_{D(v)}^{d^2/d_v^2}$. Note that this identification is independent of the choice of the algebra isomorphism $D_v \cong M_{d/d_v}(D(v))$ because of Skolem–Noether theorem. Therefore, we have

$$\nu_{D_{\mathbf{A}}} = \prod_{v\in\mathfrak{V}} \nu_{D_v} = \prod_{v\in\mathfrak{V}} \nu_{D(v)}^{d^2/d_v^2}.$$

As was shown in the proof of [T, Theorem 2], $\nu_{D_{\mathbf{A}}}$ is the Tamagawa measure of $D_{\mathbf{A}}$, namely $\nu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D) = 1$.

We define another invariant measure $\mu_{D_{\mathbf{A}}}$ on $D_{\mathbf{A}}$. If $v \in \mathfrak{V}_1$, *i.e.*, $D(v) = k_v$, then we put $\mu_{D(v)} = \mu_v$, where μ_v is the measure on k_v introduced in Notations. For $v \in \mathfrak{V}_2$, $\mu_{D(v)}$ is defined to be the invariant measure on $D(v)$ normalized so that $\mu_{D(v)}(\mathfrak{O}_{D(v)}) = 1$ if $v \in \mathfrak{V}_{f,2}$ and $\mu_{D(v)}(\{x \in D(v) : \mathrm{Nr}_{D(v)/k_v}(x) \le 1\}) = 4\pi^2$ if $v \in \mathfrak{V}_{\mathbf{R},2}$. For every $v \in \mathfrak{V}$, we set $\mu_{D_v} = \mu_{D(v)}^{d^2/d_v^2}$, which gives an invariant measure on $D_v \cong M_{d/d_v}(D(v))$. By Skolem–Noether Theorem, μ_{D_v} is independent of the choice of the algebra isomorphism $D_v \cong M_{d/d_v}(D(v))$. In particular, one has $\mu_{D_v}(\mathfrak{O}_{D_v}) = 1$ for $v \in \mathfrak{V}_f$. The product measure $\mu_{D_{\mathbf{A}}} = \prod_{v\in\mathfrak{V}} \mu_{D_v}$ is an invariant measure on $D_{\mathbf{A}}$. For every $v \in \mathfrak{V}$, there is the positive constant κ_v such that $\mu_{D(v)} = \kappa_v \nu_{D(v)}$. One has $\mu_{D_v} = \kappa_v^{d^2/d_v^2} \nu_{D_v}$.

Lemma 1 $\mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D) = \prod_{v\in\mathfrak{V}} \kappa_v^{d^2/d_v^2} = \mu_{\mathbf{A}}(\mathbf{A}/k)^{d^2} \mathrm{N}\mathfrak{d}_{D/k}^{1/2}$.

Proof. We define the Schwartz–Bruhat function $\Phi_{\mathbf{A}} = \prod_{v\in\mathfrak{V}} \Phi_v$ on $D_{\mathbf{A}}$ as follows: If $v \in \mathfrak{V}_f$, let Φ_v be the characteristic function of $\mathfrak{O}_{D_v}$. If $v \in \mathfrak{V}_\infty$, we set $\Phi_v(x) = e^{-[k_v:\mathbf{R}]d_v\pi \mathrm{Tr}(x^*x)}$, where $\mathrm{Tr}(x^*x)$ denotes the trace of the Hermitian matrix x^*x. One hand, we have

$$\int_{D_{\mathbf{A}}} \Phi_{\mathbf{A}}(x)d\mu_{D_{\mathbf{A}}}(x) = 1.$$

On the other hand, by [T, §II, Propositions 1 and 2],

$$\int_{D_{\mathbf{A}}} \Phi_{\mathbf{A}}(x)d\nu_{D_{\mathbf{A}}}(x) = \mu_{\mathbf{A}}(\mathbf{A}/k)^{-d^2} \mathrm{N}\mathfrak{d}_{D/k}^{-1/2},$$

which proves the lemma.

3.3 A formula of $C_{G,P}$

Let M_P be the Levi subgroup of P consisting of diagonal matrices in G and S be the maximal k-split torus of M_P, *i.e.*,

$$\begin{aligned} M_P(k) &= \left\{ \mathrm{diag}(a_1, \cdots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \ : \ a_1, \cdots, a_n \in D^\times \right\} \\ S(k) &= \{ \mathrm{diag}(a_1, \cdots, a_n) \ : \ a_1, \cdots, a_n \in k^\times \}. \end{aligned}$$

Let $\Sigma(G)$ be the relative root system of G with respect to S and $\Sigma^+(G)$ be the set of positive roots of $\Sigma(G)$ corresponding to P. For each $\alpha \in \Sigma(G)$, U_α denotes the root subgroup of G. We fix an isomorphism $U_\alpha(k) \cong D$ and define the invariant measures $\nu_{U_\alpha(k_v)}$ on $U_\alpha(k_v)$ for $v \in \mathfrak{V}$ and $\nu_{U_\alpha(\mathbf{A})}$ on $U_\alpha(\mathbf{A})$ as

$$\nu_{U_\alpha(k_v)} = \nu_{D_v}, \qquad \nu_{U_\alpha(\mathbf{A})} = \prod_{v \in \mathfrak{V}} \nu_{U_\alpha(k_v)} = \nu_{D_\mathbf{A}}.$$

We set

$$\nu_{U_P^-(k_v)} = \prod_{\alpha \in \Sigma^+(G)} \nu_{U_{-\alpha}(k_v)}, \qquad \nu_{U_P^-(\mathbf{A})} = \prod_{\alpha \in \Sigma^+(G)} \nu_{U_{-\alpha}(\mathbf{A})} = \prod_{v \in \mathfrak{V}} \nu_{U_P^-(k_v)}.$$

Since $\nu_{D_\mathbf{A}}$ is the Tamagawa measure on $D_\mathbf{A}$, $\nu_{U_P^-(\mathbf{A})}$ coincides with the Tamagawa measure on the unipotent group $U_P^-(\mathbf{A})$, *i.e.*, $\omega_\mathbf{A}^{U_P^-} = \nu_{U_P^-(\mathbf{A})}$.

For $v \in \mathfrak{V}$, we define the local integral I_v by

$$I_v = \int_{U_P^-(k_v)} \eta_v(u_v) d\nu_{U_P^-(k_v)}(u_v),$$

where the function $\eta_v : G(k_v) \longrightarrow \mathbf{R}_+$ is defined by

$$\eta_v(u \cdot \mathrm{diag}(a_1, \cdots, a_n) \cdot h) = \prod_{i=1}^n |\mathrm{Nr}_{D_v/k_v}(a_i)|_v^{d(n-2i+1)}$$

for $u \in U_P(k_v)$, $a_1, \cdots, a_n \in D_v^\times$ and $h \in K_v$. Since

$$\frac{\sigma_k(M_P)}{\sigma_k(G)} = \rho_k^{n-1}, \qquad \frac{L_v(1, \sigma_G)}{L_v(1, \sigma_{M_P})} = (1 - q_v^{-1})^{n-1}$$

and

$$\omega_\mathbf{A}^{U_P^-} = \mu_\mathbf{A}(\mathbf{A}/k)^{-\dim U_P} \prod_{v \in \mathfrak{V}} \omega_v^{U_P^-} = \prod_{v \in \mathfrak{V}} \nu_{U_P^-(k_v)},$$

Theorem 1 leads us to

$$C_{G,P} = \rho_k^{n-1} \prod_{v \in \mathfrak{V}_\infty} I_v \prod_{v \in \mathfrak{V}_f} (1 - q_v^{-1})^{n-1} I_v. \tag{8}$$

3.4 Reduction of I_v to the case of $GL_2(D(v))$

We fix a place $v \in \mathfrak{V}$. Let $S^{(v)}$ be the maximal k_v-split torus in M_P and $P^{(v)}$ be a minimal k_v-parabolic subgroup of G such that $S^{(v)} \subset P^{(v)} \subset P$. The unipotent radical of $P^{(v)}$ is denoted by $U^{(v)}$. The centralizer $M^{(v)}$ of $S^{(v)}$ in G is a Levi subgroup of $P^{(v)}$. As in §2.3, we set $P^{(v)}_{M_P} = P^{(v)} \cap M_P$, $U^{(v)}_{M_P} = U^{(v)} \cap M_P$ and $U^{(v)-}_{M_P} = U^{(v)-} \cap M_P$. Let $\Sigma_v(G)$ be the relative root system of G with respect to $S^{(v)}$ and $\Sigma_v^+(G)$ be the set of positive roots of $\Sigma_v(G)$ corresponding to $P^{(v)}$. For every $\beta \in \Sigma_v(G)$, $U_{(\beta)}$ stands for the root subgroup of G. We fix an isomorphism $U_{(\beta)}(k_v) \cong D(v)$ and define the invariant measures $\nu_{U_{(\beta)}(k_v)}$ on $U_{(\beta)}(k_v)$, $\nu_{U^{(v)-}(k_v)}$ on $U^{(v)-}(k_v)$ and $\nu_{U^{(v)-}_{M_P}(k_v)}$ on $U^{(v)-}_{M_P}(k_v)$ as

$$\nu_{U_{(\beta)}(k_v)} = \nu_{D(v)}, \qquad \nu_{U^{(v)-}(k_v)} = \prod_{\beta \in \Sigma_v^+(G)} \nu_{U_{(-\beta)}(k_v)}, \qquad \nu_{U^{(v)-}_{M_P}(k_v)} = \prod_{\substack{\beta \in \Sigma_v^+(G) \\ \beta|_S = 0}} \nu_{U_{(-\beta)}(k_v)}\,.$$

For a k-root $\alpha \in \Sigma(G)$, one has

$$U_\alpha(k_v) = \prod_{\substack{\beta \in \Sigma_v(G) \\ \beta|_S = \alpha}} U_{(\beta)}(k_v)\,.$$

From $\nu_{D_v} = \nu_{D(v)}^{d^2/d_v^2}$, it follows

$$\nu_{U_\alpha(k_v)} = \prod_{\substack{\beta \in \Sigma_v(G) \\ \beta|_S = \alpha}} \nu_{U_{(\beta)}(k_v)}\,.$$

This implies the relation $\nu_{U^{(v)-}(k_v)} = \nu_{U_P^-(k_v)} \cdot \nu_{U^{(v)-}_{M_P}(k_v)}$. Therefore, if we set

$$\begin{aligned} I_v^G(s) &= \int_{U^{(v)-}(k_v)} \eta^G_{P^{(v)}}(u)^{s+1/2} d\nu_{U^{(v)-}(k_v)}(u)\,, \\ I_v^{M_P}(s) &= \int_{U^{(v)-}_{M_P}(k_v)} \eta^{M_P}_{P^{(v)}_{M_P}}(u)^{s+1/2} d\nu_{U^{(v)-}_{M_P}(k_v)}(u) \end{aligned}$$

for $\Re(s) > 0$ with the notations in §2.3, then $I_v \cdot I_v^{M_P}(1/2) = I_v^G(1/2)$ holds similarly as (6).

Let $K_v^{GL_2}$ be a maximal compact subgroup of $GL_2(D(v))$ defined by the same way as K_v. We define the function $\eta_v^{GL_2} : GL_2(D(v)) \longrightarrow \mathbf{R}_+$ as follows:

$$\eta_v^{GL_2}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} h \right) = |\mathrm{Nr}_{D(v)/k_v}(a_1)|_v^{d_v} |\mathrm{Nr}_{D(v)/k_v}(a_2)|_v^{-d_v}$$

for $b \in D(v)$, $a_1, a_2 \in D(v)^\times$ and $h \in K_v^{GL_2}$. We set

$$I_v^{GL_2}(s) = \int_{D(v)} \eta_v^{GL_2}\left(\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \right)^{s+1/2} d\nu_{D(v)}(b)$$

for $\Re(s) > 0$. Then, by the Gindikin–Karpelevič formula,

$$\begin{aligned} I_v^G(s) &= \prod_{\beta\in\Sigma_v^+(G)} \int_{U_{(-\beta)}(k_v)} \xi_\beta^G(u)^{(\rho_v^G,\beta^\vee)s}\eta_\beta(u)^{1/2}d\nu_{U_{(-\beta)}(k_v)}(u) \\ &= \prod_{\beta\in\Sigma_v^+(G)} I_v^{GL_2}((\rho_v^G,\beta^\vee)s/d_v^2) \\ &= \prod_{1\le i<j\le nd/d_v} I_v^{GL_2}((j-i)s), \end{aligned}$$

and, in a similar fashion,

$$I_v^{M_P}(s) = \left(\prod_{1\le i<j\le d/d_v} I_v^{GL_2}((j-i)s)\right)^n .$$

Therefore,

$$I_v = \left(\prod_{1\le i<j\le d/d_v} I_v^{GL_2}((j-i)/2)\right)^{-n} \prod_{1\le i<j\le nd/d_v} I_v^{GL_2}((j-i)/2)\,. \tag{9}$$

3.5 Computations of $I_v^{GL_2}(s)$

An Iwasawa decomposition of the unipotent matrix $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in GL_2(D(v))$ is given as follows:

- If $v \in \mathfrak{V}_f$,

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} & (x \in \mathfrak{O}_{D(v)}). \\ \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix} & (x \notin \mathfrak{O}_{D(v)}). \end{cases}$$

- If $v \in \mathfrak{V}_{\mathbf{R},1}$,

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x}{1+x^2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & 0 \\ 0 & \sqrt{1+x^2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} \\ \frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}.$$

- If $v \in \mathfrak{V}_{\mathbf{C}}$,

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\bar{x}}{1+|x|_v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+|x|_v}} & 0 \\ 0 & \sqrt{1+|x|_v} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+|x|_v}} & -\frac{\bar{x}}{\sqrt{1+|x|_v}} \\ \frac{x}{\sqrt{1+|x|_v}} & \frac{1}{\sqrt{1+|x|_v}} \end{pmatrix}.$$

- If $v \in \mathfrak{V}_{\mathbf{R},2}$,

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\bar{x}}{1+|x|^2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+|x|^2}} & 0 \\ 0 & \sqrt{1+|x|^2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+|x|^2}} & -\frac{\bar{x}}{\sqrt{1+|x|^2}} \\ \frac{x}{\sqrt{1+|x|^2}} & \frac{1}{\sqrt{1+|x|^2}} \end{pmatrix},$$

  where $|x| = \mathrm{Nr}_{D(v)/k_v}(x)^{1/2}$ for $x \in D(v)$.

Lemma 2

$$I_v^{GL_2}(s) = \kappa_v^{-1} \times \begin{cases} \dfrac{1-q_v^{-2d_v s - d_v}}{1-q_v^{-2d_v s}} & (v \in \mathfrak{V}_f). \\ \pi^{1/2}\dfrac{\Gamma(s)}{\Gamma(s+1/2)} & (v \in \mathfrak{V}_{\mathbf{R},1}). \\ \pi/s & (v \in \mathfrak{V}_{\mathbf{C}}). \\ \dfrac{\pi^2}{s(4s+1)} & (v \in \mathfrak{V}_{\mathbf{R},2}). \end{cases}$$

Proof. Let $v \in \mathfrak{V}_f$ and $\pi_{D(v)}$ be a prime element of $D(v)$. Since $\kappa_v \nu_{D(v)} = \mu_{D(v)}$, one has

$$\begin{aligned} \kappa_v I_v^{GL_2}(s) &= 1 + \sum_{t=1}^{\infty} \int_{\pi_{D(v)}^{-t}\mathfrak{O}_{D(v)}^{\times}} \left| N_{D(v)/k_v}(x) \right|_v^{-2d_v s - d_v} d\mu_{D(v)}(x) \\ &= 1 + \sum_{t=1}^{\infty} q_v^{-(2s+1)td_v} \int_{\pi_{D(v)}^{-t}\mathfrak{O}_{D(v)}^{\times}} d\mu_{D(v)}(x) \\ &= 1 + \sum_{t=1}^{\infty} q_v^{-2td_v s}(1-q_v^{-d_v}) \\ &= 1 + (1-q_v^{-d_v}) \frac{q_v^{-2d_v s}}{1-q_v^{-2d_v s}} \\ &= \frac{1-q_v^{-2d_v s - d_v}}{1-q_v^{-2d_v s}}. \end{aligned}$$

If $v \in \mathfrak{V}_{\mathbf{R},2}$,

$$\begin{aligned} \kappa_v I_v^{GL_2}(s) &= \int_{D(v)} (1+|x|^2)^{-4s-2} d\mu_{D(v)}(x) \\ &= 4\int_0^{\infty} (1+r^2)^{-4s-2} r^3 dr \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi d\varphi \int_{-\pi/2}^{\pi/2} (\cos\psi)^2 d\psi \\ &= \frac{\pi^2}{s(4s+1)}. \end{aligned}$$

The other cases are also easy.

3.6 An explicit formula of $C_{G,P}$

To describe I_v, we define functions $F_1(s), F_2(s), F_3(s)$ in $s \in \mathbf{C}$ as

$$F_1(s) = \pi^{-s/2}\Gamma(s/2), \qquad F_2(s) = (2\pi)^{1-s}\Gamma(s), \qquad F_3(s) = (2\pi)^{2-s}\Gamma(s)\,.$$

By the formula (9) and Lemma 2, we have the following conclusion.

Lemma 3 *Notations being as above, we have*

$$I_v = \kappa_v^{-\frac{1}{2}\frac{d^2}{d_v^2}n(n-1)} \times \begin{cases} \prod\limits_{\substack{1\le i\le d \\ i\equiv 0\ (d_v)}} (1-q_v^{-i})^{-(n-1)} \prod\limits_{\substack{d+1\le i\le nd \\ i\equiv 0\ (d_v)}} (1-q_v^{-i})^{-1} & (v\in\mathfrak{V}_f) \\ \prod\limits_{1\le i\le d} F_1(i)^{n-1} \prod\limits_{d+1\le i\le nd} F_1(i)^{-1} & (v\in\mathfrak{V}_{\mathbf{R},1}) \\ \prod\limits_{1\le i\le d} F_2(i)^{n-1} \prod\limits_{d+1\le i\le nd} F_2(i)^{-1} & (v\in\mathfrak{V}_{\mathbf{C}}) \\ \prod\limits_{\substack{1\le i\le d \\ i\equiv 0\ (2)}} F_3(i)^{n-1} \prod\limits_{\substack{d+1\le i\le nd \\ i\equiv 0\ (2)}} F_3(i)^{-1} & (v\in\mathfrak{V}_{\mathbf{R},2}). \end{cases}$$

It is convenient to introduce a zeta function of D in order to formulate an explicit formula of $C_{G,P}$. We first define the constant C_D as follows:

- If $\mathrm{ch}(k)=0$,

$$\begin{aligned} C_D &= \rho_k \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D) \prod_{2\le i\le d} \zeta_k(i) F_1(i)^{r_1+r_3} F_2(i)^{r_2} \\ &\quad \times \prod_{v\in\mathfrak{V}_{f,2}} \left(\prod_{\substack{1\le i\le d-1 \\ i\not\equiv 0\ (d_v)}} 1-q_v^{-i} \right) \cdot \prod_{\substack{1\le i\le d-1 \\ i\not\equiv 0\ (2)}} i^{r_3}, \end{aligned}$$

  where r_1, r_2 and r_3 denote the cardinality of $\mathfrak{V}_{\mathbf{R},1}$, $\mathfrak{V}_{\mathbf{C}}$ and $\mathfrak{V}_{\mathbf{R},2}$, respectively.

- If $\mathrm{ch}(k)>0$,

$$C_D = (\log q)\rho_k \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D) \prod_{2\le i\le d} \zeta_k(i) \cdot \prod_{v\in\mathfrak{V}_{f,2}} \prod_{\substack{1\le i\le d-1 \\ i\not\equiv 0\ (d_v)}} (1-q_v^{-i}).$$

Then the zeta function of D is defined by

$$\begin{aligned} Z_D(s) &= C_D^{-1} \prod_{0\le i\le d-1} \zeta_k(s-i) F_1(s-i)^{r_1+r_3} F_2(s-i)^{r_2} \\ &\quad \times \prod_{v\in\mathfrak{V}_{f,2}} \left(\prod_{\substack{1\le i\le d-1 \\ i\not\equiv 0\ (d_v)}} (1-q_v^{-(s-i)}) \right) \cdot \prod_{\substack{1\le i\le d-1 \\ i\not\equiv 0 (2)}} (s-i)^{r_3}. \end{aligned}$$

By [T, Propositions 7 and 8], $Z_D(s)$ has a simple pole at $s=d$ with the residue

$$\rho_D = \begin{cases} \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D)^{-1} & (\mathrm{ch}(k)=0) \\ (\log q)^{-1}\mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D)^{-1} & (\mathrm{ch}(k)>0) \end{cases}$$

By the formula (8) and Lemmas 1 and 3, the constant $C_{G,P}$ is expressed in terms of $Z_D(s)$.

Theorem 2 *If $G(k)=GL_n(D)$ and P a minimal k-parabolic subgroup of G, then*

$$C_{G,P} = \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D)^{-n(n-1)/2} \rho_D^{n-1} \prod_{2\le i\le n} Z_D(id)^{-1}.$$

We take positive integers $n_1, \cdots, n_t$ such that $n = n_1 + \cdots + n_t$. For such $n_1, \cdots, n_t$, $R_{(n_1,\cdots,n_t)}$ denotes the standard k-parabolic subgroup of G whose Levi subgroup $M_{R_{(n_1,\cdots,n_t)}}(k)$ is isomorphic with $GL_{n_1}(D) \times \cdots \times GL_{n_t}(D)$.

Corollary 1 *Let $R = R_{(n_1,\cdots,n_t)}$ be a standard k-parabolic subgroup of G. Then we have*

$$C_{G,R} = \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D)^{-\frac{1}{2}(n^2-\sum_{1\le j\le t} n_j^2)} \rho_D^{t-1} \frac{\prod_{1\le j\le t}\prod_{2\le i\le n_j} Z_D(id)}{\prod_{2\le i\le n} Z_D(id)}.$$

This is a consequence of Theorem 2 and the relation $C_{G,R} = C_{G,P}/C_{M_R,M_R\cap P}$.

4 Applications

4.1 Fundamental Hermite constants of $GL_n(D)$

We use the same notations as in §3. For $1 \le m \le n-1$, Q_m denotes the standard maximal k-parabolic subgroup $R_{(m,n-m)}$ of G. We recall the fundamental Hermite constants $\gamma(G, Q_m, k)$ introduced in [Wa].

In the following, we fix m and write Q for Q_m. The Levi subgroup M_Q is given by

$$M_Q(k) = \left\{ \mathrm{diag}(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,:\, a \in GL_m(D),\ \ b \in GL_{n-m}(D) \right\}.$$

Denote by Z_G and Z_Q the central maximal k-split tori of G and M_Q, respectively, *i.e.*,

$$Z_G(k) = \{\lambda I_n \,:\, \lambda \in k^\times\} \quad \text{and} \quad Z_Q(k) = \{\mathrm{diag}(\lambda I_m, \mu I_{n-m}) \,:\, \lambda, \mu \in k^\times\}.$$

We define the k-rational characters $\alpha_Q \in \mathbf{X}_k^*(Z_Q)$ and $\widehat{\alpha}_Q \in \mathbf{X}_k^*(M_Q)$ as follows:

$$\alpha_Q(\mathrm{diag}(\lambda I_m, \mu I_{n-m})) = \lambda\mu^{-1}$$

for $\mathrm{diag}(\lambda I_m, \mu I_{n-m}) \in Z_Q(k)$ and

$$\widehat{\alpha}_Q(\mathrm{diag}(a,b)) = \mathrm{Nr}_{M_m(D)/k}(a)^{(n-m)/\gcd(m,n-m)} \mathrm{Nr}_{M_{n-m}(D)/k}(b)^{-m/\gcd(m,n-m)}$$

for $\mathrm{diag}(a,b) \in M_Q(k)$. Then α_Q (resp. $\widehat{\alpha}_Q$) is trivial on Z_G and forms a $\mathbf{Z}$-basis of the module $\mathbf{X}_k^*(Z_G \backslash Z_Q)$ (resp. $\mathbf{X}_k^*(Z_G \backslash M_Q)$).

Define the unimodular subgroups $G(\mathbf{A})^1$, $M_Q(\mathbf{A})^1$ and $Q(\mathbf{A})^1$ as follows:

$$\begin{aligned} G(\mathbf{A})^1 &= \{g \in G(\mathbf{A}) \,:\, |\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbf{A}} = 1\}, \\ M_Q(\mathbf{A})^1 &= \{\mathrm{diag}(a,b) \in M_Q(\mathbf{A}) \,:\, |\mathrm{Nr}_{M_m(D)/k}(a)|_{\mathbf{A}} = |\mathrm{Nr}_{M_{n-m}(D)/k}(b)|_{\mathbf{A}} = 1\}, \\ Q(\mathbf{A})^1 &= U_Q(\mathbf{A}) M_Q(\mathbf{A})^1. \end{aligned}$$

The height function $H_Q : G(\mathbf{A}) \longrightarrow \mathbf{R}_+$ is well defined by

$$H_Q(u \cdot \mathrm{diag}(a,b) \cdot h) = |\widehat{\alpha}_Q(\mathrm{diag}(a,b))|_{\mathbf{A}}^{-1}$$

for $u \in U_Q(\mathbf{A}), \mathrm{diag}(a,b) \in M_Q(\mathbf{A})$ and $h \in K$, and this is left $Z_G(\mathbf{A})Q(\mathbf{A})^1$ and right K invariant. We set $X_Q = Q(k)\backslash G(k)$ and $Y_Q = Q(\mathbf{A})^1 \backslash G(\mathbf{A})^1$. Then X_Q is a subset of Y_Q and the natural map $Y_Q \longrightarrow (Z_G(\mathbf{A})Q(\mathbf{A})^1)\backslash G(\mathbf{A})$ is injective. Thus H_Q is restricted to Y_Q. Then the Hermite constants $\gamma(G,Q,k)$ and $\widetilde{\gamma}(G,Q,k)$ are defined to be

$$\gamma(G,Q,k) = \max_{g\in G(\mathbf{A})^1} \min_{x \in X_Q} H_Q(xg).$$

We write $\gamma_{n,m}(D)$ for $\gamma(G, Q_m, k)$, and especially $\gamma_n(D)$ for $\gamma(G, Q_1, k)$ since it is an analogue of Hermite–Rankin's constant.

4.2 An explicit lower bound of $\gamma_{n,m}(D)$

Since $Q = Q_m$ is maximal, there is a positive constant $\widehat{e}_Q$ such that $\delta_Q(g) = |\widehat{\alpha}_Q(g)|_{\mathbf{A}}^{\widehat{e}_Q}$ holds for all $g \in M_Q(\mathbf{A})$. It was proved in [Wa] that

$$\left(\frac{D_{G,Q} \cdot E_Q}{C_{G,Q}} \cdot \frac{\tau(G)}{\tau(Q)} \right)^{1/\widehat{e_Q}} \leq \gamma(G, Q, k)\,, \tag{10}$$

where $D_{G,Q}$ and E_Q are given as follows with the notations in §1.1:

$$D_{G,Q} \;=\; \begin{cases} [\mathbf{X}_k^*(Z_G) : \mathbf{X}_k^*(G)]/[\mathbf{X}_k^*(Z_Q) : \mathbf{X}_k^*(M_Q)] & (\mathrm{ch}(k) = 0), \\ d_G^*/d_{M_Q}^* & (\mathrm{ch}(k) > 0), \end{cases}$$

$$E_Q \;=\; \begin{cases} \widehat{e}_Q[\mathbf{X}_k^*(Z_Q/Z_G) : \mathbf{X}_k^*(M_Q/Z_G)] & (\mathrm{ch}(k) = 0). \\ (1 - q_0^{-\widehat{e}_Q}) & (\mathrm{ch}(k) > 0). \end{cases}$$

Here, $q_0 > 1$ stands for the generator of the subgroup $|\widehat{\alpha}_Q(M_Q(\mathbf{A}) \cap G(\mathbf{A})^1)|_{\mathbf{A}}$ of the cyclic group $q^{\mathbf{Z}}$. The inequality (10) is strict if $\mathrm{ch}(k) > 0$. It is easy to see

$$[\mathbf{X}_k^*(Z_G) : \mathbf{X}_k^*(G)] = dn, \qquad [\mathbf{X}_k^*(Z_Q) : \mathbf{X}_k^*(M_Q)] = d^2 m(n-m),$$
$$[\mathbf{X}_k^*(Z_Q/Z_G) : \mathbf{X}_k^*(M_Q/Z_G)] = dm(n-m)/\mathrm{gcd}(m, n-m), \qquad \widehat{e}_Q = d \cdot \mathrm{gcd}(m, n-m)$$
$$d_G^* = \log q, \qquad d_{M_Q}^* = (\log q)^2, \qquad q_0 = q^{n/\mathrm{gcd}(m,n-m)}.$$

Therefore,

$$D_{G,Q} \cdot E_Q = \begin{cases} dn & (\mathrm{ch}(k) = 0). \\ (1 - q^{-dn})/(\log q) & (\mathrm{ch}(k) > 0). \end{cases}$$

Since $\tau(G) = \tau(Q) = 1$ is known, Cororally 1 gives the following.

Theorem 3 *If* $\mathrm{ch}(k) = 0$, *then*

$$\left\{ dn \cdot \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D)^{m(n-m)} \cdot \rho_D^{-1} \cdot \frac{\prod_{j=n-m+1}^{n} Z_D(jd)}{\prod_{j=2}^{m} Z_D(jd)} \right\}^{\frac{1}{d \cdot \mathrm{gcd}(m,n-m)}} \leq \gamma_{n,m}(D)\,.$$

If $\mathrm{ch}(k) > 0$, *then*

$$\left\{ \frac{1 - q^{-dn}}{\log q} \cdot \mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D)^{m(n-m)} \cdot \rho_D^{-1} \cdot \frac{\prod_{j=n-m+1}^{n} Z_D(jd)}{\prod_{j=2}^{m} Z_D(jd)} \right\}^{\frac{1}{d \cdot \mathrm{gcd}(m,n-m)}} < \gamma_{n,m}(D)\,.$$

For example, if D is a quaternion division algebra over $\mathbf{Q}$ and $m = 1$, then one has $\rho_{\mathbf{Q}} = 1$, $\mu_{D_{\mathbf{A}}}(D_{\mathbf{A}}/D) = \mathrm{N}\mathfrak{d}_{D/\mathbf{Q}}^{1/2} = \prod_{p \in \mathfrak{V}_{f,2}} p$ and hence

$$\left\{ \frac{12n(2n-1)^{r_3}}{\pi^{2n+1/2}} \zeta_{\mathbf{Q}}(2n)\zeta_{\mathbf{Q}}(2n-1)\Gamma(n)\Gamma(n - \frac{1}{2}) \prod_{p \in \mathfrak{V}_{f,2}} p^{n-1} \left(\frac{1 - p^{-(2n-1)}}{1 - p^{-1}} \right) \right\}^{1/2} \leq \gamma_n(D)\,,$$

where $r_3 = 1$ or 0 according as D is definite or indefinite. We denote the value of the left-hand side by $[n, D]$. For a square-free integer $N > 1$, let D_N be a quaternion algebra over $\mathbf{Q}$ such that $\mathrm{N}\mathfrak{d}_{D_N/\mathbf{Q}}^{1/2} = N$, e.g., $D_2 = (-1,-1)$, $D_3 = (-1,-3)$, $D_5 = (-2,-5)$, $D_6 = (-1,3)$, $D_7 = (-1,-7)$ and $D_{10} = (-2,5)$, where (a,b) stands for the quaternion algebra generated by $\mathbf{i}$ and $\mathbf{j}$ with $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$ and $\mathbf{ij} = -\mathbf{ji}$. The following tables give numerical examples of $[n, D_N]$:

n	$[n, D_2]$	$[n, D_3]$	$[n, D_5]$	$[n, D_7]$
2	1.297258519	1.443456027	1.726586552	1.978704389
3	1.515273677	1.995775367	3.042255888	4.115273864
4	2.530418525	4.040765897	7.938578156	12.70444456
5	5.393737367	10.52001705	26.67683122	50.51365650
6	13.94246428	33.28151972	108.9521040	244.1035544
7	42.33203429	123.7370964	522.9445997	1386.303048
8	147.6045644	528.3922475	2882.945637	9042.800847
9	581.1565361	2547.947350	17947.12248	66607.84112
10	2549.878172	13691.81879	124505.8889	546744.5241

By [C-W], it is known $\gamma_2(D_2) = 2$, $\gamma_2(D_3) = 3$ and $\gamma_2(D_5) = 5$.

n	$[n, D_6]$	$[n, D_{10}]$	$[n, D_{14}]$	$[n, D_{15}]$
2	1.559110703	1.864926623	2.137245010	2.075098781
3	2.484720294	3.787578034	5.123474644	4.988640043
4	6.085153489	11.95502729	19.13213909	19.09070223
5	19.81735311	50.25316799	95.15640162	98.01444678
6	80.25844451	262.7381944	588.6561594	627.1722287
7	388.2457592	1640.825823	4349.756821	4796.155594
8	2182.851359	11909.79207	37356.88820	42634.46615
9	13982.96635	98492.61985	365539.4219	431818.2696
10	100515.7012	914034.6441	4013813.651	4907997.900

There is no example of the exact value of $\gamma_n(D)$ for indefinite quaternion algebras.

4.3 The asymptotic distribution of rational points on Y_Q

Let $Q = Q_m$, $X_Q = Q(k)\backslash G(k)$ and $Y_Q = Q(\mathbf{A})^1\backslash G(\mathbf{A})^1$ be the same as in §4.1. The projective variety $Q\backslash G$ is a k-form of Grassmannian and is called the Brauer–Severi variety. The set X_Q is considered as the set of k-rational points of $Q\backslash G$. For a positive real number T, let us define the subset B_T of Y_Q by

$$B_T = \{y \in Y_Q \ : \ H_Q(y) \leq T\} .$$

For $g \in G(\mathbf{A})^1$, the subset $B_T g$ is the translation of B_T by g. The constant $\gamma_{n,m}(D)$ measures the existence of rational points in $B_T g$, *i.e.*, we have $B_T g \cap X_Q \neq \emptyset$ for every $g \in G(\mathbf{A})^1$ if $\gamma_{n,m}(D) \leq T$. In the case that k is an algebraic number field, the cardinality of $B_T g \cap X_Q$ is increasing to proportion to the volume of B_T as $T \to \infty$. More precisely, it was proved in [Wa2] that

$$\lim_{T\to\infty} \sharp(B_T g \cap X_Q) \cdot \frac{D_{G,Q} \cdot E_Q}{C_{G,Q}} T^{-\widehat{e}_Q} = \frac{\tau(Q)}{\tau(G)} .$$

Therefore, we obtain the following.

Theorem 4 *We assume k is an algebraic number field. Then the asymptotic behavior*

$$\sharp(B_T g \cap X_Q) \sim \frac{T^{d\cdot \gcd(m,n-m)}}{dn|D_k|^{d^2(m(n-m)+1)/2} \mathrm{N}\mathfrak{d}_{D/k}^{(m(n-m)+1)/2}} \frac{\prod_{j=2}^{m} Z_D(jd)}{\prod_{j=n-m+1}^{n} Z_D(jd)} \quad \textit{as } T \to \infty$$

holds for all $g \in G(\mathbf{A})^1$.

For example, if $k = \mathbf{Q}$, $m = 1$ and $D = D_N$ as defined above, then we have

$$\sharp(B_T g \cap X_Q) \sim \frac{T^2}{[n, D_N]^2} \quad \text{as } T \to \infty.$$

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043 Japan

nakamura@gaia.math.wani.osaka-u.ac.jp ; watanabe@math.wani.osaka-u.ac.jp