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<u>Comment</u>. Different motivic Galois groups are attached to different realizations; what is the relation between them?

<u>Abstract answer</u>: $\mathcal{M}$ tannakian category of motives $/F = \mathbb{Q}$, say (e.g $\mathcal{M} = \mathcal{M}_{num}(k)$ under $\sim_{hom} = \sim_{num}$, or $\mathcal{M}_{\nu}$ built in terms of motivated correspondences, unconditionally if char $k=0$)

$$\mathcal{M} \overset{H}{\rightrightarrows} Vec_? \qquad \underline{Aut}^{\otimes} H$$

e.g. H_B, H_{dR}, H_ℓ (ℓ prime $\neq$ char k)
(if $k \subset \mathbb{C}$) (if char $k=0$)

Internal motivic Galois group $\pi(\mathcal{M})$: pro-object in $\mathcal{M}^{op}$ independent of H.

For any H, $\quad H(\pi(\mathcal{M})) = \underline{Aut}^{\otimes} H$

<u>Concrete answer</u> for char $k = 0$:

comparison iso. $\quad H_B(M) \otimes \mathbb{C} \cong H_{dR}(M) \otimes \mathbb{C}$
$k \subset \bar{k} \subset \mathbb{C} \quad H_B(M) \otimes \mathbb{Q}_\ell \cong H_\ell(M)$.

The motivic Galois gps $\quad G^{(H)}_{mot}(M) = \underline{Aut}^{\otimes} H|_{\langle M \rangle}$
$$\cap$$
$$GL(H(X))$$

correspond to each other via these comp. iso.

⑤ Enriched realizations of pure motives.

a) Hodge realization.

$HS_{\mathbb{Q}} = \{ \mathbb{Q}\text{- Hodge structures } V \}$

$\mathbb{Q}$ - space + bigrading on $V_{\mathbb{C}}$ s.t $V^{p,q} = \overline{V^{q,p}}$

$$\Updownarrow$$

$$\mu : \mathbb{G}_m^2 \to GL(V_{\mathbb{C}})$$

Mumford-Tate gp

$MT(V) :=$ smallest closed alg. subgp of $GL(V)$ whose cplex points contain $\mathrm{Im}\,\mu$.

connected reductive gp. (if V pol.)

$k \subset \mathbb{C}$ Hodge realization: $\mathcal{M}_{hom}(k)_{\mathbb{Q}} \xrightarrow{H_B} HS_{\mathbb{Q}}$

Hodge cj $\Leftrightarrow$ if $k = \bar{k}$, it is fully faithful.

(Under st^d cj, $\Leftrightarrow$ if $k = \bar{k}$ $MT(H_B(M))$
$= G_{mot}(M)$ $\forall M$

Without st^d cj.

$\mathcal{M}_{\mathcal{V}} \to HS_{\mathbb{Q}}$ $\rightsquigarrow$ $\boxed{MT(H_B(M)) \subseteq G_{mot}(M)}$ $\forall M \in \mathcal{M}_{\mathcal{V}}$

$(k = \bar{k})$ fully faithful $\Leftrightarrow$ every Hodge class on any $X \in \mathcal{V}$ is motivated

$\Leftrightarrow$ $MT(H_B(M)) = G_{mot}(M)$ $\forall M \in \mathcal{M}$

Ex : • in $H^2(1)$, any Hodge class is alg. (Lefschetz

• on abelian varieties, any Hodge class is motivated (provided $\mathcal{V} \ni$ compact ab. pencils

(A.).

k of char k.

b) Tate realization.

$\mathrm{Rep}_{\mathbb{Q}_\ell}\, \mathrm{Gal}(\bar{k}/k) = \{$ continuous f.d.! $\mathbb{Q}_\ell$-rep's of $\mathrm{Gal}(\bar{k}/k)\}$

Tate realization : $\mathcal{M}_{hom}(k)_{\mathbb{Q}_\ell} \xrightarrow{\text{"}H_\ell\text{"}} \mathrm{Rep}_{\mathbb{Q}_\ell}\, \mathrm{Gal}(\bar{k}/k)$

$G_\ell(M) :=$ Zariski closure of $\mathrm{Im}\left(\mathrm{Gal}(\bar{k}/k) \to GL(H_\ell(M))\right)$

k of f.t. (over its prime field)

Tate conj $\Leftrightarrow$ Tate realization is fully faithful

(under st.! conj. $\Leftrightarrow$ $G_\ell(M) = G_{mot}(M)$ $\forall M$
$\sim_{hom} = \sim_{num}$

Without st.! conj

$\mathcal{M}_{V,\mathbb{Q}_\ell} \xrightarrow{\ } \mathrm{Rep}_{\mathbb{Q}_\ell}\, \mathrm{Gal}(\bar{k}/k)$ $\rightsquigarrow$ $\boxed{G_\ell(M) \subseteq G_{mot}(M)}$ $\forall M \in \mathcal{M}_V$

k of f.t. $\mathcal{M}_{V,\mathbb{Q}_\ell}$ abelian and Tate real. fully faithful $\Leftrightarrow$ every Tate class on any X is $\mathbb{Q}_\ell$-linear combination of motivated classes

$\Leftrightarrow$ $\mathcal{M}_{V,\mathbb{Q}_\ell}$ abelian and $\boxed{G_\ell(M) = G_{mot}(M)}$ $\forall M \in \mathcal{M}_V$

Rmk: if $k \subset \mathbb{C}$, $G^{(H_\ell)}_{mot}(M) = G^{(H_B)}_{mot}(M) \otimes \mathbb{Q}_\ell$

So $G^{(H_\ell)}_{mot}(M)$ "interpolates" the $G_{\ell}(M)$ for various ℓ.

Ex. • on abelian varieties over finite fields every Tate class is $\mathbb{Q}_\ell$-lin. comb. of motivated classes

c) "period realization" (or "De Rham - Betti realization")

$k \subset \mathbb{C}$

$Vec_{k,\mathbb{Q}} = \{ (V, W, \varpi),$ $V \in Vec_{\mathbb{Q}}$, $W \in Vec_k$, $\varpi : W \otimes_k \mathbb{C} \underset{iso}{\cong} V \otimes_{\mathbb{Q}} \mathbb{C} \}$

tannakian $/\mathbb{Q}$.

period realization: $\mathcal{M}_{hom}(k)_{\mathbb{Q}} \xrightarrow{\text{"}H_{B,\varpi}\text{"}} Vec_{k,\mathbb{Q}}$

$M \longmapsto (H_B(M), H_{dR}(M), \varpi)$

canonical iso $H_{dR}(M) \otimes_k \mathbb{C} \cong H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}$.

Concretely, ϖ is given by a cplx matrix whose coefficients are called periods.

<u>Period cj.</u> (weak form)

if $k \subset \bar{\mathbb{Q}}$, the period realization is *fully* faithful.

Ex: • for $\langle \mathbb{Q}(1) \rangle_{\mathbb{Q}}$, this amounts to the transcendence of π

• for elliptic curves, this follows from known results in transcendence theory.

<u>Grothendieck's period cj.</u> (strong form) (i.e. all alg. relations with coef. in k between periods are of motivic origin.)

period torsor of M : $\mathcal{P}(M) = \underline{\mathrm{Iso}}^{\otimes}(H_{DR}, H_B \otimes k)$

(torsor under $G_{mot}^{(H_B)}(M) \otimes k$).

$\varpi_M : \operatorname{Spec} \mathbb{C} \longrightarrow \mathcal{P}(M)$

Grothendieck's period cj

$\Leftrightarrow$ $\mathrm{im}(\varpi_M)$ is the generic point of $\mathcal{P}(M)$

$\Leftrightarrow$ $\mathcal{P}(M)$ is connected and

$$\mathrm{tr.deg}_{\mathbb{Q}}\, k(\Omega_M) = \dim G_{mot}(M).$$

Rmk: as before, one can get rid of the st^d cj using motivated classes instead of alg. classes.

Ex: • (strong) period cj is known for elliptic curves with cplex multiplication (Chudnovsky).

• linear alg. relations between periods of an h^1 are of motivic origin (Wüstholz).

Comment. char $k = 0$

$$\begin{tikzcd}
 & CHM(k)_{\mathbb{Q}} = M_{rat}(k)_{\mathbb{Q}} \arrow[d] & \\
 & M_{hom}(k)_{\mathbb{Q}} \arrow[dl, "H_B"'] \arrow[d, "H_\ell"] \arrow[dr, "H_{dR}"] & \\
HS_{\mathbb{Q}} & Rep_\ell \, Gal(\bar{k}/k) & Vec_{k,\mathbb{Q}}
\end{tikzcd}$$

$M_1, M_2 \in CHM(k)_{\mathbb{Q}}$

$M_1 \simeq M_2$

$\Downarrow \quad \Uparrow$ under either "Schur finiteness" or "finite dimensionality in the sense of Kimura - O'Sullivan"

"same" underlying homological motives

$\swarrow$ $\nearrow$ Hodge cj if $k = \bar{k} \subset \mathbb{C}$ — "same" Hodge structure

$\Downarrow$ $\Uparrow$ Tate cj if k f. type — "same" Galois rep.

$\searrow$ $\nwarrow$ period cj if $k \subset \bar{\mathbb{Q}}$ — "same" periods "up to $\bar{\mathbb{Q}}$"

⑥ Techniques of computation of motivic Galois groups.

M motive$/_k$
e.g $h(X)$, $X \in \mathcal{P}(k)$.

A) char $k = 0$ case

I. $k = \mathbb{C}$ First compute the Mumford-Tate gp $MT(M)$

Reason: it is connected, $MT(M) \leftrightarrow \mathrm{Lie}\, MT(M)$.

$$MT = \overset{\text{center}}{Z} . MT^{ss}$$

$\mathbb{G}_m \xrightarrow{w} Z \subset (\mathrm{End}\, V)^\times$ weight cochar.

$V = H_B(M)$ by polarizability, $Z/_{\mathrm{im}\, w}$ is a compact torus.

$$MT^{ss} \subset SL(H_B(M)) = SL_n$$

Recall: there are only finitely many conj. classes of semi-simple subgroups of SL_n (determined by tensor invariants of effectively (?) bounded degree).

More advanced techniques:

$$\mathrm{Lie}\ MT^{ss}_{\mathbb{C}} = \bigoplus \mathfrak{g}_{j\,\text{simple}}$$

$V = H_B(M)$

$V_{\mathbb{C}} = \bigoplus V_{i\ \text{irred}}$ $\qquad V_i = \bigotimes W_{ij}$ irred rep of $\mathfrak{g}_j$

Zarhin: bounds for $\overline{\text{level}}$ (= max $p-q$, $V^{pq} \neq 0$

$\Downarrow$

bounds for weights of W_{ij}.

e.g. if level = 1, all weights are miniscule

etc...

II. Using classical invariant theory, determine generators of small coh. degree for the algebra of MT-invariant tensors (i.e. Hodge classes)

. if $M \subset h(X)$ and generators $\in H^2(X^n) \subset H(X)^{\otimes n}$

Lefschetz' thm $\Rightarrow$ Hodge cj for $\langle M\rangle_\otimes$

$\Rightarrow G_{mot}(M) = MT(M)$.

. otherwise, try to deform M to a motive which satisfies Hodge cj (see below).

III. $k \subseteq \bar{k} \hookrightarrow \mathbb{C}$

$$G_{mot}(M_{\mathbb{C}}) = G_{mot}(M_{\bar{k}}) \subset G_{mot}(M)$$

finite index

"gap" determined by Galois rep. on $H_\ell(M)$.

(B) char. $k = p$

try to replace I by study of Galois rep (replacing MT by G_ℓ, Zarhin's work by Pink's work etc....

(c) if k is transcendental over its prime field, monodromy techniques are available (see below).

Ex: X : elliptic curve $/k$

$h(X) = S(h^1(X))$ in $\mathcal{M}_{hom}(k)$

(i.e. $= \Lambda(h^1(X))$ in $\tilde{\mathcal{M}}_{hom}(k)$)

so $G_{mot}(h(X)) = \underset{\text{reductive gp}}{G_{mot}(h^1(X))} \subset GL(H^1(X)) = GL_2$.

$w : \mathbb{G}_m \to G_{mot}(h(X)) \subset GL_2$
diagonal

$k = \mathbb{C}$: MT connected red. sbgp of GL_2, $\supseteq \mathbb{G}_m$

$V = H^1_B(X)$ $\Rightarrow$ determined by $\mathrm{End}_{MT} V = \mathrm{End}\, X \otimes \mathbb{Q}$

$MT = GL_2 \longleftarrow \mathbb{Q}$ — $\mathrm{End}\, X \otimes \mathbb{Q}$

or

$MT = R_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}\, \mathbb{G}_m \longleftarrow \mathbb{Q}(\sqrt{-d})$
(complex multiplication)

in both cases, invariant tensors are generated by those in $V^{\otimes 2}(1)$ i.e. $\mathrm{End}\, X \otimes \mathbb{Q}$

hence Hodge cj holds for all powers of X and

$$G_{mot}(X) = MT(X).$$

if $\mathrm{ch}\, k = 0$, $G_{mot}(X) = GL_2$ if $X_{\bar{k}}$ has no CM

$=$ non-commut. Cartan if X has no CM but $X_{\bar{k}}$ has CM

$=$ split Cartan if X has CM.

k finite. (M. Spieß)

$G_{mot} = G_\ell(M)$ and all invariant tensors are gen. in deg. 2.

Ex: Abelian var. with cplex multiplication.

X CM ab. var / k (number field $\subset \mathbb{C}$)

$\mathrm{End}\, X \otimes \mathbb{Q} = E$ CM field $[E:\mathbb{Q}] = 2 \dim X$.

$\Omega^1(X)$ $k \otimes E$-module.

$$\det_k(1 \otimes ? \mid \Omega^1(X)) : T_E \longrightarrow T_k$$

$$x \longmapsto \prod_{s : E \hookrightarrow \mathbb{C}} s(x)^{\tau(s)}$$

τ "CM type of weight 1" attached to X $(\tau(s) + \tau(\bar{s}) = 1)$.

On the other hand,

$$\det_E(? \otimes 1 \mid \Omega^1(X)) : T_k \longrightarrow T_E$$

$$y \longmapsto \prod_{\tilde{s} : \tilde{E} \hookrightarrow \mathbb{C}} \tilde{s}(N_{k/\tilde{E}}(y))^{\tilde{\tau}(\tilde{s})}$$

where $\tilde{E}$ (reflex CM field) is the smallest CM subfield of k s.t. $T_k \to T_E$ factors through $N_{k/\tilde{E}} : T_k \to T_{\tilde{E}}$.

The image of the induced homom. $T_{\tilde{E}} \to T_E$ is $MT(X) = G_{mot}(X)$.

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⑦ Parallel transport of algebraic classes

$k \subset \mathbb{C}$ for simplicity.

$f: X \to S$ proj. smooth, S smooth connected.

Parallel transport: $H(X_s)^{\pi_1(S(\mathbb{C}),s)} \underset{\pi_{t,s}}{\xrightarrow{\sim}} H(X_t)^{\pi_1(S(\mathbb{C}),t)}$.

Conj (Grothendieck): $\pi_{t,s}$ respects alg. classes.

Stronger conj: $\pi_{t,s}$ is induced by an alg. correspondence

prop (A.): Std cj $\sim_{hom} = \sim_{num} \Rightarrow$ this conjecture.
(also true in char. p).

prop (A.): $\pi_{t,s}$ is motivated
(for U big enough).

Conseq: if for one fiber of the family, all Hodge cycles are motivated, it is the same for every fiber.

⑧ Variation of motivic Galois groups in families.

Same setting. Variation of $G_{mot}(X_s)$ with s ?

Ex: non isotrivial elliptic pencil
$X \to S$ in general $G_{mot}(X_s) = GL_2$,
except for countably many pts s
(complex multiplication).

$G_{mono}(X_s) := \left[\mathrm{Im}\left(\pi_1(S,s) \to GL(H(X_s)) \right) \right]^{Zar.}$

G^0_{mono} semisimple (Deligne).

$k = \bar{k} \subseteq \mathbb{C}$

Thm (A.) (for U big enough)
There exists a local system $(\Gamma_s)_s$ of reductive subgroups of $GL(H(X_s))$ s.t.

1) $\forall s$, $G^0_{mono}(X_s) \triangleleft \Gamma_s$, $G_{mot}(X_s) \subseteq \Gamma_s$
2) $\exists$ countable union Σ of subvarieties of S $(\neq S)$ s.t. $\forall s \notin \Sigma$, $G_{mot}(X_s) = \Gamma_s$
3) $\exists$ ∞^{ly} many $s \in S(k)$ s.t. $G_{mot}(X_s) = \Gamma_s$.

In particular, if $s \in S(\mathbb{C})$ is "Weil-generic", $G^0_{mono}(X_s) \triangleleft G_{mot}(X_s)$.

Ex: for a generic hypersurface Y in $\mathbb{P}^{2n}$ $n>0$
(moving in a Lefschetz pencil),
$G^0_{mono}(Y) = Sp$ $\to G_{mot}(Y) = GSp$.
$\mathrm{Im}\ w \neq 1$