Categories of motives

First aim: understand the diagram

$$\begin{tikzcd}
P(k) \arrow[r] \arrow[d, "op"] & C(k) \arrow[r, hook] & CHM^{eff}(k) \arrow[r, hook] \arrow[d, "op"] & CHM(k) \arrow[d, "op"] \\
Sm(k) \arrow[r] & Cor(k) \arrow[r] & DM^{eff}_{gm}(k) \arrow[r, hook] \arrow[d, hook] & DM_{gm}(k) \arrow[d, hook] \\
 & & DM^{-}_{eff}(k) \arrow[r, hook] & DM^{-}(k)
\end{tikzcd}$$

$\hookrightarrow$ full embedding

$\xrightarrow{op}$ contravariant

k ground field

§1 Pure motives

Fix an <u>adequate equivalence relation</u> $\sim$ on the category

$$\mathcal{P}(k) = \text{smooth, proj. varieties}/k,$$

i.e., an additive equ. rel. $\sim$ on all groups $Z^j(X) = \{\text{alg. cycles of codim } j \text{ on } X\}$ (all X in $\mathcal{P}(k)$, all $j \geq 0$) such that push-forward f_*, pull-back f^* and intersection of cycles are well-defined modulo $\sim$.

E.g. $\sim_{rat}$ rational equivalence

$\sim_{alg}$ algebraic "

$\sim_{num}$ numeric "

$$A^j_\sim(X) = Z^j(X)/\sim \; = Z^j(X)/Z^j(X)_\sim$$

The category of <u>pure motives</u> modulo $\sim$ over k is defined in 3 steps.

1) $C_\sim(k)$ cat. of correspondences

<u>objects</u>: $h(X)$, for every X in $P(k)$

<u>morphisms</u>:

$$\mathrm{Hom}(h(X), h(Y)) = \mathrm{Corr}^0_\sim(X,Y) = \bigoplus_\alpha A_\sim^{\dim X_\alpha}(X_\alpha \times Y)$$

(X_α = irred. components of X)

<u>composition</u>

$$\mathrm{Corr}^0(X,Y) \times \mathrm{Corr}^0(Y,Z) \to \mathrm{Corr}^0(X,Z)$$

$$(f, g) \mapsto g \circ f = (p_{XZ})_*(p_{XY}^* f \cdot p_{YZ}^* g)$$

(associative: <u>base change</u> formula)

$C_\sim(k)$ is additive $\otimes$-category: $h(X) \otimes h(Y) = h(X \times Y)$

2) effective motives $\mathcal{M}_\sim^{eff}(k) = C_\sim(k)^\natural$

$=$ pseudo-abelian envelope of $C_\sim(k)$

objects: $(h(X), p)$, $p^2 = p \in End(h(X))$

morph.: $Hom((h(X),p),(h(Y),q)) = q\, Hom(h(X), h(Y))\, p$

<u>Example:</u> $h(\mathbb{P}^1_k) = 1 \oplus L$

$$1: \quad p = 0 \times \mathbb{P}^1 \qquad\qquad L: \quad p = \mathbb{P}^1 \times 0$$

$1 \cong h(pt)$

L Lefschetz motive

$\mathcal{M}_\sim^{eff}(k)$ is a pseudo-abelian $\otimes$-category

$(h(X), p) \otimes (h(Y), q) = (h(X \times Y), p \times q)$

3) $\mathcal{M}_\sim(k)$ (cat. of pure motives) by inverting L (i.e., $- \otimes L$).

objects: $M(i)$ $\left(M \text{ in } \mathcal{M}_\sim^{\mathrm{eff}}(k),\ i \in \mathbb{Z}\right)$

morph.:

$$\mathrm{Hom}(M(i), N(j)) = \varinjlim_{\substack{n \\ n-i,\, n-j \geq 0}} \mathrm{Hom}_{\mathcal{M}_\sim^{\mathrm{eff}}}(M \otimes L^{n-i}, N \otimes L^{n-j})$$

where $L^m := L^{\otimes m}$.

$\mathcal{M}_\sim(k)$ is a rigid pseudo-abelian $\otimes$-category in which L is invertible.

$[\ M(i) = M \otimes L^{-i}\ ]$

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objects of $\mathcal{M}_\sim(k)$ are $(X, p, n) = (h(X), p)(n)$,
where X smooth, proj. variety
$p^2 = p \in \mathrm{Corr}^0(X, X)$
$n \in \mathbb{Z}$

§2 Tensor categories

A $\otimes$-category is a cat. $\mathcal{C}$ with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$
$$(A, B) \longmapsto A \otimes B$$

and

(a) an associativity constraint: functorial isos

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z \, ,$$

(b) a commutativity constraint: funct. isos

$$\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X \, ,$$

(c) a unit object: an object 1 together with an isom. $1 \xrightarrow{u} 1 \otimes 1$ s.t.

$$1 \longmapsto 1 \otimes X$$

is an equivalence of categories,

satisfying 3 (obvious) compatibilities.

<u>Examples</u> (i) Mod_R = R-modules over a commut. ring R with unit. Usual $\otimes$ + (a) - (c)

(ii) Gr_R = $\mathbb{Z}$-graded R-modules, $(M \otimes N)^n = \bigoplus_{i+j=n} M^i \otimes N^j$, $\psi(m_p \otimes n_q) = (-1)^{p \cdot q} n_q \otimes m_p$.

(iii) $sMod_R$ = "super R-modules" (as in (ii), but $\mathbb{Z}/2$-graded)

<u>Def.</u> L in $(\mathcal{C}, \otimes)$ is called <u>invertible</u>, if $X \mapsto X \otimes L$ is an equivalence of categories.

<u>Def.</u> If the functor $T \mapsto Hom(T \otimes X, Y)$ is representable, we call the representing object the <u>internal Hom</u> $\underline{Hom}(X,Y)$.

So $\quad Hom(T \otimes X, Y) = Hom(T, \underline{Hom}(X,Y))$

Yoneda lemma $\Rightarrow$

- $ev_{X,Y} : \underline{Hom}(X,Y) \otimes X \to Y$ 'evaluation' morphism

- $Hom(X,Y) \cong Hom(1, \underline{Hom}(X,Y))$

<u>Def.</u> A tensor category $\mathcal{C}$ is <u>rigid</u>, if all internal Hom's exist, and if for the dual $X^\vee := \underline{Hom}(X,1)$ the canonical morphisms

$$X \to X^{\vee\vee}$$

$$X^\vee \otimes Y^\vee \to (X \otimes Y)^\vee$$

$$X^\vee \otimes Y \to \underline{Hom}(X,Y)$$

are isomorphisms.

In a rigid tensor category $\mathcal{C}$ one has, for each object X,

(a) a trace map $tr_X : End(X) \to End(1)$

(by applying $Hom(1,-)$ to

$$\underline{Hom}(X,X) \xleftarrow{\sim} X^\vee \otimes X \xrightarrow{ev_{X,1}} 1 \quad)$$

(b) a rank $rk(X) = tr_X(id) \in End(1)$

§3 Geometric triangulated motives

Work with category $Sm(k)$ of smooth* varieties over k.

Def. For X, Y in $Sm(k)$ define

$c(X, Y)$ = free abelian group on closed integral subschemes $Z \subseteq X \times Y$ for which $Z \to X$ is finite and dominates an irreducible component of X. ("finite correspondence")

1) Define the category $Cor(k)$ of finite correspondences (of smooth varieties) over k as follows:

objects: $[X]$ for every X in $Sm(k)$

morphisms: $Hom([X], [Y]) = c(X, Y)$

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* quasi-projective

<u>composition</u>

$$c(X,Y) \times c(Y,Z) \longrightarrow c(X,Z)$$

$$(\alpha,\beta) \mapsto \beta\circ\alpha = (pr_{XZ})_*(\underbrace{pr_{XY}^*\alpha \cdot pr_{YZ}^*\beta})$$

finite

always defined! intersections are finite over component of X

$Cor(k)$ is an additive category

There is a <u>covariant</u> functor

$$Sm(k) \to Cor(k)$$
$$X \mapsto [X]$$
$$f \mapsto \Gamma_f$$

2) Let $H^b(Cor(k))$ be the category of bounded homological complexes in $Cor(k)$ modulo homotopy.

homol. notation : $\ldots \to C_n \to C_{n-1} \to \ldots$

For general reasons, this is a triangulated category.

3) Let $\mathcal{T}$ be the smallest thick triangulated subcategory of $H^b(\mathrm{Cor}(k))$ containing the complexes

(a) $[X \times \mathbb{A}^1_k] \xrightarrow{[pr_X]} [X]$ for X in $Sm(k)$,

(b) $[U \cap V] \xrightarrow{[j_U] \oplus [j_V]} [U] \oplus [V] \xrightarrow{[k_U] + [-k_V]} [U \cup V]$

where $U \cup V = X$ is an open covering of X in $Sm(k)$, and $j_U : U \cap V \to U$, $\ldots$, $k_V : V \hookrightarrow U \cup V$ are the obvious open immersions.

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thick: • $0 \in \mathcal{T}$

• $A \to B \to C \to A[1]$ exact triangel, A and B in $\mathcal{T}$ $\Rightarrow$ C in $\mathcal{T}$

• $\mathcal{T}$ closed under direct factors

Then define the triangulated category of effective geometric motives over k as

pseudo-abelian envelope

$$DM^{eff}_{gm}(k) = \left(H^b(Cor(k))/T\right)^{\natural} \quad *$$

This is a pseudo-abelian triangulated tensor category.

The idea of T is that one forces the complexes (a), (b) (extended by 0) to be <u>exact</u> in DM^{eff}_{gm}. In fact, one has an

- isomorphism $M(X \times \mathbb{A}^1) \xrightarrow{\sim} M(X)$

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* $\mathcal{D}$ triangulated, $\mathcal{C} \subseteq \mathcal{D}$ thick $\leadsto$ define $\mathcal{D}/\mathcal{C} = \mathcal{D}_S$ where S is the (localizing) set of morphisms s whose 'cone' is in $\mathcal{C}$. This is a triangulated category with the universal prop. mapping $\mathcal{C}$ to 0.

(homotopy invariance), and

(b) exact triangles

$$M(U \cap V) \to M(U) \oplus M(V) \to M(U \cup V) \to$$

in the situation of (b) above

(Mayer-Vietoris property).

Here $M(X)$ is the image of $X \in Sm(k)$ under

$$\begin{array}{ccccccc} Sm(k) & \to & Cor(k) & \to & H^b(Cor(k)) & \to & DM_{gm}^{eff}(k) \\ X & \mapsto & [X] & \longmapsto & & & M(X). \end{array}$$

4) Every rational point p of $X \in Sm(k)$ defines an idempotent

$$X \to \operatorname{Spec} k \xrightarrow{p} X$$

and hence a splitting

$$M(X) = 1 \oplus \tilde{M}(X)$$

with $1 = M(\operatorname{Spec} k)$ and $\tilde{M}(X)$

the reduced motive of (X, p) (depends on p).

Define the homological Tate motive

$$\mathbb{1}(1) = \tilde{M}(\mathbb{G}_m)[-1]$$

where the rational point is $1 \in \mathbb{G}_m(k) = k^\times$.

Then define $DM_{gm}(k)$, the triang. category of geometric motives over k, by inverting the Tate motive in $DM_{gm}^{eff}(k)$.