If we take $\mathcal{M}$ to be one of the following groupoids (over $C =$ (Schemes)):

$U \longmapsto$ (all $U$-schemes)

$U \longmapsto$ (quasicoherent $\mathcal{O}_U$-Modules)

then $\Delta_{\mathcal{M}}$ is not representable but has the weaker sheaf property:

for each $U$ and $X, Y \in \mathrm{ob}\, \mathcal{M}(U)$, the presheaf $\underline{\mathrm{Isom}}_{\mathcal{M}}(X, Y)$ is a sheaf for the étale (even for the fpqc) topology:

If $(U_i \longrightarrow U)$ is an étale covering, and $(\varphi_i : X_{U_i} \rightarrow Y_{U_i})$ is a family of isomorphisms which agree on $U_{ij} := U_i \times_U U_j$ (in the obvious sense), then they come from a unique isomorphism $\varphi : X \rightarrow Y$.

# SHEAF-THEORETIC PROPERTIES:

# PRESTACKS AND STACKS

We now assume given a Grothendieck topology on our category $C$ (e.g. $C$ = (Schemes) + étale topology).

**Definition.** A $C$-groupoid $\mathcal{M}$ is a prestack if for all $\begin{cases} U \in \mathrm{ob}\, C \\ X, Y \in \mathrm{ob}\, \mathcal{M}(U) \end{cases}$ the functor $\underline{\mathrm{Isom}}_{\mathcal{M}}(X, Y)$ is a sheaf on $C$.

Examples:

- $C$ = (Schemes), 
  - $\mathcal{M}(U)$ = (all $U$-schemes)
  - $\mathcal{M} = \mathcal{M}_{g,n}$
  - (etc.)

- $C$ arbitrary, $\mathcal{M}$ = a presheaf $F$ on $C$:
  Then: $C$ is a prestack
  $$\Updownarrow$$
  $F$ is a separated presheaf
  (first axiom of sheaves: two locally equal sections are equal).

## Stacks (gluing objects)

Let $\mathcal{M}$ be a $C$-groupoid and

$$\left(U_i \xrightarrow{\varphi_i} U\right)_{i \in I}$$

a covering family. Any $X \in \mathrm{ob}\,\mathcal{M}(U)$ determines the following data:

- $X_i := \varphi_i^* X = X|_{U_i}$ in $\mathcal{M}(U_i)$

- $\theta_{ij}:$
$$\begin{tikzcd}
X_i|_{U_{ij}} \arrow[rr, "\sim"] & & X_j|_{U_{ij}} \\
& X|_{U_{ij}} \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$
in $\mathcal{M}(U_{ij})$

satisfying the usual cocycle condition (compatibility of $\theta_{ij}$'s on $U_{ijk}$).

Similarly, any morphism $f : X \to Y$ in $\mathcal{M}(U)$ determines $f_i : X_i \to Y_i$ in $\mathcal{M}(U_i)$, compatible with the $\theta_{ij}$'s.

To say that $\mathcal{M}$ is a prestack just means that

$$\mathrm{Hom}_{\mathcal{M}(U)}(X,Y) \xrightarrow{\sim} \{\text{families } (\varphi_i) \text{ as above}\}$$

If this is the case, we can reconstruct the functor (represented by) $X$ in $\mathcal{M}(U)$ from the "descent datum" $(X_i, \theta_{ij})$.

<u>Definition</u>. A $C$-groupoid $\mathcal{M}$ is a stack if

(i) $\mathcal{M}$ is a prestack, and

(ii) for each covering $(U_i \xrightarrow{\varphi_i} U)$ in $C$, every descent datum $(X_i, \theta_{ij})$ as above is effective, i.e. associated to an object of $\mathcal{M}(U)$ (unique up to unique isomorphism, by (i)).

<u>Remarks</u>. One may think of a stack as a "sheaf of groupoids".

- One can (formally) construct the stack associated to a $C$-groupoid.

Examples:

- If $F$ is a presheaf on $C$, then:

  $F$ is a stack $\Leftrightarrow$ $F$ is a sheaf

- $C = (\text{Schemes})$ + étale topology:

  - QCOH and $\text{BUN}_n$ are stacks

    (étale descent theorem for quasicoherent sheaves)
  - $U \mapsto (\text{Schemes}/U)$ is a prestack but not a stack (étale descent is not always effective)
  - $\mathcal{M}_{g,n}$ is a stack $\Leftrightarrow$ $(g,n) \neq (1,0)$.

    Indeed:

    - if $(g,n) \neq (1,0)$ then for every $(X \to U, x_1, \dots, x_n)$ in $\mathcal{M}_{g,n}(U)$ there is a canonical relatively ample sheaf on $X$:

      - $T_{X/U}$ if $g = 0$
      - $\Omega_{X/U}$ if $g \geq 2$
      - $\mathcal{O}_X(-x_1)$ if $n \geq 1$.

For $\mathcal{M}_{1,0}$ there are [should be ?] examples of noneffective descent data.

If we "want $\mathcal{M}_{1,0}$ to be a stack" we have to relax some conditions in the definition:

$\boxed{\mathcal{M}_{g,n}(U) \text{ (NEW DEFINITION):}}$

objects = $(X \xrightarrow{f} U, x_1, \ldots, x_n)$

where $f: X \longrightarrow U$ is a smooth proper morphism of algebraic spaces, with geometric fibres connected, 1-dimensional, of genus $g$ and $x_1, \ldots, x_n : U \longrightarrow X$ are disjoint sections.

# ALGEBRAIC SPACES

**Definition.** A (quasiseparated) algebraic space is a functor

$$F : (\text{Schemes})^\circ \longrightarrow (\text{Sets})$$

with the following properties: - étale sheaf

(i) The diagonal morphism

$$\Delta_F : F \hookrightarrow F \times F$$

is representable and quasicompact.

(ii) There is a scheme $X$ and a morphism

$$p : X \longrightarrow F$$

(automatically representable) which is étale and surjective.

If $p : X \to F$ is as above, then consider the Cartesian diagram of sheaves

$$\begin{tikzcd} R := X \times_F X \arrow[r, hook] \arrow[d] & X \times X \arrow[d, "p \times p"] \\ F \arrow[r, hook, "\Delta_F"] & F \times F \end{tikzcd}$$

- By (i), $R$ is a scheme and the morphism $R \to X \times X$ is quasicompact (in fact, quasiaffine).
- By (ii), both projections $R \rightrightarrows X$ are étale.
- By construction, $R$ is an equivalence relation on $X$ (i.e. $R(U) \subset X(U) \times X(U)$ is an equivalence relation, for each scheme $U$).
- By (ii), $p$ is an epimorphism of sheaves, hence one can reconstruct $F$ from $X$ and $R$ by

$$\underbrace{X/R}_{} \xrightarrow{\sim} F.$$

(quotient in the category of étale sheaves)

CONVERSELY, any diagram of schemes

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

with the above properties $\begin{cases} \text{equiv. relation} \\ p_1, p_2 \text{ étale} \\ R \xrightarrow{(p_1,p_2)} X \times X \text{ quasicompact} \end{cases}$

defines an algebraic space $F := X/R$.

For instance, if a finite group $G$ acts freely on a scheme $X$, then $X/G$ is an algebraic space (not a scheme in general).

Other examples:

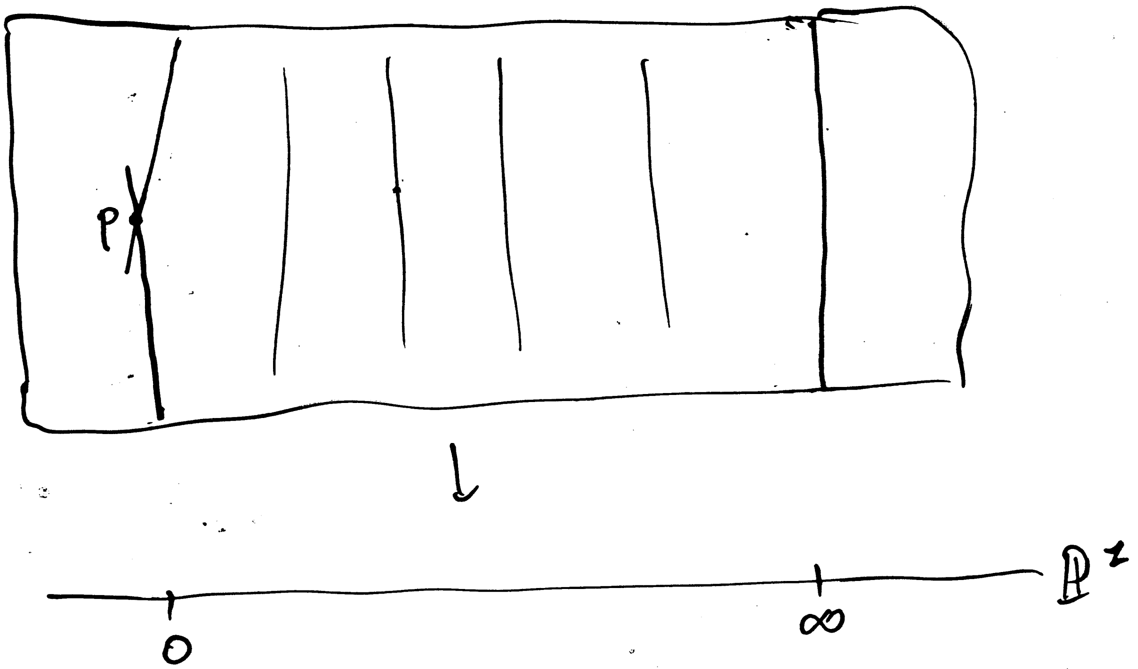
- (M. Artin) If $f: X \to U$ is a morphism of finite presentation, then the Hilbert functor on $U$-schemes

$$(T \to U) \longmapsto \left\{ \text{closed subschemes of } X_T, \text{ proper and flat over } T \right\}$$

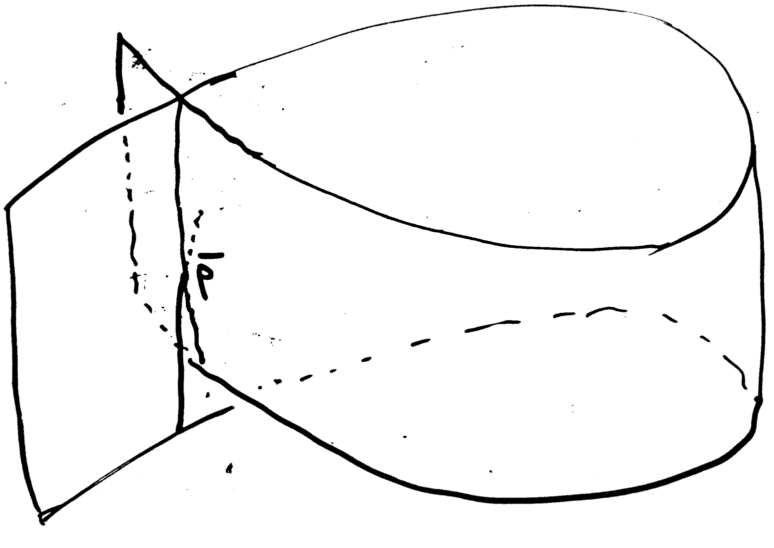
is an algebraic space (but not a scheme in general, unless $X \to U$ is quasiprojective).

Same for symmetric products $(X/U)^{(n)}$.

. Start with $\mathbb{P}^1 \times \mathbb{P}^1$, with one point blown up:




and identify the red lines. We get something like




which (exercise) . is an algebraic space

. is not a scheme ($\bar{p}$ has no affine neighbourhood).

Given a functor $F : (\text{Schemes})^\circ \to \text{Sets}$, it is in general much easier to prove that $F$ is an algebraic space than a scheme.

It is also almost as useful (except if $F$ is a quasiprojective scheme).

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Back to our new $\mathcal{M}_{g,n}$:

- if $(g,n) \neq (1,0)$ it's the old one (quasiprojective algebraic spaces are schemes)
- if $X \to U$ is a $U$-curve of genus 1 (in the new sense) then its Jacobian $E \to U$ is an elliptic curve, and $\underline{\mathrm{Isom}}_U(E,X)$ contains $X$ as a connected component, hence is not necessarily a scheme (but still an algebraic space).

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RE-DEFINITION !

If $\Phi : \mathcal{M} \to \mathcal{N}$ is a 1-morphism of stacks (over the cat. of schemes + étale topology), we shall say that $\Phi$ is representable [in the sense of algebraic spaces] if:

For every algebraic space $U$ and 1-morphism $X : U \to \mathcal{N}$, the fibre product

$$U \times_{X, \mathcal{N}, \Phi} \mathcal{M}$$

is an algebraic space.

Remarks:

- it suffices to test on affine schemes $U$;
- one could also extend $\mathcal{M}$ and $\mathcal{N}$ to stacks $\underbrace{\overline{\mathcal{M}}, \overline{\mathcal{N}}}$ over the category of algebraic spaces: the above definition is then equivalent to the representability of $\overline{\Phi} : \overline{\mathcal{M}} \to \overline{\mathcal{N}}$.

# ALGEBRAIC STACKS

[Definition. A stack $\mathcal{M}$ (over the cat. of schemes, with the étale topology) is algebraic (in Artin's sense) if:

(i) The diagonal morphism

$$\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$$

is representable, quasicompact and separated.

(ii) There exists a scheme $Y$ and a 1-morphism

$$P : Y \longrightarrow \mathcal{M}$$

which is [representable,] smooth and surjective.

A Deligne-Mumford stack is an algebraic stack $\mathcal{M}$ for which there exists $P : Y \to \mathcal{M}$ as in (ii) which is étale.]

Example: $BUN_r$ is algebraic:

(i) $E, F \in BUN_r(U)$: $\underline{\mathrm{Isom}}_U(E, F)$ is a $U$-scheme locally isomorphic to $GL_r$;

(ii) Take $Y = \mathrm{Spec}\,\mathbb{Z}$ and $P : Y \longrightarrow BUN_r$ given by the trivial bundle: then for any $U \xrightarrow{E} BUN_r$, the $U$-scheme $U \times_{E, BUN_r, P} Y = \underline{\mathrm{Isom}}(\mathcal{O}_U^r, E)$ is smooth and surjective over $U$. Hence $P : Y \longrightarrow BUN_r$ is smooth and surjective (and affine).

Example: $\mathcal{M}_{g,n}$ is algebraic:

. the representability of the diagonal is known.

If $(g,n) \neq (1,0)$, every $(X \xrightarrow{p} U, x_1, \dots, x_n)$ in $\mathcal{M}_{g,n}(U)$ carries a canonical invertible sheaf $\mathcal{L}(X)$, very ample relative to $p : X \to U$, such that $p_*\mathcal{L}(X)$ is locally free (of rank $r$, say) and commutes with base change

(e.g. $\omega_{X/U}^{\otimes 3}$ if $g \geq 2$).

Define a stack $Y$ as follows:

$$Y(U) = \text{cat. of } (\underbrace{X \to U, x_1, \dots, x_n}_{\text{in } \mathcal{M}_{g,n}(U)}, \underbrace{\beta}_{\text{basis of } p_*\mathcal{L}(X)})$$

+ isomorphisms respecting the bases.

For such an object of $Y(U)$, we get an embedding:

$$X \hookrightarrow \mathbb{P}(p_*\mathcal{L}(X)) \xrightarrow[\sim]{\beta} \mathbb{P}^{r-1}_U$$

and we can identify $Y(U)$ with the set of all subschemes of $\mathbb{P}^{r-1}_U$ satisfying certain conditions (+ additional data, such as the marked points).

Then we can use the Hilbert scheme theory to show that $Y$ is an algebraic space (in fact, a quasi-projective scheme).

. The natural 1-morphism

$$Y \longrightarrow \mathcal{M}_{g,n} \quad (\text{"forget } \beta\text{"})$$

is obviously smooth and surjective: for a 1-morphism $U \to \mathcal{M}_{g,n}$, corresponding to $(X \to U, \dots) \in \mathcal{M}_{g,n}(U)$, the fibre product

$$Y \times_{\mathcal{M}_{g,n}} U \longrightarrow U$$

is the sheaf of bases of $p_* \mathcal{L}(X)$, which is a $GL_r$-torsor on $U$, hence a smooth, affine, surjective $U$-scheme.

<u>Remark</u>: the above arguments can be put differently: we have a 1-morphism

$$\Phi : \mathcal{M}_{g,n} \longrightarrow BUN_r$$
$$(X \to U, \dots) \longmapsto p_* \mathcal{L}(X)$$

and we check (as in the case of "3K") that $\Phi$ is representable. Now use:

<u>Proposition</u> $\Phi : \mathcal{M} \to \mathcal{N}$ representable, $\mathcal{N}$ algebraic (resp. D-M) $\Rightarrow \mathcal{M}$ algebraic (resp. D-M).

If $(g,n)=(1,0)$, consider the 1-morphism

$$\Phi : \mathcal{M}_{1,1} \longrightarrow \mathcal{M}_{1,0} \quad \text{(forget base point)}$$

Since $\mathcal{M}_{1,1}$ is algebraic, there is a $P : Y \longrightarrow \mathcal{M}_{1,1}$, smooth and surjective. But $\Phi$ is also smooth and surjective, hence so is $\Phi \circ P : Y \longrightarrow \mathcal{M}_{1,0}$.

Characterising Deligne-Mumford stacks:

**Theorem.** Let $\mathcal{M}$ be an algebraic stack.
Equivalent conditions:
(i) $\mathcal{M}$ is a Deligne-Mumford stack.
(ii) The diagonal $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is unramified.
(iii) For every field $k$ and $X \in \mathcal{M}(k)$, the $k$-group scheme $\underline{\mathrm{Aut}}(X)$ is finite étale ("no infinitesimal automorphisms").

(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) easy

(ii) $\Rightarrow$ (i) a bit harder.

Corollary (characterisation of algebraic spaces)

Let $F : (\text{Schemes})^{o} \to (\text{Sets})$ be a functor.

Then:

$F$ is an algebraic stack $\Leftrightarrow$ $F$ is an algebraic space

Proof: $\Leftarrow$ trivial

$\Rightarrow$ trivial if $F$ is a Deligne-Mumford stack.

In general, if $F$ is an algebraic stack and a presheaf, the diagonal $\Delta_F$ is a monomorphism, hence unramified. By the preceding theorem, $F$ is a D-M stack, so we are done ■

Corollary.

- $\mathcal{M}_{g,n}$ is a D-M. stack $\Leftrightarrow$ $2g-2+n>0$
  $\Leftrightarrow (g,n) \notin \{(0,0),(0,1),(0,2),(1,0)\}$
- $\mathcal{M}_{g,n}$ is an algebraic space $\Leftrightarrow n > 2g+2$. ■