W. III

Recall

Canonical Map $K_n^M(F)/\ell \longrightarrow H^n_{et}(F, \mu_\ell^{\otimes n})$

$\{a_1,\dots,a_n\} \mapsto [a_1]\cup\dots\cup[a_n]$

$\frac{1}{\ell}\in F$

$$\begin{array}{ccc} K_n^M(F)/\ell & \longrightarrow & H^n_{et}(F, \mu_\ell^{\otimes n}) \\ \| & & \wr\| \\ H^n_{nis}(F, \mathbb{Z}/\ell(n)) & \longrightarrow & H^n_{et}(F, \mathbb{Z}/\ell(n)) \end{array}$$

Thm / Conjecture If $\frac{1}{\ell}\in F$, this is isomorphism

$\ell=2$ Milnor 1970 Voevodsky preprint 1996, published 2004

$\ell\neq 2$ Kato, Bloch-Kato Voevodsky-Rost preprint (V) 2003 Rost's part ??

Strategy ① Reduce to $\boxed{\text{Theorem } H^{n+1}_{et}(k, \mathbb{Z}(n))_{(\ell)} = 0 \text{ all } n}$

② Invoke construction of Rost to change k to assume $K_n^M(k)/\ell = 0$

③ Standard argument in this case.

Theorem
Derived Reformulation

$\boxed{Hom_{DM}(\mathbb{Z}_{tr}(X), B[n]) \cong H^n_{nis}(X, C_*B)}$ note $B\cong C_*B$ in DM

Proof (a) If B injective sheaf with transfers, $H^*(X,B) = \begin{cases} B(X) & *=0 \\ 0 & \text{else} \end{cases}$

$B \hookrightarrow E^\circ$ canonical flasque sheaf (a map in ST)

True for $E^\circ \Rightarrow$ true for B

(b) Both sides computed using the same injective resolution in ST_{nis}.

Passage from DM^-_{eff} to DM

"big DM"

Objects: sequences of complexes in $\underline{\underline{Ch}}^-(ST_{nis})$ $E_0, E_1, E_2, \dots$ with $E_n(1)[1] \to E_{n+1}$

$X(1)[1] = \dfrac{X \times (A^1 - \{0\})}{X \times pt}$

$\Sigma^\infty X : \; X, X(1)[1], \dots, X(n)[n], \dots$

bonding: $X(n)(1) \cong X(n+1)$

$0 \to X \to X \times (A^1 - \{0\})$ - as complex

defn DM = localization with respect to A^1-w.e.

get $DM^-_{eff} \hookrightarrow DM$

Note $X(-1)$ represented by $0, X[+1], \dots, X(n-1)[n], \dots$

Toy Model: passage from $D^-(A)$ to $D(A)$

Chain complex C_* has good truncations $\tau_{\le n} C \subseteq \tau_{\le n+1} C \subseteq C$

sequence $(\tau_{\le n} C)[-n]$ yields $[C_*]$

$$\underbrace{\to \dots \to C^{n-1} \to C^n}_{H_* \text{ same as } C} \to \underbrace{dC^n \to 0 \to 0 \dots}_{H_* = 0}$$

Can talk about motivic cohomology of objects of DM

Cancellation Theorem $H^p(X, \mathbb{Z}(q)) \xrightarrow{\cong}$ ~~$H^{p+1}(X, \mathbb{Z}(q))$~~

$H^{p+1}(X(1)[1], \mathbb{Z}(q+1))$

or $\mathrm{Hom}_{DM}(X, \mathbb{Z}(q)[p])$

$\cong \mathrm{Hom}_{DM}(X(1)[1], \mathbb{Z}(q+1)[p+1])$

$\otimes \mathbb{Z}(1)[1]$ is invertible so this big DM contains "localization" of DM^-_{eff} "little DM"

$\Rightarrow$ contains DM_{gm}

Deglise: the heart of DM is the category of "homotopy modules"

Each E_n is in $\mathcal{H}$ (h.i. S.T.'s)

Coefficients $0 \to \mathbb{Z}(i)_{(\ell)} \xrightarrow{\ell} \mathbb{Z}(i)_{(\ell)} \to \mathbb{Z}/\ell(i) \to 0$

localize at ℓ

$$\begin{tikzcd}
 & & & & 0 \\
H^n(F,\mathbb{Z}(n))_{(\ell)} \arrow[r, "\ell"] \arrow[d] & H^n(F,\mathbb{Z}(n))_{(\ell)} \arrow[r] \arrow[d] & H^n(F,\mathbb{Z}/\ell(n)) \arrow[r] \arrow[d, "\text{iso}\,?"'] & H^{n+1}(F,\mathbb{Z}(n))_{(\ell)} \arrow[ur] & \\
H^n_{et}(F,\mathbb{Z}(n))_{(\ell)} \arrow[r, "\ell"] & H^n_{et}(F,\mathbb{Z}(n))_{(\ell)} \arrow[r] & H^n_{et}(F,\mu_\ell^{\otimes n}) \arrow[r] & \boxed{H^{n+1}_{et}(F,\mathbb{Z}(n))_{(\ell)}} \arrow[r, "\ell"] & {}
\end{tikzcd}$$

ℓ-torsion group

Note: $H^n(F,\mathbb{Z}(n)) \cong K^M_n(F)$ so $H^n(F,\mathbb{Z}/\ell(n)) = K^M_n(F)/\ell$

Thm $\forall n$ $\boxed{K^M_n(k)/\ell \cong H^n_{et}(F,\mu_\ell^{\otimes n})} \Longleftrightarrow \boxed{H^{n+1}_{et}(F,\mathbb{Z}(n))_\ell = 0}$ $\forall F/k$

Suslin-Voevodsky 1995 $\qquad$ $\underbrace{\text{(Banff Proc., 2000)}}_{\text{This is what is proven!}}$

Corollary ① $H^p(X,\mathbb{Z}(q)) \cong H^p_{et}(X,\mathbb{Z}(q))$ $\forall p \le q+1$

② $H^p(X,\mathbb{Z}/\ell^\nu(q)) \cong \begin{cases} H^p_{et}(X,\mu_{\ell^\nu}^{\otimes q}) & p \le q \\ 0 & p > q+d \end{cases}$ $\forall p \le q$

Useful for calculating K-theory of

$\mathbb{C}$ varieties (Pedrini-Weibel)

$\mathbb{R}$ varieties (Karoubi-Weibel)

Best Possible $H^2(X,\mathbb{Z}/\ell(1)) = \mathrm{Pic}(X)/\ell$ $\qquad$ Kummer seq. calculated $H^*(X,\mathbb{Z}(1))$

$$0 \to \mathrm{Pic}(X)/\ell \to H^2_{et}(X,\mu_\ell^{\otimes}) \to {}_\ell\mathrm{Br}(X) \to 0$$

Passage from DM^-_{eff} to DM

"big DM"

<u>Objects</u>: sequences of complexes in $\underline{Ch}^-(ST_{nis})$ $E_0, E_1, E_2, \dots$
with $E_n(1)[1] \to E_{n+1}$

$X(1)[1] = \dfrac{X \times (\mathbb{A}^1 - \{0\})}{X \times pt}$

$\Sigma^\infty X$: $X, X(1)[1], \dots, X(n)[n], \dots$

bonding: $X(n)(1) \cong X(n+1)$

$0 \to X \to X \times (\mathbb{A}^1 - \{0\}) \to$
as complex

<u>defn</u> DM = localization with respect to $\mathbb{A}^1$-w.e.

get $DM^-_{eff} \hookrightarrow DM$

<u>Note</u> $X(-1)$ represented by $0, X[+1], \dots, X(n-1)[n], \dots$

<u>Toy Model</u>: passage from $D^-(A)$ to $D(A)$

Chain complex C_* has good truncations $\tau_{\le n} C \subseteq \tau_{\le n+1} C \subseteq C$

sequence $(\tau_{\le n} C)[-n]$ yields $[C_*]$

$$\underbrace{\to \cdots \to C^{n-1} \to C^n}_{H_* \text{ same as } C} \to \underbrace{dC^n \to 0 \to 0 \cdots}_{H_* = 0}$$

Can talk about motivic cohomology of objects of DM

<u>Cancellation Theorem</u> $H^p(X, \mathbb{Z}(q)) \xrightarrow{\cong} H^{p+1}(X(1)[1], \mathbb{Z}(q+1))$

or $\mathrm{Hom}_{DM}(X, \mathbb{Z}(q)[p])$

$\cong \mathrm{Hom}_{DM}(X(1)[1], \mathbb{Z}(q+1)[p+1])$

$\otimes \mathbb{Z}(1)[1]$ is invertible
so this big DM contains
"localization" of DM^-_{eff} "little DM"
$\Rightarrow$ contains DM_{gm}

<u>Deglise</u>: the heart of DM
is the category of
"homotopy modules"
Each E_n is in $\mathcal{H}$ (h.i. S.T.'s)

Construction for $l=2$

defn: variety $X = X_a$ generic splitting variety for a

$a = a_1, \dots, a_n$ units of k — $K_a = k(X)$ generic splitting field for a

$$\boxed{\{a_1, \dots, a_n\} = 0 \text{ in } K^M(K_a) \Leftrightarrow X(K) \neq \emptyset}$$

$n=1$: $K = k(\sqrt{a_1})$

merkurjev-Suslin

$n=2$: $\boxed{K = k(t,x,y),\ t^2 = a_1 x^2 + a_2 y^2}$ plane curve (Brauer-Severi)

$X =$ variety of right ideals in $A = k\{x,y\}$, $x^2 = a_1$, $y^2 = a_2$, $yx + xy = 0$

Quaternion Division Alg

$n>1$ Pfister quadric $\langle\langle a_1 \dots a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$

quadratic form in 2^n variables — $x^2 - a_n y^2$

X_a hypersurface in $\mathbb{P}^{2^{n-1}}$ $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle = a_n t^2$

($l=2$) **Theorem** This is generic splitting variety for $a = (a_1, \dots, a_n)$

Prove that $\boxed{H^{n+1}_{et}(k, \mathbb{Z}(n))_{(2)} \hookrightarrow H^{n+1}_{et}(K_a, \mathbb{Z}(n))_{(2)}}$ (localized at $l=2$)

Since $\{a_1, \dots, a_n\}$ dies in K_a, embed $k \subset F$ so that

- $K^M_n(F)$ is l-divisible
- $H^{n+1}_{et}(k, \mathbb{Z}(n)) \hookrightarrow H^{n+1}_{et}(F, \mathbb{Z}(n))$ localized at $l=2$

Lemma $\boxed{\text{If } K^M_n(F) \text{ is } l\text{-divisible, } H^{n+1}_{et}(F, \mathbb{Z}(n))_{(l)} = 0}$

idea $\beta \in H^{n+1}(F, \mathbb{Z}/l(n))$ factors as $b \cup \alpha$, $b \in F^*$ (RHS = torsion)

But $\alpha \in H^n(F, \mathbb{Z}/l(n-1))$, which $= 0$ by induction on n.

This proves V's Theorem that $H^{n+1}_{et}(k, \mathbb{Z}(n))_{(2)} = 0$. //

Modifications for $\ell \neq 2$

- At present, we don't have nice geometric models for splitting varieties
- Need $DM(\mathcal{X})$ for $\mathcal{X}$ a simplicial scheme; $\mathcal{X} \xrightarrow{\pi} pt$ because previous reductions don't simplify $DM(\mathcal{X}) \xrightarrow[\pi_*]{} DM(k)$

  Interpret degree formula in $DM(\mathcal{X})$ and apply π_*
- Rost's result not yet written up. (☺)

<u>definition</u> $H_{-1,-1}(X) = \mathrm{Hom}_{DM}(\mathbb{Z}, X \otimes \mathbb{G}_m) = \mathrm{Hom}_{DM}(\mathbb{Z}(-1)[-1], X)$

$\mathbb{G}_m = \mathbb{Z}(1)[1]$

<u>Example</u> $X = \mathrm{Spec}(k)$ $\boxed{H_{-1,-1}(k) = k^*} = \mathrm{Hom}_{DM}(\mathbb{Z}_{tr}(k), \overbrace{\mathbb{Z}(1)[1]}^{\mathcal{O}^*}) = H^1_{nis}(k, \mathcal{O}^*) = H^1_{nis}(k, \mathbb{Z}(1))$

<u>Duality</u> X smooth projective $d = \dim X$ $\boxed{H_{-1,-1}(X) \cong H^{2d+1}(X, \mathbb{Z}(d+1))}$ Friedlander-Voevodsky

describe using "cycles with coefficients in K^M_*" (Rost's description)

6.3 $\boxed{\begin{array}{l}\text{"Theorem" (Rost) } \forall a_1, \dots, a_n \in k^* \ \exists X \text{ so that} \\ (1)\ F = k(X)\quad \{a_1, \dots, a_n\} \in K^M_n(k) \text{ dies in } K^M_n(F)/\ell \\ (2)\ H_{-1,-1}(X \times X) \rightrightarrows H_{-1,-1}(X) \longrightarrow k^* \text{ exact}\end{array}}$

(3) ν_i-varieties $X_i \to X$

$X \times X \leftleftarrows X \leftarrow pt$

Cohomology Operations mod ℓ

$$H^{p,q}(X) = H^p(X, \mathbb{Z}/\ell(q))$$

defn: natural cohomology operation $\phi_X : H^{p,q}(X) \to H^{p+i, q+j}$

$\leftrightarrow$ elements of $H^{i,j}(\underline{\underline{H}})$, $\underline{\underline{H}} = \sum^{\infty}$ "$\mathbb{Z}(j)[i]$" in big DM

motivic Steenrod algebra $\boxed{\mathcal{A}^{**}}$ = $\underbrace{\text{algebra}}$ of natural coh. ops.

contains $H^{**}(k)$ $[\cong K_*^M(k)]$

Bockstein $\overset{\text{"}\beta\text{"}}{Q_0} : H^{p,q} \to H^{p+1,q}$ $\qquad$ $k^*/\ell : H^{p,q} \to H^{p+1,q+1}$

$0 \to \mathbb{Z}/\ell(q) \to \mathbb{Z}/\ell^2(q) \to \mathbb{Z}/\ell(q) \to 0$

Milnor operations Q_i of bidegree $(2\ell^i - 1, \ell^i - 1)$ (primitive if $\ell \neq 2$)

2.1 Theorem char$(k)=0$ $\boxed{\exists!\ H^{2n+1,n} \xrightarrow{\phi} H^{2n\ell+2,n\ell}}$ additive $\left(\begin{matrix}\text{up to}\\ \text{scalar}\\ \mathbb{Z}/\ell^*\end{matrix}\right)$

$H^{2n,n}(X) \to H^{2n+1}(\Sigma X) \xrightarrow{\phi} H^{\cdot\cdot}$ is zero

need motivic cohom of simplicial schemes
easy with Hom_{DM} interpretation

Realization (Betti) $k = \mathbb{C}$

$Sm/\mathbb{C} \to \mathbb{C}\text{manifolds} \to \underline{\underline{\mathrm{Top}}}$

$\underline{\underline{H}} \longmapsto$ Eilenberg MacLane spectrum

$\mathcal{A}^{**} \overset{\cong\ \ell \neq 2}{\longmapsto} \mathcal{P}$ usual Steenrod algebra

$Q_1 \longmapsto Sq^3 + Sq^2 Sq^1 = Q_1$ if $\ell = 2$

Application: Use symmetric powers of M to construct ϕ and see that it must be ~~$Q_{n-2} \cdots Q_2 Q_1$~~ $f(Q_0, Q_1, \dots, Q_{n-2})$

$\check{C}(X)$: simplicial $X \leftleftarrows X^2 \Lleftarrow X^3 \cdots$ [$\check{C}(X) \to pt$ is $\simeq$ iff $X(k) \neq \emptyset$]

(inductive assumption on BL) $H^{p,q-1}(k) \xrightarrow{\cong} H^{p,q-1}(\check{C}X)$

4.3 <u>Lemma</u> $\mathcal{X}$ pointed simplicial scheme, "good"

$$\begin{array}{cc} DM(X) & \cdot \\ \downarrow & \downarrow \\ DM(k) & [X] \end{array}$$

ν_n-variety

exact sequence $\xrightarrow{Q_n} \tilde{H}^{p,q}(\mathcal{X}) \xrightarrow{Q_n} \tilde{H}^{r,s}(\mathcal{X})$

"Margulis homology" rel. Milnor operation

$$\begin{tikzcd}
0 \arrow[r] & H^{n+1,n}(\check{C},\mathbb{Z}_{(l)}) \arrow[r] \arrow[d, "\Phi"] & H^{n+1,n}(\check{C}) \arrow[r, "Q_0"] \arrow[d, "\phi", "\text{monic}"'] & H^{n+2,n}(\check{C}) \arrow[d, "\phi"] \\
& \underbrace{H^{s,t}(\check{C},\mathbb{Z})_{(l)}}_{0\ (\text{Rost})} \arrow[r] & H^{s,t}(\check{C}) \arrow[r, "Q_0"] & H^{s+1,t}(\check{C})
\end{tikzcd}$$

Want $=0$

6.8 $\dashrightarrow$ 0 (Rost)

<u>Theorem</u> $\phi = Q_{n-2} \cdots Q_2 Q_1$ is a monomorphism on $H^{n+1,n}(\check{C})$

<u>Idea</u>: $\underbrace{H^{*,q-i}}_{=0 \text{ by induction}} \xrightarrow{Q_i} H^{\bullet\bullet} \xrightarrow[\text{monic}]{Q_i} H^{\bullet\bullet}$

$$H^{n+1,n} \xrightarrow{Q_1} H^{\cdot\cdot} \xrightarrow{Q_2} H^{\cdot\cdot} \rightarrowtail \cdots \xrightarrow{Q_{n-2}} H^{st}$$