ON "HORIZONTAL" INVARIANTS ATTACHED TO QUADRATIC FORMS

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Abstract. We introduce series of invariants related to the dimension for quadratic forms over a field, study relationships between them and prove a few results about them.

This is the TeX-ing of a manuscript from 1993 entitled *Quadratic forms and simple algebras of exponent two.* The original manuscript contained an appendix that has appeared in [K3]: I removed it and replaced references to it by references to [K3]. I also added a missing table at the end of Section 2, improved Proposition 3.3 a bit and removed a section that did not look too useful. Finally I changed the title to a better-suited one. These are the only changes to the original manuscript.

The main reasons I have to exhume it are that 1) the notion of dimension modulo I^{n+1} has recently been used very conceptually by Vishik (e.g. [Vi], to which the reader is referred for lots of highly nontrivial computations) and 2) Question 1.1 has been answered positively by Parimala and Suresh (see their article).

Everything here is anterior to Voevodsky's proof of the Milnor conjecture, which is not used.

Introduction

Let F be a field of characteristic $\neq 2$. The u-invariant $u(F)$ of F is the least integer n such that any quadratic form in more than n variables over F is isotropic, or $+\infty$ if no such integer exists. (A finer version exists for formally real fields, but for simplicity we shall not consider it.) In [Me2] (see also [Ti]), Merkurjev disproved a long-standing conjecture of Kaplansky, asserting that the u-invariant should always be a power of 2. In fact, Merkurjev produced for any integer $m \geq 3$ examples of fields of u-invariant $2m$.

A remarkable feature of Merkurjev's examples is that they can be contrived to have 2-cohomological dimension 2. This destroys a naïve belief that $u(F)$ would be $2^{\nu(F)}$, where $\nu(F)$, the ν-invariant of F, is

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the largest integer n such that $I^\nu F \neq 0$, which was the case in all previously known examples (see *e.g.* [K1, th. 1]). It hints that a good understanding of the u-invariant involves not only the ν-invariant, but also 'horizontal' invariants attached to the quotients $I^nF/I^{n+1}F$.

Introducing such invariants and starting their study of is the object of this paper. Given a quadratic form, one may approximate its anisotropic dimension, the dimension of its kernel forms, by its 'dimension modulo I^{n+1}' for every $n \geq 1$ (the fact that this actually is an approximation is a consequence of the Arason-Pfister theorem). Given an element of $I^nF/I^{n+1}F$, one may study its 'length' or 'linkage index', the smallest number of classes of Pfister forms necessary to express it. The suprema $u_n(F)$ of the former invariants ('u-invariant modulo I^{n+1}') approximate the u-invariant; the suprema $\lambda^n(F)$ of the latter help giving upper bounds for the former. More precisely, one can bound $u_n(F)$ in terms of $\lambda^n(F)$ and $u_{n-1}(F)$ (Proposition 1.2).

The only case in which I can prove a converse to these bounds is $n = 2$, where $u_2(F) = 2\lambda^2(F) + 2$. However, it is not impossible that all the $\lambda^n(F)$, as well as $u(F)$ when F is not formally real, can actually be bounded in terms of $\lambda^2(F)$ (and n for $\lambda^n(F)$). At least this is the case when $\lambda^2(F) = 1$, by a theorem of Elman-Lam [Lam, th. XI.4.10]. I partially generalize this theorem in one direction (Propositions 3.2 and 3.3), but the general case seems quite open.

In all this paper, we use Lam's [Lam] notations for Pfister forms, *i.e.* $\ll a_1, \dots, a_n \gg = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$. We write $\cong$ (resp. $\sim$) for isometry (resp. Witt-equivalence) of quadratic forms.

1. A few quadratic invariants

Recall [A] that, for a quadratic form q, $\operatorname{diman}(q)$ denotes the rank of the unique anisotropic quadratic form whose class in $W(F)$ equals the class of q (the kernel form of q). As will be seen in Proposition 1.1, Definition 1.1 generalises this definition.

Definition 1.1. Let q be a quadratic form over F and n an integer ≥ 0. The *n-dimension* of q is $\dim_n(q) = \inf\{\dim(q') \mid q' \equiv q \pmod{I^{n+1}F}\}$.[1]

Remark 1.1. $\dim_n(q)$ only depends on the class of q modulo $I^{n+1}F$. For all n, one has $\dim_n(q) \leq \operatorname{diman}(q)$.

Definition 1.2. A quadratic form q is *anisotropic modulo* $I^{n+1}F$ (or *n-anisotropic*) if $\dim_n(q) = \dim(q)$.

[1] One should be careful that Vishik's notation in [Vi, Def. 6.8] is different: our $\dim_n(q)$ is his $\dim_{n+1}(q)$.

It is clear that for two quadratic forms q, q', one has $\dim_n(q \perp q') \leq \dim_n(q) + \dim_n(q')$. The following lemma strengthens this result when $q' \in I^n F$. Despite its simplicity, it is basic in much of this section.

Lemma 1.1. *Let $(q, q') \in W(F) \times I^n F$ with $q, q' \neq 0$. Then $\dim_n(q \perp q') \leq \dim_n(q) + \dim_n(q') - 2$.*

Proof. Without loss of generality, we may assume that q and q' are anisotropic modulo $I^{n+1}F$. Since $q' \in I^n F$, $q' \equiv \langle a \rangle q' \pmod{I^{n+1}F}$ for any $a \in F^*$. For suitable a, the form $q \perp \langle a \rangle q'$ is isotropic. Hence

$$\begin{aligned}\dim_n(q \perp q') &= \dim_n(q \perp \langle a \rangle q') \leq \operatorname{diman}(q \perp \langle a \rangle q') \\ &\leq \dim(q) + \dim(q') - 2 = \dim_n(q) + \dim_n(q') - 2.\end{aligned}$$

□

Lemma 1.2. *Let $q \in W(F)$ be anisotropic modulo $I^{n+1}F$. Then its only subforms belonging to $I^n F$ are 0 and possibly q.*

Proof. This follows from Lemma 1.1. □

Definition 1.3. Let $n \geq 0$ and $x \in I^n F / I^{n+1} F$. The *length* of x is

$$\lambda(x) = \inf\{r \mid x \text{ is a sum of } r \text{ classes of } n\text{-fold Pfister forms}\}.$$

If $q \in I^n F$, we write $\lambda(q)$ for $\lambda(x)$, where x is the image of q in $I^n F / I^{n+1} F$.

Proposition 1.1. *a)* $\dim_0(q) = \begin{cases} 0 & \textit{if } \dim(q) \textit{ is even} \\ 1 & \textit{if } \dim(q) \textit{ is odd.} \end{cases}$

b) $\dim_1(q) = \begin{cases} 0 & \textit{if } \dim(q) \textit{ is even and } d_\pm(q) = 1 \\ 1 & \textit{if } \dim(q) \textit{ is odd} \\ 2 & \textit{if } \dim(q) \textit{ is even and } d_\pm(q) \neq 1. \end{cases}$

c) If $q \in I^n F - I^{n+1} F$, then $2^n \leq \dim_n(q) \leq (2^n - 2)\lambda(q) + 2$.

d) If $q \notin I^n F$, then $\dim_n(q) \leq (2^n - 2)\lambda(q') + \dim_{n-1}(q)$ for some $q' \in I^n F$.

e) If $q \in I^2 F - I^3 F$, then $\dim_2(q) = 2\lambda(q) + 2$.

f) If $2^n \geq \operatorname{diman}(q)$, then $\dim_n(q) = \operatorname{diman}(q)$.

Proof. a) is obvious. Let $\dim(q)$ be odd. Then, for the right choice of $\varepsilon = \pm 1$, $q \perp \langle \varepsilon d_\pm(q) \rangle \in I^2 F$, so $\dim_1(q) = 1$. Let $\dim(q)$ be even. Then $q \in I^2 F$ if and only if $d_\pm(q) = 1$; if $q \in I^2 F$, then $q \perp \langle -1, d_\pm(q) \rangle \in I^2 F$, so $\dim_1(q) = 2$. This proves b).

In c), the lower bound for $\dim_n(q)$ is a consequence of the Arason-Pfister theorem [AP]. The upper bounds in c) and d) follow from

Lemma 1.1 by induction on $\lambda(q)$. Let us prove equality in the case $n=2$ (*cf.* [Me1, lemma]). We may assume $\dim(q)=\dim_2(q)=2m$. We argue by induction on m. The case $m=1$ is impossible. Assume $m\geq 2$. We may write $q=\langle a,b,c\rangle \perp q'$. Then q is Witt-equivalent to $\langle a,b,c,abc\rangle \perp (\langle -abc\rangle \perp q')$. The first summand is $\langle a\rangle \ll ab,ac \gg \equiv \ll ab,ac\gg \pmod{I^3F}$, while the second one $q''=\langle -abc\rangle \perp q'$ has dimension $2m-2$, so that $\dim_2(q'')\leq 2m-2$. By induction, $2\lambda(q'')+2\leq \dim_2(q'')$, so $2\lambda(q)+2\leq 2\lambda(q'')+4\leq \dim_2(q'')+2\leq \dim_2(q)$.

Note that this argument fails for $n\geq 3$.

Let us prove f). We may assume that q is anisotropic. Assume that $\dim_n(q)<\dim(q)$. Then there exists q' with $q'\equiv q \pmod{I^{n+1}F}$ and $\dim(q')<\dim(q)$. Therefore, $q\perp -q'\in I^{n+1}F$. But $\dim(q\perp -q')<2\dim(q)\leq 2^{n+1}$: therefore, by the Arason-Pfister theorem [AP], $q\perp -q'$ is hyperbolic. This means that q is Witt-equivalent to q': this is impossible, since q is anisotropic and $\dim(q')<\dim(q)$. □

Definition 1.4. Let $n\geq 1$ and $k\leq n$. The *k-restricted u-invariant modulo I^{n+1} of F* is $u_n^k(F)=sup\{\dim_n(q)\mid q\in I^k(F)\}$. If $k=0$, we write $u_n(F)$ for $u_n^k(F)$ and call it the *u-invariant of F modulo I^{n+1}*.

Remark 1.2. When k is fixed, the $u_n^k(F)$'s form a non-decreasing sequence for increasing n. In particular the $u_n(F)$ form a non-decreasing sequence. Similarly, when n is fixed, the $u_n^k(F)$ form a non-increasing sequence for increasing k.

Definition 1.5. Let $n\geq 1$. The *n-th λ-invariant* of F is $\lambda^n(F)=\sup\{\lambda(x)\mid x\in I^nF/I^{n+1}F\}$.

Proposition 1.2. *a) For $k>0$ and $n\geq k$, $I^kF\neq 0 \iff u_n^k(F)\neq 0 \iff u_n^k(F)\geq 2^k$. If $I^nF\neq 0$, $u_n^k(F)\geq 2^n$. In particular, if $I^nF\neq 0$, $u_n(F)\geq 2^n$.*
b) For $k\geq 0$, $\sup_{n\geq k} u_n^k(F)=u_k(F):=\sup\{\mathrm{diman}(q)\mid q\in I^k(F)\}$; in particular, $\sup_{n\geq 0} u_n(F)=u(F)$.
c) $u_n^n(F)\leq (2^n-2)\lambda^n(F)+2$.
d) For any $k<n$, $u_n^k(F)\leq (2^n-2)\lambda^n(F)+u_{n-1}^k(F)$.
e) $u_0(F)=1$.
f) If $IF\neq 0$, $u_1(F)=u_1^1(F)=2$.
g) If $I^2F\neq 0$, then $u_2(F)=u_2^2(F)=2\lambda^2(F)+2$.

(I am indebted to O. Gabber for pointing out d).)

Proof. It is clear that $I^kF=0\Rightarrow u_n^k(F)=0$ for all $n\geq k$. By Remark 1.2, $u_n^k(F)\geq u_k^k(F)$ when $n\geq k$. Assume $I^kF=I^{k+1}F$. Then any k-fold Pfister form belongs to $I^{k+1}F$. By [AP], such a form must be hyperbolic, hence $I^kF=0$. This shows that if $I^kF\neq 0$, there exists a form $q \in I^kF - I^{k+1}F$. By Proposition 1.1 c), we have $\dim_k(q) \geq 2^k$, hence $u_k^k(F) \geq 2^k$ and $u_n^k(F) \geq 2^k$. The last two claims of a) follow by Remark 1.2 ($u_n(F) \geq u_n^k(F) \geq u_n^n(F)$ when $k \leq n$). This proves a).

To prove b), first assume that $u_k(F)$ is finite. Let n be such that $2^n \geq u_k(F)$. By Proposition 1.1 f), $\dim_n(q) = \mathrm{diman}(q)$ for any $q \in I^kF$. In particular, $u_n^k(F) = u^k(F)$ for all such n. Assume now that the sequence $(u_n^k(F))_{n \geq k}$ is bounded, say by N. Let $q \in I^kF$ and choose n such that $2^n \geq \dim(q)$. Applying Proposition 1.1 f) again, we have $\dim_n(q) = \mathrm{diman}(q)$. This shows that $\mathrm{diman}(q) \leq N$, hence that $u^k(F) \leq N$.

Part c) is an immediate consequence of Proposition 1.1 c). To prove d), we may assume that $IF \neq 0$, otherwise it is trivial. We may further assume that $I^kF \neq 0$. Let $q \in I^kF$. We distinguish two cases:

(i) $q \in I^nF$. By Proposition 1.1 d), $\dim_n(q) \leq (2^n - 2)\lambda^n(F) + \dim_{n-1}(q) \leq (2^n - 2)\lambda^n(F) + u_{n-1}^k(F)$.

(ii) $q \in I^nF$. By Proposition 1.1 c), $\dim_n(q) \leq (2^n - 2)\lambda^n(F) + 2$. This is $\leq (2^n - 2)\lambda^n(F) + u_{n-1}^k(F)$ provided $u_{n-1}^k(F) \geq 2$. If $k \geq 1, u_{n-1}^k(F) \geq 2^k \geq 2$ by a). If $k = 0$, $u_{n-1}(F) \geq u_1(F) \geq 2$ since $IF \neq 0$ (see f)).

e) and f) follow from Proposition 1.1 a) and b). It remains to prove g). First we prove that $u_2(F)$ cannot be odd, i.e. $u_2(F) = u_2^1(F)$. This is a consequence of Proposition 1.1 e), Proposition 1.2 e) and the following lemma.

Lemma 1.3. *a)* $u_2(F) = \infty$ *iff* $\lambda^2(F) = \infty$.
b) Assume that $u_2(F) < \infty$ *and* $IF \neq 0$. *Then* $u_2(F) = u_2^1(F)$.

Proof. a) is a consequence of Proposition 1.1, b) d) and e). For b), let q be such that $\dim(q) = \dim_2(q) = u_2(F)$. We show that $\dim(q)$ cannot be odd, unless $IF = 0$. If it is, then as in the proof of Proposition 1.1 b) we choose $\varepsilon = \pm 1$ such that $q' = q \perp \langle \varepsilon d_\pm(q) \rangle \in I^2F$. By assumption, $\dim_2(q') \leq \dim(q)$, so that $q' \equiv q'' \pmod{I^3F}$, where $q'' \in I^2F$ is such that $\dim(q'') \leq \dim(q)$. Then $q \equiv q'' \perp \langle -\varepsilon d_\pm(q) \rangle \pmod{I^3F}$. If $q'' = 0$, then $u_2(F) = 1$, which implies $IF = 0$ (Proposition 1.2, b) and remark 1.2). Otherwise, by Lemma 1.1 we have $\dim_2(q) \leq \dim(q'') - 1 < \dim(q)$, a contradiction. □

We now prove that $u_2^1(F) = u_2^2(F)$. By lemma 1.3 we may assume that $u_2(F) < \infty$. Let $q \in IF$ be such that $\dim_2(q) = u_2^1(F)$. We may assume that q is anisotropic modulo I^3F. Then $q' = q \perp \langle -1, d_\pm(q) \rangle \in I^2F$, with $\dim(q') = \dim(q) + 2$. Since $\dim(q) = u_2^1(F) = u_2(F)$, we have $q \perp \langle -1 \rangle \equiv q'' \pmod{I^3F}$, where q'' is such that $\dim(q'') \leq \dim(q)$. But $\dim(q'') \equiv \dim(q \perp \langle -1 \rangle) \pmod 2$, so that $\dim(q'')\langle\dim(q)$. Therefore, $q' \equiv q'''$ (mod I^3F), with $\dim(q''') \leq \dim(q) = \dim_2(q)$, and $q \equiv q''' \perp \langle 1, -d_\pm(q)\rangle$ (mod I^3F). If $q''' = 0$, $u_2^1(F) = 2$, but then $I^2F = 0$. Otherwise, since $q''' \in I^2F$, $\dim_2(q) = \dim_2(q''' \perp \langle 1, -d_\pm(q)\rangle) \leq \dim_2(q''')$ by Lemma 1.1. Hence $\dim_2(q''') = \dim_2(q)$ and $u_2^1(F) = u_2^2(F)$. □

Remark 1.3. The proof of g) is stolen from [Lam, ch XI, proof of lemma 4.9].

Remark 1.4. The statement $u_n(F) = u_n^1(F)$ is equivalent to "$u_n(F)$ is even".

Remark 1.5. These proofs prompt the definition of quadratic forms universal modulo I^{n+1}, round modulo I^{n+1}. This is left to the reader.

Corollary 1.1 (cf [Lam, ch. XI, lemma 4.9])**.** *If $u_2(F) > 1$, it is even.*

Example 1.1. For all $n \geq 0$, $u_n(\mathbf{R}) = 2^n$.

Question 1.1. For $n > 2$, can one bound $\lambda^n(F)$ in terms of n and $u_n(F)$?

Here is a possible 'generic' argument to get an affirmative answer at least when $n = 3$. Let k be a base field, m a fixed integer, $F_0 = k(T_1, \dots, T_{2m})$, q the quadratic form $\langle T_1, \dots, T_{2m}\rangle$ over F_0, $F_1 = F_0(\sqrt{(-1)^m T_1 \dots T_{2m}})$ and F_2 the function field of the Severi-Brauer variety of the Clifford algebra of q_{F_1}. Then $q_{F_2} \in I^3F_2$ and is a good candidate for a 'generic element of rank $2m$ in I^3'. It is then plausible that, for any field F containing k and any $q \in I^3F$ of rank $2m$, one has $\lambda(q) \leq \lambda(q_{F_2})$. I have not checked the details.

Question 1.2. By Elman-Lam [EL2], if F is not formally real and $\lambda^2(F) = 1$ then $u(F) = 1, 2, 4$ or 8. Is there a nonformally real field F such that $\lambda^2(F) = 2$ and $u(F) = \infty$?

In the next section, we give some evidence that the answer to this question might be 'no'.

2. Discrete valuations; iterated power series

Let A be a complete discrete (rank 1) valuation ring, E its quotient field anf F its residue field. We assume that char $F \neq 2$.

Proposition 2.1. *a) For all $n \geq 1$, $\lambda^n(E) \leq \lambda^n(F) + \lambda^{n-1}(F)$.*
b) For all $n \geq 1$ and $0 \leq k \leq n$, $u_n^k(E) \leq u_n^{k-1}(F) + u_{n-1}^{k-1}(F)$.

Note. In b), one should interpret $u_n^{-1}(F)$ as $u_n(F)$.
Proof. Let π be a prime element of E. Every quadratic form q over E can be written $q \cong q_1 \perp \pi q_2$, where q_1 and q_2 are classes of unimodular forms over A. Alternatively, q can be written up to Witt equivalence $q_1' \perp \ll \pi \gg q_2$, still with q_1' integral. If $q \in I^n E$, then $q_1' \in I^n A$ and $q_2 \in I^{n-1} A$ [S]. Since $W(A) \xrightarrow{\sim} W(F)$ is a filtred isomorphism, this proves a).

To see b), let $q \in I^n E$ and q_1', q_2 as above. Let $\bar{q}_2$ be the residue image of q_2 over F. Take $\bar{q}_2' \equiv \bar{q}_2 \pmod{I^n F}$ with $\bar{q}_2' \in I^{k-1} F$ and $\dim(\bar{q}_2') \leq u_{n-1}^{k-1}(F)$. Let q_2' be a lift of $\bar{q}_2'$ to A. Then $q_2' \equiv q_2 \pmod{I^n A}$ and $q \equiv q_1' \perp \ll \pi \gg q_2' = q_1' \perp -q_2' \perp \langle \pi \rangle q_2' \pmod{I^{n+1} F}$. Now let $q_1'' = q_1' \perp -q_2' \in I^{k-1} A$; choose $q_1''' \in I^{k-1} A$ such that $\bar{q}_1''' \equiv \bar{q}_1'' \pmod{I^{n+1} F}$ and $\dim(\bar{q}_1''') \leq u_n^{k-1}(F)$. Then $q_1''' \equiv q_1'' \pmod{I^{n+1} A}$, $q \equiv q_1''' \perp \langle \pi \rangle q_2' \pmod{I^{n+1} F}$ and $\dim(q_1''' \perp \langle \pi \rangle q_2') \leq u_n^{k-1}(F) + u_{n-1}^{k-1}(F)$. $\square$

An example. Start from a field K and set $K_d = K((t_1))\dots((t_d))$, a field of iterated power series. Then K_d is complete for a discrete valuation, with residue field K_{d-1}. Proposition 2.1 allows one to get upper bounds for the invariants of K_d in terms of d and those of K, by induction on d.

However, these inductive bounds are by no means sharp in general. Computing, or at least estimating $\lambda^n(K_d)$ and $u_n(K_d)$ turns out to be “global” in d. To illustrate this, we now consider the case where K is algebraically closed.

Definition 2.1. Let k be a field, V a d-dimensional k-vector space and n an integer $\leq d$. Let $x \in \Lambda^n(V)$ be an n-vector. The *length* of x is the smallest integer $\ell(x)$ such that x is the sum of $\ell(x)$ pure n-vectors. We denote by $N(k,d,n)$ the supremum of $\ell(x)$ when x runs through V (this is independent of V).

The following proposition is clear.

Proposition 2.2. *a) Let $V = K_d^*/K_d^{*2}$, viewed as an F_2-vector space with basis $t_1, \dots, t_d$. Then there are canonical isomorphisms*

$$I^n K_d / I^{n+1} K_d \simeq \Lambda^n(V)$$

mapping Pfister forms to pure n-vectors.
b) For $x \in I^n K_d / I^{n+1} K_d$, with image x' in $\Lambda^n(V)$, $\lambda(x) = \ell(x')$, where $\lambda(x)$ is as in Definition 1.3 and $\ell(x')$ is as in Definition 2.1.
c) $\lambda^n(K_d) = N(\mathbf{F}_2, d, n)$. $\square$

The following information on $N(k,d,n)$ is collected from [K2].

Proposition 2.3. *a)* $N(k,d,n) = N(k,d,d-n)$.
b) $N(k,d,0) = N(k,d,1) = 1$.
c) $N(k,d,2) = [d/2]$.
d) $N(k,6,3) = 3$ *for any field* k.
e) If k *is algebraically closed,* $N(k,7,3) \leq 4$ *(probably* $= 4$*);* $N(\mathbf{R},7,3) = 5$*; for any field* k*,* $N(k,7,3) \leq 6$ *(probably* ≤ 5*).*
f) If k *is algebraically closed of characteristic* 0*,* $N(k,8,3) = 5$*;* $N(\mathbf{R},8,3) \leq 8$*; for any* k*,* $N(k,8,3) \leq 10$*.*
g) If k *is algebraically closed of characteristic* 0*,* $N(k,9,3) \leq 9$*.*
h) There exists a polynomial f_n *of degree* $\leq n-2$ *such that, for any field* k*,* $N(k,d,n) \leq \dfrac{d^{n-1}}{2(n-1)!} + f_n(d)$*. In particular, for any* k*,* $\limsup_{d\to\infty} \dfrac{N(k,d,n)}{d^{n-1}} \leq \dfrac{1}{2(n-1)!}$*.*
i) If k *is infinite,* $N(k,d,n) \geq \dfrac{\binom{d}{n}}{n(d-n)+1}$*. If* k *is finite with* q *elements,* $N(k,d,n) \geq \dfrac{\binom{d}{n}}{n(d-n)+1+\varepsilon(q)}$*, where*

$$\varepsilon(q) = \log_q\Big(\prod_{i=2}^{\infty}(1-q^{-i}) - 1\Big).$$

(So $\varepsilon(2) \approx 0.75$*.) In particular, for any field* k*,* $\liminf_{d\to\infty} \dfrac{N(k,d,n)}{d^{n-1}} \geq \dfrac{1}{n.n!}$*.*

This proposition enables us to list values of $\lambda^n(K_d)$ for $d \leq 6$:

$d\backslash n$	0	1	2	3	4	5	6
1	1	1					
2	1	1	1				
3	1	1	1	1			
4	1	1	2	1	1		
5	1	1	2	2	1	1	
6	1	1	3	3	3	1	1

From this table, we see that the evaluation

$$u(K_d) \leq \sum_{n=2}^{d}(2^n-2)\lambda^n(K_d) + 2$$

from Proposition 1.2 d) quickly becomes completely off the mark.

3. Relationships between the u-invariant, the ν-invariant and the λ-invariants

Lemma 3.1 ([EL1, th. 4.5]). *Let φ be an m-fold Pfister form and ψ be an n-fold Pfister form. Assume that φ and ψ are r-linked but not $(r+1)$-linked. Then, for any $a, b \in F^*$,* $\operatorname{ind}(\langle a\rangle\varphi \perp \langle b\rangle\psi) = 2^r$. □

Proposition 3.1. *Assume that $u(F) \leq 2^n$. Then $\lambda^n(F) \leq 1$.*

Proof. Let φ and ψ be two n-fold Pfister forms. Then ψ is universal, so $\psi \equiv -\psi$, and $\operatorname{ind}(\varphi \perp \psi) = \operatorname{ind}(\varphi \perp -\psi) \geq 2^{n-1}$. By Lemma 3.1, φ and ψ are $(n-1)$-linked, so $\varphi \perp \psi$ is isometric to an n-fold Pfister form. □

Using a result of Bloch, we can deduce a nice corollary to this proposition, generalising a well-kown result for global fields:

Corollary 3.1. *Let F be a function field in n variables over an algebraically closed field, or in $n-1$ variables over a finite field. Then every element of $H^n(F, \mathbf{Z}/2)$ is a symbol.*

Proof. A generalisation of Bloch's argument in [B, Lecture 5] shows that for a field as in the statement, $H^n(F, \mathbf{Z}/2)$ is generated by symbols. It is then sufficient to prove that every element of $K_n^M(F)/2$ is a symbol. Notice that F is C_n in the sense of Lang [G], hence $u(F) \leq 2^n$. In view of Proposition 3.1, it is then sufficient to have:

Lemma 3.2 ([EL1, th. 6.1]). *Let F be a field, $n \geq 1$ and $x, y, z \in K_n^M(F)/2$ be three symbols. Assume that $\nu_n(x) + \nu_n(y) = \nu_n(z)$. Then $x + y = z$.* □

In this lemma, $\nu_n : K_n^M(F)/2 \to I^nF/I^{n+1}F$ is the homomorphism defined in [Mi].

The following is not more than [EL2, lemma 2.3 and cor. 2.5].

Proposition 3.2. *Assume that $\lambda^n(F) \leq 1$. Then $\lambda^{n+1}(F) \leq 1$. If furthermore F is not formally real, then $I^{n+2}F = 0$.* □

The next proposition is the main result of this section.

Proposition 3.3. *Assume that $\lambda^n(F) = m < \infty$ and that F is not formally real. Then $I^{n(m+1)+1}F = 0$. If -1 is not a square in F, then $I^{n(m+1)}F = 0$.*

In other words, if F is not formally real then $\nu(F) \leq n(\lambda^n(F) + 1)$ for all n. For $n = 2$, the right hand side is $u_2(F)$ by Proposition 1.2

d). When -1 is a square in F, this bound is improved to $\nu(F) \leq n(\lambda^n(F)+1)-1$.

Proof. By Milnor [Mi], there are surjective homomorphisms $\nu_n : K_n^M(F)/2 \to I^nF/I^{n+1}F$, and ν_2 is an isomorphism. Let $K'^M_*(F) = K_*^M(F)/\{-1\}K_*(F)$: as explained in [K3, Appendix], the commutative ring $K'^M_*(F)/2$ enjoys graded divided power operations $x \mapsto x^{[i]}$ which vanish on symbols: by [the argument of the proof of] [K3, Prop. 1 (8)], $K'^M_{n(m+1)}(F)/2 = 0$, hence every element of $I^{n(m+1)}F/I^{n(m+1)+1}F$ is a multiple of (the 1-fold Pfister form) $\ll 1 \gg$.

If -1 is a square in F, then $\ll 1 \gg$ is hyperbolic and $I^{n(m+1)}F/I^{n(m+1)+1}F = 0$; using the Arason-Pfister theorem, we deduce that $I^{n(m+1)}F = 0$. Assume now that -1 is not a square in F; let $E = F(\sqrt{-1})$. By [BT, Cor. 5.3], $K^M_{n(m+1)+1}(E)$ is generated by symbols $\{a_1, \dots, a_{n(m+1)+1}\}$, with $a_1, \dots, a_{n(m+1)}$ in F^*. Since every element of $K^M_{n(m+1)}(F)/2$ is a multiple of $\{-1\}$, every element in $K^M_{n(m+1)+1}(E)/2$ is a multiple of $\{-1\} = 0$, *i.e.* $K^M_{n(m+1)+1}(E)/2 = 0$, hence $I^{n(m+1)+1}E/I^{n(m+1)+2}E = 0$ and $I^{n(m+1)+1}E = 0$. If now F is not formally real, [A, Satz 3.6 (ii)] implies that $I^{n(m+1)+1}F = 0$. $\square$

Remark 3.1. For $n = 2$ Prop. 3.3 is optimal, at least when -1 is a square in F. For example, let $F = \mathbf{C}((t_1))\dots((t_d))$ be the field of iterated formal power series in d variables over $\mathbf{C}$. Then $\nu(F) = d$ and, by Section 2, $\lambda^2(F) = [d/2]$ which is also the least integer greater than $\dfrac{d-1}{2}$.

Question 3.1. Is there a universal bound for $\lambda^n(F)$ in terms of n and $\lambda^2(F)$? In view of Propositions 1.2 and 3.3, this would provide a negative answer to question 1.2. For example, it seems plausible that if $\lambda^n(F) = m$, then $\lambda^{nm}(F) \leq 1$. This is true in the test example of Remark 3.1. If one could prove it in general, then the estimate $\nu(F) \leq n(\lambda^n(F)+1)$ or $\nu(F) \leq n(\lambda^n(F)+1)-1$ of Proposition 3.3 would be improved to $\nu(F) \leq n\lambda^n(F)+1$ thanks to Proposition 3.2 (note that this is no improvement if $n = 2$ and -1 is a square in F).

It is clear that divided power operations have not been used up to their full potentialities.

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