THE MINIMAL HEIGHT OF QUADRATIC FORMS OF GIVEN DIMENSION

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In Memory of Martin Kneser

Abstract. Given an arbitrary n, we consider anisotropic quadratic forms of dimension n over all fields of characteristic $\neq 2$ and prove that the height of an n-dimensional excellent form (depending on n only) is the (precise) lower bound of the heights of all forms of dimension n.

Contents

1. Introduction

Excellent quadratic forms and their fundamental properties have been discovered by Knebusch [9] in 1977. They do occur over arbitrary fields. The simplest examples are sums of squares, which are always excellent. This shows that, over suitable fields, *anisotropic* excellent forms do occur in any dimension.

The definition of excellent forms is closely related to Pfister forms [10]. A quadratic form ϕ is a *Pfister neighbor* [9], if it is isomorphic to a subform of a quadratic form π such that π is similar to a Pfister form and $\dim \phi > (\dim \pi)/2$. In this case, the form π and the orthogonal complement of ϕ inside π are uniquely determined (up to an isomorphism) by ϕ. A quadratic form is called *excellent*, if its dimension is ≤ 1 or if it is a Pfister neighbor with an excellent complement.

If a quadratic form ϕ over a field F is anisotropic and $\dim \phi \geq 2$, its *first higher Witt index* is the Witt index of the form $\phi_{F(\phi)} = \phi \otimes_F F(\phi)$, where $F(\phi)$ denotes the function

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field of the quadric defined by ϕ. The anisotropic kernel $(\phi_{F(\phi)})_{an}$ is the first higher anisotropic kernel of ϕ (see [10] for the definitions of the usual Witt index and of the usual anisotropic kernel of a quadratic form). Repeating this construction gives, for each ϕ, a sequence of field extensions of F which is called the *(generic) splitting tower* of ϕ. It is generic in the sense that, for each extension L of F, a specialization from one of the layers K of the splitting tower of ϕ to L can be defined such that the indices of ϕ_K and ϕ_L are the same.

The length of the splitting tower of ϕ is called the *height* $\mathfrak{h}(\phi)$, and the respective anisotropic kernels of ϕ over the field extensions of its splitting tower are its *higher anisotropic kernels*.

These notions were introduced and extensively studied as well by Knebusch in [8], [9]. It turned out that, for anisotropic excellent forms ϕ of given dimension, the dimensions of the higher kernels are completely independent of the underlying excellent form and even of the ground field F. Knebusch also gave another characterization of excellent forms in [9]: A quadratic form is excellent if and only if all of its higher anisotropic kernels are already defined over F itself. (It should be mentioned here that Knebusch worked over fields of characteristic $\neq 2$, but recent research by Hoffmann and Laghribi [1] showed that the same phenomena do occur in characteristic 2.)

In [2], the notion of the *splitting pattern* of a quadratic form ϕ was defined as the sequence of all possible indices of ϕ over some field extension. From the description above it is clear that all these indices do occur already over the field extensions of the splitting tower of ϕ. In the present note we define the splitting pattern of ϕ as the set of the dimensions of its higher anisotropic kernels, or, equivalently, as the set of the integers $\dim(\phi_E)_{an}$, where E runs over all field extension of F.

Every nonnegative integer n is uniquely representable as an alternating sum of 2-powers:

$$n = 2^{p_0} - 2^{p_1} + 2^{p_2} - \cdots + (-1)^{r-1}2^{p_{r-1}} + (-1)^r 2^{p_r} \qquad (*)$$

with integers $p_0, p_1, \ldots, p_r$ satisfying $p_0 > p_1 > \cdots > p_{r-1} > p_r + 1 > 0$. For an excellent form, it was shown in [2] that its splitting pattern can be determined from $(*)$ in an easy way (see [2] for details). In particular, the height $\mathfrak{h}(\phi)$ of an anisotropic excellent quadratic form ϕ of dimension n is given by $\mathfrak{h}(\phi) = r$ if n is even and $\mathfrak{h}(\phi) = r - 1$ if n is odd.

Properties of anisotropic excellent quadratic forms suggest that the excellent forms should be a kind of "object of lowest complexity" among anisotropic forms of a given dimension. In some sense they can be considered as generalizations of Pfister forms to arbitrary dimensions. (Some aspects of their splitting behavior are described in [2, cor. 2.14], which seem to support the conjecture that anisotropic excellent forms provide examples of quadratic forms of minimal *canonical dimension* in the sense of [6], for quadratic forms of given dimension. An object of "highest complexity" for forms of given dimension clearly is the "generic quadratic form" of that dimension, whose height and canonical dimension are maximal for that dimension.) Hence the authors of [2] conjectured that, for example, the height of an anisotropic excellent form should be the minimal possible height for anisotropic forms of given dimension.

However, a proof of that conjecture so far could not be given with "classical" methods of the theory of quadratic forms. But it was observed by the first and third authors that the results in [4] (see Theorem 1.2 here) could be used to give a simple proof for forms of odd dimension, and, with a slight modification, for all forms of height ≤ 4. In order to give a general proof, the methods of [4] had to be refined, which was done by the second named author in [5], and, in addition, by proving the content of section 3. These ingredients allowed him to assemble a general proof. The combined methods yielded the proof given in this note.

We shall write $P(n)$ for the set $\{p_0, p_1, \ldots, p_r\}$ occurring in $(*)$ (note that p_r coincides with the 2-adic order $v_2(n)$ of n). Let us point out that, for $n = 0$, our representation is the empty sum, so that $P(0) = \emptyset$.

The *height* $\mathfrak{h}(n)$ of the integer n is the number of positive elements in $P(n)$ (so, $\mathfrak{h}(n)$ is the number $|P(n)|$ of all elements in the set $P(n)$ for even n, while $\mathfrak{h}(n) = |P(n)| - 1$ for odd n).

Let us state our Main Theorem:

Theorem 1.1. *For any anisotropic quadratic form ϕ over a field of characteristic $\neq 2$, one has*

$$\mathfrak{h}(\phi) \geq \mathfrak{h}(\dim \phi) \ .$$

For even $\dim \phi$, Theorem 1.1 is proved in section 4; for odd $\dim \phi$, see Corollary 2.4.

Sometimes, instead of the splitting pattern $\{\dim(\phi_E)_{an}\}$ of ϕ, it is more convenient to consider the set $\{\mathfrak{i}(\phi_E)\}$ (with E running over all field extensions of F), where $\mathfrak{i}(\phi_E)$ is the Witt index of ϕ_E. The elements of this set are called the *absolute higher Witt indices* of ϕ (the notion of the higher Witt indices has been introduced in [2]). Putting them in ascending order we assign to them the numbers $\mathfrak{j}_0 = \mathfrak{i}(\phi) < \mathfrak{j}_1 < \cdots < \mathfrak{j}_\mathfrak{h} = [(\dim \phi)/2]$ and refer to $\mathfrak{j}_i = \mathfrak{j}_i(\phi)$ as the *i-th (absolute) Witt index* of ϕ.

Sometimes it is more convenient to consider the *relative higher Witt indices* defined for $i \in [1,\ \mathfrak{h}]$ as $\mathfrak{i}_i(\phi) = \mathfrak{j}_i(\phi) - \mathfrak{j}_{i-1}(\phi)$ (for convenience, we also set $\mathfrak{i}_0(\phi) = \mathfrak{j}_0(\phi)$; this is the usual Witt index $\mathfrak{i}(\phi)$).

Illustration: Let ϕ be an anisotropic excellent form of dimension $n = 42$. Then $\mathfrak{h}(\phi) = \mathfrak{h}(n) = 5$ and the splitting pattern of ϕ is given by the set $\{\dim(\phi_E)_{an}\} = \{42, 22, 10, 6, 2, 0\}$. Accordingly the set $\{j(\phi_E)\}$ of absolute Witt indices is given by $\{0, 10, 16, 18, 20, 21\}$, and the sequence of relative Witt indices is given by $10, 6, 2, 2, 1$.

The splitting pattern $\{n_0 > n_1 > \cdots > n_\mathfrak{h}\}$ of ϕ is reconstructed from the relative higher Witt indices and the parity of $\dim \phi$ by the formulae $n_{i-1} = n_i + 2\mathfrak{i}_i$ ($i \in [1,\ \mathfrak{h}]$), taking into account that $n_\mathfrak{h}$ is 0 or 1 depending on the parity of $\dim \phi$ (note that all n_i have the same parity as $\dim \phi$). Vice versa, $\mathfrak{i}_i = (n_{i-1} - n_i)/2$ for $i \in [1,\ \mathfrak{h}]$.

The determination of possible values of splitting patterns is one of the main problems in the modern theory of quadratic forms. It is known that the splitting pattern of any quadratic form satisfies the following restriction:

Theorem 1.2 ([4])**.** *For any integer $i \in [1,\ \mathfrak{h}]$, there exists some m such that $2^m < n_{i-1}$, $\mathfrak{i}_i \in [1,\ 2^m]$, and $\mathfrak{i}_i \equiv n_{i-1} \pmod{2^m}$.*

Remark 1.3. There is the following interpretation of Theorem 1.2 in terms of the dyadic expansion of n_{i-1}: for an odd n_{i-1}, the integer $\mathfrak{i}_i$ is a proper binary suffix of n_{i-1}; for an even n_{i-1}, the integer $\mathfrak{i}_i$ is a proper binary suffix or a proper 2-power divisor of n_{i-1}.

Remark 1.4 ([8])**.** It is useful to keep in mind that if $\{n_0 > n_1 > \cdots > n_{\mathfrak{h}}\}$ with $\mathfrak{h} \geq 1$ is the splitting pattern of an anisotropic quadratic form ϕ over a field F, then $\{n_1 > n_2 > \cdots > n_{\mathfrak{h}}\}$ (and, consequently, also $\{n_i > n_{i+1} > \cdots > n_{\mathfrak{h}}\}$ for any $i \leq \mathfrak{h}$) is also the splitting pattern of some quadratic form, namely, of the quadratic form $(\phi_{F(\phi)})_{an}$. In particular, it suffices to announce Theorem 1.2 for $i = 1$ only.

It turns out (see Corollary 2.4), that Theorem 1.1 in odd dimensions is an easy consequence of Theorem 1.2. In even dimensions the situation is more complicated. So, before beginning with the proof of Theorem 1.1 in even dimension, we establish one more property of splitting patterns (see section 3).

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2. Relations between heights

In this section, the set $\{n_0 > n_1 > \cdots > n_{\mathfrak{h}}\}$ is the splitting pattern of some anisotropic quadratic form ϕ. The main lemma, due to the third author, is:

Lemma 2.1. *For an arbitrary integer $i \in [1,\ \mathfrak{h}]$, the difference $\mathfrak{d}(i) = \mathfrak{h}(n_{i-1}) - \mathfrak{h}(n_i)$ is as follows:*

(I) *If the dimension of ϕ is odd, then $|\mathfrak{d}(i)| = 1$.*

(II) *If the dimension of ϕ is even, then $|\mathfrak{d}(i)| \leq 2$ and*

- (+2) *if $\mathfrak{d}(i) = 2$, then $P(n_i) \subset P(n_{i-1})$ and $v_2(n_i) \geq v_2(n_{i-1}) + 2$;*
- (+1) *if $\mathfrak{d}(i) = 1$, then the difference $P(n_i) \setminus P(n_{i-1})$ is either empty or consists of one element p, in which case both integers $p-1$ and $p+1$ are in $P(n_{i-1})$;*
- (0) *if $\mathfrak{d}(i) = 0$, then the difference $P(n_i) \setminus P(n_{i-1})$ consists of one element p and either $p-1$ or $p+1$ is in $P(n_{i-1})$;*
- (-1) *if $\mathfrak{d}(i) = -1$, then the difference $P(n_i) \setminus P(n_{i-1})$ consists either of two elements $p-1$ and $p+1$ for some $p \in P(n_{i-1})$, or the difference consists of one element;*
- (-2) *if $\mathfrak{d}(i) = -2$, then the difference $P(n_i) \setminus P(n_{i-1})$ consists of two elements (that is, $P(n_i) \supset P(n_{i-1})$); moreover, one of these two elements is equal to $p+1$ for some $p \in P(n_{i-1})$.*

Proof. We write $p_0, p_1, \ldots, p_r$ for the elements of $P(n_{i-1})$ in descending order. We have $n_i = n_{i-1} - 2\mathfrak{i}_i$. On the other hand, by Theorem 1.2, there exists a nonnegative integer m such that $2^m < n_{i-1}$, $\mathfrak{i}_i \equiv n_{i-1} \pmod{2^m}$, and $1 \leq \mathfrak{i}_i \leq 2^m$. The condition $2^m < n_{i-1}$ means that $m < p_0$. Let us take the element p_s with maximal even s such that $m < p_s$.

If $m = p_s - 1$, then $\mathfrak{i}_i = 2^{p_s - 1} - 2^{p_{s+1}} + 2^{p_{s+2}} - \ldots$ and, therefore,

$$n_i = 2^{p_0} - 2^{p_1} + \cdots - 2^{p_{s-1}} + 2^{p_{s+1}} - 2^{p_{s+2}} + \cdots + (-1)^{r-1} 2^{p_r} .$$

If $s = r$ and $p_{r-1} + 1 = p_{r-2}$, we get that $P(n_i)$ is $P(n_{i-1})$ without p_r and p_{r-2}. Otherwise, $P(n_i)$ is $P(n_{i-1})$ without p_s.

Below in this proof, we are assuming that $m < p_s - 1$.

If $s = r$ (and $m < p_r - 1$), then $\mathfrak{i}_i = 2^m$ and $n_i = n_{i-1} - 2^{m+1}$. In the case when $m = p_r - 2$ we get that $P(n_i)$ is $P(n_{i-1})$ with p_r transformed to $p_r - 1$. In the case when $m < p_r - 2$ we get that $P(n_i)$ is $P(n_{i-1})$ with $m+1$ added.

Below in this proof, we are assuming that $s < r$.

If $p_s - 1 > m > p_{s+1}$, then $\mathfrak{i}_i = 2^m - 2^{p_{s+1}} + 2^{p_{s+2}} - \dots$ and, therefore,

$$n_i = 2^{p_0} - 2^{p_1} + \dots - 2^{p_{s-1}} + 2^{p_s} - 2^{m+1} + 2^{p_{s+1}} - 2^{p_{s+2}} + \dots + (-1)^{r+1} 2^{p_r} \ .$$

This is the right representation of n_i (and, therefore, $P(n_i)$ is $P(n_{i-1})$ with $m+1$ added).

It remains to consider the case with $m \le p_{s+1}$ (while $s < r$). For this case, let us first assume that $s = r-1$. Then $\mathfrak{i}_i = 2^m$ and $n_i = n_{i-1} - 2^{m+1}$. If $m < p_r - 2$, then $P(n_i)$ is $P(n_{i-1})$ with $p_r + 1$ and $m+1$ added. If $m = p_r - 2$, then $P(n_i)$ is $P(n_{i-1})$ with p_r removed and $p_r + 1$ and $p_r - 1$ added. In the case with $m = p_r - 1$, one has: if $p_{r-1} > p_r + 2$, then $P(n_i)$ is $P(n_{i-1})$ with p_r removed and $p_r + 1$ added; if $p_{r-1} = p_r + 2$, then $P(n_i)$ is $P(n_{i-1})$ with p_r and p_{r-1} removed while $p_r + 1$ added. Finally, in the case with $m = p_r$, we have: if $p_{r-1} = p_r + 2$, then $P(n_i)$ is $P(n_{i-1})$ without p_{r-1}, otherwise $P(n_i)$ is $P(n_{i-1})$ with $p_r + 2$ added.

We finish the proof considering the case with $m \le p_{s+1}$ and $s < r - 1$. We have: $\mathfrak{i}_i = 2^{p_{s+2}} - 2^{p_{s+3}} + \dots + (-1)^r 2^{p_r}$ and

$$n_i = 2^{p_0} - 2^{p_1} + \dots + 2^{p_s} - 2^{p_{s+1}+1} + 2^{p_{s+1}} - 2^{p_{s+2}} + \dots + (-1)^{r+1} 2^{p_r} \ .$$

So, if $p_s > p_{s+1} + 1$, then $P(n_i)$ is $P(n_{i-1})$ with $p_{s+1} + 1$ added; otherwise $P(n_i)$ is $P(n_{i-1})$ with p_s removed. □

Corollary 2.2. *For odd-dimensional ϕ, and for any $i \in [1,\ \mathfrak{h}]$, one has*

$$\mathfrak{h}(n_{i-1}) - \mathfrak{h}(n_i) \le 1 \ .$$

□

Remark 2.3. In fact, for odd-dimensional ϕ, we have $\mathfrak{h}(n_i) = \mathfrak{h}(n_{i-1}) \pm 1$, which is true as well for an even-dimensional ϕ, if $\mathfrak{i}_i$ is a proper binary suffix (and not a proper 2-power divisor) of n_{i-1} (see Remark 1.3), but we will not use this fact explicitly during our proof.

Corollary 2.4. *For the height $\mathfrak{h}(\phi)$ of an arbitrary anisotropic quadratic form ϕ of odd dimension n, one has $\mathfrak{h}(\phi) \ge \mathfrak{h}(n)$ (that is, Theorem 1.1 holds in odd dimension).*

Proof. We have: $\mathfrak{h}(n_{\mathfrak{h}}) = 0$ (simply because $n_{\mathfrak{h}} = 1$) and (by Corollary 2.2) $\mathfrak{h}(n_{i-1}) - \mathfrak{h}(n_i) \le 1$ for every $i \in [1,\ \mathfrak{h}]$. Therefore, $\mathfrak{h}(n_0) \le \mathfrak{h}$. Since ϕ is assumed to be anisotropic, $n = n_0$, and we are done. □

3. One more property of splitting patterns

In this section, $\{n_0 > n_1 > \dots > n_{\mathfrak{h}}\}$ with $\mathfrak{h} \ge 1$ is the splitting pattern of an anisotropic quadratic form ϕ of *even* dimension $n = n_0$. We write X for the projective quadric given by ϕ (see [5] for the definition of the reduced modulo 2 Chow group $\bar{\mathrm{Ch}}$ of a variety and for the terminology concerning algebraic cycles on $X^2 = X \times X$).

Proposition 3.1 ([5, th. 3.3 and th. 5.1])**.** *Let $\alpha \in \bar{\mathrm{Ch}}(X^2)$ be the minimal cycle containing $h^0 \times l_0$. Assume that α also contains $h^q \times l_q$ for some integer $q \in [1,\ [n/2])$ and take the minimal q with this property. Let $i \in [1,\ \mathfrak{h})$ be the maximal integer such* *that* $\mathfrak{j}_i \leq q$ *(then in fact* $q = \mathfrak{j}_i$ *by* [5, prop. 3.1]*). Then* $v_2(\mathfrak{i}_{i+1}) \geq v_2(\mathfrak{i}_1)$*. Moreover, if* $v_2(\mathfrak{i}_2 + \cdots + \mathfrak{i}_i) \geq v_2(\mathfrak{i}_1) + 2$ *(in particular, if* $i = 1$*, in which case the sum* $\mathfrak{i}_2 + \cdots + \mathfrak{i}_i$ *is understood to be* 0*, while by definition* $v_2(0) = \infty$*), then* $v_2(\mathfrak{i}_{i+1}) \leq v_2(\mathfrak{i}_1) + 1$ *so that* $v_2(\mathfrak{i}_{i+1})$ *is equal to* $v_2(\mathfrak{i}_1)$ *or* $v_2(\mathfrak{i}_1) + 1$ *in this case.*

A situation in which the condition on α in Proposition 3.1 is always satisfied, is described in:

Theorem 3.2 ([3], for a more elementary proof see [7])**.** *If* $n_1 + \mathfrak{i}_1$ *(*$= n_0 - \mathfrak{i}_1$*) is not a* 2 *power, then the minimal cycle on* X^2 *containing* $h^0 \times l_0$ *also contains* $h^q \times l_q$ *for some* $q \in [1,\ [n/2])$*.*

The main result of the current section, which is the basement of the proof of Theorem 1.1 for even dimension, is:

Theorem 3.3. *Assume that* $v_2(n_i) \geq v_2(n_{i-1}) + 2$ *for some* $i \in [1,\ \mathfrak{h})$*. Then the open interval* $(i,\ \mathfrak{h})$*, contains an integer* $i' \in (i,\ \mathfrak{h})$ *such that* $|v_2(n_{i'}) - v_2(n_{i-1})| \leq 1$*.*

Proof. It suffices to consider the case of $i = 1$ (see Remark 1.4). Note that $\mathfrak{h} \geq 2$ (otherwise $[1,\ \mathfrak{h}) = \emptyset$). We set $p = v_2(n_0)$. By assumption, we have $v_2(n_1) \geq p + 2$. Therefore $v_2(\mathfrak{i}_1) = p - 1$. Clearly, $\mathfrak{i}_1 + n_1$ is not a power of 2; therefore, by Theorem 3.2, the minimal cycle on X^2 containing $h^0 \times l_0$ also contains $h^q \times l_q$ for some $q \in [1,\ [n/2])$. Let j be the maximal integer satisfying $\mathfrak{j}_j \leq q$. We are going to show that $v_2(n_j)$ or $v_2(n_{j+1})$ is in $[p-1,\ p+1]$ for this j. Then we can take $i' = j$ in the first case and $i' = j+1$ in the second case. Note that $i' \neq 1, \mathfrak{h}$ (because of $v_2(n_1) \geq p + 2$, while $v_2(n_{\mathfrak{h}}) = \infty$; we recall that $\dim \phi$ is even in this section).

By the first part of Proposition 3.1, $v_2(\mathfrak{i}_{j+1}) \geq p - 1$. Consequently, by Theorem 1.2, $v_2(n_j) \geq p-1$ as well. Since $n_1 = 2(\mathfrak{i}_2 + \cdots + \mathfrak{i}_j) + n_j$, it follows that $v_2(\mathfrak{i}_2 + \cdots + \mathfrak{i}_j) + 1 \geq p-1$. If $v_2(\mathfrak{i}_2 + \cdots + \mathfrak{i}_j) < p+1$, then $v_2(n_j) = v_2(\mathfrak{i}_2 + \cdots + \mathfrak{i}_j) + 1 \in [p-1,\ p+1]$. So, it remains to consider the case when $v_2(\mathfrak{i}_2 + \cdots + \mathfrak{i}_j) \geq p+1$, where we may apply the second part of Proposition 3.1 as well, stating that $v_2(\mathfrak{i}_{j+1}) \in \{p-1, p\}$. Since now $v_2(n_j) \geq p+2$ while $n_j = 2\mathfrak{i}_{j+1} + n_{j+1}$, it follows that $v_2(n_{j+1}) = v_2(\mathfrak{i}_{j+1}) + 1 \in \{p, p+1\}$. $\square$

Corollary 3.4. *Under the condition of Theorem 3.3, we set* $p = v_2(n_{i-1})$ *(note that* $p \in P(n_{i-1})$*, while* $\min P(n_i) \geq p + 2$*). Then there exists* $i' \in (i,\ \mathfrak{h})$ *such that the set* $P(n_{i'})$ *contains an element* p' *with* $|p' - p| \leq 1$*.*

Proof. Take i' as in Theorem 3.3 and set $p' = v_2(n_{i'})$. $\square$

4. Proof of the Main Theorem

Proof of Theorem 1.1. We only need to prove Theorem 1.1 for even-dimensional forms (see Corollary 2.4). So, let $\{n_0 > n_1 > \cdots > n_{\mathfrak{h}}\}$ with $\mathfrak{h} \geq 1$ be the splitting pattern of an anisotropic quadratic form ϕ of even dimension $n = n_0$.

Let H be the set $\{1, 2, \ldots, \mathfrak{h}\}$ of $\mathfrak{h} = \mathfrak{h}(\phi)$ elements. For any $i \in H$, we let $\mathfrak{d}(i) = \mathfrak{h}(n_{i-1}) - \mathfrak{h}(n_i)$. Let C be the subset of H consisting of all $i \in H$ such that $\mathfrak{d}(i) = 2$ (we recall that $\mathfrak{d}(i) \leq 2$ for any $i \in H$ by Lemma 2.1). We prove Theorem 1.1 by constructing a map $f : C \to H$ such that $\mathfrak{d}(j) \leq 1 - |f^{-1}(j)|$ for any $j \in f(C)$ (in particular, $f(C) \subset H \setminus C$). Since the subsets $f^{-1}(j) \cup \{j\}$, where j runs over $H \setminus C$, are disjoint and cover H, while the average value of $\mathfrak{d}$ on each such subset is ≤ 1, the average value of $\mathfrak{d}$ on H is ≤ 1, and the statement of Theorem 1.1 follows by the same argument as the proof of the odd-dimensional case (see Corollary 2.4).

To define f, let us consider any $i \in C$. By Lemma 2.1, $v_2(n_i) \geq v_2(n_{i-1})+2$. Therefore, by Corollary 3.4, there exists $i' \in (i,\ \mathfrak{h})$ such that the set $P(n_{i'})$ contains an element p' satisfying $|p'-p| \leq 1$ for $p = v_2(n_{i-1})$. Taking the minimal i' with this property, we set $f(i) = i'$. Also we define $g(i)$ to be the minimal element of $P(n_{f(i)})$ such that $|g(i)-p| \leq 1$.

The map f is constructed. It only remains to check the stated properties of f.

First of all we note that by the very definition of f, for any $j \in f(C)$, the difference of sets $P(n_j) \setminus P(n_{j-1})$ is nonempty (and so, $\mathfrak{d}(j) \neq 2$ by Item II+2 of Lemma 2.1). Moreover, this difference contains an element p such that $\{p-1,\ p+1\} \not\subset P(n_{j-1})$ (and so, $\mathfrak{d}(j) \neq 1$ by Item II+1 of Lemma 2.1). Therefore, by Lemma 2.1, $\mathfrak{d}(j) \leq 0$.

Now let j be an element of $f(C)$ with $|f^{-1}(j)| \geq 2$. Let $i_1 < i_2$ be two different elements of $f^{-1}(j)$. Note that $i_1 < i_2 < j$ and $|p_2 - p_1| > 1$ (where $p_1 = v_2(n_{i_1-1})$, $p_2 = v_2(n_{i_2-1})$) by definition of $f(i_1)$. We are going to show that $\mathfrak{d}(j) \leq -1$. As we already know, $\mathfrak{d}(j) \leq 0$. If $\mathfrak{d}(j) = 0$, then by Item II-0 of Lemma 2.1 the difference $P(n_j) \setminus P(n_{j-1})$ consists of one element p' and either $p'-1$ or $p'+1$ is in $P(n_{j-1})$. Since the difference $P(n_j) \setminus P(n_{j-1})$ consists of one element p', we have $p' = g(i_1) = g(i_2)$, and it follows that $\{p_1, p_2\} = \{p'-1, p'+1\}$ (in particular, $|p_2 - p_1| = 2$). Consequently, the set $P(n_{j-1})$ contains neither $p'-1$ nor $p'+1$, a contradiction.

Now let j be an element of $f(C)$ with $|f^{-1}(j)| \geq 3$. Let i_1, i_2, i_3 be three different elements of $f^{-1}(j)$. The equalities $g(i_1) = g(i_2) = g(i_3)$ do not hold simultaneously (simply because the three conditions $|p_2 - p_1| = 2$, $|p_3 - p_2| = 2$, and $|p_1 - p_3| = 2$ can not be satisfied simultaneously). On the other hand, the difference $P(n_j) \setminus P(n_{j-1})$ can have at most two elements. Therefore we may assume that $g(i_1) = g(i_2)$ and that $g(i_3)$ is different from $g(i_1) = g(i_2)$. We set $p' = g(i_1) = g(i_2)$. We are going to show that $\mathfrak{d}(j) = -2$. As we already know, $\mathfrak{d}(j) \leq -1$. If $\mathfrak{d}(j) = -1$, then by Item II-1 of Lemma 2.1 the difference $P(n_j) \setminus P(n_{j-1})$ consists of $\tilde{p}-1$ and $\tilde{p}+1$ for some $\tilde{p} \in P(n_{j-1})$. However p' is neither $\tilde{p}-1$ nor $\tilde{p}+1$, a contradiction.

We finish the proof by showing that $|f^{-1}(j)|$ is never ≥ 4. Indeed, if $|f^{-1}(j)| \geq 4$, then the difference $P(n_j) \setminus P(n_{j-1})$ contains two elements p' and p'' such that none of $p' \pm 1$ and of $p'' \pm 1$ is in $P(n_{j-1})$, contradicting Lemma 2.1. □

5. Final Example

We close with an illustration: Let ϕ be an anisotropic quadratic form of dimension n over a field of characteristic not 2. Above we proved that $\mathfrak{h}(\phi) \in [\mathfrak{h}(n), [n/2]]$. For n odd one can be more specific. It is a nice exercise to deduce that $\mathfrak{h}(\phi) \equiv \mathfrak{h}(n) \mod 2$ for forms of odd dimension n. So,

$$\mathfrak{h}(\phi) \in \{\mathfrak{h}(n), \mathfrak{h}(n)+2, \ldots, (n-1)/2\}.$$

Example: For $n = 21$ we have $\mathfrak{h}(n) = 4$, hence $\mathfrak{h}(\phi) \in \{4, 6, 8, 10\}$ and, in fact, each of the four values is the height $h(\phi)$ of some 21-dimensional anisotropic form ϕ, cf. [11].

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