Witt kernels of quadratic forms for purely inseparable multiquadratic extensions in characteristic 2

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Abstract. The aim of this note is to give a complete answer to the hyperbolicity of nonsingular quadratic forms over purely inseparable multiquadratic extensions in characteristic 2. This completes partial computations in [**MM**].

1. Introduction

Let F denote a field of characteristic 2, and let $W_q(F)$ denote the Witt group of nonsingular quadratic forms over F [**B2**]. For a field extension K/F, there exists a group homomorphism $W_q(F) \longrightarrow W_q(K)$ induced by the inclusion $F \subset K$. An important problem in the algebraic theory of quadratic forms is to compute the kernel $W_q(K/F)$ of this homomorphism.

Let us recall some examples of field extensions K/F where $W_q(K/F)$ was computed. It is well-known that $W_q(K/F)$ is trivial for K purely transcendental over F. Recently in [**L2**] some general results have been proved on $W_q(K/F)$ when K is given by the function field of a quadratic form. Moreover, very few results are known for K/F algebraic of finite degree, and the case of multiquadratic extensions aroused a lot of interest. In characteristic 2, it is well-known that a quadratic extension K/F is either separable and thus it is given by $K = F(\wp^{-1}(\alpha))$ where $\wp : K \longrightarrow K$ is the homomorphism defined by $\wp(x) = x^2 + x$, or inseparable and thus it is given by $K = F(\sqrt{\alpha})$ for some $\alpha \in F$.

For $\alpha_1, \alpha_2 \in F$, we know the kernels for bi-quadratic extensions:

$$W_q\left(F(\wp^{-1}(\alpha_1), \wp^{-1}(\alpha_2))/F\right) = W(F) \otimes [1, \alpha_1] + W(F) \otimes [1, \alpha_2] \tag{1}$$

$$W_q\left(F(\wp^{-1}(\alpha_1), \sqrt{\alpha_2})/F\right) = W(F) \otimes [1, \alpha_1] + \langle 1, \alpha_2\rangle \otimes W_q(F) \tag{2}$$

$$W_q\left(F(\sqrt{\alpha_1}, \sqrt{\alpha_2})/F\right) = \langle 1, \alpha_1\rangle \otimes W_q(F) + \langle 1, \alpha_2\rangle \otimes W_q(F) \tag{3}$$

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where $[\alpha, \beta]$ denotes the quadratic form $\alpha x^2 + xy + \beta y^2$; $W(F)$ is the Witt ring of regular symmetric bilinear forms over F; $\otimes$ is the action of $W(F)$ on $W_q(F)$ [**B2**]; and $\langle \alpha_1, \cdots, \alpha_n \rangle$ is the bilinear form $\sum_{i=1}^n \alpha_i x_i y_i$ for $\alpha_1, \cdots, \alpha_n \in F^*$.

The kernel (1) is due to Baeza [**B2**, Cor. 4.16, Page 128]. The "mixed" kernel (2) is due to Ahmad [**A**]. The kernel (3) is also due to Baeza in the case of quadratic extensions [**B2**, Lem. 4.3, Page 182], and due to Mammone and Moresi in the case of bi-quadratic extensions [**MM**, Th. 2(i)]. Moreover, Mammone and Moresi proved that, in general, the kernel (1) does not generalize to separable tri-quadratic extensions [**MM**, Proposition 1], and the kernel (3) generalizes to inseparable multiquadratic extensions provided that the $W(F)$-submodule $I^3W_q(F)$ of $W_q(F)$ is trivial [**MM**, Th. 2(ii)] (see below for the definition of this submodule).

Our aim in this note is to prove that the kernel (3) generalizes to inseparable multiquadratic extensions without additional hypothesis on the ground field F. More precisely, we will prove the following theorem:

THEOREM 1. *Let F be a field of characteristic 2. Then for any scalars $\alpha_1, \cdots, \alpha_n \in F$ ($n \geq 1$), we have:*

$$W_q\left(F(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_n})/F\right) = \langle 1, \alpha_1 \rangle \otimes W_q(F) + \cdots + \langle 1, \alpha_n \rangle \otimes W_q(F).$$

Let us make a comment on the situation when the characteristic is different from 2. Obviously, in this case a quadratic extension is of the form $F(\sqrt{\alpha})$ for some $\alpha \in F^*$. An analogue result of the kernel (1) has been proved by Elman, Lam and Wadsworth [**ELW**, Th. 2.10, page 137]. Still in this case, Elman, Lam, Tignol and Wadsworth constructed an example of a field for which the kernel (1) does not generalize to the case of tri-quadratic extensions [**ELTW**, § 5, Page 1142].

For the proof of Theorem 1, we will begin by a generalization to the case of purely inseparable multiquadratic extensions of a recent result by Aravire and Baeza [**AB**] concerning the behaviour of differential forms under inseparable quadratic extensions (Proposition 1). We also use a result by Kato [**K**] which establishes the connection between nonsingular quadratic forms and differential forms of F over F^2. We first reduce the proof of Theorem 1 to the case of a field which admits a finite 2-basis, and then give a proof in this case.

We suppose that the reader is familiar with the algebraic theory of quadratic forms in characteristic 2. For any unexplained notation and terminology we refer to [**B2**] or [**HL**]. However, some definitions used here are taken from [**HL**], and some of them differ from those introduced in [**B2**]. For this reason, we fix again some rappels: A nonsingular quadratic form over F is just an orthogonal sum of binary quadratic forms of type $[a, b]$ with $a, b \in F$. The hyperbolic plane, denoted by $\mathbb{H}$, is the quadratic form $[0, 0]$. For $n \geq 1$, an n-fold Pfister form is a nonsingular quadratic form of type $\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_{n-1} \rangle \otimes [1, a_n]$ for $a_1 \neq 0, \cdots, a_{n-1} \neq 0, a_n \in F$. We denote it by $\langle\langle a_1, \cdots, a_n]]$. Let P_nF (resp. GP_nF) denote the set of n-fold Pfister forms up to isometry (resp. the set $\{\alpha\pi \mid \alpha \in F^*, \pi \in P_nF\}$). We denote by $I^nW_q(F)$ the $W(F)$-submodule of $W_q(F)$ generated by GP_nF.

Let Ω_F^n denote the vector space of n-differential forms of F over F^2 (with $\Omega_F^0 = F$), and let $d : \Omega_F^n \longrightarrow \Omega_F^{n+1}$ denote the differential operator defined by:

$$d\left(\alpha da_1 \wedge da_2 \wedge \cdots \wedge da_n\right) = d\alpha \wedge da_1 \wedge da_2 \wedge \cdots \wedge da_n.$$

For $\{b_i \mid i \in I\}$ a 2-basis of F, and after choosing an ordering on the set I, we get that $\left\{\frac{db_{i_1}}{b_{i_1}} \wedge \cdots \wedge \frac{db_{i_n}}{b_{i_n}} \mid i_1 < \cdots < i_n\right\}$ is an F-basis of Ω_F^n. A result by Cartier [**C**] asserts the existence of a homomorphism $\wp_n : \Omega_F^n \longrightarrow \Omega_F^n/d\Omega_F^{n-1}$ given on generators by:

$$\wp_n\left(x\frac{db_{i_1}}{b_{i_1}} \wedge \cdots \wedge \frac{db_{i_n}}{b_{i_n}}\right) = \overline{(x^2-x)\frac{db_{i_1}}{b_{i_1}} \wedge \cdots \wedge \frac{db_{i_n}}{b_{i_n}}}.$$

We denote by $\mathcal{H}^nF$ (*resp.* $\mathcal{Q}_n(F)$) the cokernel of $\wp_n$ (*resp.* the quotient $I^nW_q(F)/I^{n+1}W_q(F)$).

An important result that we will use in this note is due to Kato [**K**], and asserts that for any integer $n \geq 1$, there exists an isomorphism $k_n : \mathcal{Q}_n(F) \longrightarrow \mathcal{H}^{n-1}F$ given on generators by:

$$k_n\left(\overline{\langle\langle a_1, \cdots, a_n]]}\right) = \overline{a_n\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_{n-1}}{a_{n-1}}}.$$

2. A preliminary result

To prepare the proof of Theorem 1, we begin by a generalization of [**AB**, Lem. 2.18] to the case of purely inseparable multiquadratic extensions:

Proposition 1. *Let F be a field of characteristic 2 and $\alpha_1, \cdots, \alpha_m \in F^*$ $(m \geq 1)$. Then we have:*

$$\mathrm{Ker}\,(\mathcal{H}^nF \longrightarrow \mathcal{H}^nF(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_m})) = \sum_{i=1}^{m} \overline{\Omega_F^{n-1} \wedge d\alpha_i}.$$

Proof. Without loss of generality we may suppose that $[F(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_m}) : F] = 2^m$. We proceed by induction on m. The case $m = 1$ has been proved by Aravire and Baeza [**AB**, Lem. 2.18]. Suppose $m \geq 2$ and that the proposition is true as soon as we have a purely inseparable multiquadratic extension of degree $< 2^m$ over a field of characteristic 2. Let $\omega \in \Omega_F^n$ be such that $\overline{\omega} \in \mathrm{Ker}(\mathcal{H}^nF \longrightarrow \mathcal{H}^nF(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_m}))$. Since $\overline{\omega}_{F(\sqrt{\alpha_1})} \in \mathrm{Ker}(\mathcal{H}^nF(\sqrt{\alpha_1}) \longrightarrow \mathcal{H}^nF(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_m}))$, it follows from induction hypothesis that

$$\overline{\omega}_{F(\sqrt{\alpha_1})} = \sum_{i=2}^{m} \overline{\omega_i \wedge d\alpha_i} \tag{4}$$

for suitable $\omega_2, \cdots, \omega_m \in \Omega_{F(\sqrt{\alpha_1})}^{n-1}$.

Claim: In the equality (4), we may suppose $\omega_2, \cdots, \omega_m$ in Ω_F^{n-1}, and thus again by induction hypothesis applied to the field $F(\sqrt{\alpha_1})$ we may conclude.

Hence, it suffices to justify the claim. Since $\alpha_1 \notin F^{*2}$, we may choose a 2-basis B of F such that $B \cup \{\sqrt{\alpha_1}\}$ is a 2-basis of $F(\sqrt{\alpha_1})$.

Let us now fix $i \in \{2, \cdots, m\}$. We have $\omega_i = \sum_{\text{finite}} x_j^i \omega_j^i$ with $x_j^i \in F(\sqrt{\alpha_1})^*$, and

$$\omega_j^i = dc_{j_1}^i \wedge \cdots \wedge dc_{j_{n-1}}^i$$

not zero with $c_{j_1}^i, \cdots, c_{j_{n-1}}^i \in B \cup \{\sqrt{\alpha_1}\}$ $(j_1 < \cdots < j_{n-1})$.

• On the one hand, since the equality (4) is taken modulo $\wp_n(\Omega_{F(\sqrt{\alpha_1})}^n)$, we may suppose, after changing if necessary $x_j^i \omega_j^i$ by $(x_j^i)^2 \omega_j^i$, that $x_j^i \in F^*$.

• One the other hand, if there exists an indice j_k such that in the expression of ω_j^i (see above) we have $c^i_{j_k} = \sqrt{\alpha_1}$, says $j_k = j_1$, then $c^i_{j_2}, \cdots, c^i_{j_{n-1}} \in B$ since ω_j^i is not zero. But by using the differential operator d we get:

$$d\left[\left(x_j^i\sqrt{\alpha_1}\right)dc^i_{j_2}\wedge\cdots\wedge dc^i_{j_{n-1}}\wedge d\alpha_i\right] = \begin{array}{l} x_j^i\omega_j^i \wedge d\alpha_i \ + \\ \sqrt{\alpha_1}\left(dx_j^i\wedge dc^i_{j_2}\wedge\cdots\wedge dc^i_{j_{n-1}}\wedge d\alpha_i\right). \end{array}$$

Hence, in $\mathcal{H}^nF(\sqrt{\alpha_1})$ we get

$$\begin{array}{rcl} \overline{x_j^i\omega_j^i\wedge d\alpha_i} & \overset{\mathbf{(1)}}{=} & \overline{(\sqrt{\alpha_1})\left(dx_j^i\wedge dc^i_{j_2}\wedge\cdots\wedge dc^i_{j_{n-1}}\wedge d\alpha_i\right)} \\ & \overset{\mathbf{(2)}}{=} & \overline{\alpha_1\left(dx_j^i\wedge dc^i_{j_2}\wedge\cdots\wedge dc^i_{j_{n-1}}\wedge d\alpha_i\right)} \end{array}$$

where for **(1)** (*resp.* for **(2)**) we proceed modulo $d\Omega^{n-1}_{F(\sqrt{\alpha_1})}$ (*resp.* modulo $\wp_n(\Omega^n_{F(\sqrt{\alpha_1})})$). Hence the claim. $\square$

As a corollary of Kato's result (cited before) and Proposition 1 we get the following:

Corollary 1. *With the same notation as in Proposition 1, we have:*

$$\mathrm{Ker}\left(\mathcal{Q}_n(F)\longrightarrow\mathcal{Q}_n\left(F(\sqrt{\alpha_1},\cdots,\sqrt{\alpha_m})\right)\right) = \sum_{i=1}^m \overline{\langle 1,\alpha_i\rangle\, I^{n-1}W_q(F)}.$$

Proof. Set $L = F(\sqrt{\alpha_1},\cdots,\sqrt{\alpha_m})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_n(F) & \longrightarrow & \mathcal{Q}_n(L) \\ \downarrow & & \downarrow \\ \mathcal{H}^{n-1}(F) & \longrightarrow & \mathcal{H}^{n-1}(L) \end{array}$$

where the vertical arrows are given by the isomorphism k_n (k_n as before), and the horizontal arrows are induced by the inclusion $F\subset L$. Hence, $x\in\mathrm{Ker}(\mathcal{Q}_n(F)\longrightarrow\mathcal{Q}_n(L))$ implies that $k_n(x)\in\mathrm{Ker}(\mathcal{H}^{n-1}F\longrightarrow\mathcal{H}^{n-1}L)$. By Proposition 1 we deduce that $k_n(x)\in\sum_{i=1}^m\overline{\Omega_F^{n-2}\wedge d\alpha_i}$, and by the isomorphism k_n it is clear that $x\in\sum_{i=1}^m\overline{\langle 1,\alpha_i\rangle\, I^{n-1}W_q(F)}$. Obviously, $\sum_{i=1}^m\overline{\langle 1,\alpha_i\rangle\, I^{n-1}W_q(F)}\subset\mathrm{Ker}(\mathcal{Q}_n(F)\longrightarrow\mathcal{Q}_n(L))$. $\square$

3. Proof of Theorem 1

With the same notation as in the theorem, let $\varphi\in W_q\left(F(\sqrt{\alpha_1},\cdots,\sqrt{\alpha_n})/F\right)$. Set $L=F(\sqrt{\alpha_1},\cdots,\sqrt{\alpha_n})$ and $\varphi=[a_1,b_1]\perp\cdots\perp[a_m,b_m]$. We will give the proof in two steps.

1) Reduction to the case of a field with a finite 2-basis: Let $\mathbb{F}_2$ be the finite field with two elements and $F_0=\mathbb{F}_2(\alpha_1,\cdots,\alpha_n;a_1,b_1,\cdots,a_m,b_m)$. The quadratic form φ is defined over the subfield F_0 of F. Since φ_L is hyperbolic, and after viewing L^{2m} as the underlying vector space of φ_L, there exist vectors $v_1,\cdots,v_m\in L^{2m}$, linearly independent over L, such that $\varphi_L(v_i)=0$ and $B_{\varphi_L}(v_i,v_j)=0$ where B_{φ_L} is the symmetric bilinear form associated to

φ_L. Set $v_i = (w_1^i, \cdots, w_{2m}^i)$ with $w_1^i, \cdots, w_{2m}^i \in L$ $(1 \leq i \leq m)$. For any $(i,k) \in \{1, \cdots, m\} \times \{1, \cdots, 2m\}$ set

$$w_k^i = \sum_{\text{finite}} c_{i_1,\cdots,i_n}^{i,k} (\sqrt{\alpha_1})^{i_1} \ldots (\sqrt{\alpha_n})^{i_n}$$

with $i_j \in \{0,1\}$ and $c_{i_1,\cdots,i_n}^{i,k} \in F$. Now let

$$K = F_0 \left(c_{i_1,\cdots,i_n}^{i,k} \mid i = 1, \cdots, m;\ k = 1, \cdots, 2m \right)$$

and

$$K' = K(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_n}).$$

Since $K' \subset L$, the vectors $v_1, \cdots, v_m$ are linearly independent over K', and thus $\varphi_{K'}$ is hyperbolic since $v_1, \cdots, v_m \in (K')^{2m}$. Clearly, K has a finite 2-basis since it is finitely generated over $\mathbb{F}_2$. Moreover, we may suppose that φ is not hyperbolic over F, and thus it is not hyperbolic over K. Since an element of $\sum_{i=1}^n \langle 1, \alpha_i \rangle W_q(K)$ can be considered as an element of $\sum_{i=1}^n \langle 1, \alpha_i \rangle W_q(F)$ after extending scalars to F, we see that the proof can be reduced to the case of a field with a finite 2-basis.

2) Case where F has a finite 2-basis: If n_0 is the number of elements of a 2-basis, we deduce that the space Ω_F^i is trivial for $i > n_0$. By Kato's result and the Hauptsatz of Arason and Pfister [**B1**, Satz 4.1], the group $I^i W_q(F)$ is also trivial for $i > n_0 + 1$. Let $r \geq 1$ be such that $\varphi \in I^r W_q(F)$. Since φ_L is hyperbolic, we deduce that $\overline{\varphi} \in \mathrm{Ker}(\mathcal{Q}_r(F) \longrightarrow \mathcal{Q}_r(L))$. It follows from Corollary 1 that $\overline{\varphi} \in \sum\limits_{i=1}^n \overline{\langle 1, \alpha_i \rangle I^{r-1} W_q(F)}$. Let $\rho_1, \cdots, \rho_n \in I^{r-1} W_q(F)$ be such that

$$\varphi \perp \sum_{i=1}^n \langle 1, \alpha_i \rangle \otimes \rho_i \in I^{r+1} W_q(F) \tag{5}$$

It is clear that the form $\varphi' := \varphi \perp \sum\limits_{i=1}^n \langle 1, \alpha_i \rangle \otimes \rho_i$ is hyperbolic over L. If φ' is hyperbolic over F, then we are done. If not, we reproduce the same argument as above for φ' to get $\varphi' \perp \sum\limits_{i=1}^n \langle 1, \alpha_i \rangle \otimes \rho_i' \in I^{r+2} W_q(F)$ for some quadratic forms $\rho_1', \cdots, \rho_n' \in I^r W_q(F)$, i.e. $\varphi'' := \varphi \perp \sum\limits_{i=1}^n \langle 1, \alpha_i \rangle \otimes (\rho_i' \perp \rho_i) \in I^{r+2} W_q(F)$. If φ'' is hyperbolic over F, then $\varphi \in \sum\limits_{i=1}^n \langle 1, \alpha_i \rangle \otimes W_q(F)$ and we are done. If not, we continue the process in order to get a quadratic form $\varphi \perp \sum\limits_{i=1}^n \langle 1, \alpha_i \rangle \otimes \delta_i \in I^k W_q(F)$ for some $\delta_1, \cdots, \delta_n \in W_q(F)$ and $k > n_0 + 1$, and thus to get a hyperbolic quadratic form. $\square$

4. A question

Before we formulate a general question, let us recall that the function field of a symmetric bilinear form B, denoted by $F(B)$, is the function field of the quadratic form $\widetilde{B}$ defined by $\widetilde{B}(v) = B(v,v)$ for $v \in V$, where V denotes the underlying vector space of B ($\widetilde{B}$ is uniquely determined by the isometry class of B). In particular, the function field of the bilinear form $\langle 1, \alpha \rangle$, $\alpha \in F^*$, is the field $F(x)(\sqrt{\alpha})$ for x a variable over F. Hence, Theorem 1 describes the kernels $W_q\left(F(B_1) \cdots (B_n)/F\right)$

for $B_i = \langle 1, \alpha_i \rangle$ $(1 \leq i \leq n)$. So in view of this it is natural to ask the following question:

QUESTION 1. *Let F be a field of characteristic 2, and let $B_1, \cdots, B_n$ be symmetric bilinear Pfister forms of dimension ≥ 2 $(n \geq 1)$. Is it true that the kernel $W_q(F(B_1) \cdots (B_n)/F)$ equals $B_1 \otimes W_q(F) + \cdots + B_n \otimes W_q(F)$?*

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