

GALOIS MODULE STRUCTURE OF GALOIS COHOMOLOGY

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ABSTRACT. Let F be a field containing a primitive p th root of unity, and let U be an open normal subgroup of index p of the absolute Galois group G_F of F . We determine the structure of the cohomology group $H^n(U, \mathbb{F}_p)$ as an $\mathbb{F}_p[G_F/U]$ -module for all $n \in \mathbb{N}$. Previously this structure was known only for $n = 1$, and until recently the structure even of $H^1(U, \mathbb{F}_p)$ was determined only for F a local field, a case settled by Borevič and Faddeev in the 1960s.

Let F be a field containing a primitive p th root of unity ξ_p . Let G_F be the absolute Galois group of F , U an open normal subgroup of G_F of index p , and $G = G_F/U$. Let E be the fixed field of U in the separable closure F_{sep} of F . Fix $a \in F$ such that $E = F(\sqrt[p]{a})$, and let $\sigma \in G$ satisfy $\sqrt[p]{a}^{\sigma^{-1}} = \xi_p$.

In the 1960s Z. I. Borevič and D. K. Faddeev classified the possible G -module structures of the first cohomology groups $H^1(U, \mathbb{F}_p)$ in the case F a local field [B]. Quite recently this result was extended for all fields F as above [MS]. For the study of Galois cohomology it is important to extend these results to all cohomology groups $H^n(U, \mathbb{F}_p)$, $n \in \mathbb{N}$, and a solution of this problem was out of reach until now.

Recently, based on earlier work of A. S. Merkurjev, M. Rost and A. A. Suslin, V. Voevodsky established the Bloch-Kato Conjecture [V1, V2], and it turns out that some of the main theorems in his proof are sufficient to determine the structure of all G -Galois modules $H^n(U, \mathbb{F}_p)$, using only simple arithmetical invariants attached to the field extension E/F . The theorems we use (quoted as Theorems 3 and 4 in section 1 below) had, in fact, been standard conjectures on Galois cohomology.

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It is interesting to point out, however, that the case $n = 2$ could have already been settled some 20 years ago, thanks to the work of Merkurjev and Suslin [MeSu].

The main ingredient for our determination of the G -module structure of $H^n(U, \mathbb{F}_p)$ is Milnor K -theory. (See [Mi] and [FV, Chap. IX].) For $i \geq 0$, let $K_i F$ denote the i th Milnor K -group of the field F , with standard generators denoted by $\{f_1, \dots, f_i\}$, $f_1, \dots, f_i \in F \setminus \{0\}$. For $\alpha \in K_i F$, we denote by $\bar{\alpha}$ the class of α modulo p , and we use the usual abbreviation $k_n F$ for $K_n F / pK_n F$. We write $N_{E/F}$ for the norm map $K_n E \rightarrow K_n F$, and we use the same notation for the induced map modulo p . We denote by i_E the natural homomorphism in the reverse direction. We also apply the same notation $N_{E/F}$ and i_E for the corresponding homomorphisms between cohomology groups. The image of an element $\alpha \in K_i F$ in $H^i(G_F, \mathbb{F}_p)$ we also denote by α . Voevodsky's proof of the Bloch-Kato Conjecture establishes a G_F -isomorphism $H^n(U, \mathbb{F}_p) \cong k_n E$. We formulate our results in terms of Galois cohomology for intended applications, but we use Milnor K -theory in our proof.

We concentrate upon the case when F is a perfect field, and at the end of the paper we indicate how one may reduce the case of an imperfect field F to the case of characteristic 0.

Our decomposition depends on four arithmetic invariants Υ_1 , Υ_2 , y , z , which we define as follows. First, for an element $\bar{\alpha}$ of $k_i F$, let

$$\text{ann}_{k_{n-1}F} \bar{\alpha} = \text{ann} \left(k_{n-1}F \xrightarrow{\bar{\alpha} \cdot -} k_{n-1+i}F \right)$$

denote the annihilator of the product with $\bar{\alpha}$. When the domain of $\bar{\alpha}$ is clear, we omit the subscript on the map and write simply $\text{ann } \bar{\alpha}$. Because we will often use the elements $\overline{\{a\}}$, $\overline{\{\xi_p\}}$, $\overline{\{a, a\}}$, and $\overline{\{a, \xi_p\}}$, we omit the bars for these elements. We also omit the bar in the element $\overline{\{\sqrt[p]{a}\}} \in k_n E$.

Fix $n \in \mathbb{N}$ and U an open normal subgroup of G_F of index p with fixed field E . Define invariants associated to E/F and n as follows:

$$\begin{aligned} d &:= \dim_{\mathbb{F}_p} k_n F / N_{E/F} k_n E \\ e &:= \dim_{\mathbb{F}_p} N_{E/F} k_n E \\ \Upsilon_1 &:= \dim_{\mathbb{F}_p} \text{ann}\{a, \xi_p\} / \text{ann}\{a\} \\ \Upsilon_2 &:= \dim_{\mathbb{F}_p} k_{n-1} F / \text{ann}\{a, \xi_p\} \end{aligned}$$

$$y := \begin{cases} \dim_{\mathbb{F}_p} (N_{E/F}k_n E) / \{a\} \cdot k_{n-1}F, & p > 2 \\ \dim_{\mathbb{F}_2} (N_{E/F}k_n E) / \{a\} \cdot \text{ann}_{k_{n-1}F}\{a, -1\}, & p = 2 \end{cases}$$

$$z := \begin{cases} \dim_{\mathbb{F}_p} (k_n F) / (\{\xi_p\} \cdot W + N_{E/F}k_n E), & p > 2 \\ \dim_{\mathbb{F}_2} (k_n F) / (\{a\} \cdot k_{n-1}F + N_{E/F}k_n E), & p = 2, \end{cases}$$

where W is a complement of $\text{ann}\{a, \xi_p\}$ in $k_{n-1}F$. (In Lemma 1 we show that z is independent of the choice of W .)

Our main results are then the following.

Theorem 1. *If $p > 2$, F is perfect and $n \in \mathbb{N}$ then*

$$H^n(U, \mathbb{F}_p) \cong X_1 \oplus X_2 \oplus Y \oplus Z$$

where

- (1) X_1 is a trivial $\mathbb{F}_p[G]$ -module of dimension Υ_1
- (2) X_2 is a direct sum of Υ_2 cyclic $\mathbb{F}_p[G]$ -modules of dimension 2
- (3) Y is a free $\mathbb{F}_p[G]$ -module of rank y
- (4) Z is a trivial $\mathbb{F}_p[G]$ -module of dimension z .

Further we have

- (5) $Y^G = i_E N_{E/F} H^n(U, \mathbb{F}_p)$
- (6) $N_{E/F}: X_1 \oplus X_2 \rightarrow \{a\} \cdot H^{n-1}(G_F, \mathbb{F}_p)$ is surjective
- (7) $\Upsilon_1 + \Upsilon_2 + y = e$
- (8) $\Upsilon_2 + z = d$

Theorem 2. *If $p = 2$ and $n \in \mathbb{N}$ then*

$$H^n(U, \mathbb{F}_2) \cong X_1 \oplus Y \oplus Z$$

where

- (1) X_1 is a trivial $\mathbb{F}_2[G]$ -module of dimension Υ_1
- (2) Y is a free $\mathbb{F}_2[G]$ -module of rank y .
- (3) Z is a trivial $\mathbb{F}_2[G]$ -module of dimension z .

Further we have

- (4) $Y^G = i_E N_{E/F} H^n(U, \mathbb{F}_2)$
- (5) $N_{E/F}: X_1 \rightarrow \{a\} \cdot \text{ann}\{a, -1\}$ is an isomorphism
- (6) $\Upsilon_1 + y = e$
- (7) $\Upsilon_2 + z = d$

Remark. As explained at the end of section 5, if $n \leq 2$ in Theorem 1, we may remove the hypothesis that F is perfect. The case $n = 1$ in the two theorems recovers the results of [MS].

1. BLOCH-KATO AND MILNOR K -THEORY

Our proof relies on the following two results in Voevodsky's proof of the Bloch-Kato Conjecture. Because we apply Voevodsky's results in the case when the base field contains a primitive p th root of unity we shall formulate Voevodsky's results restricted to this case. The first is the Bloch-Kato Conjecture itself:

Theorem 3 ([V1, Def. 5.1] and [V2, Thm. 7.1]).

- (1) *Let F be a field containing a primitive p th root of unity and $m \in \mathbb{N}$. Then the norm residue homomorphism*

$$k_m F \rightarrow H^m(G_F, \mu_p)$$

is an isomorphism.

- (2) *For any cyclic extension E/F of degree p , the sequence*

$$K_m E \xrightarrow{\sigma-1} K_m E \xrightarrow{N_{E/F}} K_m F$$

is exact.

The second result establishes an exact sequence connecting $k_m F$ and $k_m E$ for consecutive m . (We translate the statement of the original result to K -theory using the previous theorem.) In the following result a is chosen to satisfy $E = F(\sqrt[p]{a})$.

Theorem 4 ([V1, Prop. 5.2]). *Let F be a field containing a primitive p th root of unity with no extensions of degree prime to p . Then for any cyclic extension E/F of degree p and $m \geq 1$, the sequence*

$$k_{m-1} E \xrightarrow{N_{E/F}} k_{m-1} F \xrightarrow{\{a\}\cdot-} k_m F \xrightarrow{i_E} k_m E$$

is exact.

Now if F is a perfect field, we observe that we may remove the hypothesis that the field F has no extensions of degree prime to p . It is precisely to ensure that the sequence above is exact that we require the hypothesis that F is perfect in Theorem 1. We give a proof of the following theorem in section 5.

Theorem 5 (Modification of Theorem 4). *Let F be a field containing a primitive p th root of unity, and if $p > 2$ assume that F is perfect. Then for any cyclic extension E/F of degree p and $m \geq 1$ the sequence*

$$k_{m-1}E \xrightarrow{N_{E/F}} k_{m-1}F \xrightarrow{\{a\}\cdot-} k_mF \xrightarrow{i_E} k_mE$$

is exact.

2. NOTATION AND LEMMAS

For the remainder of the paper, except for section 6, we assume that F is perfect if $p > 2$. We fix $n \in \mathbb{N}$ and the cyclic extension $E = F(\sqrt[p]{a})$, and we write $k_{n-1}F = \text{ann}\{a\} \oplus V \oplus W$, where $\text{ann}\{a, \xi_p\} = \text{ann}\{a\} \oplus V$. Observe that $\Upsilon_1 = \dim_{\mathbb{F}_p} V$ and $\Upsilon_2 = \dim_{\mathbb{F}_p} W$. We show that the invariant z is independent of the choice of W :

Lemma 1. *The invariant z is independent of the particular complement W of $\text{ann}\{a, \xi_p\}$ in $k_{n-1}F$.*

Proof. This is obvious for $p = 2$, so assume that $p > 2$. Let W' be another \mathbb{F}_p -subspace of $k_{n-1}F$ with $k_{n-1}F = \text{ann}\{a, \xi_p\} \oplus W'$. Then we may choose bases $\{w_i\}_{i \in \mathcal{I}}$ and $\{w'_i\}_{i \in \mathcal{I}}$ of W and W' , respectively, such that $w'_i = w_i + u_i$ for $u_i \in \text{ann}\{a, \xi_p\}$. Then $\{\xi_p\} \cdot w'_i = \{\xi_p\} \cdot w_i + \{\xi_p\} \cdot u_i$, and $\{\xi_p\} \cdot u_i \in \text{ann}\{a\}$. Because $\text{ann}\{a\} = N_{E/F}k_nE$ by Theorem 5, we see that

$$\{\xi_p\} \cdot W + N_{E/F}k_nE = \{\xi_p\} \cdot W' + N_{E/F}k_nE$$

and therefore z does not depend on the choice of W . \square

We denote by $i_E: K_nF \rightarrow K_nE$ the map induced by the inclusion of F in E . In what follows we will frequently refer to the element $\sqrt[p]{a}$, and so we abbreviate it by A . We will also often use the observation that if $p = 2$ then $\{a, \xi_p\} = \{a, -1\} = \{a, a\} \in k_2F$, while if $p > 2$ then $\{a, a\} = 0 \in k_2F$. Finally, we will use the projection formula for taking the norms of standard generators of K_iF (see [FW, p. 81]).

Lemma 2. *We have the vector space isomorphism*

$$V \oplus W \xrightarrow{\{a\}\cdot-} \{a\} \cdot k_{n-1}F$$

and, if $p > 2$, the compositum of the maps $\{\xi_p\} \cdot -$ and i_E

$$W \xrightarrow{\{\xi_p\}\cdot-} \{\xi_p\} \cdot W \xrightarrow{i_E} i_E(\{\xi_p\} \cdot W)$$

is a vector space isomorphism as well.

Proof. The first isomorphism follows from the fact that $V \oplus W$ is a complement in $k_{n-1}F$ of the kernel of multiplication by $\{a\}$. For the second, assume $p > 2$. Suppose that $\bar{w} \in W$ and $\bar{\alpha} = \{\xi_p\} \cdot \bar{w} \in \ker i_E$. Then by Theorem 5, $\bar{\alpha} = \{a\} \cdot \bar{c}$ for $c \in K_{n-1}F$. Since $\{a, a\} = 0$ we see that $\{a\} \cdot \bar{\alpha} = 0$. But then $\bar{w} \in \text{ann}\{a, \xi_p\}$ and so $\bar{w} = 0$. \square

For $\gamma \in K_n E$, let $l(\gamma)$ denote the dimension of the cyclic $\mathbb{F}_p[G]$ -submodule $\langle \bar{\gamma} \rangle$ of $k_n E$ generated by $\bar{\gamma}$. Then we have

$$(\sigma - 1)^{l(\gamma)-1} \langle \bar{\gamma} \rangle = \langle \bar{\gamma} \rangle^G \neq 0 \quad \text{and} \quad (\sigma - 1)^{l(\gamma)} \langle \bar{\gamma} \rangle = 0.$$

We denote by N the map $(\sigma - 1)^{p-1}$ on $k_n E$. Because $(\sigma - 1)^{p-1} = 1 + \sigma + \cdots + \sigma^{p-1}$ in $\mathbb{F}_p[G]$, we may use $i_E N_{E/F}$ and N interchangeably on $k_n E$.

Lemma 3. *Suppose $p > 2$ and $\gamma \in K_n E$.*

- (1) *If $3 \leq l(\gamma) \leq p$, then there exists $\alpha \in K_n E$ such that*

$$\langle N\bar{\alpha} \rangle = \langle \bar{\gamma} \rangle^G.$$

- (2) *If $l(\gamma) = 2$ and*

$$\bar{\gamma} \notin \{A\} \cdot i_E(k_{n-1}F) + (k_n E)^G$$

then there exist $\alpha \in K_n E$ and $b \in K_{n-1}F$ such that

$$\langle N\bar{\alpha} \rangle = \langle \bar{\gamma} + \{A\} \cdot i_E(\bar{b}) \rangle^G.$$

Proof. Let $l = l(\gamma)$ and suppose $3 \leq l \leq i \leq p$. We show by induction on i that there exists $\alpha_i \in K_n E$ such that $\langle (\sigma - 1)^{i-1} \bar{\alpha}_i \rangle = \langle \bar{\gamma} \rangle^G$. Then setting $\alpha := \alpha_p$, the proof will be complete. If $i = l$ then $\alpha_i = \gamma$ suffices. Assume now that $l \leq i < p$ and that our statement is true for i .

Set $c = N_{E/F} \alpha_i$. Since $i_E \bar{c} = N \bar{\alpha}_i = (\sigma - 1)^{p-1} \bar{\alpha}_i$ and $i < p$, $i_E \bar{c} = 0$. By Theorem 5, $\bar{c} = \{a\} \cdot \bar{b}$ for $b \in K_{n-1}F$. Equivalently, $c = \{a\} \cdot b + pf$ for $f \in K_n F$. Then

$$N_{E/F}(\alpha_i - (\{A\} \cdot i_E(b) + i_E(f))) = 0.$$

By Theorem 3, there exists $\omega \in K_n E$ such that

$$(\sigma - 1)\omega = \alpha_i - (\{A\} \cdot i_E(b) + i_E(f)).$$

Then $(\sigma - 1)^2 \omega = (\sigma - 1)\alpha_i - \{\xi_p\} \cdot i_E(b)$. Since $i \geq 3$, $\langle (\sigma - 1)^i \bar{\omega} \rangle = \langle \bar{\gamma} \rangle^G$ and we can set $\alpha_{i+1} = \omega$.

For the second part, suppose $l = 2 = i$. Proceeding in the same way as above, we see that for $\alpha_2 = \gamma$ we have $N_{E/F}\alpha_2 = \{a\} \cdot b + pf$ for $b \in K_{n-1}F$ and $f \in K_nF$. As before, there exists $\omega \in K_nE$ such that $(\sigma - 1)\omega = \alpha_2 - (\{A\} \cdot i_E(b) + i_E(f))$. Then

$$(\sigma - 1)^2\omega = (\sigma - 1)(\alpha_2 - \{A\} \cdot i_E(b)) = (\sigma - 1)(\gamma - \{A\} \cdot i_E(b)).$$

Observe that $\bar{\gamma} - \{A\} \cdot i_E(\bar{b}) \notin (k_nE)^G$ by hypothesis. Therefore $l(\gamma - \{A\} \cdot i_E(b)) = 2$ and we can set $\alpha_3 := \omega$. We may then continue by induction on i as above, concluding that there exists an element $\alpha = \alpha_p \in K_nE$ such that $\langle N\bar{\alpha}_p \rangle = \langle (\sigma - 1)^{p-1}\bar{\alpha}_p \rangle = \langle \bar{\gamma} - \{A\} \cdot i_E(\bar{b}) \rangle^G$, as required. \square

In the following lemma we elongate the exact sequence of Theorem 5.

Lemma 4. *The following sequence is exact:*

$$0 \rightarrow \text{ann}\{a\} \rightarrow k_{n-1}F \xrightarrow{\{a\}\cdot-} k_nF \xrightarrow{i_E} (k_nE)^G \xrightarrow{N_{E/F}} \{a\} \cdot \text{ann}\{a, \xi_p\} \rightarrow 0.$$

Here the map $\text{ann}\{a\} \rightarrow k_{n-1}F$ is the natural inclusion.

Proof. We show first that $N_{E/F}((k_nE)^G) \subset \{a\} \cdot \text{ann}\{a, \xi_p\}$. Let $\bar{\alpha} \in (k_nE)^G$ and $\beta = N_{E/F}\bar{\alpha}$. Since $i_E(N_{E/F}\bar{\alpha}) = (\sigma - 1)^{p-1}\bar{\alpha} = 0$ we have that $\bar{\beta} = N_{E/F}\bar{\alpha} = \{a\} \cdot \bar{b}$ for some $b \in K_{n-1}F$ by Theorem 5.

Suppose $p = 2$. Since $\bar{\beta}$ is in the image of $N_{E/F}$, we have by Theorem 5 that $\{a\} \cdot \bar{\beta} = \{a, a\} \cdot \bar{b} = 0$. Since $\{a, a\} = \{a, -1\}$, we have $\bar{b} \in \text{ann}\{a, -1\}$.

Now suppose that $p > 2$. Write $\beta = \{a\} \cdot b + pf$ for some $f \in K_nF$. Then by the projection formula

$$N_{E/F}(\alpha - (\{A\} \cdot i_E(b) + i_E(f))) = 0.$$

By Theorem 3, there exists $\omega \in K_nE$ such that

$$(\sigma - 1)\omega = \alpha - (\{A\} \cdot i_E(b) + i_E(f)).$$

Then $(\sigma - 1)^2\bar{\omega} = \{\xi_p\} \cdot i_E(\bar{b})$.

If $(\sigma - 1)^2\bar{\omega} = 0$ then since by Theorem 5, $\ker i_E = \{a\} \cdot k_{n-1}F$,

$$\{\xi_p\} \cdot \bar{b} = \{a\} \cdot \bar{h}$$

for some $h \in K_{n-1}F$. Because $\{a, a\} = 0$, the right-hand side of the preceding equation is annihilated by $\{a\}$. Therefore $\bar{b} \in \text{ann}\{a, \xi_p\}$.

If $(\sigma - 1)^2 \bar{\omega} \neq 0$ then $l(\omega) = 3$ and Lemma 3 shows that

$$i_E(\{\xi_p\} \cdot \bar{b}) = cN\bar{\lambda} = i_E(N_{E/F}(\overline{c\lambda}))$$

for some $\lambda \in K_n E$ and $c \in \mathbb{Z}$. Since by Theorem 5, $\ker i_E = \{a\} \cdot k_{n-1} F$ we have

$$\{\xi_p\} \cdot \bar{b} = N_{E/F}(\overline{c\lambda}) + \{a\} \cdot \bar{h}$$

for some $h \in K_{n-1} F$. Now by Theorem 5 and the fact that $\{a, a\} = 0$, the right-hand side of the preceding equation is annihilated by $\{a\}$. Then $\bar{b} \in \text{ann}\{a, \xi_p\}$. Hence in all cases $N_{E/F} \bar{\alpha} \in \{a\} \cdot \text{ann}\{a, \xi_p\}$.

Exactness at the first two terms is obvious, and exactness at the third term follows from Theorem 5.

For exactness at the fourth term, suppose

$$\bar{\gamma} \in (k_n E)^G \text{ and } N_{E/F} \bar{\gamma} = 0.$$

Then $N_{E/F} \gamma = pf$ for $f \in K_n F$. Let $\beta = \gamma - i_E(f)$. Then $N_{E/F} \beta = 0$ and by Theorem 3 there exists $\alpha \in K_n E$ such that $(\sigma - 1)\alpha = \beta$. If $p = 2$ then $\bar{\beta} = i_E(N_{E/F} \bar{\alpha}) \in i_E k_n F$ and we are done. Thus assume $p > 2$.

Now suppose $\bar{\alpha} \in \{A\} \cdot i_E(k_{n-1} F) + (k_n E)^G$. Then

$$\bar{\beta} = (\sigma - 1)\bar{\alpha} \in \{\xi_p\} \cdot i_E(k_{n-1} F) \subset i_E(k_n F),$$

and hence $\bar{\gamma} = \bar{\beta} + i_E(\bar{f}) \in i_E(k_n F)$ as well. Otherwise $\bar{\alpha} \notin \{A\} \cdot i_E(k_{n-1} F) + (k_n E)^G$. Now if $(\sigma - 1)\bar{\alpha} = \bar{\beta} = 0$ we are done as then $\bar{\gamma} = i_E(\bar{f})$. Hence assume $(\sigma - 1)\bar{\alpha} \neq 0$. Then $l(\alpha) = 2$ and by Lemma 3 we see that there exist $\delta \in K_n E$, $b \in K_{n-1} F$ and $c \in \mathbb{Z}$ such that

$$cN\bar{\delta} = (\sigma - 1)(\bar{\alpha} + \{A\} \cdot i_E(\bar{b})) = \bar{\beta} + \{\xi_p\} \cdot i_E(\bar{b}).$$

Thus $\bar{\beta} = cN\bar{\delta} - \{\xi_p\} \cdot i_E(\bar{b}) \in i_E(k_n F)$ and exactness at the fourth term is established.

Finally we show the exactness at the fifth term. Since

$$\{a\} \cdot \text{ann}\{a, \xi_p\} = \{a\} \cdot V$$

it is enough to show that each element $\{a\} \cdot \bar{v}$ where $\bar{v} \in V$ can be written as $N_{E/F} \bar{\alpha}$ for some $\bar{\alpha} \in (k_n E)^G$. Observe that $(\sigma - 1)(\{A\} \cdot i_E \bar{v}) = \{\xi_p\} \cdot i_E(\bar{v})$. Also we have

$$N_{E/F}(\{A\} \cdot i_E(\bar{v})) = \begin{cases} \{a\} \cdot \bar{v} & \text{if } p > 2 \\ \{-a\} \cdot \bar{v} & \text{if } p = 2. \end{cases}$$

Therefore it is enough to show that there exists an element $\bar{\gamma} \in k_n E$ such that $(\sigma - 1)\bar{\gamma} = \{\xi_p\} \cdot i_E(\bar{v})$ and

$$N_{E/F}\bar{\gamma} = \begin{cases} 0 & \text{if } p > 2 \\ \{-1\} \cdot \bar{v} & \text{if } p = 2. \end{cases}$$

Indeed then we can set $\bar{\alpha} = \{A\} \cdot i_E(\bar{v}) - \bar{\gamma}$.

Because $\bar{v} \in \text{ann}\{a, \xi_p\}$ we see that $\{\xi_p\} \cdot i_E(\bar{v}) \in \text{ann}\{a\}$. By Theorem 5 there exists $\bar{\beta} \in k_n E$ such that

$$\{\xi_p\} \cdot \bar{v} = N_{E/F}\bar{\beta} \text{ and } i_E(N_{E/F}\bar{\beta}) = (\sigma - 1)^{p-1}\bar{\beta}.$$

Then setting $\bar{\gamma} = (\sigma - 1)^{p-2}\bar{\beta}$ we obtain our required element. The proof of our lemma has now been completed. \square

Finally, we need a general lemma about $\mathbb{F}_p[G]$ -modules.

Lemma 5 (Exclusion Lemma). *Let M_1 and M_2 be $\mathbb{F}_p[G]$ -modules contained in a common $\mathbb{F}_p[G]$ -module. Suppose that $M_1^G \cap M_2^G = \{0\}$. Then $M_1 + M_2 = M_1 \oplus M_2$.*

Proof. Let $M = M_1 \cap M_2$ and suppose that $m \in M \setminus \{0\}$. Let

$$\tilde{m} = (\sigma - 1)^{l(m)-1}(m) \neq 0.$$

Then $\tilde{m} \in M_1^G \cap M_2^G$, a contradiction. Hence $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2 = M_1 \oplus M_2$. \square

3. CONSTRUCTION OF SUBMODULES

Proposition 1. *$k_n E$ contains a submodule X_1 such that*

- X_1 is a trivial $\mathbb{F}_p[G]$ -module of dimension Υ_1
- $X_1 \cap i_E k_n F = \{0\}$
- $N_{E/F}$ restricts to an isomorphism $X_1 \rightarrow \{a\} \cdot V$.

Moreover, if $p > 2$, then $k_n E$ contains a submodule X_2 , independent of X_1 , such that

- X_2 is a direct sum of Υ_2 cyclic submodules of dimension 2 and $\dim_{\mathbb{F}_p} X_2^G = \Upsilon_2$.
- $(X_1 + X_2) \cap i_E k_n F = (\sigma - 1)X_2 = X_2^G = i_E(\{\xi_p\} \cdot W)$

- We have an exact sequence

$$0 \rightarrow \{\xi_p\} \cdot W \xrightarrow{i_E} X_1 + X_2 \xrightarrow{N_{E/F}} \{a\} \cdot k_{n-1}F \rightarrow 0$$

Proof. Let \mathcal{I} be an \mathbb{F}_p -basis for V . Let \bar{v} be an arbitrary element of \mathcal{I} , and consider $\bar{\alpha} = \{A\} \cdot i_E \bar{v}$. Now $(\sigma - 1)\bar{\alpha} = i_E(\{\xi_p\} \cdot \bar{v})$.

Since $\bar{v} \in \text{ann}\{a, \xi_p\}$ we see that $\{\xi_p\} \cdot \bar{v} \in \text{ann}_{k_n F}\{a\}$. By Theorem 5

$$\{\xi_p\} \cdot \bar{v} = N_{E/F} \bar{\beta} \text{ and } i_E(N_{E/F} \bar{\beta}) = N \bar{\beta} = (\sigma - 1)^{p-1} \bar{\beta}$$

for some $\beta \in K_n E$. Set $\gamma = (\sigma - 1)^{p-2} \beta$ and $\bar{x}_v = \bar{\alpha} - \bar{\gamma} \in k_n E$.

If $p = 2$ then

$$N_{E/F} \bar{x}_v = \{-a\} \cdot \bar{v} - N_{E/F} \bar{\gamma} = \{-a\} \cdot \bar{v} - \{-1\} \cdot \bar{v} = \{a\} \cdot \bar{v}.$$

If $p > 2$, then observe that since γ is in the image of $\sigma - 1$ we have $\overline{N_{E/F} \gamma} = 0$. Then, by the projection formula

$$N_{E/F} \bar{x}_v = \{a\} \cdot \bar{v} - \overline{N_{E/F} \gamma} = \{a\} \cdot \bar{v}.$$

Now in either case, since $(\sigma - 1)^{p-1} \bar{\beta} = i_E(N_{E/F} \bar{\beta})$,

$$(\sigma - 1)\bar{x}_v = i_E(\{\xi_p\} \cdot \bar{v}) - (\sigma - 1)^{p-1} \bar{\beta} = i_E(\{\xi_p\} \cdot \bar{v}) - i_E(\{\xi_p\} \cdot \bar{v}) = 0.$$

Set

$$X_1 := \bigoplus_{\bar{v} \in \mathcal{I}} \langle \bar{x}_v \rangle.$$

We have shown that X_1 is a trivial $\mathbb{F}_p[G]$ -module. Moreover, because $N_{E/F} \bar{x}_v = \{a\} \cdot \bar{v}$ and $\{a\} \cdot -$ is injective on V by Lemma 2,

$$N_{E/F}|_{X_1}: X_1 \rightarrow \{a\} \cdot V$$

takes a basis of X_1 to a basis of $\{a\} \cdot V$ and $\dim_{\mathbb{F}_p} X_1 = \dim_{\mathbb{F}_p} V = \Upsilon_1$. Finally, since $N_{E/F}$ is trivial on $i_E k_n F$, we have $X_1 \cap i_E k_n F = \{0\}$.

Now suppose that $p > 2$. Set

$$X_2 := (\{A\} \cdot i_E W) + i_E(\{\xi_p\} \cdot W).$$

Let $\bar{w} \in W$ and consider $\bar{x}_w = \{A\} \cdot i_E(\bar{w})$.

Since $(\sigma - 1)\bar{x}_w = i_E(\{\xi_p\} \cdot \bar{w})$ and $(\sigma - 1)(\{\xi_p\} \cdot \bar{w}) = 0$, we obtain $(\sigma - 1)X_2 = i_E(\{\xi_p\} \cdot W)$. Hence on $\{A\} \cdot i_E W$, $\sigma - 1$ acts as $i_E(\{\xi_p\} \cdot -)$, which by Lemma 2 is an isomorphism of vector spaces. Hence $\sigma - 1$ is an isomorphism as well. Moreover, if an arbitrary $\{A\} \cdot i_E(\bar{w}_1) + \{\xi_p\} \cdot i_E(\bar{w}_2) \in X_2$ lies in the kernel of $\sigma - 1$, $\bar{w}_1 = 0$. Hence $X_2^G = i_E(\{\xi_p\} \cdot$

W). Since we already observed that $X_1 \cap i_E k_n F = \{0\}$ we see that $X_2^G \cap X_1 = \{0\}$ and by Lemma 5 we conclude that $X_1 + X_2 = X_1 \oplus X_2$.

By the projection formula $N_{E/F} \bar{x}_w = \{a\} \cdot \bar{w}$ and by the definition of W , $\{a\} \cdot \bar{w} = 0$ implies $\bar{w} = 0$. Since $N_{E/F}(\{\xi_p\} \cdot i_E(\bar{w}_2)) = 0$ for all $\bar{w}_2 \in W$, we deduce that restricted to X_2 , $N_{E/F}$ surjects X_2 onto $\{a\} \cdot W$ with kernel $i_E(\{\xi_p\} \cdot W)$. By Lemma 2, $\{a\} \cdot k_{n-1} F = \{a\} \cdot (V + W)$; hence on $X_1 \oplus X_2$, $N_{E/F}$ is a surjection onto $\{a\} \cdot k_{n-1} F$ with kernel $i_E(\{\xi_p\} \cdot W)$.

Finally observe that $N_{E/F} i_E k_n F = \{0\}$. Hence

$$(X_1 + X_2) \cap i_E k_n F \subset i_E(\{\xi_p\} \cdot W).$$

Since $i_E(\{\xi_p\} \cdot W) \subset i_E k_n F$, we have equality.

Now we have shown that $\sigma - 1$ is an isomorphism of vector spaces $\{A\} \cdot i_E W \rightarrow i_E(\{\xi_p\} \cdot W)$, and by Lemma 2, we have an isomorphism $W \rightarrow i_E(\{\xi_p\} \cdot W)$. Therefore X_2 is a direct sum of cyclic submodules $\langle \bar{x}_w \rangle$ of dimension 2, with \bar{x}_w in one-to-one correspondence with basis elements of W . Hence the direct sum contains Υ_2 cyclic summands. \square

If $p = 2$, let $X = X_1$ be a submodule of $k_n E$ satisfying the conditions of the preceding proposition. If $p > 2$, let $X = X_1 + X_2$ for X_1, X_2 satisfying the conditions of the same.

Proposition 2. *$k_n E$ contains a submodule Y independent from X such that*

- Y is a free $\mathbb{F}_p[G]$ -module of rank

$$y = \begin{cases} \dim_{\mathbb{F}_p}(N_{E/F} k_n E) / \{a\} \cdot k_{n-1} F, & p > 2 \\ \dim_{\mathbb{F}_2}(N_{E/F} k_n E) / \{a\} \cdot \text{ann}_{k_{n-1} F}\{a, -1\}, & p = 2 \end{cases}$$

- $Y^G = i_E N_{E/F} k_n E$
- if $p > 2$, $\Upsilon_1 + \Upsilon_2 + y = e$
- if $p = 2$, $\Upsilon_1 + y = e$

Proof. Let \mathcal{I} be a basis for the subspace $i_E(N_{E/F} k_n E)$. For each basis element $\bar{y} \in \mathcal{I}$, let $\alpha_y \in K_n E$ satisfy $i_E(N_{E/F} \bar{\alpha}_y) = \bar{y}$. Then $\langle \bar{\alpha}_y \rangle$ is a cyclic submodule of dimension p , hence isomorphic to $\mathbb{F}_p[G]$, with

$$\langle \bar{\alpha}_y \rangle^G = (\sigma - 1)^{p-1} \langle \bar{\alpha}_y \rangle = \langle N \bar{\alpha}_y \rangle = \langle \bar{y} \rangle.$$

Set

$$Y = \sum_{\bar{y} \in \mathcal{I}} \langle \bar{\alpha}_y \rangle.$$

By Lemma 5, $Y = \bigoplus_{\bar{y} \in \mathcal{I}} \langle \bar{\alpha}_y \rangle$ and so Y is a free $\mathbb{F}_p[G]$ -module. Moreover,

$$Y^G = (\sigma - 1)^{p-1} Y = NY = \bigoplus_{\bar{y} \in \mathcal{I}} \langle \bar{y} \rangle = i_E(N_{E/F} k_n E).$$

Now the rank of Y is equal to the dimension of $i_E(N_{E/F} k_n E)$, or

$$\dim_{\mathbb{F}_p}(N_{E/F} k_n E) / ((N_{E/F} k_n E) \cap \ker i_E).$$

Now by Theorem 5, $N_{E/F}(k_n E) = \text{ann}\{a\}$, and by the same Theorem, $\ker i_E = \{a\} \cdot k_{n-1} F$. Hence

$$N_{E/F}(k_n E) \cap \ker i_E = \text{ann}_{k_n F}\{a\} \cap \{a\} \cdot k_{n-1} F.$$

Suppose that $p = 2$. Since $\{a, a\} = \{a, -1\}$ we deduce that

$$N_{E/F}(k_n E) \cap \ker i_E = \{a\} \cdot \text{ann}\{a, -1\}.$$

The dimension of this subspace is equal to $\dim_{\mathbb{F}_p} \text{ann}\{a, -1\} / \text{ann}\{a\}$, or Υ_1 .

Now suppose that $p > 2$. Since $\{a, a\} = 0$, $\{a\} \cdot k_{n-1} F \subset \text{ann}\{a\}$ and we deduce that

$$N_{E/F}(k_n E) \cap \ker i_E = \{a\} \cdot k_{n-1} F,$$

which is of dimension $\dim_{\mathbb{F}_p}(k_{n-1} F) / \text{ann}\{a\}$, or $\Upsilon_1 + \Upsilon_2$.

As $e = \dim_{\mathbb{F}_p} N_{E/F} k_n E$, we deduce that if $p = 2$ then $\Upsilon_1 + \text{rank } Y = e$ and if $p > 2$, $\Upsilon_1 + \Upsilon_2 + \text{rank } Y = e$.

Now we claim that Y is independent from X . Suppose $\bar{\beta} \in X^G \cap Y^G$. Now $Y^G = i_E(N_{E/F} k_n E) \subset i_E(k_n F)$, so $\bar{\beta} = i_E(\bar{\alpha})$ where $\bar{\alpha} = N_{E/F} \bar{\gamma}$ for $\bar{\gamma} \in K_n E$. If $p = 2$ then by Proposition 1, $i_E(k_n F) \cap X = \{0\}$, and so $X \cap Y = \{0\}$ by Lemma 5.

If $p > 2$, Proposition 1 tells us that

$$i_E(k_n F) \cap X = X_2^G = \{\xi_p\} \cdot i_E W.$$

Hence $\bar{\beta} = \{\xi_p\} \cdot i_E(\bar{w})$ for $\bar{w} \in W$. Since $i_E(\{\xi_p\} \cdot \bar{w}) = i_E(\bar{\alpha})$ and by Theorem 5, $\ker i_E = \{a\} \cdot k_{n-1} F$,

$$\{\xi_p\} \cdot \bar{w} = \bar{\alpha} + (\{a\} \cdot \bar{f}) \tag{1}$$

for $f \in K_{n-1} F$. Now because $\bar{\alpha} \in N_{E/F} k_n E$, by Theorem 5, $\{a\} \cdot \bar{\alpha} = 0$. Moreover, $\{a, a\} = 0$ since we have assumed that $p > 2$. Hence the right-hand side of (1) is annihilated by multiplication by $\{a\}$. Therefore $\bar{w} \in \text{ann}\{a, \xi_p\}$, and by the definition of W , $\bar{w} = 0$. By Lemma 5, $X + Y = X \oplus Y$. \square

Now let X and Y be submodules satisfying the conditions of the preceding propositions.

Proposition 3. $k_n E$ contains a submodule Z independent from $X + Y$ such that

- Z is a trivial $\mathbb{F}_p[G]$ -module of dimension

$$z = \begin{cases} \dim_{\mathbb{F}_p}(k_n F)/(\{\xi_p\} \cdot W + N_{E/F}k_n E), & p > 2 \\ \dim_{\mathbb{F}_2}(k_n F)/(\{a\} \cdot k_{n-1}F + N_{E/F}k_n E), & p = 2 \end{cases}$$

- $(k_n E)^G = X^G + Y^G + Z$
- $\Upsilon_2 + z = d$

Proof. Let Z be a complement of $(X^G + Y^G) \cap i_E(k_n F)$ in $i_E(k_n F)$. By Lemma 5, $(X + Y) + Z = (X + Y) \oplus Z$.

Clearly $X^G + Y^G + Z \subset (k_n E)^G$. Now suppose $\bar{\alpha} \in (k_n E)^G$ and let $\beta = N_{E/F}\alpha$. By Lemma 4, $\beta = \{a\} \cdot \bar{b}$ for some $\bar{b} \in \text{ann}\{a, \xi_p\}$.

Let $\bar{v} \in V$ be the component of \bar{b} in the decomposition $\text{ann}\{a\} \oplus V$ of $\text{ann}\{a, \xi_p\}$. By Proposition 1, there exists $\bar{\gamma} \in X_1 \subset X^G$ such that

$$N_{E/F}\bar{\gamma} = \{a\} \cdot \bar{v} = \{a\} \cdot \bar{b} = \beta.$$

Then $N_{E/F}(\bar{\alpha} - \bar{\gamma}) = 0$. By Lemma 4, $\bar{\alpha} - \bar{\gamma} \in i_E(k_n F)$. But $i_E(k_n F) \subset X^G + Y^G + Z$. Hence $\bar{\alpha} \in X^G + Y^G + Z$ and we have shown that $(k_n E)^G = X^G + Y^G + Z$.

For the dimension of Z , assume first that $p > 2$. By Theorem 5, $N_{E/F}k_n E = \text{ann}_{k_n F}\{a\}$ and $\ker i_E = \{a\} \cdot k_{n-1}F$. Since $\{a, a\} = 0$ we see that $\ker i_E \subset N_{E/F}k_n E$. Hence

$$d = \dim_{\mathbb{F}_p} \frac{k_n F}{N_{E/F}k_n E} = \dim_{\mathbb{F}_p} \frac{i_E(k_n F)}{i_E(N_{E/F}k_n E)} = \dim_{\mathbb{F}_p} \frac{i_E(k_n F)}{Y^G},$$

where in the last equation we use Proposition 2 to identify Y^G . By Propositions 1 and 2, $(X^G + Y^G) \cap i_E(k_n F) = X_2^G \oplus Y^G$. Hence $d = \dim_{\mathbb{F}_p}(X_2^G \oplus Y^G \oplus Z)/Y^G$. By Proposition 1, $\dim_{\mathbb{F}_p} X_2^G = \Upsilon_2$. Hence $\Upsilon_2 + \dim_{\mathbb{F}_p} Z = d$ for $p > 2$. Also we see that

$$\dim_{\mathbb{F}_p} Z = \dim_{\mathbb{F}_p} \frac{i_E(k_n F)}{X_2^G \oplus Y^G} = \dim_{\mathbb{F}_p} \frac{k_n F}{\{\xi_p\} \cdot W + N_{E/F}k_n E} = z.$$

Now assume $p = 2$. By Propositions 1 and 2, $i_E(k_n F) \cap (X^G + Y^G) = Y^G$. Proceeding as in the last case,

$$\begin{aligned} \dim_{\mathbb{F}_2} Z &= \dim_{\mathbb{F}_2} \frac{i_E(k_n F)}{Y^G} = \dim_{\mathbb{F}_2} \frac{i_E(k_n F)}{i_E(N_{E/F} k_n E)} \\ &= \dim_{\mathbb{F}_2} \frac{k_n F}{N_{E/F} k_n E + \ker i_E} \\ &= \dim_{\mathbb{F}_2} \frac{k_n F}{N_{E/F} k_n E + \{a\} \cdot k_{n-1} F} = z, \end{aligned}$$

since $\ker i_E = \{a\} \cdot k_{n-1} F$, by Theorem 5.

We then consider the filtration

$$k_n F \supset \left((\{a\} \cdot k_{n-1} F) + N_{E/F} k_n E \right) \supset N_{E/F} k_n E.$$

The dimension of the quotient of the first and third modules is, by definition, d . By Theorem 5, $N_{E/F} k_n E = \text{ann}_{k_n F} \{a\}$. Since $\{a, a\} = \{a, -1\}$ we see that

$$(\{a\} \cdot k_{n-1} F) \cap N_{E/F} k_n E = \{a\} \cdot V.$$

Hence

$$\dim_{\mathbb{F}_2} \frac{(\{a\} \cdot k_{n-1} F) + N_{E/F} k_n E}{N_{E/F} k_n E} = \dim_{\mathbb{F}_2} \frac{\{a\} \cdot k_{n-1} F}{\{a\} \cdot V} = \dim_{\mathbb{F}_2} \{a\} \cdot W.$$

By Lemma 2, $\dim_{\mathbb{F}_2} \{a\} \cdot W = \dim_{\mathbb{F}_2} W = \Upsilon_2$. Hence $\Upsilon_2 + \dim_{\mathbb{F}_2} Z = d$ for $p = 2$ as well. \square

4. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. By Propositions 1, 2, and 3, there exist independent submodules $X = X_1 + X_2$, Y , and Z satisfying the conditions of the theorem. All that remains is to show that $k_n E = X + Y + Z$.

We proceed by induction on the length $l(\gamma)$ of the cyclic submodule $\langle \bar{\gamma} \rangle$ of $k_n E$ generated by an arbitrary element $\bar{\gamma} \in k_n E$. If $l(\gamma) = 1$, then by Proposition 3, $\bar{\gamma} \in X^G + Y^G + Z$. Assume then that $\bar{\beta} \in X + Y + Z$ if $l(\beta) \leq i < p$ and that $l(\gamma) = i + 1$.

Suppose first that $l(\gamma) = 2$ and

$$\bar{\gamma} \in \{A\} \cdot i_E(k_{n-1} F) + (k_n E)^G.$$

Then $(\sigma - 1)\bar{\gamma} = i_E(\{\xi_p\} \cdot \bar{b})$ for some $b \in K_{n-1} F$. In the decomposition $\text{ann}\{a, \xi_p\} \oplus W$ of $k_{n-1} F$, write $\bar{b} = \bar{f} + \bar{w}$. By Proposition 1 there exists

$\bar{\omega} \in X_2$ such that $(\sigma - 1)\bar{\omega} = i_E(\{\xi_p\} \cdot \bar{\omega})$. We also have $\{\xi_p\} \cdot \bar{f} \in \text{ann}\{a\}$ and therefore by Theorem 5 and Proposition 2 there exists $\bar{y} \in Y$ such that $i_E(\{\xi_p\} \cdot \bar{f}) = i_E(N_{E/F}(\bar{y}))$. Hence there exists $\bar{y}' \in Y$ such that $(\sigma - 1)\bar{\gamma} = (\sigma - 1)\bar{\omega} + (\sigma - 1)\bar{y}'$. Hence $l(\gamma - \omega - y') \leq 1$ and by the inductive hypothesis $\bar{\gamma} \in X + Y + Z$.

Now since by the preceding arguments $\{A\} \cdot i_E(k_{n-1}F) \subset X + Y + Z$, in order to show that an arbitrary $\bar{\gamma}$ with $l(\gamma) = 2$ lies in $X + Y + Z$ it is enough to show that $\bar{\gamma} + \{A\} \cdot i_E(\bar{b}) \in X + Y + Z$ for any $b \in K_{n-1}F$.

Suppose then that $l(\gamma) = 2$ and

$$\bar{\gamma} \notin \{A\} \cdot i_E(k_{n-1}F) + (k_n E)^G.$$

Then, by Lemma 3, there exist $b \in K_n F$ and $\alpha \in K_n E$ such that $\bar{\beta} = \bar{\gamma} + \{A\} \cdot i_E(\bar{b})$ satisfies $l(\bar{\beta}) \leq 2$ and $\langle \bar{\beta} \rangle^G = \langle N\bar{\alpha} \rangle$. Hence $(\sigma - 1)^{l(\bar{\beta})-1}\bar{\beta} = cN\bar{\alpha}$ for some $c \in \mathbb{Z}$. But $cN\bar{\alpha} = i_E N_{E/F}(\bar{c}\bar{\alpha}) \in Y^G$, by Proposition 2. Hence there exists $\bar{\omega} \in Y$ such that $(\sigma - 1)^{p-1}\bar{\omega} = cN\bar{\alpha}$. Now $\bar{\lambda} = (\sigma - 1)^{p-l(\bar{\beta})}\bar{\omega} \in Y$ and $(\sigma - 1)^{l(\bar{\beta})-1}(\bar{\beta} - \bar{\lambda}) = 0$. Hence $l(\bar{\beta} - \bar{\lambda}) < l(\bar{\beta})$ and by the inductive hypothesis $\bar{\beta}$ and hence $\bar{\gamma}$ lie in $X + Y + Z$.

If $l(\gamma) \geq 3$ then the same argument works again. By Lemma 3 $\langle \bar{\gamma} \rangle^G = \langle N\bar{\alpha} \rangle$ and so $(\sigma - 1)^{l(\gamma)-1}\bar{\gamma} = cN\bar{\alpha}$ for some $c \in \mathbb{Z}$. But $cN\bar{\alpha} = i_E N_{E/F}(c\bar{\alpha}) \in Y^G$, by Proposition 2. Hence there exists $\bar{\omega} \in Y$ such that $(\sigma - 1)^{p-1}\bar{\omega} = cN\bar{\alpha}$. Now $\bar{\lambda} = (\sigma - 1)^{p-l(\gamma)}\bar{\omega} \in Y$ and $(\sigma - 1)^{l(\gamma)-1}(\bar{\gamma} - \bar{\lambda}) = 0$. Hence $l(\bar{\gamma} - \bar{\lambda}) < l(\bar{\gamma})$ and by the inductive hypothesis $\bar{\gamma} \in X + Y + Z$. \square

Proof of Theorem 2. By Propositions 1, 2, and 3, there exist independent submodules $X = X_1$, Y , and Z satisfying the conditions of the theorem. All that remains is to show that $k_n E = X + Y + Z$.

Let $\bar{\gamma} \in k_n E$ be arbitrary. If $l(\gamma) = 1$, then by Proposition 3, $\bar{\gamma} \in X^G + Y^G + Z$. Otherwise $(\sigma - 1)\bar{\gamma} = (\sigma + 1)\bar{\gamma} = i_E N_{E/F}\bar{\gamma} \in Y^G$, by Proposition 2. Hence there exists $\bar{\omega} \in Y$ such that $(\sigma - 1)\bar{\omega} = (\sigma - 1)\bar{\gamma}$. Therefore $l(\gamma - \omega) < 2$ and by the inductive hypothesis $\bar{\gamma} \in X + Y + Z$. \square

5. PROOF OF THEOREM 5

For the case $p = 2$ we have the long exact sequence of Galois cohomology groups due to Arason [A, Satz 4.5]. Suppose then that $p > 2$

and F is perfect. Let S be any p -Sylow subgroup of $G_F = \text{Gal}(F_{\text{sep}}/F)$, and set L to be the fixed field of S . Because F is perfect, the separable closure F_{sep} is identical to the algebraic closure \bar{F} , and hence each finite extension of L has degree a power of p . In particular, all of the hypotheses of Theorem 4 are valid for the field L in place of F . Furthermore, $([L : F], p) = 1$. (Here we use basic properties of supernatural numbers and Sylow p -subgroups. See [Se, Chapter 1].) Therefore if $E = F(\sqrt[p]{a})$ is a cyclic extension of F of degree p , so is $EL = L(\sqrt[p]{a})$ over L . By Theorem 4 we see that the sequence

$$k_{m-1}EL \xrightarrow{N_{EL/L}} k_{m-1}L \xrightarrow{\{a\}\cdot-} k_mL \xrightarrow{i_{EL}} k_mEL$$

is exact for each $m \in \mathbb{N}$.

We claim that $i_L: k_mF \rightarrow k_mL$ is injective. Indeed, suppose that $i_L(\alpha) = 0$ for some $\alpha \in k_mF$. Then there exists a finite subextension M/F of L/F such that $i_M(\alpha) = 0$. Then

$$0 = N_{M/F}(i_M(\alpha)) = [M : F]\alpha,$$

(see [FV, p. 300]). Because $[M : F]$ is coprime with p , we see that $\alpha = 0$ and i_L is injective as asserted. Similarly we have that $i_{EL}: k_mE \rightarrow k_mEL$ is injective.

We then have the following commutative diagram:

$$\begin{array}{ccccccc} k_{m-1}EL & \xrightarrow{N_{EL/L}} & k_{m-1}L & \xrightarrow{\{a\}\cdot-} & k_mL & \xrightarrow{i_{LE}} & k_mEL \\ \uparrow i_{EL} & & \uparrow i_L & & \uparrow i_L & & \uparrow i_{EL} \\ k_{m-1}E & \xrightarrow{N_{E/L}} & k_{m-1}F & \xrightarrow{\{a\}\cdot-} & k_mF & \xrightarrow{i_E} & k_mE \end{array}$$

Because the vertical maps are injective, we see that the bottom row of the diagram is a complex: the composition of any two consecutive maps is the zero map. We now establish exactness at the second and third terms of the complex.

Let $\alpha \in k_{m-1}F$ such that $\{a\} \cdot \alpha = 0$. Then $\{a\} \cdot i_L(\alpha) = 0$ and therefore there exists an element $\beta \in k_{m-1}EL$ such that $N_{EL/L}(\beta) = i_L(\alpha)$. Let M/F be a finite extension such that β is defined over EM . Then $N_{EM/M}(\beta) = i_M(\alpha)$, and we have

$$N_{EM/F}(\beta) = N_{M/F}(N_{EM/M}(\beta)) = N_{M/F}(i_M(\alpha)) = [M : F]\alpha$$

and $N_{EM/F}(\beta) = N_{E/F}(N_{EM/E}(\beta))$. Thus

$$N_{E/F}(N_{EM/E}(\beta)) = [M : F]\alpha.$$

Because $([M : F], p) = 1$ we see that $\alpha \in N_{E/F}(k_{m-1}E)$. Therefore we have established the exactness of our complex at $k_{m-1}F$.

Now assume that $\alpha \in k_m F$ such that $i_E(\alpha) = 0 \in k_m E$. Then arguing as above, we see that there exist a finite extension M/F and $\beta \in k_{m-1}M$ such that

$$\{a\} \cdot \beta = i_M(\alpha) \in k_m M.$$

Applying $N_{M/F}$ and using the projection formula we see that

$$\{a\} \cdot N_{M/F}(\beta) = N_{M/F}(i_M(\alpha)) = [M : F]\alpha.$$

Because $[M : F]$ is coprime with p , $\alpha \in \{a\} \cdot N_{M/F}(\gamma)$ for a suitable element $\gamma \in k_{m-1}M$. Hence we see that our complex is also exact at $k_m F$ and the full complex is exact. \square

Remark. Assuming as usual that F contains a primitive root ξ_p , then if $m \leq 2$, no further assumption on F in Theorem 5 is necessary. For $m = 1$ this claim follows from basic Kummer theory, and for $m = 2$ see [Me, Prop. 5] and [Sr, Chap. 5, Lemma 8.4].

6. REDUCTION TO THE CASE $\text{char } F = 0$

Suppose now that $\text{char } F = q > 0$ and $q \neq p$. We also assume that F is infinite, because if F is finite then $K_n F = 0$ for $n \geq 2$ and therefore this is a trivial case. (See [FV, Prop. IX.1.3].) Assume as before that F contains a primitive p th root of unity ξ_p and E/F is a cyclic extension of degree p . We shall show that there exists an explicitly defined cyclic extension J/L of degree p such that $\text{char } L = 0$, so that L is perfect, and $k_n J$ is naturally isomorphic with $k_n E$ as a $G = \text{Gal}(E/F) \cong \text{Gal}(J/L)$ module.

Recall first that there exists a discrete valuation ring A of characteristic 0 such that its maximal ideal M is generated by q and $A/M \cong F$. (Such a ring is called a q -ring. See [Ma, p. 223].) By passing to a completion \hat{A} of A with respect to M -adic topology and observing that $\hat{A}/\hat{M} \cong A/M \cong F$ and $\hat{M} = \hat{A}.q$ we see that we may and will assume that A is a complete local q -ring such that $A/M \cong F$.

It is known that a complete q -ring is uniquely determined up to its isomorphism by its residue field. (See [Ma, Cor., p. 225].) Observe further that a complete discrete valued field is henselian. (See [R, Thm. 5].)

Now following [FW, §IX.3] we have a natural construction

$$R = \varinjlim (A^{(1)} \subset A^{(2)} \subset A^{(3)} \subset \dots), \text{ where}$$

$$A^{(1)} = A \text{ as above and } A^{(n+1)} := A^{(n)}[t]/(t^p - \pi_n),$$

where π_n is a uniformizer of $A^{(n)}$ for $n \geq 1$. As was noticed in [FW], this ring R is a henselian valuation ring of characteristic 0, with value group the underlying group of the ring $\mathbb{Z}[1/p]$.

For each $i \in \mathbb{N}$ let $L^{(i)}$ be the quotient field of $A^{(i)}$, and let L be the quotient field of R . Then from Lemma IX.3.5 in [FW], there is a natural isomorphism $k_n F \cong k_n L$ for each $n \in \mathbb{N}$.

Let T be the inertia subgroup of $G_{L^{(1)}}$. We have the natural isomorphism $G_{L^{(1)}}/T \xrightarrow{\sim} G_F$. Let φ be the compositum of the natural surjections

$$G_{L^{(1)}} \rightarrow G_{L^{(1)}}/T \xrightarrow{\sim} G_F \rightarrow \text{Gal}(E/F),$$

and let $J^{(1)}$ be the fixed field of $\ker \varphi$. Then $\text{Gal}(J^{(1)}/L^{(1)})$ is naturally isomorphic to $\text{Gal}(E/F)$. (This is a special case of a more general construction about lifting certain Galois abelian extensions. See [K, Lemma 2.5].)

Thus we see that $J^{(1)}/L^{(1)}$ is a cyclic, purely inert extension of degree p . Because the tower

$$L^{(1)} \subset L^{(2)} \subset \dots \subset L^{(n)} \subset \dots$$

is a chain of totally ramified extensions $L^{(n+1)}/L^{(n)}$, we see that $J^{(1)} \cap L = L^{(1)}$. Set $J := J^{(1)}L$. Then J/L is a cyclic, purely inert extension of degree p .

Now set $J^{(i)} := J^{(1)}L^{(i)}$, and let $B^{(i)}$ be the unique valuation ring in $J^{(i)}$ such that $B^{(i)} \cap L^{(i)} = A^{(i)}$. ($B^{(i)}$ is unique because $A^{(i)}$ is henselian.) Then again following the proofs of Lemma IX.3.2 and Lemma IX.3.5 in [FW], we establish, in exactly the same way we proved that $k_n F \cong k_n L$, that $k_n E \cong k_n J$ under a G -equivariant isomorphism. Our reduction is complete.

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