CANONICAL p-DIMENSION OF ALGEBRAIC GROUPS

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Abstract. We describe a way to compute the p-relative version of the Berhuy-Reichstein canonical dimension for an arbitrary split semisimple algebraic group over an arbitrary field of an arbitrary characteristic (p is any prime integer). The canonical p-dimension is computed for all split simple groups of classical types.

Contents

The notion of the canonical dimension of an algebraic structure was introduced by Berhuy and Reichstein in [1]. The canonical dimension measures the size of generic

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splitting fields of the structure. The formal definition is given in Section 2. Here we present two basic examples:

• Let X be a scheme over a field F. A field extension L/F is called a *splitting field of* X, if X has a point over L. A splitting field L is called *generic*, if for any splitting field K of X there exists an F-place $L \to K$. The *canonical dimension of* X is the minimum of the transcendence degree (over F) of all generic splitting fields of X.

• Let G be an algebraic group over F. The *canonical dimension of* G is the maximum of the canonical dimension of all principal homogeneous varieties (G-torsors), defined over field extensions of F.

When dealing with a given algebraic structure, we usually have finitely many "significant" prime integers involved. For example, such primes associated with an algebraic group G are the *torsion prime integers of* G (see Remark 6.7). In order to locate contribution of a prime integer p to the canonical dimension, we define *canonical p-dimension* in a similar fashion.

It turns out that canonical dimension and p-dimension of an arbitrary regular complete variety X is closely related to the algebraic cycles on X (see Corollaries 4.7 and 4.12). We express canonical p-dimension of a generically cellular variety in terms of its Chow group (see Theorem 5.8).

The main result of the paper is Theorem 6.9, giving a recipe to compute canonical p-dimension of an arbitrary split semisimple algebraic group over an arbitrary field (of arbitrary characteristic). The values of the canonical p-dimension are given for all split simple groups of classical type (see §8).

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1. Agreements and preliminaries

1.1. **Varieties.** We refer as *schemes* to separated schemes of finite type over a field (there is no restrictions on the field, its characteristic is arbitrary).

Variety in the paper is an integral scheme.

For a scheme X, the integer $d(X)$ is defined as the g.c.d. of degree of all closed points on X; for a prime integer p, $d_p(X)$ is the p-primary part of $d(X)$.

1.2. **Chow groups.** Let X be a scheme over a field F. We write $\mathrm{CH}(X)$ for the integral Chow group of X (see [6]). Fixing a prime p, we write $\mathrm{Ch}(X)$ for the modulo p Chow group:

$$\mathrm{Ch}(X) = \mathrm{CH}(X)/p \cdot \mathrm{CH}(X) .$$

Furthermore, we write $\mathrm{Ch}(\overline{X})$ (resp. $\mathrm{CH}(\overline{X})$) for the colimit of $\mathrm{Ch}(X_L)$ (resp. $\mathrm{CH}(X_L)$) with L running over all field extensions L/F, and we write $\overline{\mathrm{Ch}}(X)$ (resp. $\overline{\mathrm{CH}}(X)$) for the image of the restriction homomorphism $\mathrm{res}\colon \mathrm{Ch}(X) \to \mathrm{Ch}(\overline{X})$ (resp. $\mathrm{CH}(X) \to \mathrm{CH}(\overline{X})$). The group $\overline{\mathrm{CH}}(X)$ is called the *reduced* Chow group of X; the group $\overline{\mathrm{Ch}}(X)$ is called the *modulo p reduced* Chow group of X. Note that

$$\overline{\mathrm{Ch}}(X) = \overline{\mathrm{CH}}(X)/\big(\overline{\mathrm{CH}}(X) \cap p\,\mathrm{CH}(\overline{X})\big)$$

is not the same as $\overline{\mathrm{CH}}(X)/p\,\overline{\mathrm{CH}}(X)$.

1.3. **Places.** Let K be a field. A *valuation ring* R of K is a subring $R \subset K$, satisfying $K = R \cup (R \setminus \{0\})^{-1}$. Any valuation ring is local; $R = K$ is a trivial example of a valuation ring.

Given two fields K and L, a *place* $K \to L$ is a local ring homomorphism $\pi\colon R \to L$ of a valuation ring $R \subset K$ (an embedding of fields is a trivial example of a place).

If K and L are extensions of a field F, an *F-place* (or *place over* F) is a place $K \to L$ with π defined and identical on F.

Places are composable: if $K \to L$ is a place, given by a ring homomorphism π, and $L \to E$ a place to a third field E, given by a homomorphism ρ of a ring $S \subset L$, then the composition is the place $K \to E$, given the homomorphism $\rho \circ \pi : \pi^{-1}(S) \to E$, defined on the valuation ring $\pi^{-1}(S)$. In particular, any place $L \to E$ can be restricted to any subfield $K \subset L$.

In this paper, a place $K \to L$ is said to be *geometric*, if it can be represented as a composition of places with valuation rings being discrete valuation rings.

1.4. **Places and rational morphisms.** Let X and Y be varieties over a field F. If X is complete, then for any valuation ring R of $F(X)$ there exists an F-morphism $\operatorname{Spec} R \to X$ [8, ch. II th. 4.7]; therefore an F-place $F(X) \to F(Y)$ produces a rational morphism $Y \to X$.

Vice versa, if there is a rational morphism $Y \to X$ and X is regular, then there exists a *geometric* F-place $F(X) \to F(Y)$. Indeed, since X is regular at the image x of the generic point of Y, there exists a system of local parameters around x, which produces a geometric place $F(X) \to F(x)$; composing with the embedding $F(x) \subset F(Y)$, we get a required place $F(X) \to F(Y)$.

2. Canonical dimension of determination functions

Let F be a field, $\mathbf{Fields}_F$ the category of all field extensions of F. Let $\mathbf{2^0}$ be the category of the subsets of a 1-elemental set 0. A *determination function* D over F is a continuous functor $\mathbf{Fields}_F \to \mathbf{2^0}$, where by continuity we mean that D commutes with the filtered colimits. In other words, D is a rule assigning to each $E \in \mathbf{Fields}_F$ a value $D(E) \in \{\emptyset, 0\}$ such that

- if $D(E) = 0$ for some E, then $D(E') = 0$ for any E' admitting an F-embedding $E \to E'$;
- (continuity property) if $D(E) = 0$ for some field E covered by a (possibly infinite) filtered family of subfields E_i, then $D(E_i) = 0$ for some E_i.

A field $E \in \mathbf{Fields}_F$ is called a *splitting field* of a determination function D, if $D(E) = 0$. A splitting field E of D is called *generic*, if for any splitting field L there exists an F-place $E \to L$. If D has at least one generic splitting field, *canonical dimension* $\mathrm{cd}(D)$ of D is defined as the minimum of the transcendence degree (over F) of all generic splitting fields of D; if D does not admit a generic splitting field, we set $\mathrm{cd}(D) = \infty$.

Lemma 2.1. *For a given determination function D, any its splitting field, which is a subfield of a generic splitting field, is also generic. Besides, any splitting field contains a finitely generated splitting field and* $\operatorname{cd}(D) = \infty$ *only if D does not admit generic splitting.*

Proof. If E is a generic splitting field and E' a splitting field contained in E, then for any splitting field L, restricting a place $E \to L$ to E', we get a place $E' \to L$; therefore E' is also generic.

Any splitting field contains a finitely generated splitting field by the continuity of the determination function.

If D has a generic splitting field, then, taking a finitely generated splitting subfield, we get a finitely generated generic splitting field, showing that $\operatorname{cd}(D)$ is finite. □

A determination function D over F is *split*, if $D(F) = 0$. In this case, F is a generic splitting field of D and $\operatorname{cd}(D) = 0$.

Our basic example of a determination function is the determination function associated with a scheme X over F:

$$L \mapsto \begin{cases} \emptyset, & \text{if } X(L) = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Canonical dimension $\operatorname{cd}(X)$ of an F-scheme X is defined as the canonical dimension of the associated determination function.

Example 2.2 ([12, th. 4.3])**.** Let F be a field of characteristic $\neq 2$. Let X be an anisotropic smooth projective quadric over F. Then $\operatorname{cd}(X) = \dim X - \mathfrak{i}_1(X) + 1$, where $\mathfrak{i}_1(X)$ is the first Witt index of X.

Let **pSets** be the category of pointed sets and let k be a field. A functor

$$\mathcal{F} \colon \mathbf{Fields}_k \to \mathbf{pSets}$$

is called *continuous*, if it commutes with filtered colimits. If $\mathcal{F}$ is a continuous functor, then for any $F \in \mathbf{Fields}_k$ and $\alpha \in \mathcal{F}(F)$, we get a determination function D_α over F by setting

$$D_\alpha(L) = \begin{cases} 0, & \text{if } \alpha_L \text{ is the distinguished point of the set } \mathcal{F}(L); \\ \emptyset, & \text{otherwise.} \end{cases}$$

Berhuy-Reichstein canonical dimension of a continuous functor $\mathcal{F}$ [1, §10] is the supremum of $\operatorname{cd}(D_\alpha)$ for all F and $\alpha \in \mathcal{F}(F)$. If G is an algebraic group over the field k, *canonical dimension of* G is defined as canonical dimension of the (continuous) functor Tors_G, taking a field F to the set of isomorphism classes $\operatorname{Tors}_G(F)$ of G-torsors over F. We note that canonical dimension of an algebraic group G/k is not the same as canonical dimension of the underlying variety of G (which is always 0 because $G(k) \neq \emptyset$).

3. Notion of canonical p-dimension

Let us fix an arbitrary prime p and refer to a splitting field E of a determination function D as *p-generic*,[1] if for any splitting field L of D there exists a finite field extension L'/L

[1]Our notion of p-generic splitting is based on on the notion of *p-generic splitting varieties* of symbols in a modulo p Milnor's K-group of a field, introduced in [16].

of degree prime to p admitting a place $E \to L'$. Replacing generic splitting fields by the p-generic ones in the definitions of section 2, we get a modified notion of canonical dimension which we call *canonical p-dimension* and denote cd_p.

We refer to a finite field extension as *p-coprime*, if its degree is not divisible by p.

The following two Lemmas are useful when working with cd_p.

Lemma 3.1 (cf. [12, lemma 3.3]). *Let K be an arbitrary field, p a prime, K'/K a p-coprime field extension, and L/K an arbitrary field extension. Then there exists a field L', containing K' and L, such that the extension L'/L is also p-coprime.*

Proof. We almost repeat the proof of [12, lemma 3.3], where the case of $p = 2$ was treated. We may assume that K' is generated over K by one element; let $f(t) \in F[t]$ be its minimal polynomial. Since the degree of f is coprime with p, there exists an irreducible divisor $g \in L[t]$ of f over L such that $\deg(g)$ is coprime with p as well. We set $L' = L[t]/(g)$. □

Lemma 3.2. *Let K be a field extension of F of finite transcendence degree over F; let $K \to L$ be a geometric F-place and let K' be a p-coprime field extension of K. Then there exists a p-coprime field extension L'/L such that the place $K \to L$ extends to a place $K' \to L'$.*

Proof. By Lemma 3.1 it suffices to prove Lemma 3.2 in the case where the place $K \to L$ is surjective and its valuation ring R is a discrete valuation ring. Also it is clear, that is suffices to consider only two cases: (1) K'/K is purely inseparable and (2) K'/K is separable.

In the first case, the degree $[K' : K]$ is a power of a prime $q \neq p$. We take an arbitrary valuation ring R' of K', *lying over* R, i.e., such that $R' \cap K = R$ and the embedding $R \to R'$ is local (such R' exists in the case of an arbitrary field extension K'/K, [17, ch. VI th. 5′]). Let L' be the residue field of R' so that we have a surjective place $K' \to L'$. We show that L' is also purely inseparable over L (and therefore $[L' : L]$, being a power of the same q, is coprime to p). For this, we take an element $l \in L'$ and show that $l^{q^n} \in L$ for some n: let $k \in R'$ be a preimage of l; then $k^{q^n} \in K$ for some n and consequently $l^{q^n} \in L$ for the same n.

In the second case we consider all valuation rings $R_1, \dots, R_r$ of K', lying over R (the number of such valuation rings is finite by [17, ch. VI th. 12 cor. 4]). The residue field of each R_i is a finite extension of L. Moreover, $\sum_{i=1}^{r} e_i n_i = [K' : K]$ [17, ch. VI th. 20 and p. 63] (the discrete valuation ring assumption and the separability assumption are needed for this equality), where n_i is the degree over L of the residue field of R_i, and e_i is the *reduced ramification index* of R_i over R, [17, def. on pp. 52–53]. It follows that at least one of n_i is not divisible by p. □

Let us make some first general observations on cd_p. Clearly, a generic splitting field of a determination function is also p-generic; therefore we always have $\mathrm{cd} \geq \mathrm{cd}_p$.

Also it is clear, that cd_p is not interesting, if the determination function in question has a p-coprime splitting field. More precisely, one has a simple

Lemma 3.3. *If a determination function D is split by an algebraic field extension E/F of degree non-divisible by p, then* $\mathrm{cd}_p(D) = 0$.

Proof. Taking a finitely generated splitting subfield of E, we get a p-coprime splitting field. It follows by Lemma 3.1, that F is a p-generic splitting field of D. $\square$

Example 3.4. The computation of $\mathrm{cd}(X)$ for an anisotropic smooth projective quadric X (over a field of characteristic $\neq 2$), given in [12] (see Example 2.2), shows in fact also that $\mathrm{cd}_2(X) = \mathrm{cd}(X)$.

Example 3.5. Let X be a Severi-Brauer variety. If $d(X) = d_p(X)$ (that is, $d(X)$ is a power of the prime p), then $\mathrm{cd}(X) = \mathrm{cd}_p(X) = d_p(X) - 1$, [1, th. 11.4]. Now if $d(X)$ is not a power of a prime, the value of $\mathrm{cd}(X)$ is not known, while $\mathrm{cd}_p(X)$ is still $d_p(X) - 1$ (see Example 5.10).

Example 3.6. The computation of $\mathrm{cd}(\mathbf{SO}_n)$, given in [11], also shows that $\mathrm{cd}_2(\mathbf{SO}_n) = \mathrm{cd}(\mathbf{SO}_n)$ (see also Example 5.11 as well as (8.2) and (8.4)).

Remark 3.7. Let $\mathcal{F}$ and $\mathcal{F}'$ be continuous functors $\mathbf{Fields}_k \to \mathbf{pSets}$ with a morphism $f : \mathcal{F} \to \mathcal{F}'$. If the kernel of f is trivial, then for any $F \in \mathbf{Fields}$ and any $\alpha \in \mathcal{F}(F)$ the determination function of α coincides with the determination function of $f(\alpha)$ (cf. [1, lemma 10.2(a)]); therefore $\mathrm{cd}(\mathcal{F}) \leq \mathrm{cd}(\mathcal{F}')$ (cf. [1, lemma 10.2(b)]) and $\mathrm{cd}_p(\mathcal{F}) \leq \mathrm{cd}_p(\mathcal{F}')$ (for any p). If moreover f is surjective (but may not be necessarily injective), then $\mathrm{cd}(\mathcal{F}) = \mathrm{cd}(\mathcal{F}')$ (cf. [1, lemma 10.2(c)]) and $\mathrm{cd}_p(\mathcal{F}) = \mathrm{cd}_p(\mathcal{F}')$.

4. Canonical (p-)dimension of regular complete varieties

Lemma 4.1. *The function field of a regular variety X is a generic splitting field of X; in particular,* $\mathrm{cd}(X) \leq \dim X$ *for regular X.*

Proof. The function field $F(X)$ is a splitting field of X (even in the non-regular case). To check that it is generic, we take an arbitrary splitting field L of X, find a finitely generated splitting subfield of L, take a model Y/F of it, and consider a rational morphism $Y \to X$, that exists because X has a point over the field $F(Y)$. Since X is regular, there exists a geometric place $F(X) \to F(Y)$; composing it with the embedding $F(Y) \subset L$, we get the required place $F(X) \to L$. $\square$

Remark 4.2. The place $F(X) \to L$, constructed in the end of proof of Lemma 4.1, is geometric (as defined in §1.3).

We have the following generalization of Lemma 4.1:

Lemma 4.3. *If Y is a closed subvariety of a regular variety X, admitting a dominant rational morphism $X \to Y$, then the function field of Y is a generic splitting field of X. In particular,* $\mathrm{cd}(X) \leq \dim Y$.

Proof. Clearly, $F(Y)$ is a splitting field of X. The dominant rational morphism produces an embedding of $F(Y)$ into the field $F(X)$, which by Lemma 4.1 is a generic splitting field of X. It follows by Lemma 2.1 that $F(Y)$ is a generic splitting field too. $\square$

Lemma 4.4. *Let Y be a scheme over a field F, X a variety over F.*

(1) *If Y admits a dominant rational morphism $X \to Y$, then the $F(X)$-scheme $Y_{F(X)}$ has a closed rational point.*

(2) *If the $F(X)$-scheme $Y_{F(X)}$ has a closed rational point, then there exists a closed subvariety $Y' \subset Y$, admitting a dominant rational morphism $X \to Y'$.*

Proof. Existence of a rational morphism $X \to Y$ is equivalent to existence of a closed rational point on $Y_{F(X)}$. To prove the second statement, we take as Y' the closure of the image of the rational morphism $X \to Y$. □

Proposition 4.5. *Any regular complete variety X has a closed subvariety $Y \subset X$ of dimension $\dim Y = \operatorname{cd}(X)$, admitting a dominant rational morphism $X \to Y$.*

Proof. Let us take a generic splitting field of X, having the transcendence degree $\operatorname{cd}(X)$ over F, replace it by a finitely generated splitting subfield (Lemma 2.1), and consider a projective model T/F of it. Since X is regular, $F(X)$ is a generic splitting field of X (Lemma 4.1) and there exists a place $F(X) \to F(T)$. Since the variety X is complete, such a place produces a rational morphism $T \to X$. Replacing T by the closure in $T \times X$ of the graph of such a rational morphism, we get another model of the same field, but now we have a regular morphism $T \to X$ instead of the rational one we had before.

Since $F(T)$ is a generic splitting field of X, there exists a place $F(T) \to F(X)$. By completeness of T, it produces a rational morphism $X \to T$. Composing it with the regular morphism $T \to X$, we get a rational morphism $X \to X$; let Y be the closure of its image. Clearly, $\dim Y \leq \dim T = \operatorname{cd}(X)$. On the other hand, by Lemma 4.3, $\dim Y \geq \operatorname{cd}(X)$. Therefore, $\dim Y = \operatorname{cd}(X)$. □

Combining Lemma 4.3 and Proposition 4.5, we get

Corollary 4.6. *Canonical dimension of a regular complete variety X is the minimum of dimension of closed subvarieties $Y \subset X$, admitting a dominant rational morphism $X \to Y$.* □

Taking into account Lemma 4.4, we get the following variation of Corollary 4.6:

Corollary 4.7. *Canonical dimension of a regular complete variety X is the minimum of dimension of closed subvarieties $Y \subset X$, satisfying $Y(F(X)) \neq \emptyset$.* □

Now we establish variations of Lemma 4.3, Proposition 4.5, and Corollaries 4.6 and 4.7, related to the canonical p-dimension.

Let Y be a scheme over a field F and let X be a variety over F. We say that Y satisfies condition $(*)$ (with respect to X), if there exists an F-variety X', a dominant rational morphism $X' \to Y$, and a dominant rational morphism $X' \to X$ such that the field extension $F(X')/F(X)$ is p-coprime.

Lemma 4.8. *If Y is a closed subvariety of a regular variety X, satisfying condition $(*)$, then the function field of Y is a p-generic splitting field of X. In particular,*

$$\operatorname{cd}_p(X) \leq \dim Y \ .$$

Proof. Clearly, $F(Y)$ is a splitting field of X. Let now L/F be any splitting field of X. Then we can find a geometric place $F(X) \to L$ (see Lemma 4.1 with Remark 4.2). Applying Lemma 3.2 to this place and the field extension $F(X')/F(X)$, we get a place $F(X') \to L'$ for some p-coprime field extension L'/L. Restricting the latter place to the subfield $F(Y) \subset F(X')$, we get a place $F(Y) \to L'$; therefore, the splitting field $F(Y)$ is p-generic. □

Lemma 4.9. *Let Y be a scheme over a field F, X a variety over F.*

1. *If Y satisfies condition $(*)$, then $d_p(Y_{F(X)}) = 1$ (see §1.1 for definition of d_p).*
2. *If $d_p(Y_{F(X)}) = 1$, then there exists a closed subvariety $Y' \subset Y \subset X$, satisfying condition $(*)$.* □

Proposition 4.10. *Any regular complete variety X has a closed subvariety $Y \subset X$ of $\dim Y = \operatorname{cd}_p(X)$, satisfying condition $(*)$.*

Proof. Let us take a finitely generated p-generic splitting field of X, having the transcendence degree $\operatorname{cd}_p(X)$ over F, and consider a projective model T/F of it. Since $F(X)$ is a generic splitting field of X, while $F(T)$ is a splitting field, there exists a place $F(X) \to F(T)$. Since the variety X is complete, such a place produces a rational morphism $T \to X$. Replacing T by the closure in $T \times X$ of the graph of this morphism, we get another model of the same field, but now we have a regular morphism $T \to X$ instead of the rational one.

Since $F(T)$ is a p-generic splitting field of X, there exists a place $F(T) \to L'$, where $L' \supset F(X)$ is a field, p-coprime over $F(X)$. Let X'/F be a model of the field L'. By completeness of T, the place $F(T) \to F(X')$ produces a rational morphism $X' \to T$. Composing it with the regular morphism $T \to X$, we get a rational morphism $X' \to X$; let Y be the closure of its image. Clearly, $\dim Y \le \dim T = \operatorname{cd}_p(X)$. On the other hand, by Lemma 4.8, $\dim Y \ge \operatorname{cd}_p(X)$. Therefore, $\dim Y = \operatorname{cd}_p(X)$. □

Lemma 4.8 and Proposition 4.10 together produce

Corollary 4.11. *Canonical p-dimension of a regular complete variety X is the minimum of dimension of the closed subvarieties $Y \subset X$, satisfying $(*)$.* □

By Lemma 4.9, the following variation of Corollary 4.11 also holds:

Corollary 4.12. *Canonical p-dimension of a regular complete variety X is the minimum of dimension of the closed subvarieties $Y \subset X$ with $d_p(Y_{F(X)}) = 1$.* □

5. Generically p-split varieties

In this section X stands for a smooth complete absolutely irreducible variety over a field F.

Lemma 5.1. *Degree homomorphism $\deg\colon \operatorname{Ch}_0(\overline{X}) \to \mathbb{F}_p$ is an isomorphism if and only if $\dim_{\mathbb{F}_p} \operatorname{Ch}_0(\overline{X}) = 1$.*

Proof. The degree homomorphism is non-zero and therefore surjective. □

Lemma 5.2. *Assume that $\dim_{\mathbb{F}_p} \operatorname{Ch}_0(\overline{X}) = 1$. Let T be an arbitrary F-scheme, let E_1 and E_2 be field extensions of F, and let $f_1 : \operatorname{Spec} E_1 \to X$ and $f_2 : \operatorname{Spec} E_2 \to X$ be*

F-morphisms. Then the diagram

$$\begin{array}{ccccc} & & \overline{\mathrm{Ch}}(T \times X) & & \\ & \swarrow^{(\mathrm{id}_T \times f_1)^*} & & \searrow^{(\mathrm{id}_T \times f_2)^*} & \\ \overline{\mathrm{Ch}}(T_{E_1}) & & & & \overline{\mathrm{Ch}}(T_{E_2}) \\ & \searrow & & \swarrow & \\ & & \mathrm{Ch}(\overline{T}) & & \end{array}$$

is commutative.

Proof. Let E be a field extension of F, containing E_1 and E_2. Replacing T and X by T_E and X_E, we come to the following situation: $E = E_1 = E_2 = F$ and for some closed rational points $x_1, x_2 \in X$, f_i is the embedding $T = T \times \{x_i\} \hookrightarrow T \times X$. We want to show that $f_1^* = f_2^* \colon \overline{\mathrm{Ch}}(T \times X) \to \overline{\mathrm{Ch}}(T)$. Since $pr_* \circ f_{i*} = \mathrm{id}$ for $i = 1, 2$, where pr is the projection $T \times X \to T$, it suffices to show that

$$f_{1*} \circ f_1^* = f_{2*} \circ f_2^* \colon \overline{\mathrm{Ch}}(T \times X) \to \overline{\mathrm{Ch}}(T \times X)\ .$$

The composition $f_{i*} \circ f_i^*$ coincides with the multiplication by $[T \times x_i]$. Since by the assumption on $\dim_{\mathbb{F}_p} \mathrm{Ch}_0(\overline{X})$ and Lemma 5.1, the degree homomorphism $\deg \colon \mathrm{Ch}(\overline{X}) \to \mathbb{F}_p$ is an isomorphism, we have $[x_1] = [x_2] \in \overline{\mathrm{Ch}}(X)$, and therefore $[T \times x_1] = [T \times x_2]$ in $\overline{\mathrm{Ch}}(T \times X)$. The required assertion follows. □

Let $i \colon Y \hookrightarrow X$ be a closed subvariety of X. The closed embedding

$$(\mathrm{id}_Y \times i \times \mathrm{id}_X) \colon Y \times X \to Y \times X \times X$$

is regular, and we define a paring

$$\overline{\mathrm{Ch}}(Y) \otimes \overline{\mathrm{Ch}}(X \times X) \to \overline{\mathrm{Ch}}(Y \times X)\ ,$$

by the formula $\alpha \otimes \beta \mapsto (\mathrm{id}_Y \times i \times \mathrm{id}_X)^*(\alpha \times \beta)$.

Proposition 5.3. *Let Y be a closed subvariety of X. Assume that $\overline{\mathrm{Ch}}(X_{F(X)}) = \mathrm{Ch}(\overline{X})$ and $\dim_{\mathbb{F}_p} \mathrm{Ch}_0(\overline{X}) = 1$. Then the above paring is surjective.*

Proof. We proceed by induction on $\dim Y$. We have a commutative diagram

$$\begin{array}{ccccccc} \bigoplus_{Y'} \overline{\mathrm{Ch}}(Y') \otimes \overline{\mathrm{Ch}}(X \times X) & \longrightarrow & \overline{\mathrm{Ch}}(Y) \otimes \overline{\mathrm{Ch}}(X \times X) & \longrightarrow & \overline{\mathrm{Ch}}(X \times X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{Y'} \overline{\mathrm{Ch}}(Y' \times X) & \longrightarrow & \overline{\mathrm{Ch}}(Y \times X) & \longrightarrow & \overline{\mathrm{Ch}}(X_{F(Y)}) & \longrightarrow & 0 \end{array}$$

where Y' runs over closed subvarieties of codimension 1 in Y (the upper row is the obvious exact sequence $\bigoplus_{Y'} \overline{\mathrm{Ch}}(Y') \to \overline{\mathrm{Ch}}(Y) \to \mathbb{Z} \to 0$, tensored by $\overline{\mathrm{Ch}}(X \times X)$ over $\mathbb{F}_p$). The rows are exact and the left vertical map is surjective by the induction hypothesis. The right vertical map is surjective because the rhombus

$$\begin{array}{ccc} & \overline{\mathrm{Ch}}(X \times X) & \\ \swarrow & & \searrow\!\!\!\!\!\searrow \\ \overline{\mathrm{Ch}}(X_{F(Y)}) & & \overline{\mathrm{Ch}}(X_{F(X)}) \\ \hookrightarrow\!\!\searrow & & /\!\!/ \\ & \mathrm{Ch}(\overline{X}) & \end{array}$$

is commutative (as guaranteed by the assumption on $\mathrm{Ch}_0(\overline{X})$ and Lemma 5.2 applied to $T = X$). □

Corollary 5.4. *Under condition of Proposition 5.3, if the push-forward*

$$(i \times \mathrm{id}_X)_* \colon \overline{\mathrm{Ch}}(Y \times X) \to \overline{\mathrm{Ch}}(X \times X)$$

is non-zero, then the push-forward $i_* \colon \overline{\mathrm{Ch}}(Y) \to \overline{\mathrm{Ch}}(X)$ *is also non-zero and, in particular,* $\overline{\mathrm{Ch}}_i(X) \neq 0$ *for at least one* $i \leq \dim Y$.

Proof. The square

$$\begin{array}{ccc} \overline{\mathrm{Ch}}(Y) \otimes \overline{\mathrm{Ch}}(X \times X) & \twoheadrightarrow & \overline{\mathrm{Ch}}(Y \times X) \\ \downarrow{\scriptstyle i_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle (i \times \mathrm{id})_*} \\ \overline{\mathrm{Ch}}(X) \otimes \overline{\mathrm{Ch}}(X \times X) & \longrightarrow & \overline{\mathrm{Ch}}(X \times X) \end{array}$$

is commutative. □

Definition 5.5. We say that a (complete smooth absolutely irreducible) variety X over F as *p-balanced*, if the symmetric bilinear form

$$\mathrm{Ch}(\overline{X}) \times \mathrm{Ch}(\overline{X}) \to \mathbb{F}_p\,, \quad (\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)$$

is non-degenerate (in the sense that its radical is trivial; note that $\dim_{\mathbb{F}_p} \mathrm{Ch}(\overline{X})$ can be infinite).

A variety X over F is called *cellular*, if there is a filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

by closed subschemes such that for every $i = 0, 1, \dots, n-1$ the scheme $X_{i+1} \setminus X_i$ is isomorphic to an affine space over F.

Remark 5.6. Let X be a geometrically cellular variety, that is, X_E is cellular for some field extension E/F. We claim that X is p-balanced (for any p). Indeed, the Chow-motive of the cellular variety X_E decomposes into a finite direct sum of twists of the motive of the point (see, e.g., [10, th. 6.5]). Therefore $\mathrm{CH}(X_E) = \mathrm{CH}(\overline{X})$. Moreover, the mutually inverse isomorphisms of the motive of X_E with the above direct sum are given by certain sequences $e_0, \dots, e_n$ and $e'_0, \dots, e'_n$ of homogeneous elements in $\mathrm{CH}(X_E)$, that are bases of $\mathrm{CH}(X_E)$ mutually dual with respect to the $\mathbb{Z}$-bilinear form $(\alpha,\beta)\mapsto \deg(\alpha\cdot\beta)$ (simply because they define mutually inverse isomorphisms of motives).

Note that for any p-balanced X and any integer i, one has $\dim_{\mathbb{F}_p}\mathrm{Ch}^i(\overline{X})=\dim_{\mathbb{F}_p}\mathrm{Ch}_i(\overline{X})$, if at least one of these dimensions is finite. Since $\dim_{\mathbb{F}_p}\mathrm{Ch}^0(\overline{X})$=1, the above relation, taken with $i=0$, implies that $\dim_{\mathbb{F}_p}\mathrm{Ch}_0(\overline{X})=1$ for a p-balanced X.

Definition 5.7. A p-balanced variety X over F is called *p-split*, if $\overline{\mathrm{Ch}}(X)=\mathrm{Ch}(\overline{X})$.

A cellular variety is p-split.

We say that a variety X has a property *generically*, if X over its own function field has this property. This way we get a notion of *generically p-split* variety. According to above remarks, a *generically cellular* variety is generically p-split.

We are ready to prove the main result of the first half of the paper, interpreting the canonical p-dimension of a generically p-split variety in terms of its modulo p reduced Chow group:

Theorem 5.8. *If X is a generically p-split variety (see Definitions 5.7 and 5.5), then*

$$\mathrm{cd}_p(X)=\min\{i\mid\ \overline{\mathrm{Ch}}_i(X)\neq 0\}\ .$$

In particular, the formula holds for a generically cellular X.

Proof. Two inequalities are proved separately.

$\boxed{\leq}$ Let i be an integer such that the group $\overline{\mathrm{Ch}}_i(X)$ is non-zero. Then $[Y]\neq 0$ for a closed i-dimensional subvariety $Y\subset X$. We are going to show that $d_p(Y_{F(X)})=1$ for such Y, what is enough for our aim by Corollary 4.12.

Since the variety $X_{F(X)}$ is p-split, there exists a prime cycle $Z\subset X_{F(X)}$ such that $\deg([Y_{F(X)}]\cdot[Z])\neq 0$. Since the product $[Y_{F(X)}]\cdot[Z]$ can be represented by a cycle on the intersection $Y_{F(X)}\cap Z$ (see [6, §8.1]), the scheme $Y_{F(X)}$ has a closed p-coprime point, meaning that $d_p(Y_{F(X)})=1$.

$\boxed{\geq}$ Let now Y be a closed subvariety of X, satisfying condition $(*)$ We will show that $\overline{\mathrm{Ch}}_i(X)\neq 0$ for some $i\leq\dim Y$. The desired inequality will then follow by Proposition 4.10.

Let Z be the closure of the image of the rational morphism $X'\to Y\times X$ (the variety X' with the rational morphisms $X'\to Y$ and $X'\to X$ comes from condition $(*)$). The cycle $(i\times\mathrm{id}_X)_*([Z])\in\overline{\mathrm{Ch}}(X^2)$ is non-zero, because for the second projection $pr\colon X^2\to X$, we have

$$pr_*(i\times\mathrm{id}_X)_*[Z]=[F(X'):F(X)]\cdot[X]\neq 0\in\overline{\mathrm{Ch}}(X)\ .$$

It follows by Corollary 5.4 that $\overline{\mathrm{Ch}}_i(X)\neq 0$ for some $i\leq\dim Y$. □

Remark 5.9. If we take Y with $\dim Y=\mathrm{cd}_p(X)$ in the beginning of the $(\geq)$-part of the proof of Theorem 5.8, then, since we have already proved the $(\leq)$-part of the theorem, we come to the conclusion that the $(\dim Y)$-dimensional component of the homomorphism $i_*\colon\overline{\mathrm{Ch}}(Y)\to\overline{\mathrm{Ch}}(X)$ is non-zero. Since the $(\dim Y)$-dimensional component of the image of i_* is generated by $[Y]\in\overline{\mathrm{Ch}}(X)$, we see that in fact the class in $\overline{\mathrm{Ch}}(X)$ of Y itself is non-zero.

Example 5.10. Let X be the Severi-Brauer variety of a central simple F-algebra A. Since by Theorem 5.8, $\mathrm{cd}_p(X) = \mathrm{cd}_p(X_L)$ for any p-coprime field extension L/F, $\mathrm{cd}_p(X) = \mathrm{cd}_p(Y)$, where Y is the Severi-Brauer variety of a division algebra, Brauer-equivalent to the p-primary part of A. Furthermore, $\overline{\mathrm{Ch}}(Y) = \overline{\mathrm{Ch}}^0(Y)$ by [9, prop. 2.1.1]. Therefore, by Theorem 5.8, $\mathrm{cd}_p(Y) = \dim Y$, so that we get

$$\mathrm{cd}_p(X) = \mathrm{cd}_p(Y) = \dim Y = d_p(X) - 1 .$$

Example 5.11. In this example $p = 2$. Let X/F be the orthogonal grassmannian of n-dimensional totally isotropic subspaces of a $(2n+1)$-dimensional quadratic form. If $d_2(X) = 2^n$, then $\overline{\mathrm{Ch}}(X) = \overline{\mathrm{Ch}}^0(X)$ by [11, prop. 1.4] and therefore

$$\mathrm{cd}_2(X) = \dim(X) = n(n+1)/2 .$$

Without any restriction on $d_2(X)$, canonical 2-dimension of X can be expressed as the sum of all i such that the i-th special Schubert class $e_i \in \mathrm{CH}^i(\overline{X})$ is *non-rational*, i.e, does not lie in $\overline{\mathrm{CH}}(X)$ (cf. [18, conj. 6.6]): indeed, by [18, main th. 5.7], the product of all *rational* e_i, reduced modulo 2, is a non-zero element of $\overline{\mathrm{Ch}}(X)$ of the smallest possible dimension.

6. Canonical p-dimension of algebraic groups

If P is an algebraic group over a field F, we write $\mathrm{CH}(BP)$ for the P-equivariant Chow ring $\mathrm{CH}_P(\operatorname{Spec} F)$ of the point $\operatorname{Spec} F$ (see [5]).

Let G be a connected algebraic group over F and let P be a subgroup of G. Consider the homomorphism

$$\varphi_G = \varphi_{G,P} : \mathrm{CH}(BP) = \mathrm{CH}_P(\operatorname{Spec} F) \xrightarrow{q^*} \mathrm{CH}_P(G) = \mathrm{CH}(G/P),$$

where $q : G \to \operatorname{Spec} F$ is the structure morphism.

Remark 6.1. If G is a subgroup of a group G', then $\varphi_G = i^* \circ \varphi_{G'}$, where $i : G/P \to G'/P$ is the morphism, induced by embedding of the groups.

Proposition 6.2. *Let $G = \mathbf{GL}_n$. Then the map φ_G is surjective and the left G-action on G/P induces the trivial action on* $\mathrm{CH}(G/P)$.

Proof. The group G is embedded into the affine space of $\mathrm{End}(F^n)$ as a G-equivariant open subset. The map q^* factors as the composite

$$\mathrm{CH}_P(\operatorname{Spec} F) \to \mathrm{CH}_P(\mathrm{End}(F^n)) \to \mathrm{CH}_P(G),$$

where the first pull-back map is an isomorphisms by the homotopy invariance property and the second restriction map is surjective by the localization. Hence φ_G is surjective.

For a rational point g of G, let $\lambda_g : G \to G$ is the morphism of the left multiplication by g. It follows from $q \circ \lambda_g = q$ that $\lambda_g^* \circ q^* = q^*$. Since q^* is surjective, λ_g^* is the identity, i.e., G acts trivially on $\mathrm{CH}(G/P)$. □

Recall that we write $\mathrm{CH}(\overline{G/P})$ for the colimit of $\mathrm{CH}(G_L/P_L)$ over all field extensions L/F. We define a homomorphism $\overline{\varphi}_G$ as the composite

$$\overline{\varphi}_G : \mathrm{CH}(BP) \xrightarrow{\varphi_G} \mathrm{CH}(G/P) \xrightarrow{\mathrm{res}} \mathrm{CH}(\overline{G/P}) .$$

Let E be a (right) G-torsor over a field extension F' of F. Set $K = F'(E)$. Let $\psi_E\colon \mathrm{CH}(E/P) \to \mathrm{CH}(G_K/P_K)$ be pull-back map with respect to the morphism $G_K/P_K \to E/P$, induces by the G-equivariant morphism $G_K \to E$, taking the identity of G to the generic point of E. We define a homomorphism $\overline{\psi}_E$ as the composite

$$\overline{\psi}_E : \mathrm{CH}(E/P) \xrightarrow{\psi_E} \mathrm{CH}(G_K/P_K) \xrightarrow{\mathrm{res}_K} \mathrm{CH}(\overline{G/P}) \ .$$

We identify G with a subgroup of $S = \mathbf{GL}_n$ for some n.

Lemma 6.3. *Suppose that there is a G-equivariant morphism $E \to S$ over F and let $h : E/P \to S/P$ be the induced morphism. Then $\overline{\varphi}_G = \overline{\psi}_E \circ h^* \circ \varphi_S$.*

Proof. The composition $G_K \to E_K \to S_K$ of the morphisms induced by $G_K \to E$ and $E \to S$, differs from the inclusion $G_K \hookrightarrow S_K$ by a left multiplication by an element of $S(K)$. By Proposition 6.2, the induced pull-back homomorphisms $\mathrm{CH}(S_K/P_K) \to \mathrm{CH}(G_K/P_K)$ coincide. Composing with the restriction homomorphism $\mathrm{CH}(S/P) \to \mathrm{CH}(S_K/P_K)$, we get $\psi_E \circ h^* = \mathrm{res}_{K/F} \circ i^*$, where $i : G/P \to S/P$ is the morphisms, induced by the embedding $G \hookrightarrow S$. We have:

$$\overline{\varphi}_G = \mathrm{res} \circ \varphi_G = \mathrm{res}_K \circ \mathrm{res}_{K/F} \circ i^* \circ \varphi_S = \mathrm{res}_K \circ \psi_E \circ h^* \circ \varphi_S = \overline{\psi}_E \circ h^* \circ \varphi_S$$

(for the second equality, see Remark 6.1). ☐

Theorem 6.4. *1. For any G-torsor E (over any field extension of F) we have*

$$\mathrm{Im}(\overline{\varphi}_G) \subset \mathrm{Im}(\overline{\psi}_E) \ .$$

2. There exists a G-torsor E (over a field extension of F) such that $\mathrm{Im}(\overline{\varphi}_G) = \mathrm{Im}(\overline{\psi}_E)$.

Proof. 1. We may assume that E is a G-torsor over F. By the Hilbert theorem 90, the S-torsor $(E \times S)/G$ is trivial (where $(E \times S)/G$ stands for the quotient of $E \times S$ by the action $(e,s)\cdot g = (eg, g^{-1}s)$ of G; the action of G on this quotient is defined by the formula $(e,s) \cdot g = (e, sg)$, so that the embedding $E = E \times 1 \hookrightarrow E \times S$ induces a G-equivariant morphism $E \to (E \times S)/G$). In particular, there is a G-equivariant morphism $E \to S$. By Lemma 6.3, $\mathrm{Im}(\overline{\varphi}_G) \subset \mathrm{Im}(\overline{\psi}_E)$.

2. Let $X = S/G$ and $K = F(X)$. Denote by $E \to \operatorname{Spec} K$ the generic fiber of the projection $S \to X$. Clearly, E is a G-torsor over K. Denote by $h : E/P_K \to S/P$ the morphism induced by the canonical G-equivariant morphism $E \to S$. Since E/P_K is a localization of S/P, the pull-back homomorphism h^* is surjective. By Proposition 6.2, φ_S is also surjective. It follows from Lemma 6.3 that $\mathrm{Im}(\overline{\varphi}_G) = \mathrm{Im}(\overline{\psi}_E)$. ☐

Let G be an algebraic group over a field F and let Tors_G be the functor $\mathbf{Fields}_F \to \mathbf{pSets}$, taking a field K to the set of isomorphism classes $\mathrm{Tors}_G(K)$ of G-torsors over K. For a G-torsor E/K, the determination function of $E \in \mathrm{Tors}_G(K)$ coincides with the determination function of the K-variety E.

Let $P \subset G$ be a subgroup. We assume that P is a special group, that is, the functor Tors_P is trivial.

Lemma 6.5. *The determination functions of the varieties E and E/P coincide.*

Proof. Suppose E/P has a point over K. We need to show that $E(K) \neq \emptyset$. The fiber of the natural morphism $E \to E/P$ over the point is a P-torsor. Since P is special, this torsor is trivial, i.e., the fiber has a point over K. $\square$

Remark 6.6. Let G be a split semisimple algebraic group and let P be a parabolic subgroup. The variety G/P is cellular (see, e.g., [2]), therefore $\mathrm{CH}(\overline{G/P}) = \mathrm{CH}(G/P)$.

Remark 6.7. Suppose further that P is a Borel subgroup of G. The image of the composite

$$\mathrm{CH}(BP) \xrightarrow{\varphi_G} \mathrm{CH}(G/P) \xrightarrow{\deg} \mathbb{Z}$$

is a subgroup $t_G\mathbb{Z}$ with a positive integer t_G known as the *torsion index of* G (see [7]). It follows from Theorem 6.4 and Lemma 6.5 that t_G is the l.c.m. of the numbers $d(E)$ over all G-torsors over all field extension. This statement is known as Grothendieck's theorem [7, th. 2]. The prime divisors of the torsion index t_G are called the *torsion primes of* G.

It follows that the canonical dimension of G (resp. $\mathrm{cd}_p(G)$) is the supremum of the canonical dimension of E/P (resp. $\mathrm{cd}_p(E/P)$) over all G-torsors over all field extensions of F.

Let G be a split semisimple group and let $P \subset G$ be a parabolic subgroup. Suppose P is a special group (for example, P is a Borel subgroup of G). Let E be a G-torsor. Note that the variety E/P is projective. In order to apply Theorem 5.8 to the variety E/P, we need the following

Corollary 6.8. *The variety E/P is generically cellular.*

Proof. By Lemma 6.5, the torsor E is split over the function field $L = F(E/P)$, hence $E_L \simeq G_L$ and therefore $(E/P)_L \simeq (G/P)_L$. The latter variety is cellular. $\square$

Theorems 5.8 and 6.4 yield

Theorem 6.9. *Let G be a split semisimple group and let $P \subset G$ be a special parabolic subgroup (for example, a Borel subgroup). Denote by $\widetilde{\mathrm{CH}}(G/P)$ the image of the graded ring homomorphism $\varphi_G : \mathrm{CH}(BP) \to \mathrm{CH}(G/P)$. Then $\mathrm{cd}_p(G)$ for a prime p, is equal to the smallest integer i such that $\widetilde{\mathrm{CH}}_i(G/P)$ is not contained in $p\,\mathrm{CH}_i(G/P)$.* $\square$

Remark 6.10. The canonical dimension $\mathrm{cd}_p(G)$ is positive if and only if p is a torsion prime of G (see Remark 6.7). Indeed, by Theorem 6.9, $\mathrm{cd}_p(G) = 0$ if and only if $\widetilde{\mathrm{CH}}_0(G/P)$ is not divisible by p in $\mathrm{CH}_0(G/P)$, where P is a Borel subgroup of G. Since $\mathrm{CH}_0(G/P)$ is an infinite cyclic group generated by the class of a rational point, the latter is equivalent to the condition that p does not divide t_G, i.e., p is not a torsion prime of G.

7. Remarks on $\widetilde{\mathrm{CH}}(G/P)$

Let P be an arbitrary subgroup of an algebraic group G. Let $P \to \mathbf{GL}(V)$ be a finite-dimensional representation. The group P acts (on the right) on the product $G \times V$ by $(g, v) \cdot p = (g \cdot p,\ p^{-1} \cdot v)$. The factor variety $(G \times V)/P$ is a vector bundle over G/P, we denote it by $\mathrm{Bun}(V)$.

We can view V as a P-equivariant vector bundle over the point $\operatorname{Spec} F$. For any $n \geq 0$, the n-th P-equivariant Chern class $c_n(V)$ is an element of $\mathrm{CH}^n(BP)$ (see [5]).

Let T be a split torus. There is a canonical isomorphism

$$S(\widehat{T}) \xrightarrow{\sim} \mathrm{CH}(BT),$$

(where $\widehat{T}$ is the character group of T, S stands for the symmetric algebra) defined by the property that the image of a character χ is the first Chern class $c_1(\chi)$ where χ is considered as a 1-dimensional representation of T [5, 3.2].

Let P be a special parabolic subgroup of a split semisimple algebraic group G. Let T be a maximal split torus contained in P and let W_P be the Weyl group of P. Since P is special, the canonical homomorphism

$$\mathrm{CH}(BP) \to \mathrm{CH}(BT)^{W_P} = S(\widehat{T})^{W_P}$$

is an isomorphism [5, prop. 6]. Identifying $\mathrm{CH}(BP)$ with $S(\widehat{T})^{W_P}$, we get a homomorphism

$$\varphi_G : S(\widehat{T})^{W_P} \to \mathrm{CH}(G/P)$$

with the image the subring $\widetilde{\mathrm{CH}}(G/P)$.

Lemma 7.1. *Let $\chi_1, \chi_2, \ldots, \chi_m \in \widehat{T}$ be all characters (with multiplicities) of a representation $P \to \mathbf{GL}(V)$. Let $s_n \in S^n(\widehat{T})^{W_P}$ be the standard symmetric functions on the characters χ_i. Then $\varphi_G(s_n) = c_n\big(\mathrm{Bun}(V)\big)$.*

Proof. By naturality of the Chern classes, we have $\varphi_G\big(c_n(V)\big) = c_n\big(\mathrm{Bun}(V)\big)$. On the other hand, $c_n(V)$ is the n-th standard symmetric functions on the characters of V. □

Remark 7.2. Let G be a split semisimple group over an arbitrary field (of an arbitrary characteristic), $B \subset G$ a Borel subgroup, $T \subset B$ a split maximal torus, W the Weyl group of G. The closures X_w of the cells BwB/B of the cellular variety G/B are indexed by the elements $w \in W$ and called *generalized Schubert varieties* of G/B; moreover, $\dim X_w = l(w)$, where $l\colon W \to \mathbb{Z}_{\geq 0}$ is the length function. Taking the unique maximal length element $w_0 \in W$ and setting $X^w = X_{w_0 w}$, we get a different (preferable for us) indexation of the same varieties, for which $\operatorname{codim} X^w = l(w)$. The group $\mathrm{CH}(G/B)$ is free and the classes $[X^w]$, called *generalized Schubert classes*, form its basis.

The following formula for the product of a 1-codimension Schubert class with an arbitrary Schubert class is given in [4, §4.4 cor.2]:

$$[X^{s_\alpha}] \cdot [X^w] = \sum_\beta \langle \beta^\vee, \omega_\alpha \rangle \cdot [X^{w \cdot s_\beta}] \; ,$$

where α is a simple root, ω_α its fundamental weight, $s_\alpha \in W$ the reflection with respect to α; β runs over the set of positive roots such that $l(w \cdot s_\beta) = l(w) + 1$, and $\beta^\vee$ is the dual to β root. Note that the coefficients of this formula depend only on the root system; in particular, they do not depend on the base field and its characteristic. Moreover, this formula completely determines the multiplication table of the basis $[X^w]$, $w \in W$, because the $\mathbb{Q}$-algebra $\mathrm{CH}(G/B) \otimes \mathbb{Q}$ is generated by $\mathrm{CH}^1(G/B)$ [4].

Remark 7.3. Let $P = B$ be a Borel subgroup of G. We have $W_B = 1$ and therefore the subring $\widetilde{\mathrm{CH}}(G/B)$ is generated by $\phi_G(\widehat{T})$. In the case of simply connected G, for the weight ω_α of a simple root α, one has the formula

$$\phi_G(\omega_\alpha) = -[X^{s_\alpha}] \; , \qquad [4, §4 \text{ formula } (7)] \; ,$$

which determines ϕ_G also in the non simply connected case. This formula also shows that if the group G is simply connected, then $\phi_G(\widehat{T}) = \mathrm{CH}^1(G/B)$, and therefore $\widetilde{\mathrm{CH}}(G/B)$ is the subring of $\mathrm{CH}(G/B)$, generated by $\mathrm{CH}^1(G/B)$.

Remark 7.4. From Theorem 6.9 and Remark 7.3, we see that:

(1) if G_1 and G_2 are split semisimple groups, then $\mathrm{cd}_p(G_1 \times G_2) = \mathrm{cd}_p(G_1) + \mathrm{cd}_p(G_2)$;
(2) if $G' \to G$ is a central isogeny of split semisimple groups, then $\mathrm{cd}_p(G') \leq \mathrm{cd}_p(G)$.

Remark 7.5. Let us consider pairs (Φ, A), consisting of a root system Φ and a subgroup of the quotient of the weight lattice of Φ by its root lattice. An isomorphism of pairs $(\Phi, A) \to (\Phi', A')$ is an isomorphism of the root systems $\Phi \to \Phi'$ such that the induced isomorphism of the lattice quotients maps A to A'. To any split semisimple algebraic group G one attaches an isomorphism class of above pairs, to which we refer as *extended type* of G. Theorem 6.9 with Remarks 7.2 and 7.3 shows that $\mathrm{cd}_p(G)$ (for any p) depends only on the extended type of G; in particular, it does not depend on the base field and especially on its characteristic (so that computing $\mathrm{cd}_p(G)$ one may always assume that G is defined over $\mathbb{C}$).

8. Canonical p-dimension of split simple groups of classical types

In this section we compute canonical p-dimension of all split simple groups of classical types. We will need the following

Lemma 8.1. *Let R be a commutative ring, $r \in R$, and let A be the factor ring of the polynomial ring $R[x_1, x_2, \dots, x_n]$ modulo the ideal generated by the polynomial $x_1 + x_2 + \cdots + x_n - r$. The symmetric group $W = S_n$ acts on A by permuting the x_i. If R has trivial $\mathbb{Z}$-torsion, then $A^W = R[s_2, s_3, \dots, s_n]$, where s_i are the standard symmetric functions.*

Proof. Consider the natural W-action on the ring $R[x] = R[x_1, x_2, \dots, x_n]$. We have the exact sequence $0 \to R[x] \xrightarrow{f} R[x] \to A \to 0$, where the first map is the multiplication by $f = x_1 + x_2 + \cdots + x_n - r$. Applying the W-invariants we get an exact sequence

$$0 \to R[x]^W \xrightarrow{f} R[x]^W \to A^W \to H^1(W, R[x]).$$

The ring $R[x]^W$ coincides with $R[s] = R[s_1, s_2, \dots, s_n]$. The monomials in the variables x_i form a permutation basis of the R-module $R[x]$. By the Faddeev-Shapiro lemma, the group $H^1(W, R[x])$ is a direct sum of the groups $H^1(W', R) = \mathrm{Hom}(W', R)$ for certain subgroups $W' \subset W$. Since R has trivial $\mathbb{Z}$-torsion, the latter group is trivial. Therefore,

$$A^W = R[s]/(f) = R[s_2, s_3, \dots, s_n]. \qquad \square$$

8.1. **Type A_{n-1}.** A split simple group of type A_{n-1} is isomorphic to $G = \mathbf{SL}(n)/\boldsymbol{\mu}_l$, where l is a divisor of n. Let $P \subset \mathbf{SL}(n)$ be the stabilizer of the line $U = [1 : 0 : \dots : 0] \in \mathbb{P}^{n-1}$ with respect to the natural action of $\mathbf{SL}(n)$ on $\mathbb{P}^{n-1}$. The semisimple part of P is $\mathbf{SL}(n-1)$ and it intersects $\boldsymbol{\mu}_l$ trivially. Hence the parabolic subgroup $P_l = P/\boldsymbol{\mu}_l$ of G is special. We have $G/P_l = \mathbb{P}^{n-1}$.

The intersection T of the group of diagonal matrices $\mathbf{D}(n)$ of $\mathbf{GL}(n)$ with $\mathbf{SL}(n)$ is a maximal torus of $\mathbf{SL}(n)$. The character group $\widehat{T}$ is identified with the factor group of $\mathbb{Z}^n = \widehat{\mathbf{D}(n)}$ with the standard basis $x_1, x_2, \dots, x_n$ by the subgroup generated by $x_1 + x_2 + \cdots + x_n$.

The character group of the maximal torus $T_l = T/\boldsymbol{\mu}_l$ of G is the subgroup of $\widehat{T}$ consisting of all sums $\sum a_i x_i$ such that $\sum a_i$ is divisible by l. Hence, $\widehat{T_l}$ is generated by lx_1 and $x_i - x_1$ for all $i = 2, \dots, n$ with the relation $\sum_{i\geq 2}(x_i - x_1) = -nx_1$.

The Weyl group $W = W_{P_l}$ is the symmetric group S_{n-1}, permuting $x_2, \dots, x_n$. Applying Lemma 8.1 to the ring $R = \mathbb{Z}[lx_1]$, the element $r = -nx_1$, the variables $x_i - x_1$ and the group W, we get $S(\widehat{T_l})^W = \mathbb{Z}[lx_1, s_2, s_3, \dots, s_{n-1}]$, where the s_i are the standard symmetric functions on the $x_i - x_1$, $i \geq 2$.

The group P acts naturally on the space $V = F^n$. The characters of this representation are $x_1, x_2, \dots, x_n$. The corresponding vector bundle $\operatorname{Bun}(V)$ over $\mathbb{P}^{n-1} = \mathbf{SL}(n)/P = G/P_l$ is the trivial vector bundle of rank n. The line U can be viewed as a 1-dimensional representation of P given by the character x_1. We have $\operatorname{Bun}(U) = L^\vee$, where L is the canonical line bundle on $\mathbb{P}^{n-1}$ (with the sheaf of sections $\mathcal{O}(1)$). Consider the representation $M = (V/U) \otimes U^\vee$ of the group P with the characters $x_i - x_1$ for all $i = 2, \dots, n$. Note that the group $\boldsymbol{\mu}_l$ is contained in the kernel of the representation, hence M is a representation of P_l.

By Lemma 7.1, we have $\varphi_G(lx_1) = lc_1(L^\vee) = -lh$, where $h \in \operatorname{CH}_1(\mathbb{P}^{n-1})$ is the class of a hyperplane, and also $\varphi_G(s_i) = c_i\big(\operatorname{Bun}(M)\big)$ for all i. Hence the subring $\widetilde{\operatorname{CH}}(\mathbb{P}^{n-1})$ of $\operatorname{CH}(\mathbb{P}^{n-1}) = \mathbb{Z}[h]/(h^n)$ is generated by lh and the Chern classes $c_i\big(\operatorname{Bun}(M)\big)$. Since

$$\operatorname{Bun}(M) = \big(\operatorname{Bun}(V)/\operatorname{Bun}(U)\big) \otimes \operatorname{Bun}(U^\vee) = (\operatorname{Bun}(V)/L^\vee) \otimes L,$$

the class $[\operatorname{Bun}(M)]$ is equal to $n[L] - 1$ in $K_0(\mathbb{P}^{n-1})$. Hence, $c_\bullet(\operatorname{Bun}(M)) = c_\bullet(L)^n = (1+h)^n$. Thus the subring $\widetilde{\operatorname{CH}}(\mathbb{P}^{n-1})$ is generated by lh and $\binom{n}{i}h^i$ for $i = 2, \dots, n-1$.

Let p a be prime integer and let p^k be the largest power of p dividing n. Note that the binomial coefficient $\binom{n}{i}$ is divisible by p unless i is divisible by p^k. The largest value of $i < n$ such that $\binom{n}{i}$ is not divisible by p is $n - p^k$. By Theorem 6.9,

$$\operatorname{cd}_p\big(\mathbf{SL}(n)/\boldsymbol{\mu}_l\big) = \begin{cases} p^k - 1, & \text{if } p \text{ divides } l; \\ 1, & \text{otherwise.} \end{cases}$$

Denote by $\operatorname{CSA}_{n,l}(K)$ the set of isomorphism classes of central simple K-algebras of degree n and exponent dividing l. The exact sequence $1 \to \boldsymbol{\mu}_l \to \mathbf{SL}(n) \to \mathbf{SL}(n)/\boldsymbol{\mu}_l \to 1$ yields a surjective map $\operatorname{Tors}_{\mathbf{SL}(n)/\boldsymbol{\mu}_l}(K) \to \operatorname{CSA}_{n,l}(K)$ with trivial kernel. By Remark 3.7,

$$\operatorname{cd}_p\big(\operatorname{CSA}_{n,l}\big) = \begin{cases} p^k - 1, & \text{if } p \text{ divides } l; \\ 1, & \text{otherwise.} \end{cases}$$

8.2. **Type B_n.** The only torsion prime is $p = 2$.

Taking a $(2n+1)$-dimensional vector space, endowed with a completely split quadratic form, let a vector g together with vectors e_i, f_i, $i = 1, 2, \dots, n$ form a basis such that $\{e_i, f_i\}$ are pairwise orthogonal hyperbolic pairs, while g is orthogonal to all e_i, f_i. Let $G = \mathbf{SO}(2n+1)$ be the corresponding special orthogonal group. The inclusion of $\mathbf{D}(n)$ into $\mathbf{SO}(2n+1)$ given by $t(e_i) = t_i e_i$, $t(f_i) = t_i^{-1} f_i$ and $t(g) = g$, where $t = \operatorname{diag}(t_1, \dots, t_n)$, identifies $\mathbf{D}(n)$ with a maximal torus T of $\mathbf{SO}(2n+1)$. In particular, the group $\widehat{T}$ is identified with $\mathbb{Z}^n = \widehat{\mathbf{D}(n)}$. We write $x_1, x_2, \dots, x_n$ for the standard basis of $\mathbb{Z}^n$.

Let V be the totally isotropic subspace of dimension n generated by all the e_i. Denote by P the stabilizer of V in G, so that $X = G/P$ is the variety of all dimension n

totally isotropic subspaces. The characters of the natural representation $P \to \mathbf{GL}(V)$ are $x_1, x_2, \dots, x_n$. The vector bundle $\mathrm{Bun}(V)$ over X is the tautological vector bundle.

The group $W = W_P$ is the symmetric group S_n permuting the x_i. The semisimple part of P is $\mathbf{SL}(n)$, so that P is special.

We have $S(\widehat{T})^W = \mathbb{Z}[s_1, s_2, \dots, s_n]$, where s_k are the standard symmetric functions on the x_i. By Lemma 7.1, the subring $\widetilde{\mathrm{CH}}(X)$ of $\mathrm{CH}(X)$ is generated by the Chern classes of $\mathrm{Bun}(V)$. These Chern classes are divisible by 2 in $\mathrm{CH}(X)$ [15, ch. III th. 6.11]. Thus, $\widetilde{\mathrm{Ch}}^j(X) = 0$ if $j > 0$. We conclude by Theorem 6.9 that

$$\mathrm{cd}_2\, \mathbf{SO}(2n+1) = \frac{n(n+1)}{2}$$

(see also Examples 3.6 and 5.11). The set $\mathrm{Tors}_{\mathbf{SO}(2n+1)}(K)$ is identified with the set of similarity classes $\mathrm{Q}_{2n+1}(K)$ of non-degenerate quadratic forms of dimension $2n+1$ over K. Thus,

$$\mathrm{cd}_2\, \mathrm{Q}_{2n+1} = \frac{n(n+1)}{2}.$$

Let $G = \mathbf{Spin}(2n+1)$ be the spinor group. There is an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to T' \to T \to 1,$$

where T' is a maximal torus of $\mathbf{Spin}(2n+1)$. We have $\widehat{T'} = \widehat{T} + \mathbb{Z}y = \mathbb{Z}^n + \mathbb{Z}y$, where $y = (x_1 + \cdots + x_n)/2$. By Lemma 8.1 applied to the ring $R = \mathbb{Z}[y]$, the element $r = 2y$ and the group W, the ring $S(\widehat{T})^W$ is the polynomial ring $\mathbb{Z}[y, s_2, s_3, \dots, s_n]$.

By Lemma 7.1, $\varphi_G(s_1) = c_1\big(\mathrm{Bun}(V)\big)$. The latter class coincides with $2e$ where e is a generator of $\mathrm{CH}^1(X)$ [15, ch. III th. 6.11]. Since $s_1 = 2y$ and $\mathrm{CH}^1(X)$ is torsion free, we have $\varphi_G(y) = e$.

As noted above, the images of the s_i in $\mathrm{CH}(X)$ are divisible by 2. Hence the image of $\widetilde{\mathrm{CH}}(X)$ in $\mathrm{Ch}(X) = \mathrm{CH}(X)/2$ is the subring generated by e mod 2. Let m be the smallest integer such that $2^m > n$. Then $e^{2^m} = 0$ and $e^{2^m - 1} \neq 0$ in $\mathrm{Ch}(X)$ [15, ch. III th. 6.11]. Thus,

$$\mathrm{cd}_2\, \mathbf{Spin}(2n+1) = \frac{n(n+1)}{2} - 2^m + 1.$$

Let $\overline{\mathrm{Q}}_{2n+1}(K)$ be the subset of $\mathrm{Q}_{2n+1}(K)$ consisting of all classes of forms with trivial even Clifford invariant. The exact sequence $1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(2n+1) \to \mathbf{SO}(2n+1) \to 1$ yields a surjective map $\mathrm{Tors}_{\mathbf{Spin}(2n+1)}(K) \to \overline{\mathrm{Q}}_{2n+1}(K)$ with trivial kernel. In particular,

$$\mathrm{cd}_2\, \overline{\mathrm{Q}}_{2n+1} = \frac{n(n+1)}{2} - 2^m + 1.$$

8.3. **Type C_n.** The group $\mathbf{Sp}(2n)$ is special, so that $\mathrm{cd}_p\, \mathbf{Sp}(2n) = 0$ for all p.

Let $G = \mathbf{PGSp}(2n)$ be the projective symplectic group. The number $p = 2$ is the only torsion prime of G. Instead of applying the general method, we proceed as follows.

The set $\mathrm{Tors}_{\mathbf{PGSp}(2n)}(K)$ is identified with the set of isomorphism classes $\mathrm{ASI}_{2n}(K)$ of central simple K-algebras A of degree $2n$ with a symplectic involution [13, §29.22].

The forgetful functor $\mathrm{ASI}_{2n} \to \mathrm{CSA}_{2n,2}$ has trivial kernel and is surjective. Therefore, by Remark 3.7 and (8.1),

$$\mathrm{cd}_2\, \mathbf{PGL}(2n) = \mathrm{cd}_2\, \mathrm{ASI}_{2n} = \mathrm{cd}_2\, \mathrm{CSA}_{2n,2} = 2^k - 1,$$

where 2^k is the largest power of 2 dividing $2n$.

8.4. **Type** D_n**.** Let $\{e_i, f_i\}$, $i = 1, 2, \dots, n$ be pairwise orthogonal hyperbolic pairs of a hyperbolic quadratic form of dimension $2n$. The inclusion of $\mathbf{D}(n)$ into $\mathbf{SO}(2n)$ given by $t(e_i) = t_i e_i$ and $t(f_i) = t_i^{-1} f_i$, where $t = \mathrm{diag}(t_1, \dots, t_n)$, identifies $\mathbf{D}(n)$ with a maximal torus T' of $\mathbf{SO}(2n)$. In particular, the group $\widehat{T}'$ is identified with $\mathbb{Z}^n = \widehat{\mathbf{D}(n)}$. We write $x_1, x_2, \dots, x_n$ for the standard basis of $\mathbb{Z}^n$.

Let V be the totally isotropic subspace of dimension n generated by all the e_i and let U be the line Fe_1. Denote by P the stabilizer of the flag $U \subset V$ in $G = \mathbf{Spin}(2n)$ and set $X = G/P$. The semisimple part of P is isomorphic to $\mathbf{SL}(n-1)$ and intersects trivially the center of G. Hence the image of P in any simple group of type D_n (under a central isogeny of G) is a special group.

Let Y be the connected component of the scheme of maximal (n-dimensional) totally isotropic subspaces such that V is a point of Y. The natural morphism $f : X \to Y$ is the projective bundle associated with the tautological vector bundle E over Y of rank n. In particular,

$$\dim X = \dim Y + (n-1) = \frac{n(n-1)}{2} + (n-1).$$

Note that Y is isomorphic to the projective homogeneous variety of the group $\mathbf{Spin}(2n-1)$ considered in the type B_{n-1}. The Chern classes of E in $\mathrm{CH}(Y)$ are divisible by 2 (see the type B_n), hence $\mathrm{Ch}(X) = \mathrm{Ch}(Y)[h]/(h^n)$, where $h = c_1(L)$ for the canonical line bundle L over X.

Similar to the case B_n, the character group of the maximal torus T of $\mathbf{Spin}(2n)$ is equal to $\mathbb{Z}^n + \mathbb{Z}y$, where $y = (x_1 + x_2 + \dots + x_n)/2$. Set $x'_i = x_i - x_1$ for $i = 2, \dots, n$, so that $x'_2 + \dots + x'_n = 2y - nx_1$. The symmetric group $W = W_P$ permutes the x'_i and acts trivially on y and x_1. Applying Lemma 8.1 to the variables x'_i, the ring $R = \mathbb{Z}[y, x_1]$ and the element $r = 2y - nx_1$ we see that $S(\widehat{T})^W = \mathbb{Z}[y, x_1, s_2, \dots s_{n-1}]$, where the s_i are the standard symmetric functions on the x'_i.

Consider the homomorphism (reduced modulo 2)

$$\varphi_G : \mathbb{Z}[y, x_1, s_2, \dots s_{n-1}] \to \mathrm{Ch}(X) = \mathrm{Ch}(Y)[h]/(h^n).$$

As in the case A_{n-1}, we have $\mathrm{Bun}(U) = L^\vee$ and therefore $\varphi_G(x_1) = c_1(L^\vee) = -h$. Similar to the case B_n, the class $e = \varphi_G(y)$ is a generator of $\mathrm{Ch}^1(Y)$. Recall that $e^{2^m-1} \neq 0$ and $e^{2^m} = 0$ where m is the smallest integer such that $2^m \geq n$.

Similar to the case A_{n-1}, we observe by Lemma 7.1 that the images of the s_i in $\mathrm{Ch}(X)$ are the Chern classes of the vector bundle $(f^*(E)/L^\vee) \otimes L$. The class of this bundle in $K_0(X)$ is equal to $[f^*(E) \otimes L] - 1$. Since the Chern classes of E are divisible by 2, we can replace E by the trivial bundle of rank n and replace $[f^*(E) \otimes L]$ by $n[L]$. As in the case A_{n-1}, we see that $\varphi_G(s_i) = \binom{n}{i} h^i$.

The subring $\widetilde{\mathrm{Ch}}(X) = \mathrm{Im}(\varphi_G)$ is generated by h and e. The largest degree nontrivial monomial in h and e is $h^{n-1}e^{2^m-1}$. By Theorem 6.9,

$$\mathrm{cd}_2\,\mathbf{Spin}(2n) = \dim X - (n-1) - (2^m - 1) = \frac{n(n-1)}{2} - 2^m + 1.$$

Let $\overline{\mathrm{Q}}_{2n}(K)$ be the subset of the set $\mathrm{Q}_{2n}(K)$ of isomorphism classes of non-degenerate quadratic forms of dimension $2n$ consisting of all classes of forms with trivial discriminant and Clifford invariant. The exact sequence $1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(2n) \to \mathbf{SO}(2n) \to 1$ yields a surjective map $\mathrm{Tors}_{\mathbf{Spin}(2n)}(K) \to \overline{\mathrm{Q}}_{2n}(K)$ with trivial kernel. In particular,

$$\mathrm{cd}_2\,\overline{\mathrm{Q}}_{2n} = \frac{n(n-1)}{2} - 2^m + 1.$$

Now let $G = \mathbf{SO}(2n)$. Recall that the character group $\widehat{T}'$ of the maximal torus T' of G is the subgroup of $\widehat{T}$ generated by all the x_i. Thus we have $S(\widehat{T}') = \mathbb{Z}[x_1, x_2, \dots, x_n]$ and therefore, $S(\widehat{T}')^W = \mathbb{Z}[x_1, s_1, \dots, s_{n-1}]$. The subring $\widetilde{\mathrm{Ch}}(X)$ is then generated by h. The largest degree nontrivial monomial in h is h^{n-1}. By Theorem 6.9,

$$\mathrm{cd}_2\,\mathbf{SO}(2n) = \dim X - (n-1) = \frac{n(n-1)}{2}.$$

Let $\mathrm{Q}'_{2n}(K)$ be the subset of the set $\mathrm{Q}_{2n}(K)$ consisting of all classes of forms with trivial discriminant. There is a canonical bijection $\mathrm{Tors}_{\mathbf{SO}(2n)}(K) \xrightarrow{\sim} \mathrm{Q}'_{2n}(K)$. Therefore,

$$\mathrm{cd}_2\,\mathrm{Q}'_{2n} = \frac{n(n-1)}{2}.$$

Let $G = \mathbf{PGO}^+(2n)$ be the projective orthogonal group. Let $\overline{T}$ be the image of the maximal torus T under the canonical isogeny $\mathbf{Spin}(2n) \to G$. The character group $\overline{T}$ is the subgroup of $\widehat{T}$ generated by all the simple roots. Thus we have $S(\widehat{\overline{T}}) = \mathbb{Z}[2x_1, x'_2, \dots, x'_n]$ and therefore, $S(\widehat{\overline{T}})^W = \mathbb{Z}[2x_1, s_1, \dots, s_{n-1}]$. The subring $\widetilde{\mathrm{Ch}}(X)$ is then generated by $\binom{n}{i}h^i$. Let 2^k be the largest power of 2 dividing n. Note that the binomial coefficient $\binom{n}{i}$ is even unless i is divisible by 2^k. The largest value of $i < n$ such that $\binom{n}{i}$ is odd is $n - 2^k$. The largest degree nontrivial monomial in h is $\binom{n}{n-2^k}h^{n-2^k}$. By Theorem 6.9,

$$\mathrm{cd}_2\,\mathbf{PGO}^+(2n) = \dim X - (n - 2^k) = \frac{n(n-1)}{2} + 2^k - 1.$$

Let $\mathrm{AQP}_{2n}(K)$ be the set of isomorphism classes of central simple algebras of degree $2n$ with a quadratic pair with trivial discriminant [13, §29.F]. The exact sequence $1 \to \mathbf{PGO}^+(2n) \to \mathbf{PGO}(2n) \to \mathbb{Z}/2\mathbb{Z} \to 1$ yields a surjective map $\mathrm{Tors}_{\mathbf{PGO}^+(2n)}(K) \to \mathrm{AQP}_{2n}(K)$ with trivial kernel. In particular,

$$\mathrm{cd}_2\,\mathrm{AQP}_{2n} = \frac{n(n-1)}{2} + 2^k - 1.$$

Suppose now that n is even. There are two isomorphic semispinor groups. We set $\mathbf{Spin}^{\sim}(2n) = \mathbf{Spin}(2n)/H$, where H is the intersection of $\mathrm{Ker}(y)$ with the center of $\mathbf{Spin}(2n)$. Let T'' be the image of the maximal torus T under the canonical isogeny

$\mathbf{Spin}(2n) \to G$. The character group of T'' is the subgroup of $\widehat{T}$ generated by all the simple roots and y. Thus we have $S(\widehat{T}'') = \mathbb{Z}[y, 2x_1, x_2', \dots, x_n']$.

Applying Lemma 8.1 to the elements x_i', the ring $R = \mathbb{Z}[y, 2x_1]$, and the element $r = 2y - nx_1$, we see that $S(\widehat{T}'')^W = \mathbb{Z}[y, 2x_1, s_2, \dots s_{n-1}]$.

The subring $\widetilde{\mathrm{Ch}}(X)$ is then generated by e and $\binom{n}{i}h^i$. The largest degree nontrivial monomial in h and e is $\binom{n}{n-2^k}h^{n-2^k}e^{2^m-1}$. By Theorem 6.9,

$$\mathrm{cd}_2\, \mathbf{Spin}^{\sim}(2n) = \dim X - (n - 2^k) - (2^m - 1) = \frac{n(n-1)}{2} + 2^k - 2^m.$$

Let $\mathrm{AQP}'_{2n}(K)$ be the set of isomorphism classes of central simple algebras of degree $2n$ with a quadratic pair with trivial discriminant and trivial component of the Clifford algebra. The exact sequence $1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}^{\sim}(2n) \to \mathbf{PGO}^+(2n) \to 1$ yields a surjective map $\mathrm{Tors}_{\mathbf{Spin}^{\sim}(2n)}(K) \to \mathrm{AQP}'_{2n}(K)$ with trivial kernel. In particular,

$$\mathrm{cd}_2\, \mathrm{AQP}'_{2n} = \frac{n(n-1)}{2} + 2^k - 2^m.$$

8.5. **Appendix: Type G_2.** The only torsion prime is 2. Since $\mathrm{Tors}_G \simeq \overline{Q}_8$ for a split simple G of type G_2, we have $\mathrm{cd}_2(G) = \mathrm{cd}_2(\overline{Q}_8) = 3$ (see §8.4).

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