# A criterion for a central simple algebra to be split

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Abstract. In these notes we give a criterion for a central simple algebra $A$ to be split in terms of the essential dimension of the algebraic group $\mathbf{SL}_1(A)$. This criterion also provides an example of an algebraic group $G$ with $\mathrm{ed}(G) = n$ which does not possess any non-trivial cohomological invariant.

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## 1. Introduction

Let $A$ be a central simple algebra over a field $k$. For every field extension $K/k$ we will denote by $A_K$ the algebra $A \otimes_k K$. We consider the algebraic group $\mathbf{SL}_1(A)$, defined over $k$, of elements of reduced norm equal to 1. This group is defined by the exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{SL}_1(A) \longrightarrow \mathbf{GL}_1(A) \xrightarrow{\mathrm{Nrd}} \mathbb{G}_m \longrightarrow 1 \tag{1}$$

where $\mathbf{GL}_1(A)$ stands for the algebraic group defined by $\mathbf{GL}_1(A)(K) = A_K^\times$ and $\mathrm{Nrd} : A_K^\times \to K^\times$ is the reduced norm homomorphism.

Applying Galois cohomology to the exact sequence (1), one finds, for every field extension $K/k$, the exact sequence of pointed sets

$$\mathbf{GL}_1(A)(K) \xrightarrow{\mathrm{Nrd}} \mathbb{G}_m(K) \xrightarrow{\partial} H^1(K, \mathbf{SL}_1(A)) \longrightarrow H^1(K, \mathbf{GL}_1(A)) \ .$$

Since $H^1(K, \mathbf{GL}_1(A)) = 1$ for all $K/k$ (see [4], pp. 392-393), one obtains a surjection

$$\partial : \ \mathbb{G}_m(K) \twoheadrightarrow H^1(K, \mathbf{SL}_1(A)) \tag{2}$$

and, much more precisely, an isomorphism

$$H^1(K, \mathbf{SL}_1(A)) \simeq K^\times / \mathrm{Nrd}(A_K^\times) \tag{3}$$

which is functorial in $K/k$ (see [4], Corollary 28.2, p.385 for details).

We will give a criterion for $A$ to be split in terms of the triviality of the functor $K \mapsto H^1(K, \mathbf{SL}_1(A))$.

## 2. Essential dimension

Let $k$ be a field. We denote by $\mathfrak{C}_k$ the category of field extensions of $k$, i.e. the category whose objects are field extensions $K$ over $k$ and whose morphisms are field homomorphisms which fix $k$. We will consider *covariant* functors $\mathbf{F} : \mathfrak{C}_k \to \mathbf{Sets}$ from $\mathfrak{C}_k$ to the category of sets. For such a functor $\mathbf{F}$ and for a field extension $K/k$, we will write $\mathbf{F}(K)$ instead of $\mathbf{F}(K/k)$. We shall say that a morphism $\mathbf{F} \longrightarrow \mathbf{G}$ between functors is a **surjection** if, for any field extension $K/k$, the corresponding map $\mathbf{F}(K) \longrightarrow \mathbf{G}(K)$ is a surjection of sets.

Definition 2.1. *Let* $\mathbf{F} : \mathfrak{C}_k \to \mathbf{Sets}$ *be a covariant functor,* $K/k$ *a field extension and* $a \in \mathbf{F}(K)$*. For* $n \in \mathbb{N}$*, we say that the* **essential dimension of** $a$ **is** $\leq n$ *(and we write* $\mathrm{ed}(a) \leq n$*) if there exists a subextension* $E/k$ *of* $K/k$ *such that:*

*i) the transcendence degree of* $E/k$ *is* $\leq n$*,*

*ii) the element* $a$ *is in the image of the map* $\mathbf{F}(E) \longrightarrow \mathbf{F}(K)$*.*

*We say that* $\mathrm{ed}(a) = n$ *if* $\mathrm{ed}(a) \leq n$ *and* $\mathrm{ed}(a) \not\leq n-1$*. The* **essential dimension of** $\mathbf{F}$ *is the supremum of* $\mathrm{ed}(a)$ *for all* $a \in \mathbf{F}(K)$ *and for all* $K/k$*. The essential dimension of* $\mathbf{F}$ *will be denoted by* $\mathrm{ed}_k(\mathbf{F})$*.*

*Examples* 2.2.

i) Consider the trivial functor $* : \mathfrak{C}_k \to \mathbf{Sets}$ which sends each $K/k$ to a one-element set $*$. Clearly one has $\mathrm{ed}_k(*) = 0$.
ii) The functor $\mathbb{G}_m$, which assigns to each $K$ the set $K^\times$, satisfies $\mathrm{ed}_k(\mathbb{G}_m) = 1$.
iii) Let $\mathbf{F} \twoheadrightarrow \mathbf{G}$ be a surjection. Then $\mathrm{ed}_k(\mathbf{G}) \leq \mathrm{ed}_k(\mathbf{F})$.

These facts are easy consequences of the definition. See [1] or [6] for example.

Definition 2.3. *For an algebraic group* $G$ *defined over* $k$*, the essential dimension of the Galois cohomology functor* $K \mapsto H^1(K, G)$ *is denoted by* $\mathrm{ed}_k(G)$*.*

For an account on the notion of essential dimension of algebraic groups see for instance [1, 2, 6] or [7].

It follows from surjection (2) and from Examples 2.2 ii) and iii) above that $\mathrm{ed}_k(\mathbf{SL}_1(A)) \leq 1$. The question is to know when this number is equal to 0 or 1. If $A$ is split, say $A \simeq M_n(k)$, then $\mathbf{SL}_1(A) \simeq \mathbf{SL}_n$ and the reduced norm homomorphism is clearly surjective. Hence the functor $K \mapsto K^\times/\mathrm{Nrd}(A_K^\times)$ is trivial and thus $\mathrm{ed}_k(\mathbf{SL}_1(A)) = 0$. We will see that the converse is true.

## 3. Statement of the result

Let $t$ be an indeterminate and denote by $[t]$ the class of $t$ in $k(t)^\times/\mathrm{Nrd}(A_{k(t)}^\times)$.

Theorem 3.1. *Let $A$ be a central simple algebra over $k$. The following conditions are equivalent:*

1) *$A$ is split,*
2) *the functor $K \mapsto K^\times/\mathrm{Nrd}(A_K^\times)$ is trivial,*
3) $\mathrm{ed}_k(\mathbf{SL}_1(A)) = 0$*,*
4) $\mathrm{ed}([t]) = 0$*.*

**Proof.** The implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) are trivial and follow easily from the definitions. It remains to show that 4) $\Rightarrow$ 1). It will be a consequence of the following fact. $\square$

Proposition 3.2. *Let $A$ be a central simple algebra over $k$. Denote by* $\mathrm{ind}(A)$ *its index and consider the group homomorphism*

$$\theta_A:\ (A \otimes_k k(t))^\times \xrightarrow{\mathrm{Nrd}} k(t)^\times \xrightarrow{\deg} \mathbb{Z}$$

*where* deg *stands for the degree in $t$. Then* $\mathrm{im}(\theta_A) = \mathrm{ind}(A)\,\mathbb{Z}$*.*

To prove this fact, we use the following lemma.

Lemma 3.3. *Let $K$ be a commutative field and $A$ and $B$ two simple central algebras over $K$. If $A$ and $B$ are Brauer equivalent, then* $\mathrm{Nrd}(A^\times) = \mathrm{Nrd}(B^\times)$*.*

**Proof.** Let $D$ be a central division algebra over K such that $A \simeq M_r(D)$ and $B \simeq M_s(D)$. We show that $\mathrm{Nrd}(A^\times) = \mathrm{Nrd}(D^\times)$. We have that $A^\times \simeq \mathbf{GL}_r(D)$. Now for invertible matrices over a division algebra there is the notion of Dieudonné determinant $\det_D : \mathbf{GL}_r(D) \to D^\times/[D^\times, D^\times]$ (see [3], Definition 3, p. 135). One has the following commutative diagram

$$\begin{tikzcd}
\mathbf{GL}_r(D) \arrow[r, "\mathrm{Nrd}"] \arrow[d, "\det_D"'] & K^\times \\
D^\times/[D^\times, D^\times] \arrow[ur, "\mathrm{Nrd}"'] &
\end{tikzcd}$$

which shows that $\mathrm{Nrd}(A^\times) = \mathrm{Nrd}(D^\times)$ since $\det_D$ is surjective (see [3], Theorem 1, p. 146). Similarly $\mathrm{Nrd}(B^\times) = \mathrm{Nrd}(D^\times)$ which finishes the proof. $\square$

*Remark* 3.4. It follows from the above lemma and the functor isomorphism (3) that, if $A$ and $B$ are two simple central algebras over $k$ which are Brauer equivalent, then the functors $K \to H^1(K, \mathbf{SL}_1(A))$ and $K \to H^1(K, \mathbf{SL}_1(B))$ are isomorphic.

**Proof of Proposition 3.2.** By Lemma 3.3, we can suppose that $A$ is a division algebra. In this case, the index of $A$ and the degree of $A$ are equal.

Consider now the element $1 \otimes t \in (A \otimes_k k(t))^\times$. We have that $\mathrm{Nrd}(1 \otimes t) = t^{\mathrm{ind}(A)}$ and thus $\mathrm{ind}(A) \in \mathrm{im}(\theta_A)$. It follows that $\mathrm{ind}(A)\,\mathbb{Z} \subseteq \mathrm{im}(\theta_A)$.
To show the converse inclusion take $a \in (A \otimes_k k(t))^\times$ and write it $a = \frac{b}{c}$ where $c \in k[t]$ and $b \in A \otimes_k k[t]$. Since $c \in k[t]$ we have that $\mathrm{Nrd}(c) = c^{\mathrm{ind}(A)}$ and thus it suffices to show that $\mathrm{ind}(A)$ divides the degree of $\mathrm{Nrd}(b)$. We then write

$$b = b_0 \otimes 1 + b_1 \otimes t + \cdots + b_n \otimes t^n$$

where $b_i \in A$ and $b_n \neq 0$. An easy computation shows that

$$\mathrm{Nrd}(b) = \mathrm{Nrd}(b_n)\, t^{n \cdot \mathrm{ind}(A)} + \cdots$$

Since $A$ is a division algebra $\mathrm{Nrd}(b_n) \neq 0$ whence the result. □

Proposition 3.2 enables us to establish the implication 4) $\Rightarrow$ 1) of Theorem 3.1.

**Proof of Theorem 3.1.** Suppose that $\mathrm{ed}([t]) = 0$. This means that there exist an element $\lambda \in k^\times$ and an element $x \in (A \otimes_k k(t))^\times$ such that $t = \lambda\, \mathrm{Nrd}(x)$. But this implies that $1 \in \mathrm{im}(\theta_A)$. By the preceeding Proposition, this means that $\mathrm{ind}(A) = 1$, saying that $A$ is split. □

Actually Theorem 3.1 can be sharpened in order to give a more precise result:

Corollary 3.5. *Let $A$ be a central simple algebra over $k$. Let $r \in \mathbb{N}$, then*

*i) one has* $\mathrm{ed}([t^r]) = 0$ *if and only if* $\mathrm{ind}(A)$ *divides $r$;*

*ii) one has* $\mathrm{ind}(A) = r$ *if and only if* $\mathrm{ed}([t^r]) = 0$ *and* $\mathrm{ed}([t^m]) \neq 0$ *for all $m \neq r$ with $m \mid r$.*

**Proof.** By Lemma 3.3 one can suppose $A$ to be a division algebra.
i) Suppose that $\mathrm{ind}(A)$ divides $r$, say $r = \mathrm{ind}(A)m$. One has $t^r = \mathrm{Nrd}(1 \otimes t^m)$ saying that $\mathrm{ed}([t^r]) = 0$. Conversely, if $\mathrm{ed}([t^r]) = 0$ then there exists $u \in k$ and $x \in (A \otimes_k k(t))^\times$ such that $t^r = u\mathrm{Nrd}(x)$. This shows that $r \in \mathrm{im}(\theta_A)$. By Proposition 3.2, it follows that $\mathrm{ind}(A)$ divides $r$.
ii) It follows easily from i). □

## 4. A slight generalization

For two algebraic groups $G$ and $H$ very little is known about the behaviour of $\mathrm{ed}(G \times H)$ (see [1, 2] or [7] for partial results). Even when $G = H$ the computation of the essential dimension is not well understood. The preceeding discussion and the interpretation of the essential dimension of $\mathbf{SL}_1(A)$ in terms of the algebra $A$ allows to give a precise description in this case. Our aim is to prove the following result for a product of copies of $\mathbf{SL}_1(A)$.

Theorem 4.1. *Let $A$ be a central simple algebra defined over a field $k$ and let $\mathbf{SL}_1(A) \times \cdots \times \mathbf{SL}_1(A)$ the product of $n$ copies of $\mathbf{SL}_1(A)$. Then*

$$\mathrm{ed}_k(\mathbf{SL}_1(A) \times \cdots \times \mathbf{SL}_1(A)) = n\, \mathrm{ed}_k(\mathbf{SL}_1(A)).$$

**Proof.** It is enough to prove that if $A$ is a non-split algebra then we have $\mathrm{ed}_k(\mathbf{SL}_1(A) \times \cdots \times \mathbf{SL}_1(A)) = n$. Notice that, since there is no non-split central simple algebra over a finite field, we may suppose that the ground field $k$ is infinite. For simplicity we will denote the functor $H^1(\_, \mathbf{SL}_1(A) \times \cdots \times \mathbf{SL}_1(A))$, with $n$ copies of $\mathbf{SL}_1(A)$, by $\mathbf{F}_n$.

Let $t_1, \ldots, t_n$ be algebraically independent variables over $k$. In the sequel we will denote by $K_n$ the field $k(t_1, \ldots, t_n)$. Consider the element $a = (\bar{t}_1, \ldots, \bar{t}_n) \in \mathbf{F}_n(K_n)$ where $\bar{t}_i$ denotes the class of $t_i$ in the quotient $K_n^\times / \mathrm{Nrd}(A_{K_n}^\times)$. We will show, by induction on $n$, that $\mathrm{ed}(a) = n$.

Suppose that $n \geq 2$ and that there is a subextension $L/k \subset K_n/k$ and an element $b \in \mathbf{F}_n(L)$ such that $b$ maps to $a$ under the map

$$\mathbf{F}_n(L) \to \mathbf{F}_n(k(t_1, \ldots, t_n)).$$

Suppose moreover that $\mathrm{trdeg}(L : k) < n$. We write $b = (\bar{b}_1, \ldots, \bar{b}_n)$ for some elements $b_i \in L^\times$. Saying that $b$ maps to $a$ means that there exist elements $x_i \in (A \otimes_k K_n)^\times$ such that

$$\begin{cases} t_1 &= & b_1 \,\mathrm{Nrd}(x_1) \\ t_2 &= & b_2 \,\mathrm{Nrd}(x_2) \\ & \vdots & \\ t_n &= & b_n \,\mathrm{Nrd}(x_n) \end{cases}$$

Recall that for a field extension $K/k$ a $k$-valuation on $K$ is a valuation $v$ on $K$ which is trivial on $k$, that is $v(x) = 0$ for all $x \in k$. We denote by $\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$ the ring of the valuation, by $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$ its unique maximal ideal and by $\kappa_v$ the residue field, that is the quotient $\mathcal{O}_v/\mathfrak{m}_v$. Since $v$ is trivial on $k$ it follows that $\mathcal{O}_v$ is a $k$-algebra and that $\kappa_v$ is again a field extension of $k$. It is well known that

$$\mathrm{trdeg}(\kappa_v : k) \leq \mathrm{trdeg}(K : k)$$

and that the equality holds if and only if $v$ is trivial on $K$.

We consider now on $K_n$ the $(t_n - \lambda)$-valuation where $\lambda \in k$. We denote it by $v$ and its restriction to $L$ by $v'$. Since the field $k$ is infinite, we can find a $\lambda$ such that the $b_i$'s, the $x_i$'s and their inverses $y_i$ are unramified at $v$. That means that $b_i \in \mathcal{O}_v$ and that $x_i, y_i \in A \otimes_k \mathcal{O}_v$. Thus we may specialise the above equations at $t_n = \lambda$ and find

$$\begin{cases} t_1 & = & b_1(\lambda)\,\mathrm{Nrd}(x_1(\lambda)) \\ t_2 & = & b_2(\lambda)\,\mathrm{Nrd}(x_2(\lambda)) \\ & \vdots & \\ \lambda & = & b_n(\lambda)\,\mathrm{Nrd}(x_n(\lambda)) \end{cases}$$

where $b_i(\lambda)$ denotes the image of $b_i$ in the residue field $\kappa_{v'}$ and $x_i(\lambda)$ denote the image of $x_i$ in $(A \otimes_k \kappa_v)^\times = (A \otimes_k K_{n-1})^\times$.
Hence $b' = (\bar{b}_1, \ldots, \bar{b}_{n-1}) \in \mathbf{F}_{n-1}(\kappa_{v'})$ maps to $(\bar{t}_1, \ldots, \bar{t}_{n-1}) \in \mathbf{F}_{n-1}(K_{n-1})$ under the induced map $\mathbf{F}_{n-1}(\kappa_{v'}) \to \mathbf{F}_{n-1}(\kappa_v)$. Since by induction hypothesis $\mathrm{ed}((\bar{t}_1, \ldots, \bar{t}_{n-1})) = n - 1$ and since $\mathrm{trdeg}(L : k) < n$ it follows that the valuation $v'$ has to be trivial over $L$. Consequently $b_n(\lambda) = b_n \in L$ and the equation $\lambda = b_n(\lambda)\,\mathrm{Nrd}(x_n(\lambda))$ actually shows that $b_n \in K_{n-1}$. Coming back to the equation $t_n = b_n\,\mathrm{Nrd}(x_n)$ this contradicts Theorem 3.1 for the central simple algebra $A' = A \otimes_k K_{n-1}$. $\square$

## 5. A side remark

Let $\mathbf{F} : \mathfrak{C}_k \to \mathbf{Sets}$ be a covariant functor and let $M$ be any torsion $\Gamma_k$-module. A **degree $d$ cohomological invariant for F with values in $M$** is, by definition, a morphism of functors $\eta : \mathbf{F} \to H^d(\_, M)$. An invariant $\eta$ is said to be non-trivial if, for every field extension $K/k$, there exists a field extension $L/K$ and an element $a \in \mathbf{F}(L)$ such that $\eta_L(a) \neq 0 \in H^d(L, M)$. It is shown in [1, 6, 7] that if $\eta$ is a non-trivial degree $d$ cohomological invariant then $\mathrm{ed}_k(\mathbf{F}) \geq d$.
Now, since the algebra $A$ splits over some field extension $K/k$, it follows that for any $L/K$ the cohomology set $H^1(L, \mathbf{SL}_1(A))$ is reduced to one element. Thus one sees that $H^1(\_, \mathbf{SL}_1(A))$ cannot have any non-trivial cohomological invariant of degree $\geq 1$. Hence, when $A$ is non-split, $\mathbf{SL}_1(A) \times \cdots \times \mathbf{SL}_1(A)$ gives an easy example of a group without any non-trivial cohomological invariant which has essential dimension $n$.

## 6. Description of $H^1(k(t), \mathbf{SL}_1(A))$

For completeness we give a detailed study of $H^1(k(t), \mathbf{SL}_1(A))$ inspired by [8]. Let $X$ denote the set of irreducible monic polynomials in $k[t]$.

Proposition 6.1. *Let $A$ be a central simple algebra over $k$ and $x \in X$. Consider the group homomorphism*

$$\eta_x : \ (A \otimes_k k(t))^\times \xrightarrow{\mathrm{Nrd}} k(t)^\times \xrightarrow{v_x} \mathbb{Z}$$

*where $v_x$ denotes the $x$-adic valuation on $k(t)$. Then* $\mathrm{im}(\eta_x) = \mathrm{ind}(A)\mathbb{Z}$.

**Proof.** As in Lemma 3.3 we can suppose that $A$ is a division algebra. One has $\mathrm{Nrd}(1 \otimes x) = x^{\mathrm{ind}(A)}$, thus $\mathrm{ind}(A)\mathbb{Z} \subseteq \mathrm{im}(\eta_x)$. Conversely, let $a \in (A \otimes_k k(t))^\times$ and write it $a = \frac{b}{c}$ where $c \in k[t]$ and $b \in A \otimes_k k[t]$. Since $\mathrm{Nrd}(c) = c^{\mathrm{ind}(A)}$ it suffices to prove that $\mathrm{ind}(A) \mid v_x(\mathrm{Nrd}(b))$. To do this, write

$$b = \sum_{r \leq i \leq s} b_i \otimes x^i f_i$$

where $\deg(f_i) < \deg(x)$ and $b_r \neq 0$. An easy computation shows then that

$$\mathrm{Nrd}(b) = \mathrm{Nrd}(b_r) x^{r \cdot \mathrm{ind}(A)} f_r^{\mathrm{ind}(A)} + x^{r \cdot \mathrm{ind}(A)+1} g$$

where $g \in k[t]$. Since $\mathrm{Nrd}(b_r) \neq 0$ this shows that $v_x(\mathrm{Nrd}(b)) = r \cdot \mathrm{ind}(A)$. Hence $\mathrm{im}(\eta_x) \subseteq \mathrm{ind}(A)\mathbb{Z}$. This concludes the proof. $\square$

Definition 6.2. *For each $x \in X$, let*

$$\partial_x : H^1(k(t), \mathbf{SL}_1(A)) \to \mathbb{Z}/\mathrm{ind}(A)\mathbb{Z}$$

*be the group homomorphism induced by the valuation $v_x$.*

Theorem 6.3. *Let $A$ be a central simple algebra over $k$. Then there is a split exact sequence*

$$1 \to H^1(k, \mathbf{SL}_1(A)) \to H^1(k(t), \mathbf{SL}_1(A)) \to \bigoplus_{x \in X} \mathbb{Z}/\mathrm{ind}(A)\mathbb{Z} \to 0$$

*where the first map is induced by $k \to k(t)$ and the second is $\oplus \partial_x$.*

**Proof.** Though it is well known that the natural map

$$\iota : H^1(k, G) \to H^1(k(t), G)$$

is injective (see [5]), we show injectivity in this particular case. As above we may suppose that $A$ is a division algebra. So let $\lambda \in k^\times$ and $x \in (A \otimes k(t))^\times$ such that $\lambda = \mathrm{Nrd}(x)$. Write $x$ as $x = (1/p(t))y$, where $p(t)$ is a unitary polynomial and $y \in A \otimes k[t]$. One has

$$\mathrm{Nrd}(x) = (1/p(t)^{\deg(A \otimes k(t))})\mathrm{Nrd}(y),$$

and hence $\lambda p(t)^{\mathrm{ind}(A)} = \mathrm{Nrd}(y)$. Write $y = a_r \otimes t^r + \cdots + a_0 \otimes 1$, where $a_i \in A$ and $a_r \neq 0$. Then $\mathrm{Nrd}(y)$ is a polynomial of degree $\mathrm{ind}(A)r$ and its leading coefficient is $\mathrm{Nrd}(a_r)$. Comparing the leading terms in the equality $\lambda p(t)^{\mathrm{ind}(A)} = \mathrm{Nrd}(y)$, one gets $\lambda = \mathrm{Nrd}(a_r)$. This proves the injectivity.

The composition

$$k^\times/\mathrm{Nrd}(A) \to k(t)^\times/\mathrm{Nrd}(A^\times_{k(t)}) \to \bigoplus_{x\in X} \mathbb{Z}/\mathrm{ind}(A)\mathbb{Z}$$

is clearly zero. We show that $\ker(\oplus\delta_x) \subseteq \mathrm{im}(\iota)$. Let $f \in k(t)^\times$ and write it $f = \lambda \prod_{x\in X} x^{v_x(f)}$ with $\lambda \in k^\times$. Since $\partial_x(f) = 0$ one has that $\mathrm{ind}(A)$ divides $v_x(f)$ for all $x \in X$. Hence $\prod_{x\in X} x^{v_x(f)} \in \mathrm{Nrd}(A^\times_{k(t)})$. Thus $f \in \mathrm{im}(\iota)$.

To end the proof, we give a section to $\oplus\partial_x$. This will show surjectivity as well. We let $s : \bigoplus_{x\in X} \mathbb{Z}/\mathrm{ind}(A)\mathbb{Z} \longrightarrow H^1(k(t), \mathbf{SL}_1(A))$ defined by sending the element $([n_x])_{x\in X}$ to the class of $\prod_{x\in X} x^{n_x}$ in $k(t)^\times/\mathrm{Nrd}(A^\times_{k(t)})$. This clearly gives the desired section. □

Corollary 6.4. *The following conditions are equivalent:*
- i) *the algebra $A$ is split,*
- ii) *the group $H^1(k(t), \mathbf{SL}_1(A))$ is trivial,*
- iii) *the group $H^1(k(t), \mathbf{SL}_1(A))$ is finite.*

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