Annihilators of quadratic and bilinear forms over fields of characteristic two

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Abstract

Let F be a field with $2=0$, $W(F)$ the Witt ring of symmetric bilinear forms over F and $W_q(F)$ the $W(F)$-module of quadratic forms over F. Let $I_F \subset W(F)$ be the maximal ideal. We compute explicitly in I_F^m and $I^m W_q(F)$ the annihilators of n-fold bilinear and quadratic Pfister forms, thereby answering positively, in the case $2=0$, certain conjectures stated by Krüskemper in [Kr].

1 Introduction

Let F be a field with $2=0$. We denote by $W(F)$ the Witt ring of symmetric non singular bilinear forms over F and by $W_q(F)$ the $W(F)$-module of non singular quadratic forms over F (see [Sa], [Ba-1], [Ba-2]).

For $a_i \in F^* = F - \{0\}$, $1 \leq i \leq n$, we denote by $\langle a_1, \ldots, a_n \rangle$ the bilinear form with diagonal Gramm matrix and entries a_i on the diagonal. The quadratic form $x^2+xy+ay^2$, $a \in F$, is denoted by $[1,a]$. The maximal ideal I_F of $W(F)$ is additively generated by the forms $\langle 1, a\rangle = \ll a \gg$, $a \in F^*$, so that the powers I_F^n, $n \geq 1$, are additively generated by the n-fold bilinear forms $\ll a_1, \ldots, a_n \gg = \langle 1, a_1\rangle \cdots \langle 1, a_n\rangle$, $a_i \in F^*$. The

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submodules $I^nW_q(F)$, $n \geq 1$, are generated by the n-fold quadratic Pfister forms $\ll a_1, \ldots, a_n; a|] = \ll a_1, \ldots, a_n \gg \cdot [1, a]$, $a_i \in F^*$, $a \in F$.

We have the filtrations $W(F) \supset I_F \supset I_F^2 \supset \cdots$ and $W_q(F) \supset IW_q(F) \supset \cdots$. The graded objects I_F^n / I_F^{n+1} and $I^nW_q(F)/I^{n+1}W_q(F)$ are denoted by $\overline{I}_F^n$ resp. $\overline{I}^n W_q(F)$.

In this paper we will study annihilators of n-fold Pfister forms. Let $x = \ll a_1, \ldots, a_n \gg$ be an n-fold bilinear Pfister form. For any $m \geq 0$ we set

$$\mathrm{annb}_m(x) = \{y \in I_F^m \mid xy = 0\}$$

$$\mathrm{annq}_m(x) = \{y \in I^m W_q(F) \mid xy = 0\}$$

$$\overline{\mathrm{ann}}\mathrm{b}_m(x) = \{\overline{y} \in \overline{I}_F^m \mid x\overline{y} = 0\}$$

$$\overline{\mathrm{ann}}\mathrm{q}_m(x) = \{\overline{y} \in \overline{I}^m W_q(F) \mid x\overline{y} = 0\}.$$

If $x = \ll a_1, \ldots, a_n; a|]$ is a quadratic n-fold Pfister form, we set

$$\mathrm{annb}_m(x) = \{y \in I_F^m \mid yx = 0\}$$

$$\overline{\mathrm{ann}}\mathrm{b}_m(x) = \{\overline{y} \in \overline{I}_F^m \mid \overline{y}x = 0\}.$$

The main results of this paper are contained in the following two theorems.

(1.1) Theorem. (i) Let $x = \ll a_1, \ldots, a_n \gg$ be a bilinear n-fold Pfister form over F with $x \neq 0$ in $W(F)$. Then for any $m \geq 1$

$$\overline{\mathrm{ann}}\mathrm{b}_m(x) = \overline{\mathrm{ann}}\mathrm{b}_1(x)\overline{I}_F^{m-1}$$

$$\overline{\mathrm{ann}}\mathrm{q}_m(x) = \overline{I}_F^m \cdot \overline{\mathrm{ann}}\mathrm{q}_0(x) + \overline{\mathrm{ann}}\mathrm{b}_1(x)\overline{I}^{m-1}W_q(F)$$

(ii) Let $x = \ll a_1, \ldots, a_n; a|]$ be a quadratic n-fold Pfister form over F with $x \neq 0$ in $W_q(F)$. Then for $m \geq 1$

$$\overline{\mathrm{ann}}\mathrm{b}_m(x) = \overline{\mathrm{ann}}\mathrm{b}_1(x)\overline{I}_F^{m-1}.$$

and the much stronger

(1.2) Theorem. (i) Let $x = \ll a_1, \ldots, a_n \gg$ be a bilinear n-fold Pfister form over F with $x \neq 0$ in $W(F)$. Then for any $m \geq 1$

$$\mathrm{annb}_m(x) = \mathrm{annb}_1(x)I_F^{m-1}$$

$$\mathrm{annq}_m(x) = I_F^m \cdot \mathrm{annq}_0(x) + \mathrm{annb}_1(x)I^{m-1}W_q(F)$$

(ii) Let $x = \ll a_1, \ldots, a_n; a]]$ be a quadratic n-fold Pfister form over F with $x \neq 0$ in $W_q(F)$. Then for $m \geq 1$

$$\mathrm{annb}_m(x) = \mathrm{annb}_1(x) I_F^{m-1}.$$

These results were conjectured by M. Krüskemper in [Kr] for fields of characteristic different from 2. The proof of theorem (1.1) will be given in section 4 and it is based on Kato's correspondence between quadratic or symmetric bilinear forms and differential forms over F. We will shortly explain this correspondence en section 3 (see [Ka], [Ba-2]) and prove there some technical results needed in the proof of (1.1). In section 2 we show that theorem (1.2) follows from theorem (1.1).

The terminology used in this paper is standard and we refer to [Ba-2], [Mi] and [Sa] for details on basic facts needed in the paper. In any case let us mention that for $a_1, \ldots, a_n \in F^*$ the form $\ll a_1, \ldots, a_n \gg$ is anisotropic over F if and only if $a_1, \ldots, a_n$ are part of a 2-basis of F and the subfield $F^2(a_1, \ldots, a_n)$ of F consists of all elements of F represented by the form $\ll a_1, \ldots, a_n \gg$. The elements of F represented by the pure part $\ll a_1, \ldots, a_n \gg'$ of $\ll a_1, \ldots, a_n \gg$ form a subgroup denoted by $F^2(a_1, \ldots, a_n)'$. Recall that $\ll a_1, \ldots, a_n \gg'$ is defined by $\ll a_1, \ldots, a_n \gg = < 1 > \perp \ll a_1, \ldots, a_n \gg'$.

2 Proof of theorem (1.2)

We will assume theorem (1.1) and derive from it theorem (1.2). Recall that a 2-basis of a field F of characteristic 2 is a set $\mathcal{B} = \{b_i \mid i \in I\} \subset F$ such that the elements $\prod_{i \in I} b_i^{\varepsilon_i}$, $\varepsilon_i \in \{0, 1\}$ and only finitely many $\varepsilon_i \neq 0$, form a basis of F over F^2. An n-fold bilinear Pfister form $\ll a_1, \ldots, a_n \gg$ over F is $\neq 0$ in $W(F)$ if and only if $\{a_1, \ldots, a_n\}$ are part of a 2-basis of F (i.e. 2-independent). Moreover if F has a finite 2-basis $\{b_1, \ldots, b_N\}$ then $I_F^m = 0$ for all $m \geq N + 1$ (see [Mi]).

We will need the following

(2.1) Lemma. (i) Let x be an n-fold bilinear Pfister form, $x \neq 0$, and $z \in I_F$ such that $zx \in I_F^{n+2}$, i.e. $z \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$. Then

$$z = z_0 + w$$

with $z_0 \in I$, $z_0 x = 0$ and $w \in I_F^2$.

(ii) Let x be an n-fold bilinear Pfister form, $x \neq 0$, and $z \in W_q(F)$ with $xz \in I^{n+1} W_q(F)$. Then

$$z = z_0 + w$$

with $z_0 \in W_q(F)$, $xz_0 = 0$ and $w \in I W_q(F)$.

Proof: (i) For any $z \in I_F$ we can write $z =< 1, d > +w$ with $d = \det(z)$ and $w \in I_F^2$. Then $xz \in I_F^{n+2}$ implies $< 1, d > x \in I_F^{n+2}$, and since $< 1, d > x$ is $(n+1)$-fold Pfister form, it follows $< 1, d > x = 0$ in $W(F)$.

(ii) Any $z \in W_q(F)$ can be written as

$$z = [1, d] + w$$

with $d = \mathrm{Arf}(z) \in F$ and $w \in IW_q(F)$ (see [Sa]). From $xz,\ xw \in I^{n+1}W_q(F)$, it follows $x[1, d] \in I^{n+1}W_q(F)$ and hence $x[1, d] = 0$. $\square$

Let us now prove (1.2). We assume first that F has a finite 2-basis, i.e. $I_F^{N+1} = 0$ for some integer N. Let $x \neq 0$ (in $W(F)$) be an n-fold bilinear Pfister form. The contentions $\supseteq$ in (i) (and (ii)) are obvious. Let $y \in \mathrm{annb}_m(x)$, i.e. $y \in I_F^m$, $yx = 0$. Hence $\overline{y} \in \overline{\mathrm{ann}}\mathrm{b}_m(x)$ and (1.1) implies $y = \sum \overline{z}_i \overline{y}_{i,0}$ with $\overline{z}_i \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$, $y_{i,0} \in I_F^{m-1}$. Then $y - \sum z_i y_{i,0} \in I_F^{m+1}$. Using (2.1) (i) we can write $z_i = z_{i,0} + w_i$ with $z_{i,0} \in \mathrm{annb}_1(x)$ and $\mathrm{w}_i \in I_F^2$. Then $y_1 = y - \sum z_i y_{i,0} \in I_F^{m+1}$ and moreover $y_1 x = 0$. The same argument implies $y_1 - \sum z_{i,1} y_{i,1} \in I_F^{m+2}$ with elements $z_{i,1} \in \mathrm{annb}_1(x)$, $y_{i,1} \in I_F^m$. Iterating this process we obtain, for any $k \geq 0$, elements $z_{i,l} \in \mathrm{annb}_1(x)$ and $y_{i,l} \in I_F^{m+l-1}$, $0 \leq l \leq k$ such that $y - \sum_{i,l} z_{i,l} y_{i,l} \in I_F^{m+k}$. Choosing $k \geq N + 1 - m$ we obtain $y = \sum_{i,l} z_{i,1} y_{i,1} \in \mathrm{annb}_1(x) I_F^{m-1}$, since $I^{N+1} = 0$.

Let now $y \in \mathrm{annq}_m(x)$, i.e. $y \in I^m W_q(F)$ with $xy = 0$. Theorem (1.1) implies $\overline{y} = \sum \overline{y_i z_i} + \sum \overline{u_j v_j}$ with $\overline{y_i} \in \overline{I}_F^m$, $\overline{z_i} \in \overline{\mathrm{ann}}\mathrm{q}_0(x)$, $\overline{u_j} \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$, $\overline{v_i} \in \overline{I}^{m-1} W_q(F)$. Hence $y - \sum y_i z_i - \sum u_j v_j \in I^{m+1} W_q(F)$. Using lemma (2.1) we can find $z_{i,0} \in \mathrm{annq}_0(x)$, $u_{j,0} \in \mathrm{annb}_1(x)$ such that $z_i = z_{i,0} + w_i$, $w_i \in IW_q(F)$ and $u_i = u_{j,0} + t_j$, $t_j \in I_F^2$. We obtain

$$y_1 = y - \sum y_i z_{i,0} - \sum u_{j,0} v_j \in I^{m+1} W_q(F)$$

with $y_1 x = 0$. Iterating this procedure we obtain after $k \geq N + 1 - m$ steps that

$$y \in I_F^m \mathrm{annq}_0(x) + \mathrm{annb}_1(x) I^{m-1} W_q(F).$$

The proof of part (ii) of (1.2) is similar and we omit the details. Thus we have proved (1.2) in the case $I^{N+1} = 0$ for some N. Let us now consider the general case.

Let $\mathcal{B}$ be a 2-basis of F, x a bilinear n-fold Pfister form over F, $x \neq 0$ in $W(F)$. Take $y \in \mathrm{annb}_m(x)$, i.e. $y \in I_F^m$ with $yx = 0$. This relation involves only finitely many elements $\{a_1, \ldots, a_N\} \subset \mathcal{B}$ of the 2-basis. We define $F_0 = F^2(a_1, \ldots, a_N) \subset F$. Then there exist an n-fold bilinear Pfister form x_0 over F_0 and $y_0 \in I_{F_0}^m$ such that $x = x_0 \otimes F$, $y = y_0 \otimes F$ and $y_0 x_0 = 0$ in $W(F_0)$. From the first part of the proof of (1.2) we obtain $y_0 \in \mathrm{annb}_1(x_0) I_{F_0}^{m-1}$ and hence $y \in \mathrm{annb}_1(x) I_F^{m-1}$. The same argument applies for the other assertions in (1.2) and this concludes the proof of theorem (1.2). $\square$

(2.2) Remark. If x is a bilinear n-fold Pfister form over F, then one can describe explicitely the annihilators $\mathrm{annb}_1(x) \subset W(F)$ and $\mathrm{annq}_0(x) \subset W_q(F)$ as follows

(2.3) $$\mathrm{annb}_1(x) = \sum_{d \in D_F(x)^*} W(F)\,\langle 1, d\rangle$$

(2.4) $$\mathrm{annq}_0(x) = \sum_{d \in D_F(x)} W(F)[1, d].$$

Here $D_F(z)$ denotes the set in F of elements represented by the form z. The result (2.3) is shown in [Ho] and (2.4) in [Ba-Kn]. If x denotes now a quadratic n-fold Pfister form over F, $x \neq 0$ in $W_q(F)$, then (see [Kn])

(2.5) $$\mathrm{annb}_1(x) = \sum_{d \in D_F(x)^*} W(F)\,\langle 1, d\rangle\,.$$

In section 4 we will give an independent proof of these facts based on Kato's correspondence (see 3.3) and on the arguments used in this section.

3 Quadratic, symmetric bilinear and differential forms

In this section we will briefly describe Kato's correspondence between quadratic, bilinear and differential forms over a field F with $2 = 0$ and prove a technical result needed in the proof of theorem (1.1) (see [Ka], [Ba-2], [A-Ba]).

Let $\Omega^1_F = F\,\mathrm{d}\,F$ be the F-space of 1-differential forms generated over F by the symbols $\mathrm{d}\,a$, $a \in F$, with $\mathrm{d}(a+b) = \mathrm{d}\,a + \mathrm{d}\,b$, $\mathrm{d}(ab) = a\,\mathrm{d}\,b + b\,\mathrm{d}\,a$. For any $n \geq 1$ set $\Omega^n_F = \bigwedge^n \Omega^1_F$ and let $\mathrm{d} : \Omega^n_F \to \Omega^{n+1}_F$ be the differential operator $\mathrm{d}(x\,\mathrm{d}\,x_1 \wedge \cdots \wedge \mathrm{d}\,x_n) = \mathrm{d}\,x \wedge \mathrm{d}\,x_1 \wedge \cdots \wedge \mathrm{d}\,x_n$, where $\wedge$ denotes exterior multiplication.

Let $\wp : \Omega^n_F \to \Omega^n_F/\mathrm{d}\,\Omega^{n-1}_F$ be the Artin-Schreier operator defined on generators by

$$\wp\left(x\frac{\mathrm{d}\,x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\,x_n}{x_n}\right) = \left(x^2 - x\right)\frac{\mathrm{d}\,x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\,x_n}{x_n} \quad \mathrm{mod}\ \mathrm{d}\,\Omega^{n-1}_F$$

and denote by $\nu_F(n)$ its kernel and by $H^{n+1}(F)$ its cokernel (see loc. cit.). In [Ka] it is shown that there are natural isomorphisms $\alpha : \nu_F(n) \simeq \overline{I}^n_F$ and $\beta : H^{n+1}(F) \simeq \overline{I}^n W_q(F)$ given on generators by $\alpha(\frac{\mathrm{d}\,x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\,x_n}{x_n}) = \ll x_1, \ldots, x_n \gg \ \mathrm{mod}\ I^{n+1}_F$ and $\beta\left(x\frac{\mathrm{d}\,x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\,x_n}{x_n}\right) = \ll x_1, \ldots, x_n; x|] \ \mathrm{mod}\ I^{n+1}W_q(F)$. The fact that $\nu_F(n)$ is additively generated by the pure logarithmic forms $\frac{\mathrm{d}\,x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\,x_n}{x_n}$ follows from a result of Kato which we explain now. Let us fix a 2-basis $\mathcal{B}$ of F, $\mathcal{B} = \{b_i \,|\, i \in I\}$, and endow I with a total ordering. For any $j \in I$, let F_j, resp. $F_{<j}$, be the subfields of F generated over F^2 by b_i, $i \leq j$, resp. b_i, $i < j$. For any $n \geq 1$ let Σ_n be the set of maps $\alpha : \{1, \ldots, n\} \to I$ such that $\alpha(i) < \alpha(j)$ whenever $1 \le i < j \le n$, and endow Σ_n with the lexicographic ordering.

We obtain a filtration of Ω^n_F given by the subspaces $\Omega^n_{F,\alpha}$, resp. $\Omega^n_{F,<\alpha}$, which are generated by the elements $\frac{\mathrm{d}\, b_\beta}{b_\beta} = \frac{\mathrm{d}\, b_{\beta(1)}}{b_{\beta(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\beta(n)}}{b_{\beta(n)}}$ with $\beta \le \alpha$, resp. $\beta < \alpha$. An important result of Kato, named here as Kato's lemma, asserts that for any $\alpha \in \Sigma_n$, $y \in F$, if $\wp\left(y \frac{\mathrm{d}\, b_\alpha}{b_\alpha}\right) \in \Omega^n_{F,<\alpha} + \mathrm{d}\,\Omega^{n-1}_F$, then there exist $v \in \Omega^n_{F,<\alpha}$ and $a_i \in F^*_{\alpha(i)}$, $1 \le i \le n$, such that $y \frac{\mathrm{d}\, b_\alpha}{b_\alpha} = v + \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}$ (see [Ka]). This implies that any $u \in \Omega^n_{F,\alpha}$ satisfying $\wp(\alpha) \in \mathrm{d}\,\Omega^{n-1}_F$, can be written as

(3.1)
$$u = \sum\nolimits_{\gamma \le \alpha} \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(n)}}{a_{\gamma(n)}}$$

with $a_{\gamma(i)} \in F_{\gamma(i)} \backslash F_{<\gamma(i)}$. Then the following result will be used in section 4 during the proof of theorem (1.1).

(3.2) Lemma. Let $\mathcal{B} = \{b_i \,|\, i \in I\}$ be a 2-basis of F with a given ordering on I. Let $\alpha \in \Sigma_n$ and $\sum_{\gamma \le \alpha} c_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma}$ be a differential form with $c_\alpha \ne 0$ such that $\sum_{\gamma \le \alpha} c_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma} \in \mathrm{d}\,\Omega^{n-1}_F$. Then there exist elements $M_i \in F_{<\alpha(i)}$, $1 \le i \le n$, such that

$$c_\alpha = b_{\alpha(1)} M_1 + \cdots + b_{\alpha(n)} M_n$$

Proof: Let $k \in I$ be the index with $c_\alpha \in F_k \backslash F_{<k}$. We claim that $k = \alpha(i)$ for some $1 \le i \le n$. Otherwise we have $k > \alpha(n)$ or $k < \alpha(1)$ or $\alpha(j) < k < \alpha(j+1)$ for some $1 \le j \le n$. From the choice of k we have $c_\alpha = b_k A + B$ with $A, B \in F_{<k}$, $A \ne 0$. Then

$$\mathrm{d}\, t = (b_k A + B) \frac{\mathrm{d}\, b_\alpha}{b_\alpha} + \sum_{\gamma < \alpha} c_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma}$$

and applying the differential operator to this form, we get

$$b_k A \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}\, b_k}{b_k} + b_k A \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}\, A}{A} + B \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}\, B}{B} + \sum_{\gamma < \alpha} \sum_{i \in I} b_i D_i(c_\gamma) \frac{\mathrm{d}\, b_\gamma}{b_\gamma} \wedge \frac{\mathrm{d}\, b_i}{b_i} = 0$$

where $D_i(c_\gamma)$ is the derivative of c_γ with respect to b_i (see [A-Ba]). Looking at the coefficient of $\frac{\mathrm{d}\, b_\alpha}{b_\alpha} \wedge \frac{\mathrm{d}\, b_k}{b_k}$ we obtain

$$b_k A = \sum_{(\alpha,k)=(\gamma_i,i)} b_i D_i(c_\gamma)$$

where (α, k) resp. (γ_i, i) denotes the unique $\lambda \in \Sigma_{n+1}$ with $\mathrm{Im}(\lambda) = \mathrm{Im}(\alpha) \cup \{k\}$ resp. $\mathrm{Im}(\lambda) = \mathrm{Im}(\gamma_i) \cup \{i\}$. Since for those i we have $i > k$, $A \in F_{<k}$ and $D_i(D_i(c_{\gamma_i})) = 0$, we conclude $A = 0$, which is a contradiction. Thus $k = \alpha(i)$ for some $1 \le i \le n$.

Let $c_\alpha = b_{\alpha(i)} M_i + B$ with $M_i,\, B \in F_{<\alpha(i)}$. Then

$$\mathrm{d}\, t = \big(b_{\alpha(i)} M_i + B\big) \frac{\mathrm{d}\, b_\alpha}{b_\alpha} + \sum_{\gamma<\alpha} c_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma}.$$

But

$$\begin{aligned} b_{\alpha(i)} M_i \frac{\mathrm{d}\, b_\alpha}{b_\alpha} &= b_{\alpha(i)} M_i \frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(i)}}{b_{\alpha(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(n)}}{b_{\alpha(n)}} \\ &= \mathrm{d}(b_{\alpha(i)} M_i) \wedge \frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(i-1)}}{b_{\alpha(i-1)}} \wedge \frac{\mathrm{d}\, b_{\alpha(i+1)}}{b_{\alpha(i+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(n)}}{b_{\alpha(n)}} \\ &+ b_{\alpha(i)} M_i \frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, M_i}{M_i} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(n)}}{b_{\alpha(n)}} \end{aligned}$$

so that replacing t by $t' = t + b_{\alpha(i)} M_i \frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(i-1)}}{b_{\alpha(i-1)}} \wedge \frac{\mathrm{d}\, b_{\alpha(i+1)}}{b_{\alpha(i+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(n)}}{b_{\alpha(n)}}$, and since $b_{\alpha(i)} M_i \frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, M_i}{M_i} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(n)}}{b_{\alpha(n)}} \in \Omega^n_{<\alpha}$, we get

$$\mathrm{d}\, t' = B \frac{\mathrm{d}\, b_\alpha}{b_\alpha} + \sum_{\gamma<\alpha} c'_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma}$$

with certain $c'_\gamma \in F$ and $B \in F_{<\alpha(i)}$. We proceed again as before with B instead of c_α and the lemma follows by induction. $\square$

An immediate generalization of (3.2) is

(3.3) Proposition. Let

$$\sum_{\gamma \le \alpha} c_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma} = \mathrm{d}(t) + \wp(w)$$

with $c_\alpha \neq 0$, where $\mathcal{B} = \{b_i \,|\, i \in I\}$ is a given 2-basis of F (and a fixed ordering in I) and $t \in \Omega_F^{n-1}$, $w \in \Omega_F^n$. Then there exist elements $u \in F$, $M_i \in F_{<\alpha(i)}$, $1 \le i \le n$, such that

$$c_\alpha = \wp u + b_{\alpha(1)} M_1 + \cdots + b_{\alpha(n)} M_n.$$

4 Annihilators of differential forms in $\nu_F(m)$ and $H^{m+1}(F)$

The groups $\nu_F(m)$ act on the groups $H^{n+1}(F)$ through exterior multiplication

$$\wedge : \nu_F(m) \times H^{n+1}(F) \longrightarrow H^{m+n+1}(F)$$

$$\wedge : \nu_F(m) \times \nu_F(n) \longrightarrow \nu_F(m+n)$$

and we can define for any $x \in \nu_F(n)$ the annihilators

$$\operatorname{annb}_m(x) = \{y \in \nu_F(m) \mid xy = 0 \quad \text{in } \ \nu_F(m+n)\}$$

$$\operatorname{annq}_m(x) = \{y \in H^{m+1}(F) \mid xy = 0 \quad \text{in } \ H^{n+m+1}(F)\}.$$

Also if $x \in H^{n+1}(F)$, we define

$$\operatorname{annb}_m(x) = \{y \in \nu_F(m) \mid yx = 0 \quad \text{in } \ H^{n+m+1}(F)\}.$$

Through Kato's isomorphisms (see § 3) these annihilators are isomorphic to the corresponding graded annihilators of bilinear and quadratic forms, namely, if $x \in \nu_F(n)$

$$\alpha : \operatorname{annb}_m(x) \simeq \overline{\operatorname{annb}}_m(\alpha(x))$$

$$\beta : \operatorname{annq}_m(x) \simeq \overline{\operatorname{annq}}_m(\alpha(x))$$

and if $x \in H^{n+1}(F)$,

$$\alpha : \operatorname{annb}_m(x) \simeq \overline{\operatorname{annb}}_m(\beta(x)).$$

Thus, theorem (1.1) is equivalent to the following

(4.1) Theorem. (i) Let $x = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \in \nu_F(n)$ be a pure logarithmic differential form, $x \neq 0$. Then for any $m \geq 1$

$$\operatorname{annb}_m(x) = \operatorname{annb}_1(x) \wedge \nu_F(m-1)$$

$$\operatorname{annq}_m(x) = \nu_F(m) \wedge \operatorname{annq}_0(x) + \operatorname{annb}_1(x) \wedge H^m(F).$$

(ii) If $x = \overline{a\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}} \neq 0$ in $H^{n+1}(F)$, then in $\nu_F(m)$

$$\operatorname{annb}_m(x) = \operatorname{annb}_1(x) \wedge \nu_F(m-1).$$

Proof: Let $\mathcal{B} = \{b_i \mid i \in I\}$ be a 2-basis of F such that $a_1, \ldots, a_n \in \mathcal{B}$ are the first elements in some ordering of I. Let $y \in \operatorname{annb}_m(x)$. Using Kato's lemma we can write

$$y = \sum_{\gamma \in \Sigma_m} \varepsilon_\gamma \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}}$$

with $a_{\gamma(i)} \in F_{\gamma(i)} \backslash F_{<\gamma(i)}$, $\varepsilon_\gamma \in \{0, 1\}$. Let $\alpha \in \Sigma_m$ be maximal with $\varepsilon_\alpha \neq 0$. Then

$$y \equiv \frac{\mathrm{d}\, a_\alpha}{a_\alpha} \mod \Omega^m_{F,<\alpha}.$$

The assumption $xy = 0$ means

$$\left(\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}\right) \wedge \frac{\mathrm{d}\, a_\alpha}{a_\alpha} + \left(\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}\right) \wedge \sum_{\gamma<\alpha} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} = 0.$$

Assume first $\alpha(1) > n$ and define $\delta = (1, \dots, n, \alpha(1), \dots, \alpha(m)) \in \Sigma_{n+m}$. It follows $\delta > (1, \dots, n, \gamma)$ for all $\gamma \in \Sigma_m$ with $\gamma < \alpha$. From the last relation we conclude

$$\mathrm{d}\, a_1 \wedge \cdots \wedge \mathrm{d}\, a_n \wedge \mathrm{d}\, a_{\alpha(1)} \wedge \cdots \wedge \mathrm{d}\, a_{\alpha(m)} = 0$$

which is a contradiction to the fact that $a_1, \dots, a_n, a_{\alpha(1)}, \dots, a_{\alpha(m)}$ are 2-independent. Thus we have $\alpha(1) \le n$, and this implies $x \wedge \frac{\mathrm{d}\, a_{\alpha(1)}}{a_{\alpha(1)}} = 0$, i.e. $\frac{\mathrm{d}\, a_{\alpha(1)}}{a_{\alpha(1)}} \in \mathrm{annb}_1(x)$. Hence $y - \frac{\mathrm{d}\, a_\alpha}{a_\alpha} \in \mathrm{annb}_m(x)$ and moreover $y - \frac{\mathrm{d}\, a_\alpha}{a_\alpha} \in \Omega^m_{F,<\alpha}$. Proceeding by induction on α we get the first assertion in (i).

Take now $\overline{y} \in \mathrm{annq}_m(x) \subset H^{m+1}(F)$. Then

$$y \equiv \sum_{\gamma \in \Sigma_m} c_\gamma \frac{\mathrm{d}\, b_\gamma}{b_\gamma} \mod \wp\Omega^m_F + \mathrm{d}\, \Omega^{m-1}_F$$

with $x \wedge y \in \wp\Omega^{m+n}_F + \mathrm{d}\, \Omega^{m+n-1}_F$, i.e.

$$\textbf{(4.2)} \qquad \sum_{\gamma \in \Sigma_m} c_\gamma \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \frac{\mathrm{d}\, b_\gamma}{b_\gamma} \in \wp\Omega^{m+n}_F + \mathrm{d}\, \Omega^{m+n-1}_F.$$

(Here the elements $b_{\gamma(i)}$ belong to $\mathcal{B}$). Let $\alpha \in \Sigma_m$ be maximal with $c_\alpha \neq 0$. If $\alpha(1) \le n$, then $\frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \in \mathrm{annb}_1(x)$ and $c_\alpha \frac{\mathrm{d}\, b_\alpha}{b_\alpha} = \frac{\mathrm{d}\, b_{\alpha(1)}}{b_{\alpha(1)}} \wedge c_\alpha \frac{\mathrm{d}\, b_{\alpha(2)}}{b_{\alpha(2)}} \wedge \cdots \wedge \frac{\mathrm{d}\, b_{\alpha(m)}}{b_{\alpha(m)}} \in \mathrm{annb}_1(x) \wedge H^m(F)$, and $y - c_\alpha \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \in \Omega^m_{F,<\alpha}$. Hence we may proceed by induction on α. Thus we can assume $\alpha(1) > n$ and we define $\delta = (1, \dots, n, \alpha(1), \dots, \alpha(m)) \in \Sigma_{n+m}$. We see in (4.2) that δ is the maximal multi-index with coefficient $c_\alpha \neq 0$. Using now proposition (3.3), we conclude from (4.2) that

$$c_\alpha = \wp(u) + E_\alpha$$

with $E_\alpha = \sum_{i=1}^n a_i M_i + \sum_{j=1}^m b_{\alpha(j)} M_{\alpha(j)}$ and $M_k \in F_{<k}$. Here we have chosen the ordering of $\mathcal{B}$ such that $a_1, \dots, a_n$ are the first elements.

Inserting c_α in y we get

$$y \equiv c_\alpha \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \mod \wp\Omega^m_F + \mathrm{d}\, \Omega^{m-1}_F + \Omega^m_{F,<\alpha}$$

$$y \equiv \left[\wp(u) + \sum_{i=1}^n a_i M_i + \sum_{j=1}^m b_{\alpha(j)} M_{\alpha(j)}\right] \frac{\mathrm{d}\, b_\alpha}{b_\alpha}$$

$$y \equiv \left[\sum_{i=1}^n a_i M_i\right] \frac{\mathrm{d}\, b_\alpha}{b_\alpha} + \left[\sum_{j=1}^m b_{\alpha(j)} M_{\alpha(j)}\right] \frac{\mathrm{d}\, b_\alpha}{b_\alpha}.$$

Since $M_k \in F_{<k}$, we have $a_i M_i \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \in \nu_F(m) \wedge \operatorname{annq}_0(x)$ because $a_i M_i \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} = \mathrm{d}\left(a_i M_i \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \overset{i}{\wedge} \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \right) \in \mathrm{d}\, \Omega_F^{n-1}$ implies $a_i M_i \in \operatorname{annq}_0(x)$ (we have used $\mathrm{d}\, M_i \wedge x = 0$). The same argument shows, since $M_{\alpha(j)} \in F_{<\alpha(j)}$, that

$$b_{\alpha(j)} M_{\alpha(j)} \frac{\mathrm{d}\, b_{\alpha(j)}}{b_{\alpha(j)}} = \mathrm{d}\left(b_{\alpha(j)} M_{\alpha(j)} \right) + b_{\alpha(j)} M_{\alpha(j)} \frac{\mathrm{d}\, M_{\alpha(j)}}{M_{\alpha(j)}} \in \mathrm{d}\, F + \Omega^1_{F,<\alpha(j)}$$

and hence

$$\left(\sum_{j=1}^{m} b_{\alpha(j)} M_{\alpha(j)} \right) \frac{\mathrm{d}\, b_\alpha}{b_\alpha} \in \mathrm{d}\, \Omega_F^{m-1} + \Omega^m_{F,<\alpha}.$$

Thus we have

$$y = y' + z \mod \wp\Omega_F^m + \mathrm{d}\, \Omega_F^{m-1}$$

with $y' \in \Omega^m_{F,<\alpha}$, $y' \in \operatorname{annq}_m(x)$ and $z \in \nu_F(m) \wedge \operatorname{annq}_0(x)$. Applying now the above procedure to y' we get our second assertion by induction on α. This proves (i).

(ii) Let $x = \overline{a \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}} \in H^{n+1}(F)$ be a pure element, $x \neq 0$. We fix as before a 2-basis $\mathcal{B} = \{ b_i \,|\, i \in I \}$ of F such that $a_1, \ldots, a_n$ are the first elements in $\mathcal{B}$ in some ordering of I. Let $y \in \operatorname{annb}_m(x) \subset \nu_F(m)$. From Kato's lemma we have $y = \sum_{\gamma \in \Sigma_m} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma}$ with $\varepsilon_\gamma \in \{0,1\}$ and $a_{\gamma(i)} \in F_{\gamma(i)} \backslash F_{<\gamma(i)}$, $1 \le i \le m$. We write

$$y = \sum_{\substack{\gamma \in \Sigma_m \\ \gamma(1) \le n}} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} + \sum_{\substack{\gamma \in \Sigma_m \\ \gamma(1) > n}} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma}.$$

For $\gamma \in \Sigma_m$ with $\gamma(1) \le n$ we have $\frac{\mathrm{d}\, a_\gamma}{a_\gamma} \in \operatorname{annb}_1(x)$ since $a_{\gamma(1)} \in F_n = F^2(a_1, \ldots, a_n)$ and hence the first summand in this decomposition is in $\operatorname{annb}_1(x) \wedge \nu_F(m-1)$. Thus the second summand is in $\operatorname{annb}_m(x)$ and we can assume $y = \sum_{\gamma \in \Sigma_m} \varepsilon_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma}$ with all γ such that $\gamma(1) > n$. Let α be maximal in this sum with $\varepsilon_\alpha \neq 0$. We can replace $\mathcal{B}$ by a new 2-basis $\mathcal{B}' = \{ c_i \,|\, i \in I \}$ such that $c_{\alpha(j)} = a_{\alpha(j)}$, $1 \le j \le m$ and $c_i = b_i$ for all $i \notin \{\alpha(1), \ldots, \alpha(m)\}$. Let $\delta = (1, \ldots, n, \alpha(1), \ldots, \alpha(m)) \in \Sigma_{n+m}$. Hence

$$0 \equiv y \wedge x \equiv a \frac{\mathrm{d}\, c_\delta}{c_\delta} \mod \wp\Omega_F^{n+m} + \mathrm{d}\, \Omega_F^{n+m-1} + \Omega^{n+m}_{F,<\delta}.$$

Then proposition (3.3) implies

$$a = \wp(u) + \sum_{i=1}^{n} c_i M_i + \sum_{j=1}^{m} c_{\alpha(j)} M_{\alpha(j)}$$

with $M_k \in F_{<k}$. Let $s \in \{1,\ldots,m\}$ be maximal with $M_{\alpha(s)} \neq 0$ and set $Q = a + \wp u + \sum_{i=1}^n c_i M_i$ i.e. $Q = \sum_{j=1}^m c_{\alpha(j)} M_{\alpha(j)}$. Then $c_{\alpha(s)} = M_{\alpha(s)}^{-1}\left(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}\right)$. Inserting in y we get modulo $\nu_{F,<\alpha}(m)$

$$\begin{aligned}
y &\equiv \frac{\mathrm{d}\, c_{\alpha(1)}}{c_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(s)}}{c_{\alpha(s)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \quad \mathrm{mod}\ \nu_{F,<\alpha}(m) \\
&\equiv \frac{\mathrm{d}\, c_{\alpha(1)}}{c_{\alpha(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, M_{\alpha(s)}^{-1}\left(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}\right)}{M_{\alpha(s)}^{-1}\left(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}\right)} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \\
&\equiv \frac{\mathrm{d}\left(c_{\alpha(1)} M_{\alpha(1)}\right)}{\left(c_{\alpha(1)} M_{\alpha(1)}\right)} \wedge \cdots \wedge \frac{\mathrm{d}\left(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}\right)}{Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}}.
\end{aligned}$$

Here we have inserted $M_{\alpha(j)}$ whenever it is $\neq 0$, without altering the congruence modulo $\nu_{F,<\alpha}(m)$. Use now the relation $\frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, b}{b} = \frac{\mathrm{d}(ab)}{ab} \wedge \frac{\mathrm{d}(a+b)}{a+b}$ to conclude

$$\begin{aligned}
y &\equiv \frac{\mathrm{d}\left(c_{\alpha(1)} M_{\alpha(1)}\right)}{\left(c_{\alpha(1)} M_{\alpha(1)}\right)} \wedge \cdots \wedge \frac{\mathrm{d}\left(Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}\right)}{Q + \sum_{j=1}^{s-1} c_{\alpha(j)} M_{\alpha(j)}} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \quad \mathrm{mod}\ \nu_{F,<\alpha}(m) \\
&\equiv \frac{\mathrm{d}\, f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}\, Q}{Q} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}}
\end{aligned}$$

with certain $f_1, \ldots, f_{s-1} \in F$. Since $\frac{\mathrm{d}\, Q}{Q} \in \mathrm{annb}_1(x)$ (we can assume $a \in F^2$ without restriction), we get $\frac{\mathrm{d}\, f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}\, Q}{Q} \wedge \cdots \wedge \frac{\mathrm{d}\, c_{\alpha(m)}}{c_{\alpha(m)}} \in \mathrm{annb}_1(x) \wedge \nu_F(m-1)$. Thus we have shown $y \in \mathrm{annb}_1(x) \wedge \nu_F(m-1) + \nu_{F,<\alpha}(m)$. We apply now induction on α to conclude the proof of (ii). $\square$

Let us briefly compute the annihilators $\mathrm{annb}_1(x)$ and $\mathrm{annq}_0(x)$ for $x = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \in \nu_F(n)$ and $\mathrm{annb}_1(x)$ for $x = \overline{a \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}} \in H^{n+1}(F)$.

(4.3) Proposition. (i) Let $x = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \in \nu_F(n)$, $x \neq 0$. Then

$$\mathrm{annb}_1(x) = \left\{ \frac{\mathrm{d}\, z}{z} \mid z \in F^2(a_1, \ldots, a_n)^* \right\}$$

$$\mathrm{annq}_0(x) = \left\{ \overline{z} \in F/\wp F \mid z \in F^2(a_1, \ldots, a_n)' \right\}$$

where $F^2(a_1, \ldots, a_n)'$ are the pure elements in $F^2(a_1, \ldots, a_n)$ (notice that $H^1(F) = F/\wp F$).

(ii) Let $x = \overline{a \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}} \in H^{n+1}(F)$, $x \neq 0$. Then

$$\mathrm{annb}_1(x) = \left\{ \frac{\mathrm{d}\, z}{z} \mid z \in D_F(\ll a_1, \ldots, a_n; |]\!])^* \right\}$$

where $D_F(q)$ denotes the elements represented in F by the quadratic form q.

Proof: (i) Let $x = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \neq 0$ in $\nu_F(n)$. If $\frac{\mathrm{d}\, z}{z} \in \operatorname{annb}_1(x) \subset \nu_F(1)$, then

$$\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \frac{\mathrm{d}\, z}{z} = 0$$

in $\nu_F(n+1)$, which means that $a_1, \ldots, a_n, z$ are 2-dependent, and since $a_1, \ldots, a_n$ are 2-independent, this means $z \in F^2(a_1, \ldots, a_n)^*$ (which is the set in F^* of elements represented by the n-fold Pfister form $\overline{\ll a_1, \ldots, a_n \gg}$).

Let now $\overline{y} \in H^1(F) = F/\wp F$ be in $\operatorname{annq}_0(x)$. Then $\overline{y\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}} = 0$ in $H^{n+1}(F)$, and this means

$$y\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \in \wp\Omega_F^n + \mathrm{d}\, \Omega_F^{n-1}.$$

Taking a 2-basis of F so that $a_1, \ldots, a_n$ are the first elements of it (in some ordering), we conclude from proposition (3.3)

$$y = \wp u + b$$

with $u \in F$ and $b \in F^2(a_1, \ldots, a_n)'$. This proves (i).

(ii) Let $x = \overline{a\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n}} \in H^{n+1}(F)$, $x \neq 0$ and take $\frac{\mathrm{d}\, z}{z} \in \operatorname{annb}_1(x) \subset \nu_F(1)$. This means

$$a\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \frac{\mathrm{d}\, z}{z} \in \wp\Omega_F^{n+1} + \mathrm{d}\, \Omega_F^{n}.$$

If $\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \frac{\mathrm{d}\, z}{z} = 0$, then we get as before $z \in F^2(a_1, \ldots, a_n)^* \subset D_F(\ll a_1, \ldots, a_n; |])^*$. Assume $\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \frac{\mathrm{d}\, z}{z} \neq 0$. Then we can assume that $a_1, \ldots, a_n, z$ are the first elements of some 2-basis of F (in some ordering), and applying now proposition (3.3) we obtain $a = \wp u + b$ with $b \in F^2(a_1, \ldots, a_n, z)'$, i.e. $b = z \cdot h + g$ with $h \in F^2(a_1, \ldots, a_n)^*$ and $g \in F^2(a_1, \ldots, a_n)'$.

Thus $z = h^{-1}(\wp u + a + g) \in D_F(\ll a_1, \ldots, a_n; |])^*$. This proves (ii). □

The isomorphisms $\nu_F(m) \simeq \overline{I}_F^m$ and $H^{m+1}(F) \simeq \overline{I}^m W_q(F)$ enable us to translate this result into the language of bilinear and quadratic forms.

Let $x = \ll a_1, \ldots, a_n \gg$ be a bilinear anisotropic n-fold Pfister form. Then we have

$$\overline{\operatorname{ann}}\mathrm{b}_1(x) = \left\{ \overline{\ll z \gg} \mid z \in D_F\,(x)^* \right\}$$

$$\overline{\operatorname{ann}}\mathrm{q}_0(x) = \left\{ \overline{z} \in F/\wp F \mid z \in D_F\,(x')^* \right\}$$

where we identify $\overline{I}^0 W_q(F)$ with $F/\wp F$ through the Arf-invariant.

If $x = \ll a_1, \ldots, a_n; a]]$ is a quadratic anisotropic n-fold Pfister form, then

$$\overline{\mathrm{ann}}\mathrm{b}_1(x) = \left\{ \overline{\ll z \gg} \mid z \in D_F(x)^* \right\}$$

Now the technique used in section 2 enables us to compute the full annihilators $\mathrm{annb}_1(x)$, $\mathrm{annq}_0(x)$ if $x = \ll a_1, \ldots, a_n \gg$ and $\mathrm{annb}_1(x)$ if $x = \ll a_1, \ldots, a_n; a]]$, thereby obtaining the results (2.3), (2.4) and (2.5). Let us prove for example (2.3) (the others cases are left as exercises). Let $x = \ll a_1, \ldots, a_n \gg$ and take $y \in \mathrm{annb}_1(x) \subset I_F$. Then $\overline{y} \in \overline{\mathrm{ann}}\mathrm{b}_1(x)$ and hence $\overline{y} = \overline{\ll z \gg}$ for some $z \in D_F(x)^*$. Thus $y - \ll z \gg \in I^2$ and $(y - \ll z \gg)\, x = 0$ i.e. $y - \ll z \gg \in \mathrm{annb}_2(x) = \mathrm{annb}_1(x) \cdot I_F$. Write $y - \ll z \gg = \sum y_i v_i$ with $y_i \in \mathrm{annb}_1(x)$, $v_i \in I_F$. Then $y_i - \ll z_i \gg \in I_F^2$ for some $z_i \in D_F(x)^*$ and hence

$$y - \ll z \gg - \sum \ll z_i \gg v_i \in I_F^3.$$

Iterating this procedure and assuming $I_F^{N+1} = 0$ for some N, we get (2.3). The general case can be reduced to the assumption $I_F^{N+1} = 0$ using the trick of section 2. This proves (2.3). The same argument applies for (2.4) and (2.5). Thus we have a complete description of the annihilators of Pfister forms over a field F with $2 = 0$.

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