Engel-like characterization of radicals in finite dimensional Lie algebras and finite groups

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Preliminary report on joint projects:

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Abelian Lie algebras:

$$[x, y] \equiv 0$$

What about other classes of Lie algebras?

$L^0 = L,\ L^{i+1} = [L^i, L^i], \ldots$

If $L^n = 0$, L is <u>solvable</u>.

$L_0 = L,\ L_{i+1} = [L, L_i], \ldots$

If $L_n = 0$, L is <u>nilpotent</u>.

Nilpotency $\Longrightarrow$ Solvability

1. Characterization of f.d. nilpotent Lie algebras

Define $e_1(x,y) = [x,y]$,
$e_{n+1}(x,y) = [e_n(x,y), y], \ldots$

Th. (Engel, 19th century)
Let L be a f.d. Lie algebra over a field k. Then L is nilpotent if and only if for some n

$$e_n(x,y) \equiv 0$$

in L.

2. Characterization of finite nilpotent groups

Denote $[x,y] = xyx^{-1}y^{-1}$.

Define $e_1(x,y) = [x,y]$,

$e_{n+1}(x,y) = [e_n(x,y), y]$.

<u>Th.</u> (Zorn, 1936).

Let G be a finite group. Then G is nilpotent if and only if for some n

$$e_n(x,y) \equiv 1$$

in G.

3. Characterization of f.d. solvable Lie algebras

Define $w_1(x,y) = [x,y]$,

$w_{n+1}(x,y) = [[w_n(x,y),x],[w_n(x,y),y]], \ldots$

<u>Th.</u> (Grunewald, Nikolova, Kunyavskii, Plotkin, 2000)

Let L be a f.d. Lie algebra over an infinite field k $(\operatorname{char} k \neq 2,3,5)$. Then L is solvable if and only if for some n

$$w_n(x,y) \equiv 0$$

in L.

4. Characterization of finite solvable groups

Denote $[x,y]=xyx^{-1}y^{-1}$.

Define $u_1(x,y)=x^{-2}y^{-1}x$,

$u_{n+1}(x,y)=[x u_n(x,y)x^{-1},\ y u_n(x,y)y^{-1}], \ldots$

Th. (Bandman, Greuel, Grunewald, Kunyavskii, Pfister, Plotkin, 2003)

Let G be a finite group. Then G is solvable if and only if for some n

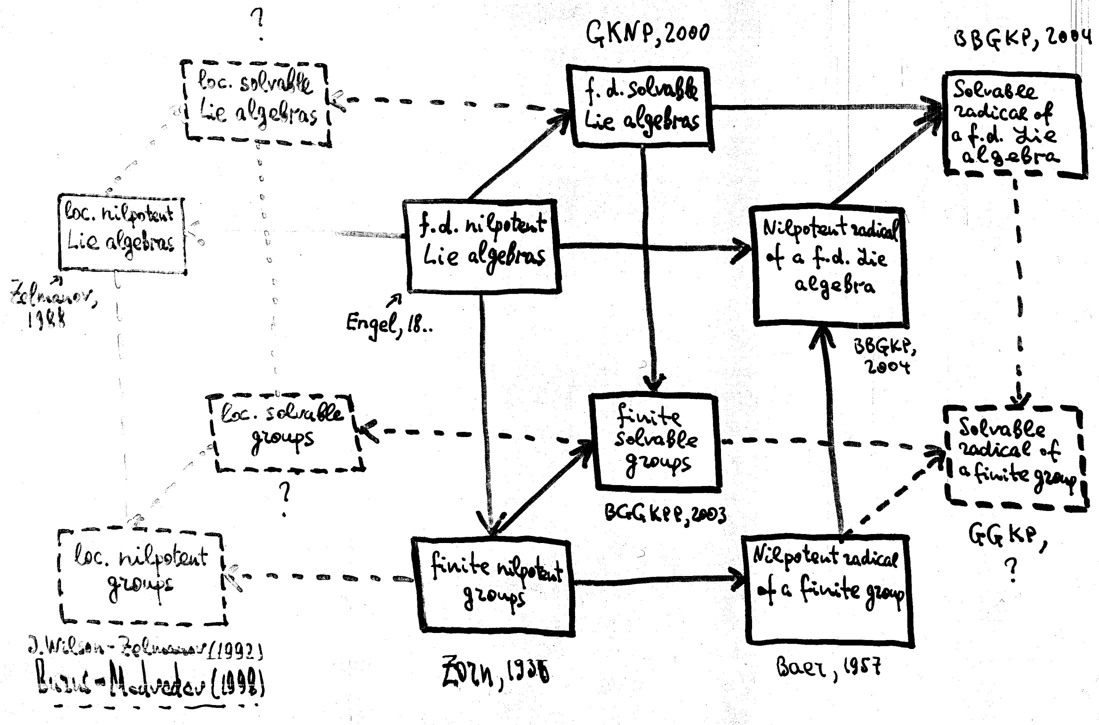
$$u_n(x,y)\equiv 1$$

in G.

Corollary. Let K be a field, $G \subset GL(m, K)$. Then G is solvable if and only if for some n

$$U_n(x, y) \equiv 1$$

in G.


?
GKNP, 2000
BBGKP, 2004
loc. solvable Lie algebras
f. d. solvable Lie algebras
Solvable radical of a f.d. Lie algebra
loc. nilpotent Lie algebras
f.d. nilpotent Lie algebras
Nilpotent radical of a f.d. Lie algebra
Zelmanov, 1988
Engel, 18..
BBGKP, 2004
loc. solvable groups
finite solvable groups
Solvable radical of a finite group
?
BGGKPP, 2003
GGKP, ?
loc. nilpotent groups
finite nilpotent groups
Nilpotent radical of a finite group
J. Wilson - Zelmanov (1992)
Zorn, 1936
Baer, 1957

Def. The nilpotent (solvable) radical of a finite group (f.d. Lie algebra) is the biggest nilpotent (solvable) normal subgroup (ideal).

Goal: to characterize these radicals using Engel-type sequences.

Def. Let G be a group. We say that $y \in G$ is an <u>Engel element</u> if

$$(\forall x \in G)(\exists n = n(x,y)) \quad e_n(x,y) = 1.$$

5. Nilpotent radical of a finite group

<u>Th.</u> (Baer, 1957)

The nilpotent radical of a finite group coincides with the set of its Engel elements.

Goals: to extend Baer's theorem to

a) the nilpotent radical of f.d. Lie algebras;

b) the solvable radical of f.d. Lie algebras;

c) the solvable radical of finite groups.

Engel-like sequences

Def. Let $F_2 = F(x,y)$ denote the free two generator group. We say that a sequence

$$\vec{u} = u_1, u_2, \dots, u_n, \dots$$

of elements of F_2 is <u>correct</u> if the following conditions hold:

(i) $u_n(a,1) = u_n(1,g) = 1$ for all sufficiently big n, every group G, and all elements $a, g \in G$;

(ii) $u_n(a,g) = 1 \Rightarrow (\forall m > n)\ u_m(a,g) = 1$.

Def. $G(\vec{u}) := \{ g \in G : (\forall a \in G)(\exists n = n(a,g))$
$u_n(a,g) = 1 \}$ –

the set of $\vec{u}$-Engel-like elements (for brevity – $\vec{u}$-Engel elements).

Solvable radical of a Lie algebra

Define $v_1(x,y)=x$,

$$v_{n+1}(x,y)=[v_n(x,y),[x,y]],\ \ldots$$

<u>Th. 1</u> Let L be a f.d. Lie algebra over a field k of characteristic 0. Then the solvable radical of L coincides with the set $L(\vec{v})$ of $\vec{v}$-Engel elements.

Define $w_1(x,y)=[x,y]$,

$$w_{n+1}(x,y)=[[w_n(x,y),x],[w_n(x,y),y]].$$

<u>Th. 2</u>. Let L be a f.d. Lie algebra over an uncountable field k of characteristic 0. Then the solvable radical of L coincides with the set $L(\vec{w})$ of $\vec{w}$-Engel elements.

If char $k > 0$,

there are counter-examples to the statements of Th. 1 and Th. 2

In the case of Th. 2, there is some hope to extend it (partially) to positive characteristics.

Strategy of proof

(1) Reduce to the case of simple L.

(2) Reduce to the case $k = \bar{k}$ using uniformity: prove that n in the definition of Engel-like element may be chosen independent on x.

(3) Treat the case of a simple algebra over alg. closed field using properties of root systems.

Nilpotent radical of Lie algebras

a direct analogue of Baer's theorem does not hold:

$sl(2)$ is simple (hence its nilpotent radical is 0) but contains Engel elements: $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

A way out:

__Def.__ $y \in L$ is __strictly Engel__ if it is Engel and $[x,y]$ is Engel for all $x \in L$.

__Th. 3.__ The nilpotent radical of a f.d. Lie algebra over a field k of characteristic 0 coincides with the set of its strictly Engel elements.

Group case

Main Question:

Does there exist a correct sequence

$$\vec{u} = u_1, u_2, \ldots, u_n$$

such that for every finite group G the solvable radical of G coincides with the set $G(\vec{u})$ of $\vec{u}$-Engel elements ?

Results: reduction to the case of simple groups

Less ambitious approach

Def. Let $s \in \mathbb{N}$, $s \geq 2$. Let G be a finite group. We say that $y \in G$ is an s-radical element if for any $x_1, x_2, \dots, x_s \in G$ the group

$$\langle [x_1, y], \dots, [x_s, y] \rangle$$

is solvable.

Conj. 1. There exists k such that for every finite group G the solvable radical of G coincides with the set of k-radical elements.

Conj. 2. In Conj. 1, one can take $k=3$.

$k=2$ is not enough: if $G=S_6$, the solvable radical is trivial but the element $y=(12)$ is 2-radical.

However, for most simple groups we expect the absence of 2-radical elements.

"Th.4". Let G be a finite simple group of Lie type over a field of characteristic different from 2 and 3. Then G contains no 2-radical elements.