NONCOMMUTATIVE LOCALIZATION IN ALGEBRA AND TOPOLOGY

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- 2002 Edinburgh conference

- Proceedings will appear in 2005, with papers by Beachy, Cohn, Dwyer, Linnell, Neeman, Ranicki, Reich, Sheiham and Skoda.

Noncommutative localization

- Given a ring A and a set Σ of elements, matrices, morphisms, ..., it is possible to construct a new ring $\Sigma^{-1}A$, the localization of A inverting all the elements in Σ. In general, A and $\Sigma^{-1}A$ are noncommutative.

- Original algebraic motivation: construction of noncommutative analogues of the classical localization

  $$A = \text{integral domain} \hookrightarrow \Sigma^{-1}A = \text{fraction field}$$
  $$\text{with } \Sigma = A - \{0\} \subset A \ .$$

  Ore (1933), Cohn (1970), Bergman (1974), Schofield (1985).

- Topological applications use the algebraic K- and L-theory of A and $\Sigma^{-1}A$, with A a group ring or a triangular matrix ring.

Ore localization

- The <u>Ore localization</u> $\Sigma^{-1}A$ is defined for a multiplicatively closed subset $\Sigma \subset A$ with $1 \in \Sigma$, and such that for all $a \in A$, $s \in \Sigma$ there exist $b \in A$, $t \in \Sigma$ with $ta = bs \in A$.

- E.g. central, $sa = as$ for all $a \in A$, $s \in \Sigma$.

- The Ore localization is the ring of fractions

$$\Sigma^{-1}A \ = \ (\Sigma \times A)/\!\sim \ ,$$

  $(s,a) \sim (t,b)$ iff there exist $u,v \in A$ with

$$us \ = \ vt \in \Sigma \ , \ ua \ = \ vb \in A \ .$$

- An element of $\Sigma^{-1}A$ is a noncommutative fraction

  $s^{-1}a \ =$ equivalence class of $(s,a) \in \Sigma^{-1}A$

  with addition and multiplication more or less as usual.

Ore localization is flat

- An Ore localization $\Sigma^{-1}A$ is a flat A-module, i.e. the functor

$$\{A\text{-modules}\} \to \{\Sigma^{-1}A\text{-modules}\} ;$$
$$M \mapsto \Sigma^{-1}A \otimes_A M = \Sigma^{-1}M$$

is exact.

- For an Ore localization $\Sigma^{-1}A$ and any A-module M

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, M) = 0 \quad (i \geqslant 1) .$$

- For an Ore localization $\Sigma^{-1}A$ and any A-module chain complex C

$$H_*(\Sigma^{-1}C) = \Sigma^{-1}H_*(C) .$$

The universal localization of P.M.Cohn

- A = ring, Σ = a set of morphisms $s : P \to Q$ of f.g. projective A-modules. A ring morphism $A \to B$ is Σ-inverting if each $1 \otimes s : B \otimes_A P \to B \otimes_A Q$ $(s \in \Sigma)$ is a B-module isomorphism.

- The universal localization $\Sigma^{-1}A$ is a ring with a Σ-inverting morphism $A \to \Sigma^{-1}A$ such that any Σ-inverting morphism $A \to B$ has a unique factorization $A \to \Sigma^{-1}A \to B$.

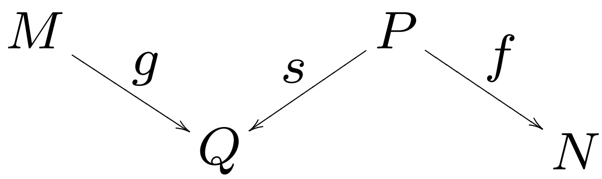
- The universal localization $\Sigma^{-1}A$ exists (and it is unique); but it could be 0 – e.g if $0 \in \Sigma$.

- In general, $\Sigma^{-1}A$ is not a flat A-module. $\Sigma^{-1}A$ is a flat A-module if and only if $\Sigma^{-1}A$ is an Ore localization (Beachy, Teichner, 2003).

The normal form (I)

- (Gerasimov, Malcolmson, 1981) Assume Σ consists of all the morphisms $s : P \to Q$ of f.g. projective A-modules such that $1 \otimes s : \Sigma^{-1}P \to \Sigma^{-1}Q$ is a $\Sigma^{-1}A$-module isomorphism. (Can enlarge any Σ to have this property). Then every element $x \in \Sigma^{-1}A$ is of the form $x = fs^{-1}g$ for some $(s : P \to Q) \in \Sigma$, $f : P \to A$, $g : A \to Q$.

- For f.g. projective A-modules M, N every $\Sigma^{-1}A$-module morphism $x : \Sigma^{-1}M \to \Sigma^{-1}N$ is of the form $x = fs^{-1}g$ for some $(s : P \to Q) \in \Sigma$, $f : P \to N$, $g : M \to Q$.



Addition by

$$fs^{-1}g + f's'^{-1}g' = (f \oplus f')(s \oplus s')^{-1}(g \oplus g') : \Sigma^{-1}M \to \Sigma^{-1}N$$

Similarly for composition.

The normal form (II)

- For f.g. projective M, N, a $\Sigma^{-1}A$-module morphism $fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N$ is such that $fs^{-1}g = 0$ if and only if there is a commutative diagram of A-module morphisms

$$\begin{array}{ccc} P \oplus P_1 \oplus P_2 \oplus M & \xrightarrow{\begin{pmatrix} s & 0 & 0 & g \\ 0 & s_1 & 0 & 0 \\ 0 & 0 & s_2 & g_2 \\ f & f_1 & 0 & 0 \end{pmatrix}} & Q \oplus Q_1 \oplus Q_2 \oplus N \\ & \searrow \begin{pmatrix} p & p_1 & p_2 & m \end{pmatrix} \qquad \begin{pmatrix} q & q_1 & q_2 & n \end{pmatrix}^T \nearrow & \\ & L & \end{array}$$

  with $s, s_1, s_2, \begin{pmatrix} p & p_1 & p_2 \end{pmatrix}, \begin{pmatrix} q & q_1 & q_2 \end{pmatrix}^T \in \Sigma$.
  (Exercise: diagram $\Longrightarrow fs^{-1}g = 0$).

- The condition generalizes to express

$$fs^{-1}g \;=\; f's'^{-1}g' : \Sigma^{-1}M \to \Sigma^{-1}N$$

  in terms of A-module morphisms.

The K_0-K_1 localization exact sequence

- Assume each $(s : P \to Q) \in \Sigma$ is injective and $A \to \Sigma^{-1}A$ is injective. The torsion exact category $T(A, \Sigma)$ has objects A-modules T with $\Sigma^{-1}T = 0$, hom. dim. $(T) = 1$. E.g., $T = \text{coker}(s)$ for $s \in \Sigma$.

- Theorem (Bass, 1968 for central, Schofield, 1985 for universal $\Sigma^{-1}A$). Exact sequence

$$K_1(A) \to K_1(\Sigma^{-1}A) \xrightarrow{\partial}$$
$$K_0(T(A,\Sigma)) \to K_0(A) \to K_0(\Sigma^{-1}A) \quad \text{with}$$
$$\partial\Big(\tau(fs^{-1}g : \Sigma^{-1}M \to \Sigma^{-1}N)\Big)$$
$$= \Big[\text{coker}(\begin{pmatrix} f & 0 \\ s & g \end{pmatrix} : P \oplus M \to N \oplus Q)\Big]$$
$$-\Big[\text{coker}(s : P \to Q)\Big] \quad (M, N \text{ based f.g. free}).$$

- Theorem (Quillen, 1972, Grayson, 1980) Higher K-theory localization exact sequence for Ore localization $\Sigma^{-1}A$, by flatness.

Universal localization is not flat

- In general, if M is an A-module and C is an A-module chain complex

$$\mathrm{Tor}_*^A(\Sigma^{-1}A, M) \neq 0 \ ,$$
$$H_*(\Sigma^{-1}C) \neq \Sigma^{-1}H_*(C) \ .$$

True for Ore localization $\Sigma^{-1}A$, by flatness.

- Example The universal localization $\Sigma^{-1}A$ of $A = \mathbb{Z}\langle x_1, x_2\rangle$ inverting $\Sigma = \{x_1\}$ is not flat. The 1-dimensional f.g. free A-module chain complex

$$d_C = (x_1 \ x_2) \ : \ C_1 = A \oplus A \to C_0 = A$$

is a resolution of $H_0(C) = \mathbb{Z}$ and

$$\begin{aligned} H_1(\Sigma^{-1}C) &= \mathrm{Tor}_1^A(\Sigma^{-1}A, H_0(C)) = \Sigma^{-1}A \\ &\neq \Sigma^{-1}H_1(C) = 0 \ . \end{aligned}$$

The lifting problem for chain complexes

- A <u>lift</u> of a f.g. free $\Sigma^{-1}A$-module chain complex D is a f.g. projective A-module chain complex C with a chain equivalence $\Sigma^{-1}C \simeq D$.

- For an Ore localization $\Sigma^{-1}A$ one can lift every n-dimensional f.g. free $\Sigma^{-1}A$-module chain complex D, for any $n \geqslant 0$.

- For a universal localization $\Sigma^{-1}A$ one can only lift for $n \leqslant 2$ in general.

- For $n \geqslant 3$ there are lifting obstructions in $\text{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geqslant 2$.

  ($\text{Tor}_1^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0$ always).

Chain complex lifting = algebraic transversality

- Typical example: the boundary map in the Schofield exact sequence

  $$\partial : K_1(\Sigma^{-1}A) \to K_0(T(A,\Sigma)); \tau(D) \mapsto [C]$$

  sends the Whitehead torsion $\tau(D)$ of a contractible based f.g. free $\Sigma^{-1}A$-module chain complex D to class $[C]$ of any f.g. projective A-module chain complex C such that $\Sigma^{-1}C \simeq D$.

- "Algebraic and combinatorial codimension 1 transversality", e-print AT.0308111, Proc. Cassonfest, Geometry and Topology Monographs (2004).

Stable flatness

- A universal localization $\Sigma^{-1}A$ is <u>stably flat</u> if

$$\text{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) \;=\; 0 \quad (i \geqslant 2)\ .$$

- For stably flat $\Sigma^{-1}A$ have stable exactness:

$$H_*(\Sigma^{-1}C) \;=\; \varinjlim_{B}\ \Sigma^{-1}H_*(B)$$

with maps $C \to B$ such that $\Sigma^{-1}C \simeq \Sigma^{-1}B$.

- Flat $\Longrightarrow$ stably flat. If $\Sigma^{-1}A$ is flat (i.e. an Ore localization) then

$$\text{Tor}_i^A(\Sigma^{-1}A, M) \;=\; 0 \quad (i \geqslant 1)$$

for every A-module M. The special case $M = \Sigma^{-1}A$ gives that $\Sigma^{-1}A$ is stably flat.

A localization which is not stably flat

- Given a ring extension $R \subset S$ and an S-module M let $K(M) = \mathrm{ker}(S \otimes_R M \to M)$.

- Theorem (Neeman, R. and Schofield)
  (i) The universal localization of the ring

$$A = \begin{pmatrix} R & 0 & 0 \\ S & R & 0 \\ S & S & R \end{pmatrix} = P_1 \oplus P_2 \oplus P_3 \text{ (columns)}$$

  inverting $\Sigma = \{P_3 \subset P_2, P_2 \subset P_1\}$ is

$$\Sigma^{-1}A = M_3(S) \ .$$

  (ii) If S is a flat R-module then

$$\mathrm{Tor}^A_{n-1}(\Sigma^{-1}A, \Sigma^{-1}A) = M_n(K^n(S)) \ (n \geqslant 3).$$

  (iii) If R is a field and $\dim_R(S) = d$ then

$$K^n(S) = K(K(\ldots K(S)\ldots)) = R^{(d-1)^n d} \ .$$

  If $d \geqslant 2$, e.g. $S = R[x]/(x^d)$, then $\Sigma^{-1}A$ is not stably flat. (e-print RA.0205034, Math. Proc. Camb. Phil. Soc. 2004).

Theorem of Neeman + R.

If $A \to \Sigma^{-1}A$ is injective and stably flat then :

- 'fibration sequence of exact categories'

  $$T(A,\Sigma) \to P(A) \to P(\Sigma^{-1}A)$$

  with $P(A)$ the category of f.g. projective A-modules, and every finite f.g. free $\Sigma^{-1}A$-module chain complex can be lifted,

- there are long exact localization sequences

  $$\cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A,\Sigma)) \to \ldots$$

  $$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(T(A,\Sigma)) \to \ldots$$

  e-print RA.0109118,
  Geometry and Topology (2004)

- Quadratic L-theory L_* sequence obtained by Vogel (1982) without stable flatness; symmetric L-theory L^* needs stable flatness.

Noncommutative localization in topology

- Applications to spaces X with infinite fundamental group $\pi_1(X)$, e.g. amalgamated free products and HNN extensions.

- The surgery classification of high-dimensional manifolds and Poincaré complexes, finite domination, fibre bundles over S^1, open books, circle-valued Morse theory, Morse theory of closed 1-forms, rational Novikov homology, codimension 1 and 2 splitting, homology surgery, knots and links.

- Survey: e-print AT.0303046 (to appear in the proceedings of the Edinburgh conference).

The splitting problem in topology

- A homotopy equivalence $h : V \to W$ <u>splits</u> at a subspace $X \subset W$ if the restriction $h| : h^{-1}(X) \to X$ is also a homotopy equivalence. In general homotopy equivalences do not split, not even up to homotopy.

- For a homotopy equivalence of n-dimensional manifolds $h : V \to W$ and a codimension 1 submanifold $X \subset W$ there are algebraic K- and L-theory obstructions to splitting h at X up to homotopy. For $n \geqslant 6$ splitting up to homotopy is possible if and only if these obstructions are zero.

- For connected X, W and injective $\pi_1(X) \to \pi_1(W)$ the splitting obstructions can be recovered from the algebraic K- and L-theory exact sequences of appropriate universal localizations expressing $\mathbb{Z}[\pi_1(W)]$ in terms of $\mathbb{Z}[\pi_1(X)]$ and $\mathbb{Z}[\pi_1(W - X)]$.

Generalized free products

Seifert-van Kampen Theorem For any space

$$W = X \times [0,1] \cup_{X \times \{0,1\}} Y$$

such that W and X are connected the complement Y has either 1 or 2 components, and the fundamental group $\pi_1(W)$ is a generalized free product :

1. If Y is connected then $\pi_1(W)$ is an HNN extension

$$\begin{aligned} \pi_1(W) &= \pi_1(Y) *_{i_1,i_2} \{z\} \\ &= \pi_1(Y) * \{z\}/\{i_1(x)z = zi_2(x) \,|\, x \in \pi_1(X)\} \end{aligned}$$

with $i_1, i_2 : \pi_1(X) \to \pi_1(Y)$ induced by the two inclusions $i_1, i_2 : X \to Y$.

2. Y is disconnected, $Y = Y_1 \cup_X Y_2$, then $\pi_1(W)$ is an amalgamated free product

$$\pi_1(W) = \pi_1(Y_1) *_{\pi_1(X)} \pi_1(Y_2)$$

with $i_1 : \pi_1(X) \to \pi_1(Y_1)$, $i_2 : \pi_1(X) \to \pi_1(Y_2)$ induced by the inclusions $i_1 : X \to Y_1$, $i_2 : X \to Y_2$.

Mayer-Vietoris in homology and K-theory

- Let $W = X \times [0,1] \cup Y$. Homology groups fit into the Mayer-Vietoris exact sequence
$$\cdots \to H_n(X) \xrightarrow{i_1 - i_2} H_n(Y)$$
$$\to H_n(W) \xrightarrow{\partial} H_{n-1}(X) \to \ldots .$$

- The algebraic K-groups of $\mathbb{Z}[\pi_1(W)]$ for $W = X \times [0,1] \cup Y$ with $\pi_1(X) \to \pi_1(W)$ injective fit into almost-Mayer-Vietoris exact sequence (Waldhausen, 1972)
$$\cdots \to K_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{i_1 - i_2} K_n(\mathbb{Z}[\pi_1(Y)]) \to$$
$$K_n(\mathbb{Z}[\pi_1(W)]) \xrightarrow{\partial} \widetilde{\mathrm{Nil}}_{n-1} \oplus K_{n-1}(\mathbb{Z}[\pi_1(X)]) \to \ldots$$
Also L-theory: UNil-groups (Cappell, 1974).

- The almost-Mayer-Vietoris sequences are the localization exact sequences for the "Mayer-Vietoris localizations" $\Sigma^{-1}A$ of triangular matrix rings A.

The Seifert-van Kampen localization (I)

- Let $W = X \times [0,1] \cup Y$. The expression of $\pi_1(W)$ as generalized free product motivates an expression of the $k \times k$ matrix ring of $\mathbb{Z}[\pi_1(W)]$ as a universal localization

$$M_k(\mathbb{Z}[\pi_1(W)]) = \Sigma^{-1}A \quad (k = 2 \text{ or } 3)$$

  of a triangular matrix ring A.

- If Y is connected take $k = 2$,

$$A = \begin{pmatrix} \mathbb{Z}[\pi_1(X)] & 0 \\ \mathbb{Z}[\pi_1(Y)]_1 \oplus \mathbb{Z}[\pi_1(Y)]_2 & \mathbb{Z}[\pi_1(Y)] \end{pmatrix}$$

  (Σ defined in "*HNN* extensions" below).

- If $Y = Y_1 \cup Y_2$ is disconnected take $k = 3$,

$$A = \begin{pmatrix} \mathbb{Z}[\pi_1(X)] & 0 & 0 \\ \mathbb{Z}[\pi_1(Y_1)] & \mathbb{Z}[\pi_1(Y_1)] & 0 \\ \mathbb{Z}[\pi_1(Y_2)] & 0 & \mathbb{Z}[\pi_1(Y_2)] \end{pmatrix}$$

  (Σ defined in "Amalgamated free products").

The Seifert-van Kampen localization (II)

- A map $h : V^n \to W = X \times [0,1] \cup Y$ on an n-manifold V is <u>transverse</u> at $X \subset W$ if

  $$T^{n-1} = h^{-1}(X)\ ,\ U^n \ =\ h^{-1}(Y) \subset V^n$$

  are submanifolds, so $V = T \times [0,1] \cup U$.

- The localization functor

  $$\{A\text{-modules}\} \to \{\Sigma^{-1}A\text{-modules}\}\ ;\ M \mapsto \Sigma^{-1}M$$

  is an algebraic analogue of the forgetful functor

  $$\{\text{transverse maps } V \to W\} \to \{\text{maps } V \to W\}\ .$$

- For any map $V \to W$ $C(\widetilde{V})$ is a $\Sigma^{-1}A$-module chain complex, up to Morita equivalence. For a transverse map $h : V = T \times [0,1] \cup U \to W$ the Mayer-Vietoris presentation of $C(\widetilde{V})$ is an A-module chain complex Γ with assembly $\Sigma^{-1}\Gamma = C(\widetilde{V})$.

Morita theory

- For any ring R and $k \geqslant 1$ let $M_k(R)$ be the ring of $k \times k$ matrices in R.

- Proposition The functors

$$\{R\text{-modules}\} \to \{M_k(R)\text{-modules}\} ;$$

$$M \mapsto \begin{pmatrix} R \\ R \\ \vdots \\ R \end{pmatrix} \otimes_R M ,$$

$$\{M_k(R)\text{-modules}\} \to \{R\text{-modules}\} ;$$

$$N \mapsto (R \ R \ \ldots \ R) \otimes_{M_k(R)} N$$

are inverse equivalences of categories.

- Proposition $K_*(M_k(R)) = K_*(R)$.

Algebraic K-theory of triangular rings

Given rings A_1, A_2 and an (A_2, A_1)-bimodule B define the triangular matrix ring

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

with f.g. projectives $P_1 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix}$.

Proposition (i) The category of A-modules is equivalent to the category of triples

$$M = (M_1, M_2, \mu : B \otimes_{A_1} M_1 \to M_2)$$

with M_i A_i-module, μ A_2-module morphism.
(ii) $K_*(A) = K_*(A_1) \oplus K_*(A_2)$.
(iii) If $A \to S$ is a ring morphism such that there is an S-module isomorphism $S \otimes_A P_1 \cong S \otimes_A P_2$ then $S = M_2(R)$ with $R = \mathrm{End}_S(S \otimes_A P_1)$, and

$$\{A\text{-modules}\} \to \{S\text{-modules}\} \approx \{R\text{-modules}\};$$
$$M \mapsto (R\ R) \otimes_A M$$
$$= \mathrm{coker}(R \otimes_{A_2} B \otimes_{A_1} M_1 \to R \otimes_{A_1} M_1 \oplus R \otimes_{A_2} M_2)$$

is an assembly map, i.e. local-to-global.

The stable flatness theorem

- <u>Theorem</u> Let

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix} \to \Sigma^{-1}A = M_2(R)$$

with Σ a set of A-module morphisms $s : P_2 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \to P_1 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$ with $R = \mathrm{End}(\Sigma^{-1}P_i)$ $(i = 1, 2)$. If B and R are flat A_1-modules and R is a flat A_2-module then $\Sigma^{-1}A$ is stably flat.

- <u>Proof</u> The A-module $M = \begin{pmatrix} R \\ R \end{pmatrix}$ has a 1-dimensional flat A-module resolution

$$0 \to \begin{pmatrix} 0 \\ B \end{pmatrix} \otimes_{A_1} R$$
$$\to \begin{pmatrix} A_1 \\ B \end{pmatrix} \otimes_{A_1} R \oplus \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \otimes_{A_2} R \to M \to 0$$

and hence so does $\Sigma^{-1}A = M \oplus M$.

- <u>Remark</u> $\mathrm{Tor}_1^A((0\ A_2), E) = \ker(B \otimes_{A_1} R \to R)$, so in general $\Sigma^{-1}A$ is not flat.

HNN **extensions**

The HNN extension of ring morphisms $i_1, i_2 : R \to S$ is the ring

$$S *_{i_1,i_2} \{z\} \;=\; S * \mathbb{Z}/\{i_1(x)z = zi_2(x) \,|\, x \in R\}\ .$$

Let $S_j = S$ with (S,R)-bimodule structure

$$S \times S_j \times R \to S_j\ ;\ (s,t,u) \mapsto sti_j(u)\ .$$

The S-vK localization of $A = \begin{pmatrix} R & 0 \\ S_1 \oplus S_2 & S \end{pmatrix}$ inverts the inclusions

$$\Sigma = \{s_1, s_2 : \begin{pmatrix} 0 \\ S \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \oplus S_2 \end{pmatrix}\}$$

with $\Sigma^{-1}A = M_2(S *_{i_1,i_2} \{z\})$.

Corollary 1. If $i_1, i_2 : R \to S$ are split injections and S_1, S_2 are flat R-modules then $A \to \Sigma^{-1}A$ is injective and stably flat. The algebraic K-theory localization exact sequence has

$$\begin{aligned} K_n(A) \;&=\; K_n(R) \oplus K_n(S)\ , \\ K_n(\Sigma^{-1}A) \;&=\; K_n(S *_{i_1,i_2} \{z\})\ , \\ K_n(T(A,\Sigma)) \;&=\; K_n(R) \oplus K_n(R) \oplus \widetilde{\mathrm{Nil}}_n\ . \end{aligned}$$

Amalgamated free products

The amalgamated free product $S_1 *_R S_2$ is defined for ring morphisms $R \to S_1$, $R \to S_2$. The S-vK localization of $A = \begin{pmatrix} R & 0 & 0 \\ S_1 & S_1 & 0 \\ S_2 & 0 & S_2 \end{pmatrix}$ inverts the inclusions

$$\Sigma = \{s_1 : \begin{pmatrix} 0 \\ S_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \\ S_2 \end{pmatrix}, \; s_2 : \begin{pmatrix} 0 \\ 0 \\ S_2 \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \\ S_2 \end{pmatrix}\}$$

with

$$\Sigma^{-1}A = M_3(S_1 *_R S_2) .$$

<u>Corollary 2.</u> If $R \to S_1$, $R \to S_2$ are split injections with S_1, S_2 flat R-modules then $A \to \Sigma^{-1}A$ is injective and stably flat. The algebraic K-theory localization exact sequence has

$$\begin{aligned} K_n(A) &= K_n(R) \oplus K_n(S_1) \oplus K_n(S_2) , \\ K_n(\Sigma^{-1}A) &= K_n(S_1 *_R S_2) , \\ K_n(T(A, \Sigma)) &= K_n(R) \oplus K_n(R) \oplus \widetilde{\mathrm{Nil}}_n . \end{aligned}$$

The algebraic L-theory of a triangular ring

- If A_1, A_2, B have involutions then $A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ may not have an involution.

- Involutions on A_1, A_2 and a symmetric isomorphism $\beta : B \to \text{Hom}_{A_2}(B, A_2)$ give a "chain duality" involution on the derived category of A-module chain complexes.

- The dual of an A-module $M = (M_1, M_2, \mu)$ is the A-module chain complex

  $d = (0, \beta^{-1}\mu^*) :$

  $C_1 = (0, M_2^*, 0) \to C_0 = (M_1^*, B \otimes_{A_1} M_1^*, 1)$

- The quadratic L-groups of A are just the relative L-groups in the sequence

$$\cdots \to L_n(A_1) \to^{\otimes(B,\beta)} L_n(A_2) \to L_n(A) \to L_{n-1}(A_1) \to \ldots .$$

The algebraic L-theory of a noncommutative localization

- Theorem Let $\Sigma^{-1}A$ be the localization of a triangular ring $A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ with chain duality inverting a set Σ of A-module morphisms $s : P_1 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \to P_2 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$, so that

$$\Sigma^{-1}A = M_2(D)$$

with $D = \mathrm{End}(\Sigma^{-1}P_1)$. If B and D are flat A_1-modules and D is a flat A_2-module then $\Sigma^{-1}A$ is stably flat,

$$L_*(\Sigma^{-1}A) \ = \ L_*(D) \text{ (Morita)}$$

and there is an exact sequence

$$\cdots \to L_n(A) \to L_n(D) \to L_n(T(A,\Sigma)) \to L_{n-1}(A) \to \ldots \ .$$

The UNil groups are the torsion groups of a noncommutative localization

- Theorem Let $D = S_1 *_R S_2$ be the amalgamated free product of split injections $R \to S_1$, $R \to S_2$ of rings with involution, and let $A \to \Sigma^{-1}A = M_3(D)$ be the S-vK localization. If S_1, S_2 are flat R-modules then

$$L_n(\Sigma^{-1}A) = L_n(D) = L_n(A) \oplus L_n(T(A,\Sigma)) \ ,$$
$$L_n(T(A,\Sigma)) = \mathrm{UNil}_n(R; S_1, S_2) \ .$$

- Similarly for the UNil-groups of an HNN extension $D = S *_{i_1,i_2} \{z\}$ of split injective morphisms $i_1, i_2 : R \to S$ of rings with involution with S_1 and S_2 flat R-modules, and the S-vK localization $\Sigma^{-1}A = M_2(D)$.

A polynomial extension is a noncommutative localization

- A particularly simple example!

- For any ring R define triangular matrix ring

$$A = \begin{pmatrix} R & 0 \\ R \oplus R & R \end{pmatrix} .$$

An A-module is a quadruple

$$M = (K , L , \mu_1, \mu_2 : K \to L)$$

with K, L R-modules and μ_1, μ_2 R-module morphisms. The localization of A inverting

$$\Sigma = \{\sigma_1, \sigma_2 : \begin{pmatrix} 0 \\ R \end{pmatrix} \to \begin{pmatrix} R \\ R \oplus R \end{pmatrix}\}$$

is a ring morphism

$A \to \Sigma^{-1}A = M_2(S)$, $S = R[z, z^{-1}]$ such that $\{A\text{-modules}\} \to \{M_2(S)\text{-modules}\} \approx \{S\text{-modules}\}$ sends an A-module M to the assembly S-module

$$(S\ S) \otimes_A M$$

$$= \text{coker}(\mu_1 - z\mu_2 : K[z, z^{-1}] \to L[z, z^{-1}]) .$$

Manifolds over S^1

- Given a map $f : V^n \to S^1$ on an n-manifold V which is transverse at $\{\text{pt.}\} \subset S^1$ cut V along the codimension 1 submanifold $T^{n-1} = f^{-1}(\{\text{pt.}\}) \subset V$ to obtain

$$V \;=\; T \times [0,1] \cup_{T \times \{0,1\}} U \;.$$

The cobordism $(U; T_1, T_2)$ is a fundamental domain for the infinite cyclic cover $\overline{V} = f^*\mathbb{R}$ of V, with T_1, T_2 copies of T.

- $A \;=\; \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $\;\Sigma^{-1}A \;=\; M_2(\mathbb{Z}[z, z^{-1}])$.

The A-module chain complex

$$\Gamma \;=\; (C(T), C(U), \mu_1, \mu_2 : C(T) \to C(U))$$

induces the assembly $\mathbb{Z}[z, z^{-1}]$-module chain complex

$$\begin{aligned} &(\mathbb{Z}[z, z^{-1}] \;\; \mathbb{Z}[z, z^{-1}]) \otimes_A \Gamma \\ &=\; \text{coker}(\mu_1 - z\mu_2 : C(T)[z, z^{-1}] \to C(U)[z, z^{-1}]) \\ &=\; C(\overline{V}) \;. \end{aligned}$$