Cup-product and Iwasawa theory for K_2

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Notations

p is an odd prime number F is a number field S is the set of places above (p) $G_S(F) = Gal(F_S/F)$ is the galois group of the maximal extension of F unramified outside S. $H^q_S(F, \bullet) = H^q(G_S(F), \bullet)$ the usual discrete galois cohomological fonctors. $H^q_S(F, \mathbb{Z}_p(i)) = \varprojlim H^q_S(F, \mu_{p^k}^{\otimes i})$

For a
$$\mathbb{Z}_p$$
-extension F_{∞}/F :
 $\Gamma_n = Gal(F_{\infty}/F), \ \Gamma = \Gamma_0$
 $G_n = Gal(F_n/F)$
 $\Lambda = \varprojlim \mathbb{Z}_p[G_n]$
 $X_{\infty} = \varprojlim K_2^{et} \mathcal{O}_{F_n}$
 $(X_{\infty})^0$ is the max finite Λ -submodule of X_{∞}
 $cap(F_{\infty}/F_n) = ker(K_2^{et} \mathcal{O}_{F_n} \to (K_2^{et} \mathcal{O}_{F_{\infty}})^{\Gamma_n})$

Motivation

For $F = \mathbb{Q}(\mu_p)$ W. McCallum and R.Sharifi recently proposed the following conjecture, which they shown to be true for p < 10000.

Conjecture. (MCS) $K_2\mathcal{O}_F \otimes \mathbb{Z}_p$ is generated in $K_2F \otimes \mathbb{Z}_p$ by symbols $\{b, a\}$ with $b, a \in \mathcal{O}_{S,F}^{\times}$.

Amotivation for this talk is to link this conjecture to the following weak form of the Greenberg generalized conjecture in Iwasawa theory:

Conjecture. (WGG) Let F be any number field containing μ_p , \tilde{F} the compositum of all \mathbb{Z}_p -extensions of F, then $K_2^{et}\mathcal{O}_F$ capitulates in \tilde{F} .

From now on, F is any number field and we want to study connections between the following hypotheses for i = 0, 1:

H1(*i*): The following cup-product is surjective: $\cup : H^1_S(F, \mathbb{Z}_p(2-i)) \otimes H^1_S(F, \mathbb{Z}_p(i)) \to H^2_S(F, \mathbb{Z}_p(2))$

SH1(*i*): There exists $a \in H^1_S(F, \mathbb{Z}_p(i))$ such that the following map

$$\cup a: H^1_S(F, \mathbb{Z}_p(2-i)) \to H^2_S(F, \mathbb{Z}_p(2))$$

H2: $K_2^{et} \mathcal{O}_F$ capitulates in \tilde{F} .

SH2: There exists a \mathbb{Z}_p extension F_{∞}/F such that $K_2^{et}\mathcal{O}_F$ capitulates in F_{∞} .

The aim of this talk is to present the following result

Theorem. Let F be any number field, then we have the following inclusion

 $K_{\mathsf{3}}^{et}\mathcal{O}_F \subset cap(\tilde{F}/F)$

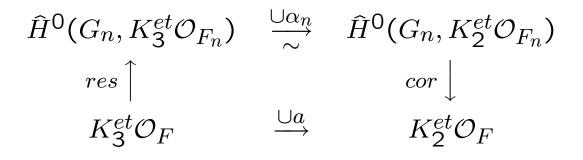
In particular, under Vandiver conjecture for $F = \mathbb{Q}(\mu_p)$, one has

 $(MCS) \Rightarrow (WGG)$

i = 0: An "up and down" description of the map $\cup a : K_3^{et} \mathcal{O}_F \to K_2^{et} \mathcal{O}_F$

Here we fix $a \in H^1_S(F, \mathbb{Z}_p)$ not divisible by p. This defines a \mathbb{Z}_p -extension F_{∞}/F . Let F_n be its n^{th} layer, $F_0 = F$. Let $\alpha_n \in H^1_S(F_n, \mathbb{Z}_p)$ so that $cor \alpha_n = a, \alpha_0 = a$.

Theorem. If p^n kills $K_2^{et}\mathcal{O}_F$ then $\cup \alpha_n$ induces an isomorphism in the following well defined commutative diagram:



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Again a is fixed. Passing to the infinite level, one has

Proposition.

$$(K_3\mathcal{O}_F\otimes\mathbb{Z}_p)\cup a=cor(X_\infty)^{\mathsf{\Gamma}}$$

On the other hand, the capitulation in F_{∞}/F_n is given by:

Proposition. We have an exact sequence

 $(X^{0}_{\infty})_{\Gamma} \hookrightarrow K^{et}_{2}\mathcal{O}_{F} \overset{res}{\twoheadrightarrow} (K^{et}_{2}\mathcal{O}_{F_{\infty}})^{\Gamma}$

Comparing both , we get:

Theorem. We have an inclusion

 $(K_3\mathcal{O}_F\otimes\mathbb{Z}_p)\cup a\subset cap(F_\infty/F)$

and the equality stands if and only if $(X_{\infty})^0 = (X_{\infty})^{\Gamma}$.

By allowing a to run in $H^1_S(F, \mathbb{Z}_p)$, one obtains the first part of the theorem stated in the introduction.

Conclusion

We have shown the following implications:

$SH1(0) \Rightarrow SH2$ $H1(0) \Rightarrow H2$

The reverse is only asymptotical:

 $SH2(F_{\infty}/F) \Rightarrow SH1(F_{\infty}/F_n), n >> 0$

To link with H1(1) we have to assume that F contains sufficiently many roots of unity:

Lemma. Let $\mu_p \subset F$, let $F_n = F(\mu_{p^n})$ and $F_{\infty} = F(\mu_{p^{\infty}})$. If $n \geq v_p((X_{\infty})^0)$ then for F_n , one has

 $H1(1) \Leftrightarrow H1(0)$

In the case $F = \mathbb{Q}(\mu_p)$ one has, under Vandiver conjecture, $(X_{\infty})^0 = 0$ so that we obtain the second part in the theorem of the introduction.