Cup-product and Iwasawa theory for K_2

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Notations

p is an odd prime number
F is a number field
S is the set of places above (p)
$G_S(F) = Gal(F_S/F)$ is the galois group of the maximal extension of F unramified outside S.
$H^q_S(F, \bullet) = H^q(G_S(F), \bullet)$ the usual discrete galois cohomological fonctors.
$H^q_S(F, \mathbb{Z}_p(i)) = \varprojlim H^q_S(F, \mu_{p^k}^{\otimes i})$

For a $\mathbb{Z}_p$-extension F_∞/F:
$\Gamma_n = Gal(F_\infty/F)$, $\Gamma = \Gamma_0$
$G_n = Gal(F_n/F)$
$\Lambda = \varprojlim \mathbb{Z}_p[G_n]$
$X_\infty = \varprojlim K_2^{et}\mathcal{O}_{F_n}$
$(X_\infty)^0$ is the max finite Λ-submodule of X_∞
$cap(F_\infty/F_n) = ker(K_2^{et}\mathcal{O}_{F_n} \to (K_2^{et}\mathcal{O}_{F_\infty})^{\Gamma_n})$

Motivation

For $F = \mathbb{Q}(\mu_p)$ W. McCallum and R.Sharifi recently proposed the following conjecture, which they shown to be true for $p < 10000$.

Conjecture. *(MCS) $K_2\mathcal{O}_F \otimes \mathbb{Z}_p$ is generated in $K_2F \otimes \mathbb{Z}_p$ by symbols $\{b, a\}$ with $b, a \in \mathcal{O}_{S,F}^{\times}$.*

Amotivation for this talk is to link this conjecture to the following weak form of the Greenberg generalized conjecture in Iwasawa theory:

Conjecture. *(WGG) Let F be any number field containing μ_p, $\tilde{F}$ the compositum of all $\mathbb{Z}_p$-extensions of F, then $K_2^{et}\mathcal{O}_F$ capitulates in $\tilde{F}$.*

From now on, F is any number field and we want to study connections between the following hypotheses for $i = 0, 1$:

H1(i): The following cup-product is surjective:

$$\cup : H^1_S(F, \mathbb{Z}_p(2-i)) \otimes H^1_S(F, \mathbb{Z}_p(i)) \to H^2_S(F, \mathbb{Z}_p(2))$$

SH1(i): There exists $a \in H^1_S(F, \mathbb{Z}_p(i))$ such that the following map

$$\cup a : H^1_S(F, \mathbb{Z}_p(2-i)) \to H^2_S(F, \mathbb{Z}_p(2))$$

H2: $K_2^{et}\mathcal{O}_F$ capitulates in $\tilde{F}$.

SH2: There exists a $\mathbb{Z}_p$ extension F_∞/F such that $K_2^{et}\mathcal{O}_F$ capitulates in F_∞.

The aim of this talk is to present the following result

Theorem. *Let F be any number field, then we have the following inclusion*

$$K_3^{et}\mathcal{O}_F \subset cap(\tilde{F}/F)$$

In particular, under Vandiver conjecture for $F = \mathbb{Q}(\mu_p)$, one has

$$(MCS) \Rightarrow (WGG)$$

$i = 0$: **An "up and down" description of the map** $\cup a : K_3^{et}\mathcal{O}_F \to K_2^{et}\mathcal{O}_F$

Here we fix $a \in H^1_S(F, \mathbb{Z}_p)$ not divisible by p. This defines a $\mathbb{Z}_p$-extension F_∞/F. Let F_n be its n^{th} layer, $F_0 = F$. Let $\alpha_n \in H^1_S(F_n, \mathbb{Z}_p)$ so that $cor\, \alpha_n = a$, $\alpha_0 = a$.

Theorem. *If p^n kills $K_2^{et}\mathcal{O}_F$ then $\cup\alpha_n$ induces an isomorphism in the following well defined commutative diagram:*

$$\begin{array}{ccc} \hat{H}^0(G_n, K_3^{et}\mathcal{O}_{F_n}) & \xrightarrow[\sim]{\cup\alpha_n} & \hat{H}^0(G_n, K_2^{et}\mathcal{O}_{F_n}) \\ res \uparrow & & cor \downarrow \\ K_3^{et}\mathcal{O}_F & \xrightarrow{\cup a} & K_2^{et}\mathcal{O}_F \end{array}$$

Iwasawa theory for K_2

Again a is fixed. Passing to the infinite level, one has

Proposition.

$$(K_3\mathcal{O}_F \otimes \mathbb{Z}_p) \cup a = cor(X_\infty)^\Gamma$$

On the other hand, the capitulation in F_∞/F_n is given by:

Proposition. *We have an exact sequence*

$$(X_\infty^0)_\Gamma \quad \hookrightarrow \quad K_2^{et}\mathcal{O}_F \quad \overset{res}{\twoheadrightarrow} \quad (K_2^{et}\mathcal{O}_{F_\infty})^\Gamma$$

Comparing both , we get:

Theorem. *We have an inclusion*

$$(K_3\mathcal{O}_F \otimes \mathbb{Z}_p) \cup a \subset cap(F_\infty/F)$$

and the equality stands if and only if $(X_\infty)^0 = (X_\infty)^\Gamma$.

By allowing a to run in $H^1_S(F, \mathbb{Z}_p)$, one obtains the first part of the theorem stated in the introduction.

Conclusion

We have shown the following implications:

$$SH1(0) \Rightarrow SH2$$

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The reverse is only asymptotical:

$$SH2(F_\infty/F) \Rightarrow SH1(F_\infty/F_n), n >> 0$$

To link with $H1(1)$ we have to assume that F contains sufficiently many roots of unity:

Lemma. *Let $\mu_p \subset F$, let $F_n = F(\mu_{p^n})$ and $F_\infty = F(\mu_{p^\infty})$. If $n \geq v_p((X_\infty)^0)$ then for F_n, one has*

$$H1(1) \Leftrightarrow H1(0)$$

In the case $F = \mathbb{Q}(\mu_p)$ one has, under Vandiver conjecture, $(X_\infty)^0 = 0$ so that we obtain the second part in the theorem of the introduction.