References.


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(3) I.Panin, S.Yagunov *Poincaré Duality for Projective Algebraic Varieties.*
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For us cohomology theory is a contravariant functor $E^*\colon Sm^2/k \to \mathcal{A}b$ given on the category $Sm^2/k$ of pairs $(X,U)$, where $X$ is a smooth algebraic variety over a field $k$ and $U$ is an open subscheme in $X$.

*Eilenberg – Steenrod Type Axioms*

(**1**) **Localization.** Let $(U,\emptyset) \xrightarrow{f} (X,\emptyset) \xrightarrow{j} (X,U)$ be morphisms in $Sm^2/k$ such that $j$ is induced by $X \xrightarrow{\mathrm{id}} X$. Then we have the following long exact sequence:

$$\dots \xrightarrow{j^*} E^*(X) \xrightarrow{f^*} E^*(U) \xrightarrow{\partial^*} E^{*+1}(X,U) \xrightarrow{j^*} \cdots$$

(**2**) **Excision.** Let $X \overset{i}{\supseteq} X_0 \supseteq Z$, where $X_0$ is open in $X$ and $Z$ is closed in $X$. Then the induced map $i^*\colon E^*(X, X-Z) \xrightarrow{\simeq} E^*(X_0, X_0 - Z)$ is an isomorphism.

(**3**) **Homotopy Invariance.** The functor $E^*$ is homotopy invariant, i.e. for every $X \in Sm/k$ the map $p_X^*\colon E^*(X) \to E^*(X \times \mathbb{A}^1)$ induced by the projection $X \times \mathbb{A}^1 \xrightarrow{p_X} X$ is an isomorphism.

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**The theory $E^*$ is called <u>orientable</u> if it satisfies the following additional axiom:**

(**4**) **Projective Bundle Theorem.** Let $\mathcal{E}$ be a vector bundle of rank $r$ over $X$. Denote by $\mathbb{P}(\mathcal{E}) \xrightarrow{p} X$ the projective bundle over $X$ associated to $\mathcal{E}$. (A fiber of this bundle over a point $\{x\}$ in $X$ is the projective space of lines in the fiber $\mathcal{E}_x$.)

Then $E^*(\mathbb{P}(\mathcal{E}))$ is a free $E^*(X)$-module with a base $1, \xi, \xi^2, \dots, \xi^{r-1}$ given by powers of the first Chern class.

Moreover, if the bundle $\mathcal{E}$ is trivial, these modules are isomorphic as rings. (In this case one has $\xi^r = 0$.)

*Transfer structure.*

For a class $\mathcal{C} \subset Mor(Sm^2/k)$ we define transfer maps $f_!\colon E^*(X) \to E^*(Y)$ provided that $(X \xrightarrow{f} Y) \in \mathcal{C}$. For orientable theories we usually set $\mathcal{C}$ to be the class induced by all projective morphisms.

(1) **Functoriality.**

We have: $(f \circ g)_! = f_! \circ g_!$ and $\mathrm{id}_! = \mathrm{id}$.

(2) **Base-change property for transversal squares.**

For any Cartesian transversal square

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{f}} & X' \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

the diagram

$$\begin{array}{ccc} E^*(Y') & \xrightarrow{\bar{f}_!} & E^*(X') \\ \uparrow{\scriptstyle \bar{g}^*} & & \uparrow{\scriptstyle g^*} \\ E^*(Y) & \xrightarrow[f_!]{} & E^*(X) \end{array}$$

commutes.

(3) **Finite additivity.**

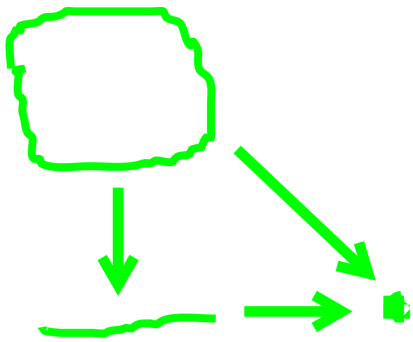
Let $X = X_0 \sqcup X_1$, $j_m\colon X_m \hookrightarrow X$ $(m = 0, 1)$ be embedding maps, and $f\colon X \to Y$ be a projective morphism. Setting $f_m = f j_m$, we have:

$$f_{0,!} j_0^* + f_{1,!} j_1^* = f_!.$$

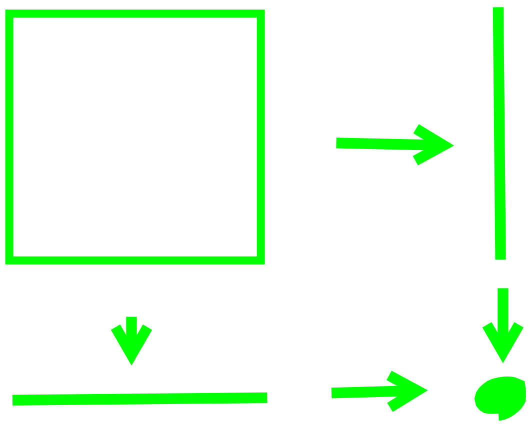
There exists a nice analogies between transfers and fiberwise integration.

**(1) Functoriality.**

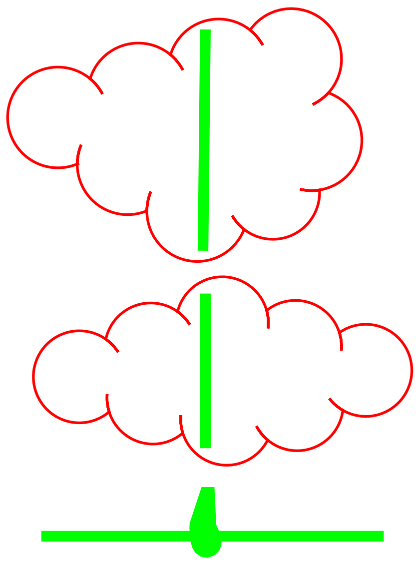
$(f \circ g)_! = f_! \circ g_!$ $\qquad\qquad \iint_{f\circ g} = \int_f(\int_g)$ (Fubini's theorem).



**(2) Base-change property for transversal squares.**



**(3) Finite additivity.=Additivity of integrals.**



*Toward transfer construction.*

Let $B = B(Y,X)$ denote the blow-up of $Y \times \mathbb{A}^1$ with center at $X \times \{0\}$. Considering the fibers of the map $B(Y,X) \to Y \times \mathbb{A}^1 \to \mathbb{A}^1$ over the points $\{0\}$ and $\{1\}$ one easily obtains two embeddings: $i_0 \colon \mathbb{P}(\mathcal{N} \oplus \mathbf{1}) \hookrightarrow B(Y,X)$ and $i_1 \colon Y \hookrightarrow B(Y,X)$. The subvariety $X \times \mathbb{A}^1$ contains the center $X \times \{0\}$ of the blow-up as a divisor. Therefore, it lifts canonically to a subvariety in $B(Y,X)$. Since $X \times \mathbb{A}^1$ crosses $\mathbb{P}(\mathcal{N} \oplus \mathbf{1})$ along $\mathbb{P}(\mathbf{1})$ and crosses $i_1(Y)$ along $i_1(X)$, one has following embeddings of the pairs:

$$(\mathcal{N}, \mathcal{N} - X) \xhookrightarrow{i_0} (B, B - X \times \mathbb{A}^1) \xhookleftarrow{i_1} (Y, Y - X) \tag{0.1}$$

Then:

$$E^{*,*}_X(\mathcal{N}) \xleftarrow[\simeq]{i_0^*} E^{*,*}_{X \times \mathbb{A}^1}(B) \xrightarrow[\simeq]{i_1^*} E^{*,*}_X(Y) \tag{0.2}$$

*Applications.*

*I. Rigidity theorems.*

**Theorem 0.1** (Orientable case). *Let $k \subset K$ be an extension of algebraically closed fields. Let also $E^{*,*}$ be an orientable functor vanishing after multiplication by $n$ mutually prime to* $\operatorname{Char} k$*. Then, for any $Y \in Sm/k$, we have:*

$$E^{*,*}(Y) \xrightarrow{\simeq} E^{*,*}(Y_K).$$

The proof is based on the following fact:

**Theorem 0.2 (The Rigidity Theorem).** *Let $\mathcal{F}\colon (Sm/k)^{\circ} \to \mathcal{A}b$ be a contravariant homotopy invariant functor with weak transfers for the class of finite projective morphisms. Assume that the field $k$ is algebraically closed and $n\mathcal{F} = 0$ for some integer $n$ relatively prime to* $\operatorname{Char} k$*. Then for every smooth affine variety $T$ and for any two $k$-rational points $t_1, t_2 \in T(k)$ the induced maps $t_1^*, t_2^*\colon \mathcal{F}(T) \to \mathcal{F}(k)$ coincide.*

**Theorem 0.3** (Henselian case)**.** *Let $E$ be such that $E(\mathbb{P}^2, l) \to E(\mathbb{P}^1, l)$ is an epimorphism (e.g. $E = MGL$, $H_{mot}$ or $K$). Then for any smooth scheme $X$ over $k$, any $P \in X(k)$ and $l$ is coprime to* $\mathrm{Char}(k)$*, we have a natural isomorphism:*

$$E(X \times_k \mathcal{O}^h_{X,P}, l) \xrightarrow{\cong} E(X, l).$$

**Theorem 0.4.** *Let $R$ be a henselian local ring with a field of fractions $Frac(R) = F$. Assume that $E = E^{**}$ is a bigraded functor on the category $Sm/k$ (of smooth schemes over an infinite field $k$) that is representable in the stable $\mathbb{A}^1$-homotopy category and that $lE = 0$ for $l \in \mathbb{Z}$ invertible in $R$. Let $f\colon M \to \operatorname{Spec} R$ be a smooth affine morphism of (pure) relative dimension $d$, $s_0, s_1\colon \operatorname{Spec} R \to M$ two sections of $f$ such that $s_0(p) = s_1(p)$, where $p$ is the closed point of* $\operatorname{Spec} R$*. Assume moreover that $E$ is normalized with respect to the field $F$. Then two composed maps $E(M) \xrightarrow{s_i^*} E(\operatorname{Spec} R) \to E(F)$ are equal $(i = 0, 1)$.*

*Products in (Co-)Homology*

$E$ is a ring-spectrum: $E \wedge E \xrightarrow{\mu} E$.

$\tau_{ij}$ is the $ij-$permutation morphism.

$S$ is the sphere spectrum.

$$E^*(X) = [X \to E] \qquad\qquad E_*(X) = [S \to X \wedge E]$$

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$$E^*(X) \otimes E^*(Y) \xrightarrow{\overline{\times}} E^*(X \wedge Y)$$

$$[X \xrightarrow{\alpha} E]\overline{\times}[Y \xrightarrow{\beta} E] = [X \wedge Y \xrightarrow{\alpha\wedge\beta} E \wedge E \xrightarrow{\mu} E]$$

---

$$E_*(X) \otimes E_*(Y) \xrightarrow{\underline{\times}} E_*(X \wedge Y)$$

$$[S \xrightarrow{a} X \wedge E]\underline{\times}[S \xrightarrow{b} Y \wedge E] =$$

$$[S \xrightarrow{\Delta} S \wedge S \xrightarrow{a\wedge b} (X \wedge E) \wedge (Y \wedge E) \xrightarrow{(1\wedge1\wedge\mu)\circ\tau_{23}} X \wedge Y \wedge E]$$

---

$$E^*(X \wedge Y) \otimes E_*(Y) \xrightarrow{/} E^*(X)$$

$$[X \wedge Y \xrightarrow{\alpha} E]/[S \xrightarrow{a} Y \wedge E] = [X \to X \wedge S \xrightarrow{1\wedge a} (X \wedge Y) \wedge E \xrightarrow{\alpha\wedge 1} E \wedge E \xrightarrow{\mu} E]$$

---

$$E^*(X) \otimes E_*(X \wedge Y) \xrightarrow{\backslash} E_*(Y)$$

$$[X \xrightarrow{\alpha} E]\backslash[S \xrightarrow{a} X \wedge Y \wedge E] =$$

$$[S \xrightarrow{a} X \wedge Y \wedge E \xrightarrow{\alpha\wedge1\wedge1} E \wedge Y \wedge E \xrightarrow{(1\wedge1\wedge\mu)\circ\tau_{12}} Y \wedge E]$$

Consider a category $Sm/k$ of smooth algebraic varieties over a field $k$. Let $E^*$ and $E_*$ be functors (cohomology and homology pretheories)

$$E^*\colon (Sm/k)^{\mathrm{op}} \to \mathbb{Z}/2\text{-}\mathcal{A}b$$

and

$$E_*\colon Sm/k \to \mathbb{Z}/2\text{-}\mathcal{A}b$$

taking their values in the category of $\mathbb{Z}/2$-graded abelian groups.

**Definition.** Let functors $E^*$ and $E_*$ (cotravariant and covariant, respectively), be endowed with a product structure consisting of two cross-products

$$\underline{\times}\colon E_p(X) \otimes E_q(Y) \to E_{p+q}(X \times Y)$$

$$\overline{\times}\colon E^p(X) \otimes E^q(Y) \to E^{p+q}(X \times Y)$$

and two slant-products

$$/\colon E^p(X \times Y) \otimes E_q(Y) \to E^{p-q}(X)$$

$$\backslash\colon E^p(X) \otimes E_q(X \times Y) \to E_{q-p}(Y).$$

Define two inner products

$$\smile\colon E^p(X) \otimes E^q(X) \to E^{p+q}(X)$$

$$\frown\colon E^p(X) \otimes E_q(X) \to E_{q-p}(X),$$

as $\alpha \smile \beta = \Delta^*(\alpha \overline{\times} \beta)$ and $\alpha \frown a = \alpha \backslash \Delta_*(a)$. We say that functors $E^*$ and $E_*$ make a **multiplicative pair** $(E^*, E_*)$ if the mentioned products satisfy the following five axioms.

(**A.1**) The cup-product makes the group $E^*(X)$ an associative skew-commutative $\mathbb{Z}/2$-graded unitary ring and this structure is functorial.

(**A.2**) The cap-product makes the group $E_*(X)$ a unital $E^*(X)$-module (we have $1 \frown a = a$ for every $a \in E_*(X)$) and this structure is functorial in the sense that

$$\alpha \frown f_*(a) = f_*(f^*(\alpha) \frown a)$$

(**A.3**) Associativity relations. For $\alpha \in E^*(X \times Y)$, $\beta \in E^*(Y)$, $\gamma \in E^*(X)$, $a \in E_*(Y)$, and $b \in E_*(X)$, we have:

(i) $\alpha/(\beta \frown a) = (\alpha \smile p_Y^*(\beta))/a$

(ii) $\gamma \smile (\alpha/a) = (p_X^*(\gamma) \smile \alpha)/a$

(iii) $(\alpha/a) \frown b = p_*^X((\alpha \frown (a \underline{\times} b))$,

where morphisms $p_X$ and $p_Y$ are corresponding projections.

(**A.4**) Functoriality for slant-product: For morphisms $f\colon X \to X'$, $g\colon Y \to Y'$, and elements
$\alpha \in E^*(X' \times Y')$ and $a \in E_*(Y)$, one has:

$$(f \times g)^*(\alpha)/a = f^*(\alpha/g_*(a))$$

(**A.5**) In the homology group of the final object pt we are given an element $[\mathrm{pt}] \in E_0(\mathrm{pt})$ such that for every $\alpha \in E^*(\mathrm{pt})$, one has:

$$\alpha/[\mathrm{pt}] = \alpha$$

We would also need a transfer structure in homology, which is an analogue of the cohomological one. These two structures are compatible in the following sense:

*For a line bundle* $\mathcal{L}$ *over* $X$ *we set* $e(\mathcal{L}) \stackrel{def}{=} z^* z_!(1)$*, where* $z : X \to \mathcal{L}$ *is the zero-section. Let* $\mathcal{L}$ *be a line bundle over* $X$*. Then, for the zero-section* $z\colon X \to \mathcal{L}$ *the relation* $z^! \circ z_* = e(\mathcal{L}) \frown\colon E_*(X) \to E_*(X)$ *holds.*

**Definition 0.5.** Let $\mathcal{E}$ be an oriented pseudo-representable theory and $X \in Sm/k$ projective variety with structure morphism $\pi\colon X \to \mathrm{pt}$. Then, we call an element $\pi^!(1) \in E_0(X)$ the **fundamental class** of $X$ and denote it by $[X]$.

**Theorem 0.6 (Poincaré Duality).** *Let $\mathcal{E}$ be an oriented pseudo-representable theory. For every projective $X \in Sm/k$, denote by $\mathcal{D}^\bullet\colon E^*(X) \to E_*(X)$ the map $\mathcal{D}^\bullet(\alpha) = \alpha \frown [X]$ and by $\mathcal{D}_\bullet\colon E_*(X) \to E^*(X)$ the map $\mathcal{D}_\bullet(a) = \Delta_!(1)/a$, where $\Delta\colon X \to X \times X$ is the diagonal morphism. Then, the maps $\mathcal{D}^\bullet$ and $\mathcal{D}_\bullet$ are mutually inverse isomorphisms.*

One can extract the following nice consequence of the Poincaré Duality theorem, which enables us to interpret trace maps in a way topologists like to.

**Corollary 0.7.** *For projective $X, Y \in Sm/k$ and a morphism $f\colon X \to Y$, one has:*

$$f_! = \mathcal{D}_\bullet^Y f_* \mathcal{D}_X^\bullet$$

$$f^! = \mathcal{D}_X^\bullet f^* \mathcal{D}_\bullet^Y,$$

*where $\mathcal{D}_X$ and $\mathcal{D}_Y$ are introduced above duality operators for varieties $X$ and $Y$, respectively.*