Motivic decomposition of anisotropic varieties of type F_4 and generalized Rost motives*

S.Nikolenko, N.Semenov, K.Zainoulline

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Abstract

We provide an explicit decomposition of the motives of two anisotropic projective homogeneous varieties of type F_4. In particular, we provide an explicit construction of a generalized Rost motive for a norm variety that corresponds to a symbol $(3, 3)$.

We also establish a motivic isomorphism between two projective homogeneous varieties of type F_4. We provide an explicit construction of this isomorphism.

All our proofs work for Chow motives with integral coefficients.

1 Introduction

The main motivation for our work was the result of N. Karpenko where he gave a shortened construction of a Rost motive for a norm quadric [Ka98]. In the present paper we provide a shortened and explicit construction of a generalized Rost motive for a norm variety that corresponds to a symbol $(3, 3)$. The latter is given by the Rost–Serre invariant g_3 for an Albert algebra and the respective norm variety is a projective homogeneous variety of type F_4. Namely, we prove the following

1.1 Theorem. *Let k be a field of characteristic different from 2 and 3. Let $X = G/P$ be a projective homogeneous variety over k, where G is an*

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anisotropic group of type F_4 *obtained by the first Tits process and* P *its maximal parabolic subgroup corresponding to the first (last) three vertices of the respective Dynkin diagram. Then*

(i) *the (integral) Chow motive of* X *decomposes as*

$$\mathcal{M}(X) \cong R \oplus R_{12} \oplus R(3) \oplus R^t \oplus R_{12}^t \oplus R^t(3),$$

where the motive $R = (X, p)$ *is the (integral) generalized Rost motive, i.e., over the separable closure* k' *of* k *it splits as the direct sum of Lefschetz motives* $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$*, the motive* $R^t = (X, p^t)$ *denotes its transpose and the motive* R_{12} *is isomorphic over* k' *to the direct sum* $R(1) \oplus R(2)$*;*

(ii) *the Chow motive of* X *with* $\mathbb{Z}/3$*-coefficients decomposes as*

$$\mathcal{M}(X, \mathbb{Z}/3) \cong \bigoplus_{i=0}^{3} (R(i) \oplus R^t(i)).$$

1.2. Observe that the part (ii) of the Theorem is the consequence of Voevodsky's results [Vo03, § 5], where the generalized Rost motive with $\mathbb{Z}/p$-coefficients (p is any prime) was constructed.

By the next result, we provide the first known "purely exceptional" example of two non-isomorphic varieties with isomorphic motives. Recall that the similar result for groups of type G_2 obtained in [Bo03] provides a motivic isomorphism between quadric and Fano variety.

1.3 Theorem. *Under the hypotheses of theorem 1.1 let* X_1 *and* X_2 *be two projective homogeneous varieties corresponding to the maximal parabolic subgroups generated by the last (first) three vertices of the Dynkin diagram respectively. Then the motives of* X_1 *and* X_2 *are isomorphic.*

1.4. Our proof is quite elementary. It uses well-known facts concerning linear algebraic groups and projective homogeneous varieties, a computer program that computes the Chow ring $\mathrm{CH}(G/P)$ for a split group G, Rost Nilpotence Theorem and several simple procedures that allow to produce rational cycles on $\mathrm{CH}(G/P \times G/P)$. Moreover, the proof works not only for projective homogeneous varieties of type F_4. Applying the similar arguments to Pfister quadrics and their maximal neighbors one obtains the well-known decompositions into Rost motives [Ro98]. For exceptional groups of type G_2 one immediately obtains the motivic decomposition of the variety G_2/P_2 together with the motivic isomorphism found by J.-P. Bonnet [Bo03].

1.5. The paper is organized as follows. In Section 2 we provide background information on Chow motives and rational cycles. Section 3 is devoted to computational matters of Chow rings. Namely, we introduce Pieri and Giambelli formulae and discuss their relationship with Hasse diagrams. In Section 4 we apply the formulae introduced in Section 3 to projective homogeneous varieties X_1 and X_2 of type F_4. In Section 5 we prove Theorem 1.1. Section 6 is devoted to the proof of Theorem 1.3. In Appendix I we provide the complete multiplication tables for Chow rings of X_1 and X_2 obtained by using the algorithm described in Appendix II.

2 Rational cycles on projective homogeneous varieties

2.1. Let k be a field and $\mathcal{V}ar_k$ be a category of smooth projective varieties over k. We define the category of Chow motives (over k) following [Ma68].

First, we define the category of *correspondences* (over k) denoted by $\mathcal{C}or_k$. Its objects are smooth projective varieties over k. For morphisms, called correspondences, we set $Mor(X,Y) := \mathrm{CH}^{\dim X}(X \times Y)$. For any two correspondences $\alpha \in \mathrm{CH}(X \times Y)$ and $\beta \in \mathrm{CH}(Y \times Z)$ we define the composition $\beta \circ \alpha \in \mathrm{CH}(X \times Z)$

$$\beta \circ \alpha = {pr_{13}}_*(pr_{12}^*(\alpha) \cdot pr_{23}^*(\beta)), \tag{1}$$

where pr_{ij} denotes the projection on the i-th and j-th factors of $X \times Y \times Z$ respectively and ${pr_{ij}}_*$, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups.

The pseudo-abelian completion of $\mathcal{C}or_k$ is called the category of *Chow motives* and is denoted by $\mathcal{M}_k$. The objects of $\mathcal{M}_k$ are pairs (X,p), where X is a smooth projective variety and p is a projector, that is, $p \circ p = p$. The motive (X, id) will be denoted by $\mathcal{M}(X)$.

2.2. By the construction $\mathcal{M}_k$ is a tensor additive category with self-duality. given by the transposition of cycles $\alpha \mapsto \alpha^t$. Moreover, the Chow functor

$\mathrm{CH} : \mathcal{V}ar_k \to \mathbb{Z}\text{-}\mathcal{A}b$ (to the category of $\mathbb{Z}$-graded abelian groups) factors through $\mathcal{M}_k$, that is, one has

$$\begin{tikzcd} \mathcal{V}ar_k \arrow[rr, "\mathrm{CH}"] \arrow[dr, "\Gamma"'] & & \mathbb{Z}\text{-}\mathcal{A}b \\ & \mathcal{M}_k \arrow[ur, "R"'] & \end{tikzcd}$$

where $\Gamma : f \mapsto \Gamma_f$ is the graph and $R : (X, p) \mapsto \mathrm{im}(p^*)$ is the realization.

2.3. Observe that the composition product $\circ$ induces the ring structure on the abelian group $\mathrm{CH}^{\dim X}(X \times X)$. The unit element of this ring is the class of the diagonal map Δ_X such that $\Delta_X \circ \alpha = \alpha \circ \Delta_X = \alpha$ for all $\alpha \in \mathrm{CH}^{\dim X}(X \times X)$.

2.4. Let G be a split linear algebraic group over k. Let X be a projective G-homogeneous variety, i.e., $X = G/P$, where P is a parabolic subgroup of G. The abelian group structure of $\mathrm{CH}(X)$, as well as its ring structure, is well-known. Namely, X has a cellular filtration and the base of this filtration generates the free abelian group $\mathrm{CH}(X)$ (see [Ka01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\mathrm{CH}^*(X \times Y) \cong \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \mathrm{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \mathrm{CH}(Y \times X)$ is given by (cf. [Bo03, Lem. 5])

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta), \tag{2}$$

where $\deg : \mathrm{CH}(Y) \to \mathrm{CH}(\{pt\}) = \mathbb{Z}$ is the degree map.

2.5. Let X be a projective variety of dimension n over a field k. Let k' be the separable closure of the field k. Consider the scalar extension $X' = X \times_k k'$. We say a cycle $J \in \mathrm{CH}(X')$ is *rational* if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X')$. For instance, there is an obvious rational cycle $\Delta_{X'}$ on $\mathrm{CH}^n(X' \times X')$ that is given by the diagonal class. Clearly, any linear combinations, intersections and correspondence products of rational cycles are rational.

2.6. Several techniques allow to produce rational cycles (cf. [Ka04, Prop. 3.3] for the case of quadrics). We shall use the following:

(i) Consider a variety Y and a morphism $X \to Y$ such that $X' = Y' \times_Y X$, where $Y' = Y \times_k k'$. Then any rational cycle on $\mathrm{CH}(Y')$ gives rise to a rational cycle on $\mathrm{CH}(X')$ by the induced pull-back $\mathrm{CH}(Y') \to \mathrm{CH}(X')$.

(ii) Consider a variety Y and a projective morphism $Y \to X$ such that $Y' = X' \times_X Y$. Then any rational cycle on $\mathrm{CH}(Y')$ gives rise to a rational cycle on $\mathrm{CH}(X')$ by the induced push-forward $\mathrm{CH}(Y') \to \mathrm{CH}(X')$.

(iii) Let X and Y be projective homogeneous varieties over k that split completely over the function fields $k(Y)$ and $k(X)$ respectively. Consider the following pull-back diagram

$$\begin{tikzcd} \mathrm{CH}^i(X \times Y) \arrow[r, "g"] \arrow[d, "f"'] & \mathrm{CH}^i(X' \times Y') \arrow[d, "f'"] \\ \mathrm{CH}^i(X_{k(Y)}) \arrow[r, "="] & \mathrm{CH}^i(X'_{k'(Y')}) \end{tikzcd}$$

where the vertical arrows are surjective by [IK00, §5]. Now take any cycle $\alpha \in \mathrm{CH}^i(X' \times Y')$, $i \leq \dim X$. Let $\beta = g(f^{-1}(f'(\alpha)))$. Then $f'(\beta) = f'(\alpha)$ and β is rational. Hence, $\beta = \alpha + J$, where $J \in ker f'$, and we conclude that $\alpha + J \in \mathrm{CH}^i(X' \times Y')$ is rational.

2.7. We will use the following fact (see [CGM, Cor. 8.3]) that follows from the Rost Nilpotence Theorem. Let p' be a non-trivial rational projector on $\mathrm{CH}^n(X' \times X')$, i.e., $p' \circ p' = p'$. Then there exists a non-trivial projector p on $\mathrm{CH}^n(X \times X)$ such that $p \times_k k' = p'$. Hence, the existence of a non-trivial rational projector p' on $\mathrm{CH}^n(X' \times X')$ gives rise to the decomposition of the Chow motive of X

$$\mathcal{M}(X) \cong (X, p) \oplus (X, 1-p)$$

We shall find such a projector in the case of an "exceptional" projective homogeneous variety "of type F_4".

3 Hasse diagrams and Chow rings

3.1. To each projective homogeneous variety X we may associate an oriented labeled graph $\mathcal{H}$ called Hasse diagram. It is known that the ring structure of $\mathrm{CH}(X)$ is determined by $\mathcal{H}$. In the present section we remind several facts concerning relations between Hasse diagrams and Chow rings. For a precise reference on this account see [De74], [Hi82a] and [Ko91].

3.2. Let G be a split simple algebraic group defined over a field k. We fix a maximal split torus T in G and a Borel subgroup B of G containing T and defined over k. Denote by Φ the root system of G, by Π the set of simple roots of Φ corresponding to B, by W the Weyl group, and by S the corresponding set of fundamental reflections.

Let $P = P_\Theta$ be a (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta \mid \theta \in \Theta \rangle$. Denote

$$W^\Theta = \{w \in W \mid \forall\, s \in \Theta \quad l(ws) = l(w) + 1\},$$

where l is the length function. The pairing

$$W^\Theta \times W_\Theta \to W \qquad (w, v) \mapsto wv$$

is a bijection and $l(wv) = l(w)+l(v)$. It is easy to see that W^Θ consists of all representatives in the cosets W/W_Θ which have minimal length. Sometimes it is also convenient to consider the set of all representatives of maximal length. We shall denote this set as ${}^\Theta W$. Observe that there is a bijection $W^\Theta \to {}^\Theta W$ given by $v \mapsto vw_\theta$, where w_θ is the longest element of W_Θ. The longest element of W^Θ corresponds to the longest element w_0 of the Weyl group.

3.3. To a subset Θ of the finite set Π we associate an oriented labeled graph, which we call a Hasse diagram and denote by $\mathcal{H}_W(\Theta)$. This graph is constructed as follows. The vertices of this graph are the elements of W^Θ. There is an edge from a vertex w to a vertex w' with a label i if and only if $l(w) < l(w')$ and $w' = ws_i$. The example of such a graph is provided in 4.4. Observe that the diagram $\mathcal{H}_W(\emptyset)$ coincides with the Cayley graph associated to the pair (W, S).

3.4 Lemma. *The assignment $\mathcal{H}_W \colon \Theta \mapsto \mathcal{H}_W(\Theta)$ is a contravariant functor from the category of subsets of the finite set Π (with embeddings as morphisms) to the category of oriented graphs.*

Proof. It is enough to embed the diagram $\mathcal{H}_W(\Theta)$ to the diagram $\mathcal{H}_W(\emptyset)$. We do this as follows. We identify the vertices of $\mathcal{H}_W(\Theta)$ with the subset of vertices of $\mathcal{H}_W(\emptyset)$ by means of the bijection $W^\Theta \to {}^\Theta W$. Then the edge from w to w' of ${}^\Theta W \subset W$ has a label i if and only if $l(w) < l(w')$ and $w' = s_i w$ (as elements of W). Clearly, the obtained graph will coincide with $\mathcal{H}_W(\Theta)$. $\square$

3.5. Now consider the Chow ring of a projective homogeneous variety G/P_Θ. It is well known that $\mathrm{CH}(G/P_\Theta)$ is a free abelian group with a basis given by the varieties $[X_w]$ that correspond to the vertices w of the Hasse diagram $\mathcal{H}_W(\Theta)$. The degree of the basis element $[X_w]$ corresponds to the minimal number of edges needed to connect the respective vertex w with w_θ (which is the longest one). The multiplicative structure of $\mathrm{CH}(G/P_\Theta)$ depends only on the root system of G and the diagram $\mathcal{H}_W(\Theta)$.

3.6 Lemma. *The contravariant functor* $\mathrm{CH}\colon \Theta \mapsto \mathrm{CH}(G/P_\Theta)$ *factors through the category of Hasse diagrams* $\mathcal{H}_W$*, i.e., the pull-back (ring inclusion)*

$$\mathrm{CH}(G/P_{\Theta'}) \hookrightarrow \mathrm{CH}(G/P_\Theta)$$

arising from the embedding $\Theta \subset \Theta'$ *is induced by the embedding of the respective Hasse diagrams* $\mathcal{H}_W(\Theta') \subset \mathcal{H}_W(\Theta)$.

3.7 Corollary. *Let* B *be a Borel subgroup of* G *and* P *its (standard) parabolic subgroup. Then* $\mathrm{CH}(G/P)$ *is a subring of* $\mathrm{CH}(G/B)$*. The generators of* $\mathrm{CH}(G/P)$ *are* $[X_w]$*, where* $w \in {}^{\Theta}W \subset W$*. The cycle* $[X_w]$ *in* $\mathrm{CH}(G/P)$ *has the codimension* $l(w_0) - l(w)$.

Proof. Apply the lemma to the case $B = P_\emptyset$ and $P = P_{\Theta'}$. □

Hence, in order to compute $\mathrm{CH}(G/P)$ it is enough to compute $\mathrm{CH}(X)$, where $X = G/B$ is the variety of complete flags. The following results provide tools to perform such computations.

3.8. In order to multiply two basis elements h and g of $\mathrm{CH}(G/P)$ such that $\deg h + \deg g = \dim G/P$ we use the following formula (see [Ko91, 1.4]):

$$[X_w] \cdot [X_{w'}] = \delta_{w, w_0 w' w_\theta} \cdot [pt].$$

3.9 (Pieri formula). In order to multiply two basis elements of $\mathrm{CH}(X)$ one of which is of codimension 1 we use the following formula (see [De74, Cor. 2 of 4.4]):

$$[X_{w_0 s_\alpha}][X_w] = \sum_{\beta \in \Phi^+,\, l(ws_\beta) = l(w) - 1} \langle \beta^\vee, \overline{\omega}_\alpha \rangle [X_{ws_\beta}],$$

where the sum runs through the set of positive roots $\beta \in \Phi^+$, s_α denotes the simple reflection corresponding to α and $\bar{\omega}_\alpha$ is the fundamental weight corresponding to α. Here $[X_{w_0 s_\alpha}]$ is the element of codimension 1.

3.10 (Giambelli formula). Let $P = P(\Phi)$ be the weight space. We denote as $\bar\omega_1, \dots \bar\omega_l$ the basis of P consisting of fundamental weights. The symmetric algebra $S^*(P)$ is isomorphic to $\mathbb{Z}[\bar\omega_1, \dots \bar\omega_l]$. The Weyl group W acts on P, hence, on $S^*(P)$. Namely, for a simple root α_i,

$$w_{\alpha_i}(\bar\omega_j) = \begin{cases} \bar\omega_i - \alpha_i, & i = j, \\ \bar\omega_j, & \text{otherwise.} \end{cases}$$

We define a linear map $c\colon S^*(P) \to \mathrm{CH}^*(G/B)$ as follows. For a homogeneous $u \in \mathbb{Z}[\bar\omega_1, \dots, \bar\omega_l]$

$$c(u) = \sum_{w \in W,\, l(w) = \deg(u)} \Delta_w(u)[X_{w_0 w}],$$

where for $w = w_{\alpha_1} \dots w_{\alpha_k}$ we denote by Δ_w the composition of derivations $\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_k}$ and the derivation $\Delta_{\alpha_i}\colon S^*(P) \to S^{*-1}(P)$ is defined by $\Delta_{\alpha_i}(u) = \frac{u - w_{\alpha_i}(u)}{\alpha_i}$. Then (see [Hi82a, ch. IV, 2.4])

$$[X_w] = c(\Delta_{w^{-1}}(\frac{d}{|W|})),$$

where d is the product of all positive roots in $S^*(P)$. In other words, the element $\Delta_{w^{-1}}(\frac{d}{|W|}) \in c^{-1}([X_w])$.

Hence, in order to multiply two basis elements $h, g \in \mathrm{CH}(X)$ take their preimages under the map c and multiply them in $S^*(P) = \mathbb{Z}[\bar\omega_1, \dots \bar\omega_l]$. Then apply c to their product.

4 Projective homogeneous varieties of type F_4

4.1. From now on, we assume that the characteristics of the base field k is not equal to 2 or 3. In the present section we remind several well-known facts concerning Albert algebras, groups of type F_4 and respective projective homogeneous varieties (see [PR94], [Inv], [Gar97]). At the end we provide partial computations of Chow rings of these varieties.

We start with the following observation concerning the Picard group of a projective homogeneous variety of type F_4

4.2 Lemma. *Let G be an anisotropic group of type* F_4 *over a field k and P be its maximal parabolic subgroup. Let $X = G/P$ be a respective projective homogeneous variety and $X' = X \times_k k'$ be its scalar extension to the separable closure k'. Then the Picard group* $\mathrm{Pic}(X')$ *is a free abelian group of rank 1 with a rational generator.*

Proof. Since P is maximal, $\mathrm{Pic}(X')$ is a free abelian group of rank 1. We use the following exact sequence (see [Ar82] and [MT95, 2.3]):

$$0 \longrightarrow \mathrm{Pic}\, X \longrightarrow (\mathrm{Pic}\, X')^{\Gamma} \xrightarrow{\alpha_X} \mathrm{Br}(k),$$

where $\Gamma = \mathrm{Gal}(k'/k)$ is the absolute Galois group and $\mathrm{Br}(k)$ the Brauer group of k. The map α_X is explicitly described in [MT95] in terms of Tits classes. Since groups of type F_4 are adjoint and simply-connected, their Tits classes are trivial and so is α_X. Since Γ acts trivially on $\mathrm{Pic}(X')$ and the image of α_X is trivial, we have $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X')$. □

4.3. It is well known that the classification of algebraic groups of type F_4 is equivalent to the classification of Albert algebras (those are 27-dimensional exceptional simple Jordan algebras). All Albert algebras can be obtained from one of the two Tits constructions.

An Albert algebra A obtained by the first Tits construction is produced from a central simple algebra of degree 3. By using the Rost-Serre invariant g_3 (if the input central simple algebra is split, then $g_3 = 0$) one can show that for the respective group $G = \mathrm{Aut}(A)$ only two Tits diagrams ([Ti66, Table II]) are allowed, namely the completely split case and the anisotropic case. This means that

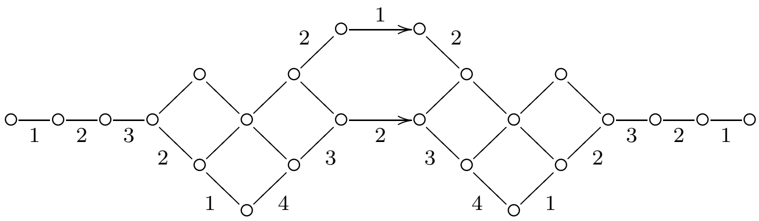
(i) anisotropic G splits completely by a cubic field extension;

(ii) for each maximal parabolic subgroup P_i, $i = 1, 2, 3, 4$, the respective projective homogeneous varieties $X = G/P_i$ split completely over the function field $k(G/P_j)$, $j = 1, 2, 3, 4$.

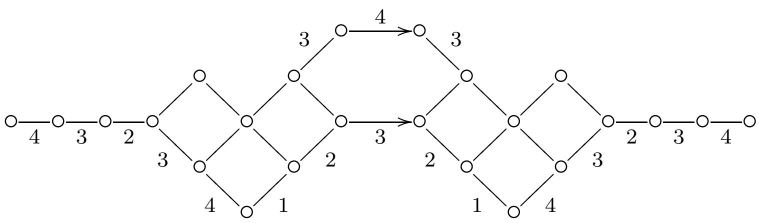
4.4. From this point on we consider a split group G of type F_4 obtained by the first Tits process. Let $X_1 = G/P_1$ and $X_2 = G/P_4$ be projective homogeneous varieties, corresponding to maximal parabolic subgroups P_1 and P_4 generated by the last $\{2, 3, 4\}$ and the first $\{1, 2, 3\}$ three vertices of the Dynkin diagram

$$\underset{1}{\circ}\!-\!\!-\!\!-\!\underset{2}{\circ}\!=\!\!>\!\!=\!\underset{3}{\circ}\!-\!\!-\!\!-\!\underset{4}{\circ}$$

Varieties X_1 and X_2 are not isomorphic and have the dimension 15. We provide the Hasse diagrams (graphs) for X_1:



and X_2:



We draw the diagrams in such a way that the labels on the opposite sides of a parallelogram are equal and in that case we omit all labels but one.

Recall that (see 3.5) the vertices of this graph correspond to the basis elements of the Chow group $\mathrm{CH}(X_1)$. The leftmost vertex is the unit class $1 = [X_{w_0}]$ and the rightmost one is the class of a 0-cycle of degree 1.

4.5. We denote the basis elements of the respective Chow groups as follows

$$\mathrm{CH}^i(X_1) = \begin{cases} \langle h_1^i \rangle, & i = 0 \ldots 3, 12 \ldots 15, \\ \langle h_1^i, g_1^i \rangle, & i = 4 \ldots 11. \end{cases}$$

$$\mathrm{CH}^i(X_2) = \begin{cases} \langle h_2^i \rangle, & i = 0 \ldots 3, 12 \ldots 15, \\ \langle h_2^i, g_2^i \rangle, & i = 4 \ldots 11. \end{cases}$$

The generators h_i correspond to the upper vertices of the respective Hasse diagrams, and g_i to the lower ones (if the corresponding rank is 2).

4.6. Applying 3.8 we immediately obtain the following partial multiplication table

$$h_k^s g_k^{15-s} = 0, \qquad h_k^s h_k^{15-s} = g_k^s g_k^{15-s} = h_k^{15},$$

where $k = 1, 2$, for all s.

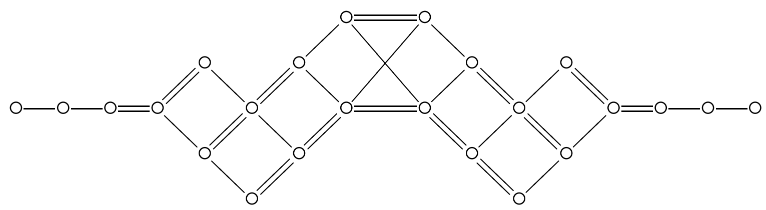
4.7. By Pieri formula 3.9 we obtain the following partial multiplication tables for $\mathrm{CH}(X_1)$:

$h_1^1h_1^1 = h_1^2$, $h_1^1h_1^2 = 2h_1^3$, $h_1^1h_1^3 = 2h_1^4 + g_1^4$, $h_1^1h_1^4 = h_1^5$,
$h_1^1g_1^4 = 2h_1^5 + g_1^5$, $h_1^1h_1^5 = 2h_1^6 + g_1^6$, $h_1^1g_1^5 = 2g_1^6$, $h_1^1h_1^6 = h_1^7 + g_1^7$,
$h_1^1g_1^6 = 2g_1^7$, $h_1^1h_1^7 = 2h_1^8 + g_1^8$, $h_1^1g_1^7 = h_1^8 + 2g_1^8$, $h_1^1h_1^8 = h_1^9$,
$h_1^1g_1^8 = h_1^9 + 2g_1^9$, $h_1^1h_1^9 = 2h_1^{10}$, $h_1^1g_1^9 = h_1^{10} + 2g_1^{10}$, $h_1^1h_1^{10} = h_1^{11} + 2g_1^{11}$,
$h_1^1g_1^{10} = g_1^{11}$, $h_1^1h_1^{11} = 2h_1^{12}$, $h_1^1g_1^{11} = h_1^{12}$, $h_1^1h_1^{12} = 2h_1^{13}$,
$h_1^1h_1^{13} = h_1^{14}$, $h_1^1h_1^{14} = h_1^{15}$.

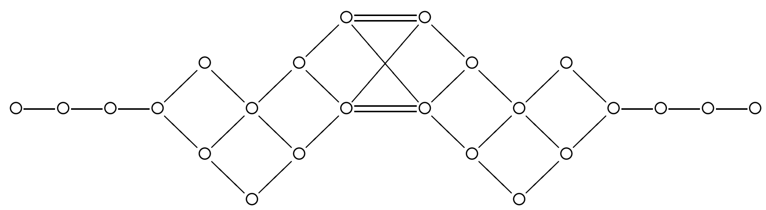
for $\mathrm{CH}(X_2)$:

$h_2^1h_2^1 = h_2^2$, $h_2^1h_2^2 = h_2^3$, $h_2^1h_2^3 = h_2^4 + g_2^4$, $h_2^1h_2^4 = h_2^5$,
$h_2^1g_2^4 = h_2^5 + g_2^5$, $h_2^1h_2^5 = h_2^6 + g_2^6$, $h_2^1g_2^5 = g_2^6$, $h_2^1h_2^6 = h_2^7 + g_2^7$,
$h_2^1g_2^6 = g_2^7$, $h_2^1h_2^7 = 2h_2^8 + g_2^8$, $h_2^1g_2^7 = h_2^8 + 2g_2^8$, $h_2^1h_2^8 = h_2^9$,
$h_2^1g_2^8 = h_2^9 + g_2^9$, $h_2^1h_2^9 = h_2^{10}$, $h_2^1g_2^9 = h_2^{10} + g_2^{10}$, $h_2^1h_2^{10} = h_2^{11} + g_2^{11}$,
$h_2^1g_2^{10} = g_2^{11}$, $h_2^1h_2^{11} = h_2^{12}$, $h_2^1g_2^{11} = h_2^{12}$, $h_2^1h_2^{12} = h_2^{13}$,
$h_2^1h_2^{13} = h_2^{14}$, $h_2^1h_2^{14} = h_2^{15}$.

4.8. Observe that the multiplication tables 4.7 can be visualized by means of the Hasse diagrams. Namely, for the variety X_1 consider the following graph which is obtained from the respective Hasse diagram by adding a few more edges and erasing all the labels:



and for X_2:



The multiplication rules can be restored from this graph as follows: for a vertex u (that corresponds to a basis element of the Chow group) we set

$$h_i^1 u = \sum_{u \to v} v,$$

where the sum runs through all the edges going from u to the right (cf. [Hi82b, Cor. 3.3]), $i = 1, 2$.

4.9. Applying Giambelli formula 3.10 we obtain the following products which will be essentially used in the next section

$$g_1^4 g_1^4 = 6h_1^8 + 8g_1^8, \qquad g_2^4 g_2^4 = 3h_2^8 + 4g_2^8.$$

5 Construction of rational projectors

The goal of the present section is to prove Theorem 1.1.

5.1. According to 2.7 in order to decompose the motive $\mathcal{M}(X)$ it is enough to construct rational projectors on $\mathrm{CH}^{15}(X' \times X')$, where $X' = X \times_k k'$. Observe that our anisotropic group G of type F_4 is obtained by the first Tits construction, so by 4.3.(i) there is a splitting field extension of degree 3. Therefore, for any basis element h of $\mathrm{CH}(X')$ the cycle $3h$ is rational (it follows immediately by transfer arguments). Hence, in order to construct rational cycles in $\mathrm{CH}(X' \times X')$ it is enough to work modulo 3. We shall write $x =_3 y$ iff $x - y = 3z$ for some cycle z. Please note that all results hold for Chow groups with integral coefficients.

5.2. Recall that the cycles h_1^1 and h_2^1 are rational by Lemma 4.2. Hence, their powers $(h_1^1)^i$ and $(h_2^1)^i$, $i = 2 \ldots 7$, are rational as well. Using multiplication tables 4.7 or their graph interpretation 4.8 we immediately obtain the following rational cycles: in codimensions 2 through 7 for $\mathrm{CH}(X_1')$:

$$h_1^2,\ h_1^3,\ h_1^4 - g_1^4,\ h_1^5 + g_1^5,\ h_1^6,\ h_1^7 + g_1^7,$$

and for $\mathrm{CH}(X_2')$:

$$h_2^2,\ h_2^3,\ h_2^4 + g_2^4,\ g_2^5 - h_2^5,\ h_2^6,\ h_2^7 + g_2^7.$$

5.3. Apply the arguments of 2.6.(iii) to $\mathrm{CH}^4(X_2' \times X_1')$ (this can be done because all the properties for X_1' and X_2' hold by 4.3.(ii)). There exists a rational cycle $\alpha_1 \in \mathrm{CH}^4(X_2' \times X_1')$ such that $f'(\alpha_1) = g_2^4 \times 1$. This cycle must have the following form:

$$\alpha_1 = g_2^4 \times 1 + a_1 h_2^3 \times h_1^1 + a_2 h_2^2 \times h_1^2 + a_3 h_2^1 \times h_1^3 + a_4 1 \times h_1^4 + a' 1 \times g_1^4,$$

where $a_i, a' \in \{-1, 0, 1\}$. We may reduce α_1 by adding cycles that are known to be rational (by 5.2) to

$$\alpha_1 = (g_2^4 \times 1) + a'(1 \times g_1^4).$$

Repeating the same procedure for a rational cycle $\alpha_2 \in \mathrm{CH}^4(X_2' \times X_1')$ such that $f'(\alpha_2) = 1 \times g_1^4$ and reducing it we obtain the rational cycle

$$\alpha_2 = b(g_2^4 \times 1) + (1 \times g_1^4),$$

where $b \in \{-1, 0, 1\}$. Hence, there is a rational cycle of the form

$$r = g_2^4 \times 1 - a(1 \times g_1^4),$$

where $a \in \{-1, 1\}$.

5.4. To obtain a rational projector p_1 we proceed as follows. First, we obtain the following rational cycles in $\mathrm{CH}(X_2' \times X_1')$ modulo 3.

$$r^2 = (g_2^4 \times 1 - a \cdot 1 \times g_1^4)^2 =_3 g_2^8 \times 1 + a(g_2^4 \times g_1^4) - 1 \times g_1^8,$$

$$\begin{aligned}
r_1 &= (1 \times (h_1^7 + g_1^7))r^2 =_3 g_2^8 \times (h_1^7 + g_1^7) - 1 \times h_1^{15} - a(g_2^4 \times (g_1^{11} + h_1^{11})),\\
r_2 &= ((h_2^7 + g_2^7) \times 1)r^2 =_3 -(h_2^7 + g_2^7) \times g_1^8 + h_2^{15} \times 1 + a((g_2^{11} - h_2^{11}) \times g_1^4),\\
r_3 &= (h_2^1 \times h_1^6)r^2 =_3 -h_2^1 \times h_1^{14} + (g_2^9 + h_2^9) \times h_1^6 + a((h_2^5 + g_2^5) \times (g_1^{10} - h_1^{10})),\\
r_4 &= (h_2^6 \times h_1^1)r^2 =_3 h_2^6 \times (g_1^9 - h_1^9) + h_2^{14} \times h_1^1 + a((h_2^{10} + g_2^{10}) \times (h_1^5 - g_1^5)),\\
r_5 &= (h_2^2 \times (h_1^5 + g_1^5))r^2 =_3 h_2^2 \times h_1^{13} + (g_2^{10} - h_2^{10}) \times (h_1^5 + g_1^5) + a((h_2^6 - g_2^6) \times g_1^9),\\
r_6 &= ((g_2^5 - h_2^5) \times h_1^2))r^2 =_3 -h_2^{13} \times h_1^2 - (g_2^5 - h_2^5) \times (h_1^{10} + g_1^{10}) + a(g_2^9 \times (h_1^6 + g_1^6)),\\
r_7 &= (h_2^3 \times (h_1^4 - g_1^4))r^2 =_3 h_2^3 \times h_1^{12} - h_2^{11} \times (h_1^4 - g_1^4) + a(h_2^7 \times (g_1^8 - h_1^8)),\\
r_8 &= ((h_2^4 + g_2^4) \times h_1^3)r^2 =_3 -h_2^{12} \times h_1^3 + (h_2^4 + g_2^4) \times h_1^{11} + a((h_2^8 - g_2^8) \times h_1^7).
\end{aligned}$$

5.5. To obtain motivic decompositions, we need to find rational projectors in $\mathrm{CH}(X_1' \times X_1')$ and $\mathrm{CH}(X_2' \times X_2')$. By compositioning (still modulo 3), we obtain the following rational cycles:

$$p_0 = -(r_2 \circ r_1^t) =_3 h_1^{15} \times 1 + (g_1^{11} + h_1^{11}) \times g_1^4 + (g_1^7 + h_1^7) \times g_1^8,$$

$$p_1 = -(r_4 \circ r_3^t) =_3 h_1^{14} \times h_1^1 + (h_1^{10} - g_1^{10}) \times (g_1^5 - h_1^5) + h_1^6 \times (h_1^9 - g_1^9),$$

$$p_2 = -(r_6 \circ r_5^t) =_3 h_1^{13} \times h_1^2 - (h_1^5 + g_1^5) \times (h_1^{10} + g_1^{10}) + g_1^9 \times (h_1^6 + g_1^6),$$

$$p_3 = -(r_8 \circ r_7^t) =_3 h_1^{12} \times h_1^3 + (h_1^8 - g_1^8) \times h_1^7 + (h_1^4 - g_1^4) \times h_1^{11} \in \mathrm{CH}^{15}(X_1' \times X_1');$$

$$q_0 = -(r_1^t \circ r_2) =_3 h_2^{15} \times 1 + (g_2^{11} - h_2^{11}) \times g_2^4 + (g_2^7 + h_2^7) \times g_2^8,$$

$$q_1 = -(r_4^t \circ r_3) =_3 h_2^1 \times h_2^{14} + (g_2^9 + h_2^9) \times h_2^6 - (h_2^5 + g_2^5) \times (h_2^{10} + g_2^{10}),$$

$$q_2 = -(r_6^t \circ r_5) =_3 h_2^2 \times h_2^{13} - (g_2^{10} - h_2^{10}) \times (g_2^5 - h_2^5) + (g_2^6 - h_2^6) \times g_2^9,$$

$$q_3 = -(r_7^t \circ r_8) =_3 h_2^{12} \times h_2^3 + (h_2^4 + g_2^4) \times h_2^{11} + (h_2^8 - g_2^8) \times h_2^7 \in \mathrm{CH}^{15}(X_2' \times X_2').$$

It remains to note that

$$p_0 \circ p_0 = p_0, \quad p_3 \circ p_3 = p_3 \quad \text{in} \quad \mathrm{CH}^{15}(X_1' \times X_1'),$$

$$q_0 \circ q_0 = q_0, \quad q_3 \circ q_3 = q_3 \quad \text{in} \quad \mathrm{CH}^{15}(X_2' \times X_2'),$$

where the equalities hold with integral coefficients (not just modulo 3).

Note that p_1, p_2, q_1, and q_2 are not projectors with integral coefficients, but the sums

$$p_{12} = p_1 + p_2^t + 3(h_1^{10} \times h_1^5 + g_1^{10} \times g_1^5),$$
$$q_{12} = q_1 + q_2^t + 3(h_2^5 \times h_2^{10} + g_2^5 \times g_2^{10})$$

are rational projectors. In fact, p_1, p_2, q_1, and q_2 are projectors modulo 3, that is, $p_1 \circ p_1 =_3 p_1$ etc.

Also note that

$$p_0 + p_{12} + p_3 + p_0^t + p_{12}^t + p_3^t = \Delta_{X_1'},$$

$$q_0 + q_{12} + q_3 + q_0^t + q_{12}^t + q_3^t = \Delta_{X_2'},$$

where $\Delta_{X_i'}$ are the diagonal cycles, and all these projectors are orthogonal to each other. Hence, by 2.7 we obtain the decomposition of the motive of X_1 and X_2:

$$\mathcal{M}(X_1) = (X_1, p_0) \oplus (X_1, p_{12}) \oplus (X_1, p_3) \oplus (X_1, p_0^t) \oplus (X_1, p_{12}^t) \oplus (X_1, p_3^t),$$

$$\mathcal{M}(X_2) = (X_2, q_0) \oplus (X_2, q_{12}) \oplus (X_2, q_3) \oplus (X_2, q_0^t) \oplus (X_2, q_{12}^t) \oplus (X_2, q_3^t),$$

It is also easy to see that over the separable closure k' the motives (X_1, p_0), (X_1, p_3), (X_2, q_0), and (X_2, q_3) split as the direct sums of Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$. Indeed, the images of the realizations of p_0, p_3, q_0, and q_3 are free abelian groups $\langle 1, g_1^4, g_1^8 \rangle$, $\langle h_1^3, h_1^7, h_1^{11} \rangle$, $\langle 1, g_2^4, g_2^8 \rangle$, and $\langle h_2^3, h_2^7, h_2^{11} \rangle$ respectively.

5.6. Since p_1, p_2, q_1, and q_2 are projectors modulo 3, we obtain here a consequence of Voevodsky's theorem [Vo03] for the case of the symbol $(3,3)$. These projectors are the generalized Rost projectors in this case.

6 Motivic isomorphism between $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$

The goal of this section is to prove Theorem 1.3.

6.1. We continue to use the notation from the previous section. Consider the following cycle in $\mathrm{CH}^{15}(X_2' \times X_1')$

$$\begin{aligned} J = {} & g_2^8 \times g_1^7 - 1 \times h_1^{15} - a g_2^4 \times g_1^{11} - a h_2^{12} \times h_1^3 + a h_2^4 \times h_1^{11} + h_2^8 \times h_1^7 \\ & - g_2^7 \times g_1^8 + h_2^{15} \times 1 + a g_2^{11} \times g_1^4 + a h_2^3 \times h_1^{12} - a h_2^{11} \times h_1^4 - h_2^7 \times h_1^8 \\ & - h_2^1 \times h_1^{14} + h_2^9 \times h_1^6 + a h_2^5 \times h_1^{10} - a g_2^5 \times g_1^{10} + a h_2^{13} \times h_1^2 - g_2^9 \times g_1^6 \\ & - h_2^6 \times h_1^9 + h_2^{14} \times h_1^1 - a h_2^2 \times h_1^{13} + g_2^6 \times g_1^9 + a g_2^{10} \times g_1^5 - a h_2^{10} \times h_1^5. \end{aligned}$$

Observe that

$$J \circ (-J^t) = \Delta_{X_1'}, \quad (-J^t) \circ J = \Delta_{X_2'}.$$

In other words, the correspondence J provides a motivic isomorphism between X_2' and X_1' with the inverse $(-J)^t$. From the other hand,

$$J =_3 (r_1 + a r_8) + (r_2 + a r_7) + (r_3 - a r_6) + (r_4 - a r_5)$$

and, hence, is rational. Theorem 1.3 is proved.

7 Further work

We plan to use our methods to obtain similar decompositions for other projective homogeneous varieties. Our closest plans include decomposing the motives of other projective homogeneous varieties of type F_4 and of other types, e.g., E_8.

Another interesting question is to determine whether the motive R_{12} obtained above is decomposable.

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Appendix I

The multiplication table for X_1:

$$
\begin{array}{llll}
h^4h^4 = h^8 + g^8, & h^4h^5 = 2h^9 + 2g^9, & h^4h^6 = 2h^{10} + g^{10}, & h^4h^7 = h^{11} + 2g^{11}, \\
h^4h^8 = h^{12}, & h^4h^9 = 2h^{13}, & h^4h^{10} = h^{14}, & h^4g^4 = 2h^8 + 3g^8, \\
h^4g^5 = h^9 + 2g^9, & h^4g^6 = 2h^{10} + 2g^{10}, & h^4g^7 = h^{11} + 3g^{11}, & h^4g^8 = 2h^{12}, \\
h^4g^9 = h^{13}, & h^4g^{10} = 0, & g^4g^4 = 6h^8 + 8g^8, & g^4g^5 = 4h^9 + 4g^9, \\
g^4g^6 = 6h^{10} + 4g^{10}, & g^4g^7 = 3h^{11} + 8g^{11}, & g^4g^8 = 6h^{12}, & g^4g^9 = 4h^{13}, \\
g^4g^{10} = h^{14}, & g^4h^5 = 5h^9 + 6g^9, & g^4h^6 = 5h^{10} + 4g^{10}, & g^4h^7 = 2h^{11} + 6g^{11}, \\
g^4h^8 = 2h^{12}, & g^4h^9 = 4h^{13}, & g^4h^{10} = 2h^{14}, & h^5h^5 = 6h^{10} + 4g^{10}, \\
h^5h^6 = 2h^{11} + 5g^{11}, & h^5h^7 = 4h^{12}, & h^5h^8 = 2h^{13}, & h^5h^9 = 2h^{14}, \\
h^5g^5 = 4h^{10} + 4g^{10}, & h^5g^6 = 2h^{11} + 6g^{11}, & h^5g^7 = 5h^{12}, & h^5g^8 = 4h^{13}, \\
h^5g^9 = h^{14}, & g^5g^5 = 4h^{10}, & g^5g^6 = 2h^{11} + 4g^{11}, & g^5g^7 = 4h^{12}, \\
g^5g^8 = 4h^{13}, & g^5g^9 = 2h^{14}, & g^5h^6 = h^{11} + 4g^{11}, & g^5h^7 = 2h^{12}, \\
g^5h^8 = 0, & g^5h^9 = 0, & h^6h^6 = 3h^{12}, & h^6h^7 = 3h^{13}, \\
h^6h^8 = h^{14}, & h^6g^6 = 3h^{12}, & h^6g^7 = 3h^{13}, & h^6g^8 = h^{14}, \\
g^6g^6 = 4h^{12}, & g^6g^7 = 4h^{13}, & g^6g^8 = 2h^{14}, & g^6h^7 = 2h^{13}, \\
g^6h^8 = 0, & h^7h^7 = 2h^{14}, & h^7g^7 = h^{14}, & g^7g^7 = 2h^{14}.
\end{array}
$$

The multiplication table for X_2:

$$
\begin{array}{llll}
h^4h^4 = 2h^8 + 2g^8, & h^4h^5 = 4h^9 + 2g^9, & h^4h^6 = 4h^{10} + g^{10}, & h^4h^7 = 2h^{11} + 2g^{11}, \\
h^4h^8 = h^{12}, & h^4h^9 = h^{13}, & h^4h^{10} = h^{14}, & h^4g^4 = 2h^8 + 3g^8, \\
h^4g^5 = h^9 + g^9, & h^4g^6 = 2h^{10} + g^{10}, & h^4g^7 = 2h^{11} + 3g^{11}, & h^4g^8 = 2h^{12}, \\
h^4g^9 = h^{13}, & h^4g^{10} = 0, & g^4g^4 = 3h^8 + 4g^8, & g^4g^5 = 2h^9 + g^9, \\
g^4g^6 = 3h^{10} + g^{10}, & g^4g^7 = 3h^{11} + 4g^{11}, & g^4g^8 = 3h^{12}, & g^4g^9 = 2h^{13}, \\
g^4g^{10} = h^{14}, & g^4h^5 = 5h^9 + 3g^9, & g^4h^6 = 5h^{10} + 2g^{10}, & g^4h^7 = 2h^{11} + 3g^{11}, \\
g^4h^8 = h^{12}, & g^4h^9 = h^{13}, & g^4h^{10} = h^{14}, & h^5h^5 = 6h^{10} + 2g^{10}, \\
h^5h^6 = 4h^{11} + 5g^{11}, & h^5h^7 = 4h^{12}, & h^5h^8 = h^{13}, & h^5h^9 = h^{14}, \\
h^5g^5 = 2h^{10} + g^{10}, & h^5g^6 = 2h^{11} + 3g^{11}, & h^5g^7 = 5h^{12}, & h^5g^8 = 2h^{13}, \\
h^5g^9 = h^{14}, & g^5g^5 = h^{10}, & g^5g^6 = h^{11} + g^{11}, & g^5g^7 = 2h^{12}, \\
g^5g^8 = h^{13}, & g^5g^9 = h^{14}, & g^5h^6 = h^{11} + 2g^{11}, & g^5h^7 = h^{12}, \\
g^5h^8 = 0, & g^5h^9 = 0, & h^6h^6 = 6h^{12}, & h^6h^7 = 3h^{13}, \\
h^6h^8 = h^{14}, & h^6g^6 = 3h^{12}, & h^6g^7 = 3h^{13}, & h^6g^8 = h^{14}, \\
g^6g^6 = 2h^{12}, & g^6g^7 = 2h^{13}, & g^6g^8 = h^{14}, & g^6h^7 = h^{13}, \\
g^6h^8 = 0, & h^7h^7 = 2h^{14}, & h^7g^7 = h^{14}, & g^7g^7 = 2h^{14}.
\end{array}
$$

Appendix II

In this appendix we describe how we obtained the necessary multiplication tables. Our root enumeration follows Bourbaki ([Bou]). We fix an orthonormal base $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{R}^4$. F_4 has the following simple roots:

$$
\begin{aligned}
&\alpha_1 = e_3 - e_2, \quad &&\alpha_2 = e_2 - e_1, \\
&\alpha_3 = e_1, &&\alpha_4 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 + \frac{1}{2}e_4.
\end{aligned}
$$

The set of fundamental weights:

$$
\begin{aligned}
&\bar{\omega}_1 = e_3 + e_4, &&\bar{\omega}_2 = e_2 + e_3 + 2e_4, \\
&\bar{\omega}_3 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{3}{2}e_4, \quad &&\bar{\omega}_4 = e_4.
\end{aligned}
$$

For the expressions of other positive roots by the base roots we refer to [Bou]. We list the expressions of these roots in the basis of fundamental weights:

$$\begin{array}{rlrl}
-\bar\omega_3+2\bar\omega_4, & -\bar\omega_2+2\bar\omega_3-\bar\omega_4, & -\bar\omega_1+2\bar\omega_2-2\bar\omega_3, & 2\bar\omega_1-\bar\omega_2,\\
-\bar\omega_2+\bar\omega_3+\bar\omega_4, & -\bar\omega_1+\bar\omega_2-\bar\omega_4, & -\bar\omega_1+2\bar\omega_3-2\bar\omega_4, & \bar\omega_1+\bar\omega_2-2\bar\omega_3,\\
-\bar\omega_1+\bar\omega_2-\bar\omega_3+\bar\omega_4, & \bar\omega_1-\bar\omega_4, & -\bar\omega_1+2\bar\omega_4, & \bar\omega_1-\bar\omega_2+2\bar\omega_3-2\bar\omega_4,\\
-\bar\omega_1+\bar\omega_3, & \bar\omega_1-\bar\omega_3+\bar\omega_4, & \bar\omega_1-\bar\omega_2+2\bar\omega_4, & \bar\omega_2-2\bar\omega_4,\\
\bar\omega_1-\bar\omega_2+\bar\omega_3, & \bar\omega_2-2\bar\omega_3+2\bar\omega_4, & \bar\omega_2-\bar\omega_3, & -\bar\omega_2+2\bar\omega_3,\\
\bar\omega_3-\bar\omega_4, & -\bar\omega_1+\bar\omega_2, & \bar\omega_4, & \bar\omega_1.
\end{array}$$

We denote by d the product of all positive roots.

Using the Giambelli formula, we obtain the preimages of g_i^4 in $S^*(P)$. Here is the list:

$$g_1^4 = c(\frac{1}{3}\bar\omega_4^4 - \frac{2}{3}\bar\omega_3\bar\omega_4^3 + \bar\omega_3^2\bar\omega_4^2 - \frac{4}{3}\bar\omega_2\bar\omega_3\bar\omega_4^2 - \frac{2}{3}\bar\omega_1\bar\omega_2\bar\omega_3^2 + \frac{2}{3}\bar\omega_2^2\bar\omega_4^2 +
\frac{2}{3}\bar\omega_1^2\bar\omega_4^2 + \bar\omega_1\bar\omega_3\bar\omega_4^2 + \frac{2}{3}\bar\omega_1\bar\omega_2\bar\omega_3\bar\omega_4 - \bar\omega_1\bar\omega_3^2\bar\omega_4 - \frac{2}{3}\bar\omega_1^2\bar\omega_3\bar\omega_4 + \frac{4}{3}\bar\omega_2\bar\omega_3^2\bar\omega_4 -
\frac{2}{3}\bar\omega_3^3\bar\omega_4 - \frac{2}{3}\bar\omega_2^2\bar\omega_3\bar\omega_4 - \frac{1}{2}\bar\omega_1^3\bar\omega_2 - \frac{2}{3}\bar\omega_2\bar\omega_3^3 - \frac{2}{3}\bar\omega_1^2\bar\omega_2\bar\omega_3 + \frac{1}{3}\bar\omega_2^2\bar\omega_3^2 +
\frac{1}{2}\bar\omega_1^4 + \frac{1}{3}\bar\omega_3^4 + \frac{2}{3}\bar\omega_1^2\bar\omega_3^2 + \frac{1}{2}\bar\omega_1^2\bar\omega_2^2 - \frac{1}{3}\bar\omega_1\bar\omega_2^2\bar\omega_3 + \frac{1}{3}\bar\omega_1\bar\omega_2\bar\omega_3^2),$$

$$g_2^4 = c(\frac{11}{6}\bar\omega_4^4 - \frac{7}{6}\bar\omega_4^3\bar\omega_3 + \frac{11}{12}\bar\omega_4^2\bar\omega_1^2 + \frac{3}{2}\bar\omega_4^2\bar\omega_3^2 - \frac{11}{6}\bar\omega_3\bar\omega_4^2\bar\omega_2 + \frac{11}{12}\bar\omega_4^2\bar\omega_2^2
- \frac{11}{12}\bar\omega_4^2\bar\omega_1\bar\omega_2 - \frac{2}{3}\bar\omega_4\bar\omega_3^3 - \frac{1}{2}\bar\omega_4\bar\omega_1^2\bar\omega_2 + \frac{1}{3}\bar\omega_4\bar\omega_3\bar\omega_1^2 + \frac{4}{3}\bar\omega_3^2\bar\omega_4\bar\omega_2 + \frac{1}{2}\bar\omega_4\bar\omega_1\bar\omega_2^2 -
\frac{2}{3}\bar\omega_3\bar\omega_4\bar\omega_2^2 - \frac{1}{3}\bar\omega_4\bar\omega_3\bar\omega_1\bar\omega_2 + \frac{1}{3}\bar\omega_3^4 - \frac{1}{3}\bar\omega_3\bar\omega_1\bar\omega_2^2 + \frac{1}{3}\bar\omega_3^2\bar\omega_1\bar\omega_2 - \frac{1}{3}\bar\omega_1^2\bar\omega_3^2 +
\frac{1}{3}\bar\omega_3\bar\omega_1^2\bar\omega_2 + \frac{1}{3}\bar\omega_3^2\bar\omega_2^2 - \frac{2}{3}\bar\omega_3^3\bar\omega_2),$$

Multiplying the correspondent polynomials and taking the c function, we find the products.