Cherednik algebras and Yangians

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Abstract

We construct a functor from the category of modules over the trigonometric (resp. rational) Cherednik algebra of type $\mathfrak{gl}_l$ to the category of integrable modules of level l over a Yangian for the loop algebra $\mathfrak{sl}_n$ (resp. over a subalgebra of this Yangian called the Yangian deformed double loop algebra) and we establish that it is an equivalence of categories if $l+2<n$.

1 Introduction

One of the most important classical results in representation theory is an equivalence, called Schur-Weyl duality, between the category of modules over the symmetric group S_l and the category of modules of level l over the Lie algebra $\mathfrak{sl}_n$ if $l \leq n-1$. When quantum groups were invented in the 1980's, it became an interesting problem to generalize the Schur-Weyl correspondence and similar equivalences were obtained between finite Hecke algebras and quantized enveloping algebras [Ji], between degenerate affine Hecke algebras and Yangians [Dr1, Ch1], between affine Hecke algebras and quantized affine Lie algebras [ChPr1], and between double affine Hecke algebras and toroidal quantum algebras [VaVa]. In this paper, we prove a similar equivalence of categories between the trigonometric (resp. rational) Cherednik algebra associated to the symmetric group S_l and a (resp. subalgebra $\mathbb{L}$ of a) Yangian LY for the loop algebra $L\mathfrak{sl}_n = \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u, u^{-1}]$.

The algebra on the other side of our equivalence from the Cherednik algebra is barely known. The author is not aware of any paper devoted to a study of Yangians of affine type; the only mention he could find of this notion anywhere in the literature is in [Va]. By contrast, there has been a recent surge of interest in the representation theory of Cherednik algebras and their relations to the geometry of Hilbert schemes, integrable systems and other important mathematical objects. Our duality theorem indicates a new route to those questions via a careful study of the Yangian and makes the study of this algebra more relevant and interesting.

Affine Hecke algebras are very important in representation theory and have been studied extensively over the last few decades, along with their degenerate (graded) version introduced in [Dr1] and in [Lu]. About fifteen years ago, I. Cherednik introduced the notion of double affine Hecke algebra [Ch2], abbreviated DAHA, which he used to prove important conjectures of I. Macdonald. His algebra also admits degenerate versions, the trigonometric one and the rational one, which are called Cherednik algebras. The trigonometric DAHA is generated by two subalgebras, one isomorphic to a degenerate affine Hecke algebra and the other one isomorphic to the group algebra of an affine Weyl group. For this reason, and because of the results mentioned above, we can expect its Schur-Weyl dual to be built from one copy of the Yangian Y for $\mathfrak{sl}_n$ and from one copy of the loop algebra $L\mathfrak{sl}_n$. This is indeed true for LY.

An epimorphic image of $\mathbb{L}$, defined in terms of operators acting on a certain space, appeared for the first time in [BHW]; this was known to P. Etingof and V. Ginzburg. However, the algebra considered in that paper is not described in a very precise way and no equivalence of categories is established. One motivation for the present article comes from our desire to find exactly the relations between the generators of the Schur-Weyl dual of a Cherednik algebra of type $\mathfrak{gl}_l$.

In the next two sections, we define Cherednik algebras and Yangians and explore some of their basic properties. The fourth section states the main result (theorem 4.2) for the trigonometric case, which is proved in the following one. After that, we look more closely at the action of certain elements of LY since this is useful in the last section, which concerns the rational case (theorem 7.1). Most of our results in the rational case follows from the observation that the rational Cherednik algebra of type $\mathfrak{gl}_l$ is contained in the trigonometric one. Furthermore, our equivalence restricts to an equivalence between two categories of BGG-type (theorem 7.2).

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2 Hecke algebras and Cherednik algebras

The definitions given in this section could be stated for any Weyl group W. However, in this paper, we will be concerned only with the symmetric group S_l, so we will restrict our definitions to the case $W = S_l$. We set $\mathfrak{h} = \mathbb{C}^l$. The symmetric group S_l acts on $\mathfrak{h}$ by permuting the coordinates; associated to $\mathfrak{h}$ are two polynomial algebras: $\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*) = \mathbb{C}[x_1, \ldots, x_l]$ and $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[y_1, \ldots, y_l]$, where $\{x_1, \ldots, x_l\}$ and $\{y_1, \ldots, y_l\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively. For $i \neq j$, we set $\alpha_{ij} = x_i - x_j, \alpha_{ij}^\vee = y_i - y_j, R = \{\alpha_{ij} | 1 \leq i \neq j \leq l\}$ and $R^+ = \{\alpha_{ij} | 1 \leq i < j \leq l\}$. The set $\Pi = \{x_i - x_{i+1} | 1 \leq i \leq n-1\}$ is a basis of simple roots. The reflection in $\mathfrak{h}$ with respect to the hyperplane $\alpha = 0$ is denoted s_α, so $s_\alpha(y) = y - \langle \alpha, y \rangle \alpha^\vee$, where $\langle \, , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the canonical pairing. We set $s_{ij} = s_{\alpha_{ij}}$.

The finite Hecke algebra $\mathcal{H}_q$ associated to S_l is a deformation of the group algebra $\mathbb{C}[S_l]$ and the affine Hecke algebra $\widetilde{\mathcal{H}}_q$ is a deformation of the group algebra of the (extended) affine Weyl group $\widetilde{S}_l = P \rtimes S_l$ where P is the lattice $\oplus_{i=1}^l \mathbb{Z}x_i \subset \mathfrak{h}^*$ (so $\mathbb{C}[\widetilde{S}_l] = \mathbb{C}[x_1^\pm, \ldots, x_l^\pm] \rtimes S_l$). The algebra $\widetilde{\mathcal{H}}_q$ admits a degenerate form $\mathtt{H}_c$ first introduced by Drinfeld [Dr1] and by Lusztig [Lu].

Definition 2.1. *The degenerate affine Hecke algebra $\mathtt{H}_c$ of type $\mathfrak{gl}_l$ is the algebra generated by the polynomial algebra* $\mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[z_1, \ldots, z_l]$ *and the group algebra* $\mathbb{C}[S_l]$ *with the relations*

$$s_\alpha \cdot z - s_\alpha(z) s_\alpha = -c\langle \alpha, z \rangle \ \ \forall z \in \mathfrak{h}, \forall \alpha \in \Pi$$

Remark 2.1. *The algebras $\mathtt{H}_{c_1}$ and $\mathtt{H}_{c_2}$ are isomorphic if $c_1 \neq 0$ and $c_2 \neq 0$. Clearly, $\mathtt{H}_0 \cong \mathbb{C}[\mathfrak{h}] \rtimes S_l$.*

The double affine Hecke algebra $\mathbb{H}$ introduced by I. Cherednik [Ch2] also admits degenerate versions: the trigonometric one and the rational one. Recall that the group $\widetilde{S}_l$ is generated by $s_\alpha \ \forall \alpha \in R$ and by the element $\pi = x_1 s_{12} s_{23} \cdots s_{l-1,l}$.

Definition 2.2 (Cherednik). *Let $t, c \in \mathbb{C}$. The degenerate (trigonometric) double affine Hecke algebra of type $\mathfrak{gl}_l$ is the algebra $\mathbf{H}_{t,c}$ generated by the group algebra of the (extended) affine Weyl* *group $\mathbb{C}[\widetilde{S}_l]$ and the polynomial algebra $\mathbb{C}[z_1, \dots, z_l] = \mathrm{Sym}(\mathfrak{h})$ subject to the following relations:*

$$s_\alpha \cdot z - s_\alpha(z) s_\alpha = -c\langle \alpha, z\rangle \ \ \forall z \in \mathfrak{h}, \forall \alpha \in \Pi$$

$$\pi z_i = z_{i+1}\pi, \ \ 1 \leq i \leq l-1 \quad \pi z_l = (z_1 - t)\pi$$

Remark 2.2. *The subalgebra generated by $\mathbb{C}[S_l]$ and the polynomial algebra $\mathbb{C}[z_1, \dots, z_l]$ is isomorphic to the degenerate affine Hecke algebra $\mathtt{H}_c$.*

The rational version of the double affine Hecke algebra has been studied quite intensively in the past few years (see, for example, [BEG1],[GGOR]) and is usually referred to as the rational Cherednik algebra.

Definition 2.3. *Let $t, c \in \mathbb{C}$. The rational Cherednik algebra $\mathsf{H}_{t,c}$ of type $\mathfrak{gl}_l$ is the algebra generated by $\mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*]$ and $\mathbb{C}[S_l]$ subject to the following relations:*

$$w \cdot x \cdot w^{-1} = w(x) \qquad w \cdot y \cdot w^{-1} = w(y) \quad \forall x \in \mathfrak{h}^*, \ \forall y \in \mathfrak{h}$$

$$[y, x] = yx - xy = t\langle y, x\rangle + c \sum_{\alpha \in R^+} \langle \alpha, y\rangle\langle x, \alpha^\vee\rangle s_\alpha$$

Remark 2.3. *The rational Cherednik algebra of type A_{l-1} is the subalgebra of $\mathsf{H}_{t,c}$ generated by $\mathbb{C}[x_i - x_j] \subset \mathbb{C}[x_1, \dots, x_l]$, by $\mathbb{C}[y_i - y_j] \subset \mathbb{C}[y_1, \dots, y_l]$ and by $\mathbb{C}[S_l]$.*

The exists a simple relation between $\mathbf{H}_{t,c}$ and $\mathsf{H}_{t,c}$. (See also [Su].)

Proposition 2.1. *The algebra $\mathbb{C}[x_1^{\pm}, \dots, x_l^{\pm}] \otimes_{\mathbb{C}[\mathfrak{h}]} \mathsf{H}_{t,c}$ is isomorphic to $\mathbf{H}_{t,c}$.*

Before giving a proof of this proposition, we need to introduce elements in $\mathsf{H}_{t,c}$ which will be very important later. For $1 \leq i \leq l$, set $\mathcal{U}_i = \frac{t}{2} + x_i y_i + c\sum_{j<i} s_{ij}$ and $\mathcal{Y}_i = \mathcal{U}_i + \frac{c}{2}\sum_{j \neq i} \mathrm{sign}(j-i) s_{ij} = \frac{t}{2} + x_i y_i + \frac{c}{2}\sum_{j\neq i} s_{ij}$.

Proposition 2.2. *[DuOp],[EtGi]*

1. $\mathcal{Y}_i = \frac{1}{2}(x_i y_i + y_i x_i)$.
2. $\mathcal{U}_i\mathcal{U}_j = \mathcal{U}_j\mathcal{U}_i$ *for any* i, j.
3. $w \cdot \mathcal{Y}_i \cdot w^{-1} = \mathcal{Y}_{w(i)}$.
4. *The elements $\mathcal{U}_i, 1 \leq i \leq l$, and $\mathbb{C}[S_l]$ generate a subalgebra of $\mathsf{H}_{t,c}$ isomorphic to the degenerate affine Hecke algebra $\mathtt{H}_c$.*

Remark 2.4. *The elements $\mathcal{Y}_i$ are not pairwise commutative if $c \neq 0$:*

$$[\mathcal{Y}_j, \mathcal{Y}_k] = \frac{c^2}{4} \sum_{\substack{i=1 \\ i \neq j,k}}^{l} [s_{ij}, s_{jk}].$$

Proof. The first statement follows from the equality

$$y_i x_i - x_i y_i \;=\; t + c \sum_{j=1, j\neq i}^{l} \langle x_i - x_j, y_i\rangle\langle x_i, y_i - y_j\rangle s_{ij} = t + c \sum_{j=1, j\neq i}^{l} s_{ij}.$$

The second part is proved in [DuOp]. The third part is obvious, so we prove only the fourth one. If $|k-i|>1$, then $s_{k,k+1}\mathcal{U}_i = \mathcal{U}_i s_{k,k+1}$, so the non-trivial relations that we have to check involve $s_{i-1,i}$ and $s_{i,i+1}$:

$$s_{i-1,i}\mathcal{U}_i = (\frac{t}{2} + x_{i-1}y_{i-1})s_{i-1,i} + c\sum_{j<i-1} s_{i-1,j}s_{i-1,i} + c = \mathcal{U}_{i-1}s_{i-1,i} + c$$

$$s_{i,i+1}\mathcal{U}_i = (\frac{t}{2} + x_{i+1}y_{i+1})s_{i,i+1} + c\sum_{j<i+1} s_{i+1,j}s_{i+1,i} - c = \mathcal{U}_{i+1}s_{i,i+1} - c$$

These two equalities, combined with the PBW-property of $\mathsf{H}_{t,c}$ [EtGi] and of $\mathtt{H}_c$, complete the proof of part 4. □

In the proof of the two main theorems, we will need the following identities.

Proposition 2.3. *1. If $i \neq j$, then $[y_j, x_i] = -cs_{ij}$ and $[x_i^{-1}, y_j] = -cx_i^{-1}x_j^{-1}s_{ij}$.*

2. $[y_i, x_i] = t + c\sum_{k\neq i} s_{ij}$ and $[x_i^{-1}, y_i] = tx_i^{-2} + c\sum_{j\neq i} x_i^{-1}x_j^{-1}s_{ij}$.

3. If $i\neq j$, then $[\mathcal{Y}_j, x_i] = -\frac{c}{2}(x_i + x_j)s_{ij}$ and $[x_i^{-1}, \mathcal{Y}_j] = -\frac{c}{2}(x_i^{-1} + x_j^{-1})s_{ij}$.

4. $[\mathcal{Y}_i, x_i] = tx_i + \frac{c}{2}\sum_{j\neq i}(x_i + x_j)s_{ij}$ and $[x_i^{-1}, \mathcal{Y}_i] = tx_i^{-1} + \frac{c}{2}\sum_{j\neq i}(x_i^{-1} + x_j^{-1})s_{ij}$.

Proof. These are all immediate consequences of the definition of $\mathsf{H}_{t,c}$. □

Proof. (of proposition 2.1) Because of proposition 2.2, part 4, and the PBW-property of $\mathsf{H}_{t,c}$ and $\mathbf{H}_{t,c}$, we only have to check the relation involving π in definition 2.2. First, assume that $i\neq l$.

$$\begin{aligned}
\pi\mathcal{U}_i &= (x_1 s_{12}\cdots s_{l-1,l})(\mathcal{U}_i) = x_1(\mathcal{U}_{i+1} - cs_{1,i+1})s_{12}\cdots s_{l-1,l}\\
&= ([x_1, \mathcal{U}_{i+1}] + \mathcal{U}_{i+1}x_1 - cx_1 s_{1,i+1})s_{12}\cdots s_{l-1,l}\\
&= \big(cx_{i+1}s_{1,i+1} + c[x_1, s_{1,i+1}] + \mathcal{U}_{i+1}x_1 - cx_1 s_{1,i+1}\big)s_{12}\cdots s_{l-1,l}\\
&= (\mathcal{U}_{i+1})x_1 s_{12}\cdots s_{l-1,l} = \mathcal{U}_{i+1}\pi
\end{aligned}$$

If $i = l$, we obtain:

$$\begin{aligned}
\pi\mathcal{U}_l &= x_1(\mathcal{U}_1 + c\sum_{j=2}^{l} s_{1,j})s_{12}\cdots s_{l-1,l} = \big(x_1[x_1, y_1] + cx_1\sum_{j=2}^{l} s_{1,j} + \mathcal{U}_1 x_1\big)s_{12}\cdots s_{l-1,l}\\
&= \big(x_1(-t - c\sum_{i\neq 1} s_{i,1}) + cx_1\sum_{j=2}^{l} s_{1,j} + \mathcal{U}_1 x_1\big)s_{12}\cdots s_{l-1,l} \;=\; (\mathcal{U}_1 - t)\pi
\end{aligned}$$

□

Corollary 2.1 (of proposition 2.1). *The algebra $\mathbf{H}_{t,c}$ can also be defined as the algebra generated by the elements $x_1^{\pm},\ldots,x_l^{\pm}$, $\mathcal{Y}_1,\ldots,\mathcal{Y}_l$ and S_l with the relations*

$$w\cdot x_i\cdot w^{-1}=x_{w(i)}\quad w\cdot\mathcal{Y}_i\cdot w^{-1}=\mathcal{Y}_{w(i)}\quad [\mathcal{Y}_j,\mathcal{Y}_k]=\frac{c^2}{4}\sum_{\substack{i=1\\ i\neq j,k}}^{l}(s_{jk}s_{ik}-s_{kj}s_{ij})$$

$$x_i\mathcal{Y}_j-\mathcal{Y}_jx_i=t\delta_{ij}x_i+\frac{c}{2}\sum_{\alpha\in R^+}\langle\alpha,y_j\rangle\langle x_i,\alpha^\vee\rangle(x_is_\alpha+s_\alpha x_i).$$

3 Finite and loop Yangians

The Yangians of finite type are quantum groups, introduced by V. Drinfeld in [Dr1], which are quantizations of the enveloping algebra of the polynomial loop algebra $\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[u]$ of a semisimple Lie algebra $\mathfrak{g}$. The second definition in [Dr2] is given in terms of a finite Cartan matrix. If we replace it with a Cartan matrix of affine type, we obtain algebras that we call loop Yangians $LY_{\beta,\lambda}$. (The only occurrence of these algebras in the literature that the author could find is in a remark in [Va].) Let $C=(c_{ij})_{1\leq i,j\leq n-1}$ ($\widehat{C}=(c_{ij})_{0\leq i,j\leq n-1}$) be a Cartan matrix of finite (resp. affine) type A_{n-1} (resp. $\widehat{A}_{n-1}$). If $n\geq 3$:

$$\widehat{C}=\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1\\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0\\ 0 & -1 & 2 & -1 & 0 & \cdots & 0\\ \vdots & & \vdots & & \vdots & & \vdots\\ 0 & \cdots & 0 & -1 & 2 & -1 & 0\\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1\\ -1 & 0 & \cdots & \cdots & 0 & -1 & 2\end{pmatrix}$$

Definition 3.1. *[Dr2], [ChPr2] Let $\lambda\in\mathbb{C}$. The Yangian Y_λ associated to C is the algebra generated by $X_{i,r}^{\pm},H_{i,r},i=1,\ldots n-1,r\in\mathbb{Z}_{\geq 0}$, which satisfy the following relations :*

$$[H_{i,r},H_{j,s}]=0,\quad [H_{i,0},X_{j,s}^{\pm}]=\pm c_{ij}X_{j,s}^{\pm}\tag{1}$$

$$[H_{i,r+1},X_{j,s}^{\pm}]-[H_{i,r},X_{j,s+1}^{\pm}]=\pm\frac{\lambda}{2}c_{ij}(H_{i,r}X_{j,s}^{\pm}+X_{j,s}^{\pm}H_{i,r})\tag{2}$$

$$[X_{i,r}^{+},X_{j,s}^{-}]=\delta_{ij}H_{i,r+s}\quad [X_{i,r}^{\pm},X_{j,s}^{\pm}]=0\ \textit{if}\ 1<|j-i|<n-1\tag{3}$$

$$[X_{i,r+1}^{\pm},X_{j,s}^{\pm}]-[X_{i,r}^{\pm},X_{j,s+1}^{\pm}]=\pm\frac{\lambda}{2}c_{ij}(X_{i,r}^{\pm}X_{j,s}^{\pm}+X_{j,s}^{\pm}X_{i,r}^{\pm})\tag{4}$$

$$\left[X_{i,r_1}^{\pm},[X_{i,r_2}^{\pm},X_{j,s}^{\pm}]\right]+\left[X_{i,r_2}^{\pm},[X_{i,r_1}^{\pm},X_{j,s}^{\pm}]\right]=0\ \ \forall r_1,r_2,s\geq 0\ \textit{if}\ j-i\equiv\pm 1\ \textit{mod}\ n\tag{5}$$

Remark 3.1. *The Yangian Y_{λ_1} is isomorphic to Y_{λ_2} if $\lambda_1\neq 0$ and $\lambda_2\neq 0$.*

Definition 3.2. *Let $\beta, \lambda \in \mathbb{C}$. The Yangian $LY_{\beta,\lambda}$ associated to $\widehat{C}$ is the algebra generated by $X_{i,r}^{\pm}, H_{i,r}, i = 0, \ldots n-1, r \in \mathbb{Z}_{\geq 0}$, which satisfy the relations of definition 3.1 for $i, j \in \{0, \ldots, n-1\}$ except that the relations (2),(4) must be modified for $i = 0$ and $j = 1, n-1$ in the following way when $n \geq 3$:*

$$[H_{1,r+1}, X_{0,s}^{\pm}] - [H_{1,r}, X_{0,s+1}^{\pm}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})H_{1,r}X_{0,s}^{\pm} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X_{0,s}^{\pm}H_{1,r} \tag{6}$$

$$[H_{0,r+1}, X_{1,s}^{\pm}] - [H_{0,r}, X_{1,s+1}^{\pm}] = (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)H_{0,r}X_{1,s}^{\pm} + (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X_{1,s}^{\pm}H_{0,r} \tag{7}$$

$$[H_{0,r+1}, X_{n-1,s}^{\pm}] - [H_{0,r}, X_{n-1,s+1}^{\pm}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})H_{0,r}X_{n-1,s}^{\pm} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X_{n-1,s}^{\pm}H_{0,r} \tag{8}$$

$$[H_{n-1,r+1}, X_{0,s}^{\pm}] - [H_{n-1,r}, X_{0,s+1}^{\pm}] = (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)H_{n-1,r}X_{0,s}^{\pm} + (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X_{0,s}^{\pm}H_{n-1,r} \tag{9}$$

$$[X_{1,r+1}^{\pm}, X_{0,s}^{\pm}] - [X_{1,r}^{\pm}, X_{0,s+1}^{\pm}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X_{1,r}^{\pm}X_{0,s}^{\pm} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X_{0,s}^{\pm}X_{1,r}^{\pm} \tag{10}$$

$$[X_{0,r+1}^{\pm}, X_{n-1,s}^{\pm}] - [X_{0,r}^{\pm}, X_{n-1,s+1}^{\pm}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X_{0,r}^{\pm}X_{n-1,s}^{\pm} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X_{n-1,s}^{\pm}X_{0,r}^{\pm} \tag{11}$$

We will also impose the relation $\sum_{i=0}^{n-1} H_{i,0} = 0$.

Remark 3.2. *We will write $X_i^{\pm}$ and H_i instead of $X_{i,0}^{\pm}$ and $H_{i,0}$. If $\beta = \frac{\lambda}{2}$, the relations defining $LY_{\beta,\lambda}$ are the same as those in definition 3.1 with $i, j \in \{0, \ldots, n-1\}$. Note also that the relations (6),(7),(8) and (9) all follow from (10) and (11) using relation (3); they were added above as a convenient reference since they will be useful later in our computations. We should also note that $LY_{\beta_1,\lambda_1} \cong LY_{\beta_2,\lambda_2}$ if $\beta_2 = \eta\beta_1$ and $\lambda_2 = \eta\lambda_1$ for some $\eta \neq 0$.*

Remark 3.3. *When $\beta = 0$, $LY_{\lambda,\beta}$ is isomorphic to the enveloping algebra of the universal central extension of $\mathfrak{sl}_n(A_\beta)$ where A_β is isomorphic to the ring of algebraic differential operators on $\mathbb{C}^*$ (see [Gu]).*

Let $\Delta = \{\epsilon_{ij}, 1 \leq i \neq j \leq n\}$ be the root system of type A_{n-1}. For a positive root $\epsilon \in \Delta^+$, we denote by $X_\epsilon^{\pm}$ the corresponding standard root vector of $\mathfrak{sl}_n$. If $\epsilon = \epsilon_{ij}, i < j$, then $X_\epsilon^+ = E_{ij}$ and $X_\epsilon^- = E_{ji}$, where E_{rs} is the matrix with 1 in the (r,s)-entry and zeros everywhere else. In particular, $X_\theta^+ = E_{1n}$ and $X_\theta^- = E_{n1}$, where θ is the longest root of $\mathfrak{sl}_n$. If $\epsilon = \epsilon_i = \epsilon_{i,i+1}$, then $X_\epsilon^{\pm} = X_i^{\pm}$.

One useful observation is that these two Yangians are generated by $X_{i,r}^{\pm}, H_{i,r}, i = 1, \ldots n-1$ (resp. $i = 0, \ldots, n-1$) with $r = 0, 1$ only. The other elements are obtained inductively by the formulas:

$$X_{i,r+1}^{\pm} = \pm\frac{1}{2}[H_{i,1}, X_{i,r}^{\pm}] - \frac{1}{2}(H_i X_{i,r}^{\pm} + X_{i,r}^{\pm}H_i), \quad H_{i,r+1} = [X_{i,r}^+, X_{i,1}^-]. \tag{12}$$

Furthermore, the subalgebra generated by the elements with $r = 0$ is isomorphic to the enveloping algebra of the Lie (resp. loop) algebra $\mathfrak{sl}_n$ (resp. $L\mathfrak{sl}_n = \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u, u^{-1}]$). We identify X_0^+ with $X_\theta^- \otimes u$ and X_0^- with $X_\theta^+ \otimes u^{-1}$. The subalgebra Y_λ^0 generated by the elements with $i \neq 0$ is a quotient of the Yangian Y_λ.

The two subalgebras Y^0_λ and $\mathfrak{U}(L\mathfrak{sl}_n)$ generate $LY_{\beta,\lambda}$. Indeed, combining the observations in the previous two paragraphs, we see that we only have to show that the subalgebra they generate contains $X^\pm_{0,1}$. From the relation (1) in definition 3.2 with $i=1$, we know that $[H_1, X^\pm_{0,1}] = \mp X^\pm_{0,1}$, so, substituting into equation (6), we obtain

$$[H_{1,1}, X^\pm_0] \pm X^\pm_{0,1} = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2}) H_1 X^\pm_0 + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta) X^\pm_0 H_1.$$

Thus $X^\pm_{0,1}$ (hence also $H_{0,1}$) belongs to the subalgebra of $LY_{\beta,\lambda}$ generated by Y^0_λ and $\mathfrak{U}(L\mathfrak{sl}_n)$. When no confusion is possible, we will write LY and Y instead of $LY_{\beta,\lambda}$ and Y_λ.

For $1 \le i \le n-1$, set

$$J(X^\pm_i) = X^\pm_{i,1} + \lambda \omega^\pm_i \text{ where } \omega^\pm_i = \pm\frac{1}{4}\sum_{\epsilon\in\Delta^+} \left([X^\pm_i, X^\pm_\epsilon]X^\mp_\epsilon + X^\mp_\epsilon[X^\pm_i, X^\pm_\epsilon]\right) - \frac{1}{4}(X^\pm_i H_i + H_i X^\pm_i)$$

$$J(H_i) = H_{i,1} + \lambda\nu_i \text{ where } \nu_i = \frac{1}{4}\sum_{\epsilon\in\Delta^+} (\epsilon,\epsilon_i)(X^+_\epsilon X^-_\epsilon + X^-_\epsilon X^+_\epsilon) - \frac{1}{2}H_i^2.$$

More explicitly, since $X^+_i = E_{i,i+1}$, $X^-_i = E_{i+1,i}$ and $H_i = E_{ii} - E_{i+1,i+1}$ for $1 \le i \le n-1$, we can write

$$\omega^+_i = \frac{1}{4}\sum_{\substack{j=1 \\ j\neq i,i+1}}^{n} \mathrm{sign}(j-i)(E_{j,i+1}E_{ij} + E_{ij}E_{j,i+1}) - \frac{1}{4}(E_{i,i+1}H_i + H_i E_{i,i+1}) \tag{13}$$

$$\omega^-_i = \frac{1}{4}\sum_{\substack{j=1 \\ j\neq i,i+1}}^{n} \mathrm{sign}(j-i)(E_{i+1,j}E_{ji} + E_{ji}E_{i+1,j}) - \frac{1}{4}(E_{i+1,i}H_i + H_i E_{i+1,i}) \tag{14}$$

It is possible to define elements $J(z) \in Y$ for any $z \in \mathfrak{sl}_n$ in such a way that $[J(z_1), z_2] = J([z_1,z_2])$: this follows from the isomorphism given in [Dr2] between two different realizations of the Yangian Y_λ: the one given above and the one first given in [Dr1] in terms of generators $z, J(z)$ $\forall z \in \mathfrak{sl}_n$ (the $J(z)$'s satisfy a"deformed" Jacobi identity).

In the proof of our first main theorem, the following algebra automorphism will be very important.

Lemma 3.1. *It is possible to define an automorphism ρ of LY by setting*

$$\rho(H_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} \left(\frac{\lambda}{2}\right)^{r-s} H_{i-1,s}, \quad \rho(X^\pm_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} \left(\frac{\lambda}{2}\right)^{r-s} X^\pm_{i-1,s} \quad \textit{for } i \neq 0,1$$

$$\rho(H_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} \beta^{r-s} H_{i-1,s}, \quad \rho(X^\pm_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} \beta^{r-s} X^\pm_{i-1,s} \quad \textit{for } i = 0,1$$

We use the convention that $X^\pm_{-1,r} = X^\pm_{n-1,r}$ and $H_{-1,r} = H_{n-1,r}$. Note that, in particular, $\rho(X^\pm_i) = X^\pm_{i-1}, \rho(H_i) = H_{i-1}$ $\forall i$ and $\rho(X^\pm_{i,1}) = X^\pm_{i-1,1} + \frac{\lambda}{2}X^\pm_{i-1}, \rho(H_{i,1}) = H_{i-1,1} + \frac{\lambda}{2}H_{i-1}$ if $i \neq 0,1$, whereas $\rho(X^\pm_{i,1}) = X^\pm_{i-1,1} + \beta X^\pm_{i-1}, \rho(H_{i,1}) = H_{i-1,1} + \beta H_{i-1}$ if $i = 0,1$ The automorphism ρ is very similar to the automorphism $\tau_{\frac{\lambda}{2}}$ (or τ_β) in [ChPr2] followed by a decrement of the indices.

Proof of lemma 3.1. We have to verify that ρ is indeed an automorphism of LY, that is, that it respects the defining relations of LY. In the case when $i,j \neq 0,1$ in the relations (1)-(5), this follows from the fact that ρ is the same as the automorphism $\tau_{\frac{\lambda}{2}}$ from [ChPr2] followed by a decrement of the indices. A short verification shows that ρ preserves the relations (1),(3) and (5) when $i = 0,1$ or $j = 0,1$. (In the case of equation (3) and $i = j$, , one has to use the identity $\sum_{a+b=k} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} = \begin{pmatrix} r+s \\ k \end{pmatrix}$.) Since the relations (6)-(9) follow from (10) and (11) by applying $[\cdot, X^{\mp}_{?,0}], ? = 0, 1, n-1$, there are three cases left that require a more detailed verification. We will use the identity $\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} r-1 \\ a \end{pmatrix} + \begin{pmatrix} r-1 \\ a-1 \end{pmatrix}$.

Case 1: We start with the relation (4) with $i = 2, j = 1$. We find that $\rho\big([X^{\pm}_{2,r+1}, X^{\pm}_{1,s}] - [X^{\pm}_{2,r}, X^{\pm}_{1,s+1}]\big)$ is equal to

$$
\begin{aligned}
= & \sum_{a=0}^{r+1} \sum_{b=0}^{s} \begin{pmatrix} r+1 \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left[\left(\frac{\lambda}{2}\right)^{r+1-a} X^{\pm}_{1,a}, \beta^{s-b} X^{\pm}_{0,b} \right] \\
& - \sum_{a=0}^{r} \sum_{b=0}^{s+1} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s+1 \\ b \end{pmatrix} \left[\left(\frac{\lambda}{2}\right)^{r-a} X^{\pm}_{1,a}, \beta^{s+1-b} X^{\pm}_{0,b} \right] \\
= & \sum_{a=0}^{r+1} \sum_{b=0}^{s} \left(\begin{pmatrix} r \\ a \end{pmatrix} + \begin{pmatrix} r \\ a-1 \end{pmatrix} \right) \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r+1-a} \beta^{s-b} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
& - \sum_{a=0}^{r} \sum_{b=0}^{s+1} \begin{pmatrix} r \\ a \end{pmatrix} \left(\begin{pmatrix} s \\ b \end{pmatrix} + \begin{pmatrix} s \\ b-1 \end{pmatrix} \right) \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s+1-b} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
= & \sum_{a=0}^{r} \sum_{b=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b} \left(\frac{\lambda}{2} - \beta\right) [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
& + \sum_{a=0}^{r+1} \sum_{b=0}^{s} \begin{pmatrix} r \\ a-1 \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a+1} \beta^{s-b} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
& - \sum_{a=0}^{r} \sum_{b=0}^{s+1} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b-1 \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b+1} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
= & \sum_{a=0}^{r} \sum_{b=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b} \left(\frac{\lambda}{2} - \beta\right) [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
& + \sum_{\tilde{a}=0}^{r} \sum_{b=0}^{s} \begin{pmatrix} r \\ \tilde{a} \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-\tilde{a}} \beta^{s-b} [X^{\pm}_{1,\tilde{a}+1}, X^{\pm}_{0,b}] \\
& - \sum_{a=0}^{r} \sum_{\tilde{b}=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ \tilde{b} \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-\tilde{b}} [X^{\pm}_{1,a}, X^{\pm}_{0,\tilde{b}+1}]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{a=0}^{r}\sum_{b=0}^{s}\begin{pmatrix} r \\ a \end{pmatrix}\begin{pmatrix} s \\ b \end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-a}\beta^{s-b}(\frac{\lambda}{2}-\beta)[X^{\pm}_{1,a},X^{\pm}_{0,b}] \\
&\quad +\sum_{\tilde{a}=0}^{r}\sum_{\tilde{b}=0}^{s}\begin{pmatrix} r \\ \tilde{a} \end{pmatrix}\begin{pmatrix} s \\ \tilde{b} \end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde{a}}\beta^{s-\tilde{b}}\big([X^{\pm}_{1,\tilde{a}+1},X^{\pm}_{0,\tilde{b}}]-[X^{\pm}_{1,\tilde{a}},X^{\pm}_{0,\tilde{b}+1}]\big) \\
&= \sum_{\tilde{a}=0}^{r}\sum_{\tilde{b}=0}^{s}\begin{pmatrix} r \\ \tilde{a} \end{pmatrix}\begin{pmatrix} s \\ \tilde{b} \end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde{a}}\beta^{s-\tilde{b}}(\frac{\lambda}{2}-\beta)[X^{\pm}_{1,\tilde{a}},X^{\pm}_{0,\tilde{b}}] \\
&\quad +\sum_{\tilde{a}=0}^{r}\sum_{\tilde{b}=0}^{s}\begin{pmatrix} r \\ \tilde{a} \end{pmatrix}\begin{pmatrix} s \\ \tilde{b} \end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde{a}}\beta^{s-\tilde{b}}\big((\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})X^{\pm}_{1,\tilde{a}}X^{\pm}_{0,\tilde{b}}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X^{\pm}_{0,\tilde{b}}X^{\pm}_{1,\tilde{a}}\big) \\
&= \sum_{\tilde{a}=0}^{r}\sum_{\tilde{b}=0}^{s}\begin{pmatrix} r \\ \tilde{a} \end{pmatrix}\begin{pmatrix} s \\ \tilde{b} \end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde{a}}\beta^{s-\tilde{b}}\left(\mp\frac{\lambda}{2}\right)(X^{\pm}_{1,\tilde{a}}X^{\pm}_{0,\tilde{b}}+X^{\pm}_{0,\tilde{b}}X^{\pm}_{1,\tilde{a}}) \\
&= \rho\left(\left(\mp\frac{\lambda}{2}\right)(X^{\pm}_{2,r}X^{\pm}_{1,s}+X^{\pm}_{1,s}X^{\pm}_{2,r})\right)
\end{aligned}
$$

Case 2: $i=1, j=0$. We have to prove that

$$\rho\big([X^{\pm}_{1,r+1},X^{\pm}_{0,s}]-[X^{\pm}_{1,r},X^{\pm}_{0,s+1}]\big)=\rho\big((\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})X^{\pm}_{1,r}X^{\pm}_{0,s}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X^{\pm}_{0,s}X^{\pm}_{1,r}\big).$$

This case is analogous to case 1, but a little bit simpler since the term $(\frac{\lambda}{2}-\beta)$ above becomes $(\beta-\beta)=0$.

Case 3: $i=0, j=n-1$. We have to show that

$$\rho\big([X^{\pm}_{0,r+1},X^{\pm}_{n-1,s}]-[X^{\pm}_{0,r},X^{\pm}_{n-1,s+1}]\big)=\rho\big((\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})X^{\pm}_{0,r}X^{\pm}_{n-1,s}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X^{\pm}_{n-1,s}X^{\pm}_{0,r}\big).$$

The computations are again very similar to those of case 1: the main difference is that the factor $(\frac{\lambda}{2}-\beta)$ gets replaced by $(\beta-\frac{\lambda}{2})$. □

4 Schur-Weyl duality functor

The Schur-Weyl duality established by M. Varagnolo and E. Vasserot [VaVa] involves, on one side, a toroidal quantum algebra (a quantized version of the enveloping algebra of the enveloping algebra of the universal central extension of the double-loop algebra $\mathfrak{sl}_n[s^{\pm},t^{\pm}]$) and, on the other side, a double affine Hecke algebra for S_l. Theorem 4.2 establishes a similar type of duality between the trigonometric DAHA $\mathbf{H}_{t,c}$ and the loop Yangian $LY_{\beta,\lambda}$, which extends the duality for the Yangian of finite type due to V. Drinfeld [Dr1].

Before stating the more classical results on the theme of Schur-Weyl duality, we have to define the notion of module of level l over $\mathfrak{sl}_n$ and the quantized enveloping algebra $\mathfrak{U}_q\mathfrak{sl}_n$. Fix a positive integer n and set $V=\mathbb{C}^n$.

Definition 4.1. *A finite dimensional representation of $\mathfrak{sl}_n$ or $\mathfrak{U}_q\mathfrak{sl}_n$ (q not a root of unity) is of level l if each of its irreducible components is isomorphic to a direct summand of $V^{\otimes l}$.*

Theorem 4.1. *[Ji, Dr1, ChPr1] Fix $l \geq 1, n \geq 2$ and assume that $q \in \mathbb{C}^\times$ is not a root of unity. Let A be one of the algebras $\mathbb{C}[S_l], \mathcal{H}_q(S_l), \mathtt{H}_1(S_l), \widetilde{\mathcal{H}}_q(S_l)$, and let B be the corresponding one (in the same order) among $\mathfrak{U}\mathfrak{sl}_n, \mathfrak{U}_q\mathfrak{sl}_n, Y(\mathfrak{sl}_n), \mathfrak{U}_q\widehat{\mathfrak{sl}}_n$. There exists a functor $\mathcal{F}$ from the category of finite dimensional right A-modules to the category of finite dimensional left B-modules which are of level l as $\mathfrak{sl}_n$-modules in the first and third case (and as $\mathfrak{U}_q\mathfrak{sl}_n$-modules in the second and fourth case) which is given by*

$$\mathcal{F}(M) = M \otimes_C V^{\otimes l}$$

where $C = \mathbb{C}[S_l]$ (first and third case) or $C = \mathcal{H}_q(S_l)$ (second and fourth case). Furthermore, this functor is an equivalence of categories if $l \leq n-1$.

The $\mathfrak{sl}_n$ module structure on $V^{\otimes l}$ commutes with the S_l-module structure obtained by simply permuting the factors in the tensor product. Let M be a right module over $\mathbf{H}_{t,c}$. Since $\mathbb{C}[S_l] \subset \mathbf{H}_{t,c}$, we can form the tensor product $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$.

On one hand, since $\mathbf{H}_{t,c}$ contains the degenerate affine Hecke algebra $\mathtt{H}_c$, M can be viewed as a right module over $\mathtt{H}_c$, so it follows from [Dr1] that $\mathcal{F}(M)$ is a module of level l over the Yangian Y_λ of $\mathfrak{sl}_n$ with $\lambda = c$. On the other hand, $\mathbf{H}_{t,c}$ also contains a copy of the group algebra of the extended affine Weyl group $\widetilde{S}_l$, so it follows from [ChPr1] (the case $q = 1$) that $\mathcal{F}(M)$ is also a module of level l over the loop algebra $L\mathfrak{sl}_n$. These two module structures can be glued together to obtain a module over LY. This is the content of our first main theorem. Before stating it, we need one definition.

Definition 4.2. *A module M over $L\mathfrak{sl}_n$ is called integrable if it is the direct sum of its integral weight spaces under the action of $\mathfrak{h}$ and if each generator $X_{i,r}^{\pm}$ acts locally nilpotently on M.*

Theorem 4.2. *Suppose that $l \geq 1, n \geq 3$ and set $\lambda = c, \beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. The functor $\mathcal{F} : M \mapsto M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ sends a right $\mathbf{H}_{t,c}$-module to an integrable $LY_{\beta,\lambda}$-module of level l (as $\mathfrak{sl}_n$-module). Furthermore, if $l + 2 < n$, this functor is an equivalence.*

Remark 4.1. *This theorem is very similar to the main result of [VaVa] where $\mathbf{H}_{t,c}$ is replaced by a double affine Hecke algebra and $LY_{\beta,\lambda}$ is replaced by a toroidal quantum algebra, under the assumption that the parameter q is not a root of unity.*

5 Proof of theorem 4.2

The proof of theorem 4.2 consists of two parts. First, we show how to obtain an integrable $LY_{\beta,\lambda}$-module structure on $\mathcal{F}(M)$, and then we prove that any integrable representation of LY of level l is of the form $\mathcal{F}(M)$. If there is no confusion possible for the values of the parameters, we will write $\mathbf{H}, \mathtt{H}, \mathsf{H}, LY$ instead of $\mathbf{H}_{t,c}, \mathtt{H}_c, \mathsf{H}_{t,c}, LY_{\beta,\lambda}$.

5.1 Proof of theorem 4.2, part 1

Fix $m \in M$, $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l} \in V^{\otimes l}$, where $\{v_1, \ldots, v_n\}$ is the standard basis of $\mathbb{C}^n$ and $1 \leq i_j \leq n$. The subalgebra $\mathfrak{sl}_n$ generated by the elements $X_i^{\pm}, H_i, 1 \leq i \leq n-1$, acts on $V^{\otimes l}$ as usual. The element $z \otimes u^{\pm} \in L\mathfrak{sl}_n$ acts on $\mathcal{F}(M)$ in the following way:

$$(z \otimes u^{\pm k})(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m x_j^{\pm k} \otimes v_{i_1} \otimes \cdots \otimes (z v_{i_j}) \otimes \cdot \otimes v_{i_l}.$$

For $z \in \mathfrak{sl}_n$, we will write $z^j(\mathbf{v})$ for $v_{i_1} \otimes \cdots \otimes (z v_{i_j}) \otimes \cdot \otimes v_{i_l}$. The elements $J(X_i^{\pm}), J(H_i)$ and $X_{i,1}^{\pm}, H_{i,1}, 1 \leq i \leq n-1$, act on $\mathcal{F}(M)$ in the following way (see [Dr1],[ChPr2]):

$$J(X_i^{\pm})(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m \mathcal{Y}_j \otimes X_i^{\pm,j}(\mathbf{v}), \quad X_{i,1}^{\pm}(m \otimes \mathbf{v}) = J(X_i^{\pm})(m \otimes \mathbf{v}) - \lambda \omega_i^{\pm}(m \otimes \mathbf{v}),$$

$$J(H_i)(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m \mathcal{Y}_j \otimes H_i^j(\mathbf{v}), \quad H_{i,1}(m \otimes \mathbf{v}) = J(H_i)(m \otimes \mathbf{v}) - \lambda \nu_i(m \otimes \mathbf{v}).$$

Following one of the main ideas in [VaVa], we define a linear automorphism T of $M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ in the following way:

$$T(m \otimes v_{i_1} \otimes \cdots \otimes v_{i_l}) = (m x_1^{-\delta_{i_1,n}} \cdots x_l^{-\delta_{i_l,n}}) \otimes v_{i_1+1} \otimes \cdots \otimes v_{i_l+1},$$

with the convention that $v_{n+1} = v_1$. (Here, δ_{ij} is the usual delta function.) One can check that $T \circ \varphi(X_{i-1}^{\pm}) = \varphi(X_i^{\pm}) \circ T$ and $T \circ \varphi(H_{i-1}) = \varphi(H_i) \circ T$ for any $0 \leq i \leq n-1$, where $\varphi : Y \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{F}(M))$ is the algebra map coming from the Y-module structure on $\mathcal{F}(M)$. Recall the automorphism ρ from section 3.

The following lemma will be crucial.

Lemma 5.1. *Let M be a module over $\mathbf{H}$. For any $2 \leq i \leq n-1$ and any $r \geq 0$, the following identities between operators on $\mathcal{F}(M)$ hold:*

$$\varphi\big(\rho(X_{i,r}^{\pm})\big) = T^{-1} \circ \varphi(X_{i,r}^{\pm}) \circ T \quad \varphi\big(\rho(H_{i,r})\big) = T^{-1} \circ \varphi(H_{i,r}) \circ T \tag{15}$$

$$\varphi\big(\rho^2(X_{1,r}^{\pm})\big) = T^{-2} \circ \varphi(X_{1,r}^{\pm}) \circ T^2 \quad \varphi\big(\rho^2(H_{1,r})\big) = T^{-2} \circ \varphi(H_{1,r}) \circ T^2 \tag{16}$$

Proof. Since the elements $X_{i,r}^{\pm}, H_{i,r}$ with $r = 0, 1$ generate LY (see equation (12)), it is enough to prove the lemma for $r = 0, 1$. First, we prove relation (15) for $X_{i,1}^{+}$ with $2 \leq i \leq n-1$. The proof for $X_{i,1}^{-}$ is exactly the same and we omit it, and the proof for $H_{i,1}$ follows from either of these two cases using identity (3). The following observation will be used repeatedly throughout this article: the action of s_{jk} on $V^{\otimes l}$ if given in terms of matrices by: $s_{jk} = \sum_{r,s=1}^{n} E_{rs}^{j} E_{sr}^{k}$.

Recall that $X_{i,1}^{+} = J(X_i^{+}) - \lambda \omega_i^{+}$. To simplify the notation, will not use φ in the proof. (We used it only to state the lemma in a convenient way.) We have to check the equality

$$(J(X_i^{+}) - \lambda \omega_i^{+})\big(T(m \otimes \mathbf{v})\big) = T\big((J(X_{i-1}^{+}) - \lambda \omega_{i-1}^{+} + \frac{\lambda}{2} X_{i-1}^{+})(m \otimes \mathbf{v})\big). \tag{17}$$

With $\mathbf{v}$ as before, suppose that $j_1 < \cdots < j_p$ are exactly the values of j for which $i_j = n$. Then $T(m \otimes \mathbf{v}) = m x_{j_1}^{-1} \cdots x_{j_p}^{-1} \otimes \mathbf{v}_{+1}$ where $\mathbf{v}_{+1} = v_{i_1+1} \otimes \cdots \otimes v_{i_l+1}$. We will use the abbreviation $x_{j_1,\ldots,j_p}^{-1}$ for the product $x_{j_1}^{-1} \cdots x_{j_p}^{-1}$.

$$
\begin{aligned}
J(X_i^+)\big(T(m \otimes \mathbf{v})\big) &= J(X_i^+)(m x_{j_1,\ldots,j_p}^{-1} \otimes \mathbf{v}_{+1}) = \sum_{k=1}^{l} m x_{j_1,\ldots,j_p}^{-1} \mathcal{Y}_k \otimes E_{i,i+1}^k(\mathbf{v}_{+1}) \\
&= \sum_{k=1}^{l} \sum_{r=1}^{p} m x_{j_1}^{-1} \cdots x_{j_{r-1}}^{-1} [x_{j_r}^{-1}, \mathcal{Y}_k] x_{j_{r+1}}^{-1} \cdots x_{j_p}^{-1} \otimes E_{i,i+1}^k(\mathbf{v}_{+1}) \qquad (18) \\
&\quad + \sum_{k=1}^{l} m \mathcal{Y}_k x_{j_1,\ldots,j_p}^{-1} \otimes E_{i,i+1}^k(\mathbf{v}_{+1}) \qquad (19)
\end{aligned}
$$

Note that $E_{i,i+1}(\mathbf{v}_{+1}) = \big(E_{i-1,i}(\mathbf{v})\big)_{+1}$ since we are assuming that $2 \leq i \leq n-1$.

$$
\sum_{k=1}^{l} m \mathcal{Y}_k x_{j_1,\ldots,j_p}^{-1} \otimes E_{i,i+1}^k(\mathbf{v}_{+1}) = \sum_{k=1}^{l} m \mathcal{Y}_k x_{j_1,\ldots,j_p}^{-1} \otimes \big(E_{i-1,i}^k(\mathbf{v})\big)_{+1} = T\big(J(X_{i-1}^+)(m \otimes \mathbf{v})\big) \qquad (20)
$$

Therefore, it follows from (18), (19) and (20) that the identity (17) that we must prove can be written as

$$
(18) - \lambda \omega_i^+\big(T(m \otimes \mathbf{v})\big) = -\lambda T\big(\omega_{i-1}^+(m \otimes \mathbf{v})\big) + \frac{\lambda}{2} T\big(X_{i-1}^+(m \otimes \mathbf{v})\big) \qquad (21)
$$

In the sum (18), note that if $k = j_s$ for some s, then $E_{i,i+1}^k(\mathbf{v}_{+1}) = 0$ since $v_{i_k+1} = v_{i_{j_s}+1} = v_1$ and $E_{i,i+1}(v_1) = 0$. Therefore, using proposition 2.3, we can simplify expression (18):

$$
(18) = \sum_{r=1}^{p} \sum_{\substack{k=1 \\ k \neq j_s \forall s}}^{l} m x_{j_1}^{-1} \cdots x_{j_{r-1}}^{-1} \left(-\frac{c}{2}\right) (x_{j_r}^{-1} + x_k^{-1}) s_{k,j_r} x_{j_{r+1}}^{-1} \cdots x_{j_p}^{-1} \otimes E_{i,i+1}^k(\mathbf{v}_{+1})
$$

$$
\begin{aligned}
(18) &= -\frac{c}{2} \sum_{r=1}^{p} \sum_{\substack{k=1 \\ k \neq j_s \forall s}}^{l} m x_{j_1}^{-1} \cdots x_{j_{r-1}}^{-1} x_{j_r}^{-1} x_{j_{r+1}}^{-1} \cdots x_{j_p}^{-1} \otimes E_{i,i+1}^{j_r}\big(s_{k,j_r}(\mathbf{v}_{+1})\big) \qquad (22) \\
&\quad -\frac{c}{2} \sum_{r=1}^{p} \sum_{\substack{k=1 \\ k \neq j_s \forall s}}^{l} m x_{j_1}^{-1} \cdots x_{j_{r-1}}^{-1} x_k^{-1} x_{j_{r+1}}^{-1} \cdots x_{j_p}^{-1} \otimes E_{i,i+1}^{j_r}\big(s_{k,j_r}(\mathbf{v}_{+1})\big) \qquad (23)
\end{aligned}
$$

We now turn to the second term on the left-hand side in (21). We distinguish two cases: when $E_{j,i+1}$ and E_{ij} act on the same tensorand, and when they act on different ones.

$$
\begin{aligned}
\lambda\omega_i^+\big(T(m\otimes\mathbf{v})\big) &= \frac{\lambda}{4}\sum_{\substack{j=1\\ j\neq i,i+1}}^{n}\sum_{k=1}^{l}\operatorname{sign}(j-i)mx^{-1}_{j_1,\dots,j_p}\otimes E^k_{i,i+1}(\mathbf{v}_{+1}) && (24)\\
&+\frac{\lambda}{2}\sum_{\substack{j=1\\ j\neq i,i+1}}^{n}\sum_{\substack{k=1\\ i_k=j-1}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}\operatorname{sign}(j-i)mx^{-1}_{j_1,\dots,j_p}\otimes E^k_{i,i+1}\big(s_{kd}(\mathbf{v}_{+1})\big) && (25)\\
&-\frac{\lambda}{4}mx^{-1}_{j_1,\dots,j_p}\otimes\big(E_{i,i+1}H_i+H_iE_{i,i+1})(\mathbf{v}_{+1}\big) && (26)
\end{aligned}
$$

The term (24) can be simplified:

$$
(24)=\frac{\lambda}{4}(n-2i)\sum_{k=1}^{l}mx^{-1}_{j_1,\dots,j_p}\otimes E^k_{i,i+1}(\mathbf{v}_{+1})=\lambda\left(\frac{n-2i}{4}\right)T\left(\sum_{k=1}^{l}m\otimes E^k_{i-1,i}(\mathbf{v})\right) \tag{27}
$$

We now consider $\lambda T\big(\omega_{i-1}^+(m\otimes v)\big)$. As for ω_i^+ above, we distinguish two cases.

$$
\begin{aligned}
\lambda T\big(\omega_{i-1}^+(m\otimes v)\big) &= T\left(\frac{\lambda}{4}\sum_{\substack{j=1\\ j\neq i-1,i}}^{n}\sum_{k=1}^{l}\operatorname{sign}(j-i+1)m\otimes E^k_{i-1,i}(\mathbf{v})\right) && (28)\\
&+T\left(\frac{\lambda}{2}\sum_{\substack{j=1\\ j\neq i-1,i}}^{n}\sum_{\substack{k=1\\ i_k=j}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}\operatorname{sign}(j-i+1)m\otimes E^k_{i-1,i}\big(s_{kd}(\mathbf{v})\big)\right) && (29)\\
&-\frac{\lambda}{4}T\big((E_{i-1,i}H_{i-1}+H_{i-1}E_{i-1,i})(m\otimes\mathbf{v})\big) && (30)
\end{aligned}
$$

The expressions (26) and (30) are identical and the difference between the sums (28) and (24) is equal to $\frac{\lambda}{2}T\left(\sum_{k=1}^{l}m\otimes E^k_{i-1,i}(\mathbf{v})\right)$ (see (27)), that is, $\frac{\lambda}{2}T\big(X^+_{i-1}(m\otimes\mathbf{v})\big)$. Therefore, the equality (21) that we have to prove simplifies to (18) = (25) − (29).

In expression (25), we consider two different cases: $j\neq 1$ and $j=1$ (hence $i_k=n$):

$$
\begin{aligned}
(25) &= \frac{\lambda}{2}\sum_{\substack{j=2\\ j\neq i,i+1}}^{n}\sum_{\substack{k=1\\ i_k=j-1}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}\operatorname{sign}(j-i)mx^{-1}_{j_1,\dots,j_p}\otimes\big(E^k_{i-1,i}(s_{kd}(\mathbf{v}))\big)_{+1} && (31)\\
&-\frac{\lambda}{2}\sum_{\substack{k=1\\ i_k=n}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}mx^{-1}_{j_1,\dots,j_p}\otimes E^k_{i,i+1}\big(s_{kd}(\mathbf{v}_{+1})\big) && (32)
\end{aligned}
$$

We can also decompose expression (29) into two sums by considering separately the cases $j\neq n$

and $j = n$.

$$(29) \quad = \quad \frac{\lambda}{2} T \left(\sum_{\substack{j=1 \\ j \neq i-1,i}}^{n-1} \sum_{\substack{k=1 \\ i_k = j}}^{l} \sum_{\substack{d=1 \\ i_d = i}}^{l} \operatorname{sign}(j-i) m \otimes E^k_{i-1,i}\big(s_{kd}(\mathbf{v})\big) \right) \tag{33}$$

$$+\frac{\lambda}{2} T \left(\sum_{\substack{k=1 \\ i_k = n}}^{l} \sum_{\substack{d=1 \\ i_d = i}}^{l} m \otimes E^k_{i-1,i}\big(s_{kd}(\mathbf{v})\big) \right) \tag{34}$$

Note that the sums (31) and (33) are equal, so the difference (25) − (29) is equal to (32) − (34). The equality (18) = (25) − (29) is now a consequence of the observation that the sums (22) and (32) are identical, and so are (23) and −(34). (Indeed, if $i_k = n$, then $k = j_r$ for some r and we have to interchange the roles of d and k to see that these expressions are indeed equal.) We also need our hypothesis that $\lambda = c$.

We now consider the case $i = 1$ in our proof of lemma 5.1. The identity (13) gives the following expression for ω_1^+ and ω_{n-1}^+:

$$\omega_1^+ = \frac{1}{4} \sum_{j=3}^{n} (E_{j2}E_{1j} + E_{1j}E_{j2}) - \frac{1}{4}(E_{12}H_1 + H_1E_{12})$$

$$\omega_{n-1}^+ = -\frac{1}{4} \sum_{j=1}^{n-2} (E_{jn}E_{n-1,j} + E_{n-1,j}E_{jn}) - \frac{1}{4}(E_{n-1,n}H_{n-1} + H_{n-1}E_{n-1,n})$$

Suppose that $j_1, \ldots, j_p$ (resp. $\gamma_1, \ldots, \gamma_e$) are exactly the values of j (resp. of γ) such that $i_j = n$ (resp. $i_\gamma = n-1$). Then $T^2(m \otimes \mathbf{v}) = m x_{j_1}^{-1} \cdots x_{j_p}^{-1} x_{\gamma_1}^{-1} \cdots x_{\gamma_e}^{-1} \otimes \mathbf{v}_{+2}$. Since $X_{1,1}^+ = J(X_1^+) - \lambda\omega_1^+$, we obtain:

$$X_{1,1}^+\big(T^2(m \otimes \mathbf{v})\big) \quad = \quad \sum_{k=1}^{l} m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \mathcal{Y}_k \otimes E^k_{12}(\mathbf{v}_{+2}) \tag{35}$$

$$-\frac{\lambda}{4} \sum_{j=3}^{n} m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes (E_{j2}E_{1j} + E_{1j}E_{j2})(\mathbf{v}_{+2}) \tag{36}$$

$$+\frac{\lambda}{4} m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes (E_{12}H_1 + H_1E_{12})(\mathbf{v}_{+2}) \tag{37}$$

In the summation (35), we can assume that $k = j_s$ for some s, since otherwise $E^k_{12}(\mathbf{v}_{+2}) = 0$.

$$(35) \quad = \quad \sum_{s=1}^{p} \sum_{u=1}^{e} m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1} \cdots x^{-1}_{\gamma_{u-1}} [x^{-1}_{\gamma_u}, \mathcal{Y}_{j_s}] x^{-1}_{\gamma_{u+1}} \cdots x^{-1}_{\gamma_e} \otimes E^{j_s}_{12}(\mathbf{v}_{+2}) \tag{38}$$

$$+\sum_{s=1}^{p} \sum_{r=1}^{p} m x^{-1}_{j_1} \cdots x^{-1}_{j_{r-1}} [x^{-1}_{j_r}, \mathcal{Y}_{j_s}] x^{-1}_{j_{r+1}} \cdots x^{-1}_{j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes E^{j_s}_{12}(\mathbf{v}_{+2}) \tag{39}$$

$$+\sum_{k=1}^{l} m \mathcal{Y}_k x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes E^k_{12}(\mathbf{v}_{+2}) \tag{40}$$

The last term is equal to $T^2\big(J(X^+_{n-1})(m\otimes\mathbf{v})\big)$. The term (37) above is equal to $\frac{\lambda}{4}T^2\big((E_{n-1,n}H_{n-1}+H_{n-1}E_{n-1,n})(m\otimes\mathbf{v})\big)$. Therefore,

$$X^+_{1,1}\big(T^2(m\otimes\mathbf{v})\big) = (38)+(39)+(36)+T^2\big(X^+_{n-1,1}(m\otimes\mathbf{v})\big)+(50).$$

(The expression (50) appears explicitly below.)

We need to decompose the sums (38) and (39).

$$\begin{aligned}
(38) &= \sum_{s=1}^{p}\sum_{u=1}^{e} mx^{-1}_{j_1,\dots,j_p}x^{-1}_{\gamma_1}\cdots x^{-1}_{\gamma_{u-1}}\left(-\frac{c}{2}\right)(x^{-1}_{\gamma_u}+x^{-1}_{j_s})s_{\gamma_u,j_s}x^{-1}_{\gamma_{u+1}}\cdots x^{-1}_{\gamma_e}\otimes E^{j_s}_{12}(\mathbf{v}_{+2})\\
&= -\frac{c}{2}\sum_{s=1}^{p}\sum_{u=1}^{e} mx^{-1}_{j_1,\dots,j_p}x^{-1}_{\gamma_1}\cdots x^{-1}_{\gamma_{u-1}}x^{-1}_{\gamma_u}x^{-1}_{\gamma_{u+1}}\cdots x^{-1}_{\gamma_e}\otimes E^{\gamma_u}_{12}\big(s_{\gamma_u,j_s}(\mathbf{v}_{+2})\big) && (41)\\
&\quad -\frac{c}{2}\sum_{s=1}^{p}\sum_{u=1}^{e} mx^{-1}_{j_1,\dots,j_p}x^{-1}_{\gamma_1}\cdots x^{-1}_{\gamma_{u-1}}x^{-1}_{j_s}x^{-1}_{\gamma_{u+1}}\cdots x^{-1}_{\gamma_e}\otimes E^{\gamma_u}_{12}\big(s_{\gamma_u,j_s}(\mathbf{v}_{+2})\big) && (42)
\end{aligned}$$

$$\begin{aligned}
(39) &= \sum_{a=1}^{p}\sum_{\substack{d=1\\ d\neq a}}^{p} mx^{-1}_{j_1}\cdots x^{-1}_{j_{d-1}}[x^{-1}_{j_d},\mathcal{Y}_{j_a}]x^{-1}_{j_{d+1}}\cdots x^{-1}_{j_p}x^{-1}_{\gamma_1,\dots,\gamma_e}\otimes E^{j_a}_{12}(\mathbf{v}_{+2})\\
&\quad +\sum_{a=1}^{p} mx^{-1}_{j_1}\cdots x^{-1}_{j_{a-1}}[x^{-1}_{j_a},\mathcal{Y}_{j_a}]x^{-1}_{j_{a+1}}\cdots x^{-1}_{j_p}x^{-1}_{\gamma_1,\dots,\gamma_e}\otimes E^{j_a}_{12}(\mathbf{v}_{+2})\\
&= \sum_{a=1}^{p}\sum_{\substack{d=1\\ d\neq a}}^{p} mx^{-1}_{j_1,\dots,j_{d-1}}\left(-\frac{c}{2}\right)(x^{-1}_{j_d}+x^{-1}_{j_a})s_{j_d,j_a}x^{-1}_{j_{d+1},\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e}\otimes E^{j_a}_{12}(\mathbf{v}_{+2})\\
&\quad +\sum_{a=1}^{p} mx^{-1}_{j_1,\dots,j_{a-1}}\left(tx^{-1}_{j_a}+\frac{c}{2}\sum_{q\neq j_a}(x^{-1}_{j_a}+x^{-1}_{q})s_{j_a,q}\right)x^{-1}_{j_{a+1},\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e}\otimes E^{j_a}_{12}(\mathbf{v}_{+2})\\
&= -\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{d=1\\ d\neq a}}^{p} mx^{-1}_{j_1,\dots,j_{d-1}}x^{-1}_{j_d}s_{j_a,j_d}(x^{-1}_{j_{d+1},\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e})\otimes E^{j_d}_{12}(s_{j_a,j_d}(\mathbf{v}_{+2})) && (43)\\
&\quad -\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{d=1\\ d\neq a}}^{p} mx^{-1}_{j_1,\dots,j_{d-1}}x^{-1}_{j_a}s_{j_d,j_a}(x^{-1}_{j_{d+1},\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e})\otimes E^{j_d}_{12}(s_{j_a,j_d}(\mathbf{v}_{+2})) && (44)\\
&\quad +t\sum_{a=1}^{p} mx^{-1}_{j_1,\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e}\otimes E^{j_a}_{12}(\mathbf{v}_{+2}) && (45)\\
&\quad +\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{q=1\\ q\neq j_a}}^{l} mx^{-1}_{j_1,\dots,j_{a-1}}x^{-1}_{j_a}s_{j_a,q}(x^{-1}_{j_{a+1},\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e})\otimes E^{q}_{12}(s_{j_a,q}(\mathbf{v}_{+2})) && (46)\\
&\quad +\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{q=1\\ q\neq j_a}}^{l} mx^{-1}_{j_1,\dots,j_{a-1}}x^{-1}_{q}s_{j_a,q}(x^{-1}_{j_{a+1},\dots,j_p}x^{-1}_{\gamma_1,\dots,\gamma_e})\otimes E^{q}_{12}(s_{j_a,q}(\mathbf{v}_{+2})) && (47)
\end{aligned}$$

We now focus on $\omega_1^+\big(T^2(m\otimes\mathbf{v})\big)$ and $T^2\big(\omega_{n-1}^+(m\otimes\mathbf{v})\big)$. We've used the equality $\omega_1^+(T^2(m\otimes\mathbf{v})) = -(36)-(37)$ earlier. We can decompose (36) by considering the cases when E_{1j} and E_{j2} act on the same tensorand and on different ones:

$$\begin{aligned}
-(36) &= \lambda\left(\frac{n-2}{4}\right) m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e}\otimes E_{12}(\mathbf{v}_{+2}) &(48)\\
&\quad +\frac{\lambda}{2}\sum_{j=3}^{n}\sum_{\substack{q=1\\ i_q+2=j}}^{l}\sum_{b=1}^{p} m x^{-1}_{\gamma_1,\ldots,\gamma_e}\otimes E^q_{12}\big(s_{q,j_b}(\mathbf{v}_{+2})\big) &(49)
\end{aligned}$$

$$\begin{aligned}
\lambda T^2\big(\omega_{n-1}^+(m\otimes\mathbf{v})\big) &= -\frac{\lambda}{4}T^2\left(\sum_{j=1}^{n-2}(E_{jn}E_{n-1,j}+E_{n-1,j}E_{jn})(m\otimes\mathbf{v})\right) &(50)\\
&\quad -\frac{\lambda}{4}T^2\big((E_{n-1,n}H_{n-1}+H_{n-1}E_{n-1,n})(m\otimes\mathbf{v})\big) &(51)
\end{aligned}$$

We observe that $-(37)=(51)$. As with (36), we can decompose (50):

$$\begin{aligned}
(50) &= -\lambda\left(\frac{n-2}{4}\right) m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e}\otimes (E_{n-1,n}(\mathbf{v}))_{+2} &(52)\\
&\quad -\frac{\lambda}{2}T^2\left(\sum_{j=1}^{n-2}\sum_{\substack{q=1\\ i_q=j}}^{l}\sum_{b=1}^{p} m\otimes E^q_{n-1,n}\big(s_{q,j_b}(\mathbf{v})\big)\right) &(53)
\end{aligned}$$

We observe that $(48)-(52)=\lambda\left(\frac{n-2}{2}\right)T^2\big(E_{n-1,n}(m\otimes\mathbf{v})\big)$.

$$(53) = -\frac{\lambda}{2}\sum_{j=1}^{n-2}\sum_{\substack{q=1\\ i_q=j}}^{l}\sum_{b=1}^{p} m x^{-1}_{j_1,\ldots,j_p}\frac{x_{j_b}}{x_q} x^{-1}_{\gamma_1,\ldots,\gamma_e}\otimes E^q_{12}(s_{q,j_b}(\mathbf{v}_{+2}))$$

To obtain the last expression, note that $i_q\neq j_a,\gamma_h$ for any a,h since $i_q=j$ and we consider values of j different from n and $n-1$.

We now decompose the sums (46) and (47) into three different sums. In the first case, $q=j_d\neq j_a$; in the second one, $q=\gamma_h$; and in the third case $q\neq j_a,\gamma_h$ for any a,h.

$$\begin{aligned}
(46) &= \frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{d=1\\ d\neq a}}^{p} m x^{-1}_{j_1,\ldots,j_{a-1}} x^{-1}_{j_a} s_{j_a,j_d}(x^{-1}_{j_{a+1},\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e})\otimes E^{j_d}_{12}\big(s_{j_a,j_d}(\mathbf{v}_{+2})\big) &(54)\\
&\quad +\frac{c}{2}\sum_{a=1}^{p}\sum_{h=1}^{e} m x^{-1}_{j_1,\ldots,j_{a-1}} x^{-1}_{j_a} x^{-1}_{j_{a+1},\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_{h-1}} x^{-1}_{j_a} x^{-1}_{\gamma_{h+1},\ldots,\gamma_e}\otimes E^{\gamma_h}_{12}(s_{j_a,\gamma_h}\big(\mathbf{v}_{+2})\big) &(55)\\
&\quad +\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{q=1\\ q\neq j_d,\gamma_h}}^{l} m x^{-1}_{j_1,\ldots,j_{a-1}} x^{-1}_{j_a} x^{-1}_{j_{a+1},\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e}\otimes E^{q}_{12}\big(s_{j_a,q}(\mathbf{v}_{+2})\big) &(56)
\end{aligned}$$

$$(47) = \frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{d=1\\ d\neq a}}^{p} m x^{-1}_{j_1,\dots,j_{a-1}} x^{-1}_{j_d} s_{j_a,j_d}(x^{-1}_{j_{a+1},\dots,j_p} x^{-1}_{\gamma_1,\dots,\gamma_e}) \otimes E_{12}^{j_d}\big(s_{j_a,j_d}(\mathbf{v}_{+2})\big) \tag{57}$$

$$+\frac{c}{2}\sum_{a=1}^{p}\sum_{h=1}^{e} m x^{-1}_{j_1,\dots,j_{a-1}} x^{-1}_{\gamma_h} x^{-1}_{j_{a+1},\dots,j_p} x^{-1}_{\gamma_1,\dots,\gamma_{h-1}} x^{-1}_{j_a} x^{-1}_{\gamma_{h+1},\dots,\gamma_e} \otimes E_{12}^{\gamma_h}\big(s_{j_a,\gamma_h}(\mathbf{v}_{+2})\big) \tag{58}$$

$$+\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{q=1\\ q\neq j_d,\gamma_h}}^{l} m x^{-1}_{j_1,\dots,j_{a-1}} x^{-1}_{q} x^{-1}_{j_{a+1},\dots,j_p} x^{-1}_{\gamma_1,\dots,\gamma_e} \otimes E_{12}^{q}\big(s_{j_a,q}(\mathbf{v}_{+2})\big) \tag{59}$$

The following equalities hold since we are assuming that $\lambda = c$:

(37) = −(51), (56) = (49), (59) = −(53), (58) = −(41), (42) = −(55), (43) = −(57), (44) = −(54)

Using our assumption that $\beta = \frac{t}{2} - \frac{\lambda(n-2)}{4}$, we can prove that $X^+_{1,1}\big(T^2(m\otimes\mathbf{v})\big) = T^2\big((X^+_{n-1,1} + 2\beta X^+_{n-1})(m\otimes\mathbf{v})\big)$:

$$X^+_{1,1}\big(T^2(m\otimes\mathbf{v})\big) - T^2\big(X^+_{n-1,1}(m\otimes\mathbf{v})\big) = (35) + (36) + (37)$$

$$\begin{aligned}
&= (38) + (39) + (41) + T^2\big(J(X^+_{n-1})(m\otimes\mathbf{v})\big) - T^2\big(X^+_{n-1,1}(m\otimes\mathbf{v})\big)\\
&= (41) + \ldots + (47) + (36) + (37) + (50) + (51)\\
&= (41) + \ldots (45) + (54) + (55) + (56) + (57) + (58) + (59) - (48) - (49)\\
&\quad +(37) + (51) + (52) + (53)\\
&= ((37) + (51)) + ((41) + (58)) + ((42) + (55)) + ((43) + (57)) + ((44) + (54)) + (45) - (48)\\
&\quad +(-(49) + (56)) + ((53) + (59)) + (52)\\
&= (45) - (48) + (52) = \left(t - \lambda\left(\frac{n-2}{2}\right)\right) T^2\big(E_{n-1,n}(m\otimes\mathbf{v})\big) = 2\beta T^2\big(X^+_{n-1}(m\otimes\mathbf{v})\big)
\end{aligned}$$

□

Using the lemma, we can now define the action of $X^\pm_{0,1}$ and of $H_{0,1}$ on $\mathcal{F}(M)$ by setting

$$X^\pm_{0,1}(m\otimes\mathbf{v}) = T^{-1}\Big(X^\pm_{1,1}\big(T(m\otimes\mathbf{v})\big)\Big) - \beta X^\pm_0(m\otimes\mathbf{v})$$

and

$$H_{0,1}(m\otimes\mathbf{v}) = T^{-1}\Big(H_{1,1}\big(T(m\otimes\mathbf{v})\big)\Big) - \beta H_0(m\otimes\mathbf{v}).$$

Note that lemma 5.1 implies that $X^\pm_{0,1}\big(T(m\otimes\mathbf{v})\big) = T\big(X^\pm_{n-1,1}(m\otimes\mathbf{v}) + \beta X^\pm_{n-1}(m\otimes\mathbf{v})\big)$ and similarly for $H_{0,1}$. In other words, and more generally, we set

$$\varphi(X^\pm_{0,r}) = T\circ\varphi\big(\rho(X^\pm_{0,r})\big)\circ T^{-1},\;\; \varphi(H_{0,r}) = T\circ\varphi\big(\rho(H_{0,r})\big)\circ T^{-1}\;\forall r\geq 0.$$

We now have to check that this indeed gives $\mathcal{F}(M)$ a structure of integrable module over LY. Choose $i,j,k\in\{0,1,\dots,n-1\}$ with $k\neq i, k\neq j$. We have to verify that $\varphi(X^\pm_{i,r}), \varphi(H_{i,r}), \varphi(X^\pm_{j,s})$

and $\varphi(H_{j,s})$ satisfy the defining relations of LY. This is true when $k=0$ from theorem 1 of [Dr1]. Using lemma 5.1, we conclude that it is also true for $k \neq 0$. This means that we have a well-defined algebra homomorphism φ from LY to $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(M))$. That $\mathcal{F}(M)$ is integrable follows from the fact that $V^{\otimes l}$ is an integrable $\mathfrak{sl}_n$-module, and that it is of level l follows from theorem 4.1 in the case of $\mathbb{C}[S_l]$ and $\mathfrak{U}\mathfrak{sl}_n$.

5.2 Proof of theorem 4.2, part 2

For the rest of this section, we assume that $l+2<n$. In the second step of the proof, we have to show that, given an integrable module $\widehat{M}$ of level l over LY, we can find a module M over $\mathbf{H}$ such that $\mathcal{F}(M)=\widehat{M}$. Integrable $\mathfrak{U}\mathfrak{sl}_n$-modules are direct sums of finite dimensional ones, so, by the results of Drinfeld [Dr1] and Chari-Pressley [ChPr1], we know that there exists modules M^1 and M^2 over, respectively, $\mathtt{H}$ and $\mathbb{C}[\widetilde{S}_l]$, such that $\widehat{M}=\mathcal{F}(M^1)$ as Y-module and $\widehat{M}=\mathcal{F}(M^2)$ as $L\mathfrak{sl}_n$-module. Since $\mathbb{C}[S_l]\subset \mathtt{H}$ and $\mathbb{C}[S_l]\subset\mathbb{C}[\widetilde{S}_l]$, we have an isomorphism $M^1\cong M^2$ of S_l-modules, so we can denote them simply by M. We have to show that M is actually a module over $\mathbf{H}$.

The following observation will be useful.

Lemma 5.2. *If $\mathbf{v}=v_{i_1}\otimes\cdots\otimes v_{i_l}$ is a generator of $V^{\otimes l}$ as a module over $\mathfrak{U}\mathfrak{sl}_n$ (that is, if $i_j\neq i_k$ for any $j\neq k$), then $m\otimes\mathbf{v}=0\Longrightarrow m=0$.*

Fix $1\leq j,k\leq l, j\neq k$. We choose $\mathbf{v}$ to be the following generator of $V^{\otimes l}$ as $\mathfrak{U}\mathfrak{sl}_n$-module: $\mathbf{v}=v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_l}$ where $i_d=d+3$ if $d<j, d\neq k$, $i_d=d+2$ if $d>j, d\neq k$, $i_j=2$ and $i_k=1$.

We can express ω_2^- as an operator on $V^{\otimes l}$ in the following way:

$$\omega_2^-|_{V^{\otimes l}}=-\frac{1}{2}\sum_{\substack{d=1\\d\neq 2,3}}^{n}\sum_{r=1}^{l}\sum_{\substack{s=1\\s\neq r}}^{l}\mathrm{sign}(2-d)(E_{3d}^rE_{d2}^s)-\left(\frac{n-2}{4}\right)\sum_{r=1}^{l}E_{32}^r-\frac{1}{2}\sum_{r=1}^{l}\sum_{\substack{s=1\\s\neq r}}^{l}E_{32}^rH_2^s$$

Therefore, $[E_{n1}^a,\omega_2^-]=-\frac{1}{2}\sum_{\substack{r=1\\r\neq a}}^{l}E_{31}^rE_{n2}^a-\frac{1}{2}\sum_{\substack{s=1\\s\neq a}}^{l}E_{31}^aE_{n2}^s$ and applying this to $m\otimes\mathbf{v}$ with $a=j,k$ gives

$$[E_{n1}^j,\omega_2^-](m\otimes\mathbf{v})=-\frac{1}{2}E_{31}^kE_{n2}^j(m\otimes\mathbf{v})\ \text{ and }\ [E_{n1}^k,\omega_2^-](m\otimes\mathbf{v})=-\frac{1}{2}E_{31}^kE_{n2}^j(m\otimes\mathbf{v}).$$

$$\begin{aligned}(X_{2,1}^-X_0^+-X_0^+X_{2,1}^-)(m\otimes\mathbf{v}) &= \sum_{r=1}^{l}\sum_{s=1}^{l}\big(mx_r\mathcal{Y}_s\otimes X_{2,0}^{-,s}E_{n1}^r(\mathbf{v})-m\mathcal{Y}_sx_r\otimes E_{n1}^rX_{2,0}^{-,s}(\mathbf{v})\big)\\ &\quad -\lambda[\omega_2^-,X_0^+](m\otimes\mathbf{v})\\ &= \sum_{r=1}^{l}\sum_{s=1}^{l}m[x_r,\mathcal{Y}_s]\otimes E_{32}^sE_{n1}^r(\mathbf{v})+\lambda\sum_{a=1}^{s}mx_a\otimes[E_{n1}^a,\omega_2^-](\mathbf{v})\end{aligned}$$

$$
\begin{aligned}
(X_{2,1}^- X_0^+ - X_0^+ X_{2,1}^-)(m\otimes \mathbf{v}) &= m[x_k,\mathcal{Y}_j]\otimes E_{32}^j E_{n1}^k(\mathbf{v}) - \frac{\lambda}{2} m x_j \otimes E_{31}^k E_{n2}^j(\mathbf{v}) \\
&\quad -\frac{\lambda}{2} m x_k \otimes E_{31}^k E_{n2}^j(\mathbf{v}) \\
&= m\big([x_k,\mathcal{Y}_j] - \frac{\lambda}{2}(x_j + x_k)s_{jk}\big)\otimes \widetilde{\mathbf{v}}
\end{aligned}
$$

where $\widetilde{\mathbf{v}} = E_{32}^j E_{n1}^k(\mathbf{v}) = v_{a_1}\otimes\cdots\otimes v_{a_l}$ with $a_d = i_d$ if $d \neq j,k,\ a_j = 3$ and $a_k = n$. We know from relation (3) that $[X_{2,1}^-, X_0^+] = 0$, so the last expression is equal to 0. Since $\widetilde{\mathbf{v}}$ is a generator of $V^{\otimes l}$ as a $\mathfrak{Usl}_n$-module, it follows, from lemma 5.2 and our assumption that $\lambda = c$, that $m\big([x_k,\mathcal{Y}_j] - \frac{c}{2}(x_j + x_k)s_{jk}\big) = 0$.

We consider now the relation between x_k and $\mathcal{Y}_k$. From the definition of ν_1:

$$
\nu_1 = \frac{1}{4}\sum_{d=3}^{n}(E_{1d}E_{d1} + E_{d1}E_{1d}) + \frac{1}{2}(E_{12}E_{21} + E_{21}E_{12}) - \frac{1}{4}\sum_{d=3}^{n}(E_{2d}E_{d2} + E_{d2}E_{2d}) - \frac{1}{2}H_1^2
$$

whence, as an operator on $V^{\otimes l}$, it is equal to

$$
\nu_1|_{V^{\otimes l}} = \frac{1}{2}\sum_{d=3}^{n}\sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}(E_{1d}^j E_{d1}^s - E_{2d}^j E_{d2}^s) + \sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l} E_{12}^j E_{21}^s - \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l} H_1^j H_1^s + \left(\frac{n-2}{4}\right)\sum_{j=1}^{l} H_1^j.
$$

Therefore,

$$
[E_{n1}^r,\nu_1] = \frac{1}{2}\sum_{d=3}^{n-1}\sum_{\substack{s=1\\ s\neq r}}^{l} E_{nd}^r E_{d1}^s + \frac{1}{2}\sum_{\substack{s=1\\ s\neq r}}^{l}(H_0^r E_{n1}^s + E_{21}^r E_{n2}^s) + \sum_{\substack{s=1\\ s\neq r}}^{l} E_{n2}^r E_{21}^s - \sum_{\substack{s=1\\ s\neq r}}^{l} E_{n1}^r H_1^s + \left(\frac{n-2}{4}\right)E_{n1}^r.
$$

Fix k, $1 \leq k \leq l$. We now choose $\mathbf{v}$ to be equal to $\mathbf{v} = v_{i_1}\otimes\cdots\otimes v_{i_l}$ with $i_d = d+2$ if $d < k$, $i_d = d+1$ if $d > k$ and $i_k = 1$. Note that $v_{i_d} \neq 2, n, n-1\ \forall d$ since $l+1 < n-1$ by assumption. Applying the previous expression for $[E_{n1}^r,\nu_1]$ to $\mathbf{v}$, we obtain the following:

$$
[E_{n1}^r,\nu_1](\mathbf{v}) = \frac{1}{2}\sum_{d=3}^{n} E_{nd}^r E_{d1}^k(\mathbf{v}) = \frac{1}{2}s_{kr}E_{n1}^k(\mathbf{v}) \text{ if } r \neq k \quad [E_{n1}^k,\nu_1](\mathbf{v}) = \left(\frac{n-2}{4}\right)E_{n1}^k(\mathbf{v}). \tag{60}
$$

Note that $H_{1,0}^s(\mathbf{v}) = 0$ if $s \neq k$. We need (60) to obtain equation (61) below.

$$
\begin{aligned}
(H_{1,1}X_0^+ - X_0^+ H_{1,1})(m\otimes\mathbf{v}) &= \sum_{r=1}^{l}\sum_{s=1}^{l} m x_r \mathcal{Y}_s \otimes H_{1,0}^s E_{n1}^r(\mathbf{v}) \\
&\quad -\sum_{s=1}^{l}\sum_{r=1}^{l} m\mathcal{Y}_s x_r \otimes E_{n1}^r H_{1,0}^s(\mathbf{v}) - \lambda[\nu_1, X_0^+](m\otimes\mathbf{v}) \\
&= -m\mathcal{Y}_k x_k \otimes E_{n1}^k H_{1,0}^k(\mathbf{v}) + \lambda\sum_{r=1}^{l} m x_r \otimes [E_{n1}^r,\nu_1](\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
(H_{1,1}X_0^+ - X_0^+H_{1,1})(m\otimes \mathbf{v}) &= -m\mathcal{Y}_k x_k \otimes E_{n1}^k(\mathbf{v}) + \frac{\lambda}{2}\sum_{\substack{r=1\\ r\neq k}}^{l} m x_r \otimes s_{kr}E_{n1}^k(\mathbf{v}) \\
&\quad +\lambda\left(\frac{n-2}{4}\right) m\otimes E_{n1}^k(\mathbf{v}) \\
&= -m\mathcal{Y}_k x_k\otimes \widetilde{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{r=1\\ r\neq k}}^{l} m x_r s_{kr}\otimes \widetilde{\mathbf{v}} + \lambda\left(\frac{n-2}{4}\right) m\otimes \widetilde{\mathbf{v}} \quad (61)
\end{aligned}
$$

where $\widetilde{\mathbf{v}} = E_{n1}^k(\mathbf{v}) = v_{a_1}\otimes\cdots\otimes v_{a_l}$ with $a_d = i_d$ if $d\neq k$ and $a_k = n$. We want to obtain a similar relation with $H_{1,1}$ replaced by $H_{n-1,1}$.

From the definition of ν_{n-1},

$$
\begin{aligned}
\nu_{n-1} &= \frac{1}{4}\sum_{d=1}^{n-2}(E_{dn}E_{nd} + E_{nd}E_{dn}) + \frac{1}{2}(E_{n-1,n}E_{n,n-1} + E_{n,n-1}E_{n-1,n}) \\
&\quad -\frac{1}{4}\sum_{d=1}^{n-2}(E_{d,n-1}E_{n-1,d} + E_{n-1,d}E_{d,n-1}) - \frac{1}{2}H_{n-1}^2
\end{aligned}
$$

whence, as an operator on $V^{\otimes l}$, it is equal to

$$
\begin{aligned}
\nu_{n-1}|_{V^{\otimes l}} &= \frac{1}{2}\sum_{d=1}^{n-2}\sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}(E_{dn}^jE_{nd}^s - E_{d,n-1}^jE_{n-1,d}^s) + \sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}(E_{n-1,n}^jE_{n,n-1}^s) \\
&\quad -\frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}H_{n-1}^jH_{n-1}^s - \left(\frac{n-2}{4}\right)\sum_{j=1}^{l}H_{n-1}^j.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[E_{n1}^r, \nu_{n-1}] &= -\frac{1}{2}\sum_{d=2}^{n-2}\sum_{\substack{s=1\\ s\neq r}}^{l}E_{d1}^rE_{nd}^s + \frac{1}{2}\sum_{\substack{s=1\\ s\neq r}}^{l}(H_0^rE_{n1}^s - E_{n,n-1}^rE_{n-1,1}^s) \\
&\quad -\sum_{\substack{s=1\\ s\neq r}}^{l}E_{n-1,1}^rE_{n,n-1}^s - \sum_{\substack{s=1\\ s\neq r}}^{l}E_{n1}^rH_{n-1}^s - \left(\frac{n-2}{4}\right)E_{n1}^r.
\end{aligned}
$$

Applying the previous expression for $[E_{n1}^r, \nu_{n-1}]$ to $\mathbf{v}$, we conclude that $[E_{n1}^r, \nu_{n-1}](\mathbf{v}) = 0$ if $r\neq k$ and

$$
[E_{n1}^k, \nu_{n-1}](\mathbf{v}) = -\frac{1}{2}\sum_{d=2}^{n-2}\sum_{\substack{s=1\\ s\neq k}}^{l}E_{d1}^kE_{nd}^s(\mathbf{v}) - \left(\frac{n-2}{4}\right)E_{n1}^k(\mathbf{v}) = -\frac{1}{2}\sum_{\substack{s=1\\ s\neq k}}^{l}s_{ks}E_{n1}^k(\mathbf{v}) - \left(\frac{n-2}{4}\right)E_{n1}^k(\mathbf{v}) \quad (62)
$$

The equation (62) allows us to compute $[H_{n-1,1}, X_0^+](m \otimes \mathbf{v})$:

$$\begin{aligned}
(H_{n-1,1}X_0^+ - X_0^+ H_{n-1,1})(m\otimes \mathbf{v}) &= \sum_{r,s=1}^{l} \big(mx_r\mathcal{Y}_s \otimes H^s_{n-1,0}E^r_{n1}(\mathbf{v}) - m\mathcal{Y}_s x_r \otimes E^r_{n1}H^s_{n-1,0}(\mathbf{v})\big) \\
&\quad -\lambda[\nu_{n-1}, X_0](m\otimes \mathbf{v}) \\
&= mx_k\mathcal{Y}_k \otimes H^k_{n-1,0}E^k_{n1}(\mathbf{v}) + \lambda \sum_{r=1}^{s} mx_r \otimes [E^r_{n1}, \nu_{n-1}](\mathbf{v}) \\
&= -mx_k\mathcal{Y}_k \otimes E^k_{n1}(\mathbf{v}) - \frac{\lambda}{2}\sum_{\substack{s=1\\ s\neq k}}^{l} mx_k\left(s_{ks} + \frac{n-2}{2}\right)\otimes E^k_{n1}(\mathbf{v}) \\
&= -mx_k\mathcal{Y}_k \otimes E^k_{n1}(\mathbf{v}) - \frac{\lambda}{2}\sum_{\substack{s=1\\ s\neq k}}^{l} mx_k\left(s_{ks} + \frac{n-2}{2}\right)\otimes \widetilde{\mathbf{v}} \qquad (63)
\end{aligned}$$

From the relations (1), (6) and (9) in LY, we know that

$$\begin{aligned}
-X^+_{0,1} &= [H_{1,1}, X_0^+] + ((\lambda-\beta)H_1X_0^+ + \beta X_0^+H_1) && (64)\\
&= [H_{n-1,1}, X_0^+] + (\beta H_{n-1}X_0^+ + (\lambda-\beta)X_0^+H_{n-1}) && (65)
\end{aligned}$$

Applying these two expressions for $-X^+_{0,1}$ to $m\otimes\mathbf{v}$, using equalities (61),(63) and the fact that $H_1X_0^+(\mathbf{v}) = 0$ and $X_0^+H_{n-1}(\mathbf{v}) = 0$ because of our choice of $\mathbf{v}$, we obtain:

$$-m\mathcal{Y}_kx_k \otimes \widetilde{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{r=1\\ r\neq k}}^{l} mx_r \otimes s_{kr}\widetilde{\mathbf{v}} + \lambda\left(\frac{n-2}{4}\right)mx_k\otimes\widetilde{\mathbf{v}} + \beta X_0^+H_1(m\otimes\mathbf{v}) =$$

$$-mx_k\mathcal{Y}_k \otimes \widetilde{\mathbf{v}} - \frac{\lambda}{2}\sum_{\substack{s=1\\ s\neq k}}^{l} mx_ks_{ks} \otimes \widetilde{\mathbf{v}} - \lambda\left(\frac{n-2}{4}\right)mx_k\otimes\widetilde{\mathbf{v}} + \beta H_{n-1}X_0^+(m\otimes\mathbf{v})$$

$$\Longrightarrow m[x_k,\mathcal{Y}_k]\otimes\widetilde{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{r=1\\ r\neq k}}^{l} m(x_r+x_k)s_{kr}\otimes\widetilde{\mathbf{v}} + \lambda\left(\frac{n-2}{2}\right)mx_k\otimes\widetilde{\mathbf{v}} + 2\beta mx_k\otimes\widetilde{\mathbf{v}} = 0$$

Since $\widetilde{\mathbf{v}}$ is a generator of $V^{\otimes l}$ as a $\mathfrak{U}\mathfrak{sl}_n$-module, it follows from lemma 5.2 and our assumptions that $2\beta + \frac{\lambda(n-2)}{2} = t, \lambda = c$ that

$$m\big([x_k,\mathcal{Y}_k] + \frac{c}{2}\sum_{\substack{r=1\\ r\neq k}}^{l}(x_r+x_k)s_{kr} + tx_k\big) = 0$$

We proved above that $m([x_k,\mathcal{Y}_j] - \frac{c}{2}(x_j+x_k)s_{jk}) = 0$ if $j\neq k$. These last two equalities imply that M is a right module over $\mathbf{H}$.

Therefore, we have shown that the $\mathtt{H}$- and the $\mathbb{C}[\widetilde{S}_l]$-module structure on M can be glued to yield a module over $\mathbf{H}$. To prove that $\mathcal{F}$ is an equivalence, we are left to show that it is fully faithful. That $\mathcal{F}$ is injective on morphisms is true because this is true for the Schur-Weyl duality functor between $\mathbb{C}[\widetilde{S}_l]$ and $\mathfrak{U}(L\mathfrak{sl}_n)$, so suppose that $f:\mathcal{F}(M_1)\longrightarrow\mathcal{F}(M_2)$ is a LY-homomorphism. From the main results of [ChPr1] and [Dr1], f is of the form $f(m_1\otimes\mathbf{v})=g(m_1)\otimes\mathbf{v},\forall m_1\in M_1$, where $g\in\mathrm{Hom}_{\mathbb{C}}(M_1,M_2)$ is a linear map which is also a homomorphism of right $\mathbb{C}[\widetilde{S}_l]$- and $\mathtt{H}$-modules. Since $\mathbf{H}$ is generated by its two subalgebras $\mathbb{C}[\widetilde{S}_l]$ and $\mathtt{H}$, g is even a homomorphism of $\mathbf{H}$-modules. Therefore, $f=\mathcal{F}(g)$ and this completes the proof of theorem 4.2. □

6 Action of the elements $X^{\pm}_{0,1}, H_{0,1}$

Now that we know that $\mathcal{F}(M)$ is a module over LY, it may be interesting to see explicitly how the elements $X^{\pm}_{0,1}$ and $H_{0,1}$ act on it. What we will discover will be useful in the next section. We will assume throughout this section that $\lambda=c$ and $\beta=\frac{t}{2}-\frac{nc}{4}+\frac{c}{2}$.

6.1 Action of $X^{+}_{0,1}$

Equations (64) and (65) yield

$$X^+_{0,1}=-\frac{1}{2}[H_{1,1}+H_{n-1,1},X^+_0]-\frac{1}{2}\big(((\lambda-\beta)H_1+\beta H_{n-1})X^+_0+X^+_0(\beta H_1+(\lambda-\beta)H_{n-1})\big).$$

We will use the notation $K_r(z)$ to denote the element $z\otimes u^r\in L\mathfrak{sl}_n$ for $z\in\mathfrak{sl}_n$; in particular, $K_1(E_{n1})=X^+_0$ and $K_{-1}(E_{1n})=X^-_0$. The element $K_r(z)$ maps to the operator in $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(M))$ given by $K_r(z)(m\otimes\mathbf{v})=\sum_{k=1}^{l}mx_k^r\otimes z^k(\mathbf{v})$. Writing $H_{1,1}$ as $H_{1,1}=J(H_1)-\lambda\nu_1$, and similarly for $H_{n-1,1}$, we can express $X^+_{0,1}$ in the following way. (We will use that $[H_{n-1}-H_1,X^+_0]=0$.)

$$\begin{aligned}
X^+_{0,1} \;=\; & -\frac{1}{2}[J(H_1+H_{n-1}),X^+_0]-\frac{\lambda}{8}\sum_{d=3}^{n-1}\big(K_1(E_{nd})E_{d1}+E_{d1}K_1(E_{nd})\big)\\
&+\frac{\lambda}{8}\Big(\big(K_1(E_{11})-K_1(E_{nn})\big)E_{n1}+E_{n1}\big(K_1(E_{11})-K_1(E_{nn})\big)\Big)+\frac{\lambda}{4}(H_1X^+_0+X^+_0H_1)\\
&-\frac{\lambda}{4}\big(K_1(E_{n2})E_{21}+E_{21}K_1(E_{n2})\big)-\frac{\lambda}{8}\big(K_1(E_{21})E_{n2}+E_{n2}K_1(E_{21})\big)\\
&+\frac{\lambda}{8}\sum_{d=2}^{n-2}\big(K_1(E_{d1})E_{nd}+E_{nd}K_1(E_{d1})\big)+\frac{\lambda}{8}\Big(\big(K_1(E_{11})-K_1(E_{nn})\big)E_{n1}\\
&+E_{n1}\big(K_1(E_{11})-K_1(E_{nn})\big)\Big)+\frac{\lambda}{4}\big(K_1(E_{n-1,1})E_{n,n-1}+E_{n,n-1}K_1(E_{n-1,1})\big)\\
&+\frac{\lambda}{8}\big(K_1(E_{n,n-1})E_{n-1,1}+E_{n-1,1}K_1(E_{n,n-1})\big)\\
&+\frac{\lambda}{4}(X^+_0H_{n-1}+H_{n-1}X^+_0)-\frac{1}{2}\big(((\lambda-\beta)H_1+\beta H_{n-1})X^+_0+X^+_0(\beta H_1+(\lambda-\beta)H_{n-1})\big)
\end{aligned}$$

$$
\begin{aligned}
X_{0,1}^{+} &= -\frac{1}{2}[J(H_1+H_{n-1}),X_0^+] - \frac{\lambda}{8}\sum_{d=3}^{n-2}\big(K_1(E_{nd})E_{d1}+E_{d1}K_1(E_{nd})\big)\\
&\quad -\frac{\lambda}{4}\big(K_1(E_{n2})E_{21}+E_{21}K_1(E_{n2})\big)+\frac{\lambda}{8}\sum_{d=3}^{n-2}\big(K_1(E_{d1})E_{nd}+E_{nd}K_1(E_{d1})\big)\\
&\quad +\frac{\lambda}{4}\Big(\big(K_1(E_{11})-K_1(E_{nn})\big)E_{n1}+E_{n1}\big(K_1(E_{11})-K_1(E_{nn})\big)\Big)\\
&\quad +\frac{\lambda}{4}\big(K_1(E_{n-1,1})E_{n,n-1}+E_{n,n-1}K_1(E_{n-1,1})\big)\\
&= -\frac{1}{2}[J(E_{11}-E_{nn}),X_0^+]+\frac{1}{2}[H_{2,1}+\cdots H_{n-2,1},X_0^+]+\frac{1}{2}[\nu_2+\cdots+\nu_{n-2},X_0^+] \qquad (66)\\
&\quad -\frac{\lambda}{8}\sum_{d=2}^{n-2}\big(K_1(E_{nd})E_{d1}+E_{d1}K_1(E_{nd})\big)+\frac{\lambda}{8}\sum_{d=3}^{n-1}\big(K_1(E_{d1})E_{nd}+E_{nd}K_1(E_{d1})\big)\\
&\quad +\frac{\lambda}{8}\big(K_1(E_{n2})E_{21}+E_{21}K_1(E_{n2})\big)-\frac{\lambda}{4}\big(K_1(H_0)E_{n1}+E_{n1}K_1(H_0)\big)\\
&\quad +\frac{\lambda}{8}\big(K_1(E_{n-1,1})E_{n,n-1}+E_{n,n-1}K_1(E_{n-1,1})\big)\\
&= \frac{1}{2}[J(E_{nn}-E_{11}),X_0^+]-\frac{\lambda}{8}\sum_{d=2}^{n-1}\big(K_1(E_{nd})E_{d1}+E_{d1}K_1(E_{nd})\big)\\
&\quad +\frac{\lambda}{8}\sum_{d=2}^{n-1}\big(K_1(E_{d1})E_{nd}+E_{nd}K_1(E_{d1})\big)-\frac{\lambda}{4}\big(K_1(H_0)E_{n1}+E_{n1}K_1(H_0)\big)\\
&= J(X_0^+)-\frac{\lambda}{8}\sum_{\epsilon\in\Delta^+}\big([X_0^+,X_\epsilon^+]X_\epsilon^-+X_\epsilon^-[X_0^+,X_\epsilon^+]\big)-\frac{\lambda}{8}\big(K_1(H_0)E_{n1}+E_{n1}K_1(H_0)\big).
\end{aligned}
$$

We define $J(X_0^+)$ to be equal to $\frac{1}{2}[J(E_{nn}-E_{11}),X_0^+]$. In line (66), we used the fact that

$$
\begin{aligned}
\frac{1}{2}[\nu_2+\cdots+\nu_{n-2},X_0^+] &= \frac{1}{8}[E_{12}E_{21}+E_{21}E_{12}-E_{1,n-1}E_{n-1,1}+E_{2n}E_{n2}+E_{n2}E_{2n}\\
&\quad -E_{n-1,1}E_{1,n-1}-E_{n,n-1}E_{n-1,n}-E_{n-1,n}E_{n,n-1},X_0^+]
\end{aligned}
$$

Recall that $H_0=E_{nn}-E_{11}=-H_1-\ldots-H_{n-1}$. Set $\widetilde{\mathcal{Y}}_j=\frac{1}{2}(x_j\mathcal{Y}_j+\mathcal{Y}_jx_j)$. The element $J(X_0^+)$ acts on $\mathcal{F}(M)$ in the following way:

$$
\begin{aligned}
J(X_0^+)(m\otimes\mathbf{v}) &= \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l}m[x_k,\mathcal{Y}_j]\otimes H_0^jE_{n1}^k(\mathbf{v})+\frac{1}{2}\sum_{j=1}^{l}m(x_j\mathcal{Y}_j+\mathcal{Y}_jx_j)\otimes E_{n1}^j(\mathbf{v})\\
&= \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l}m(x_k+x_j)s_{jk}\otimes H_0^jE_{n1}^k(\mathbf{v})+\sum_{j=1}^{l}m\widetilde{\mathcal{Y}}_j\otimes E_{n1}^j(\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
J(X_0^+)(m\otimes \mathbf{v}) &= \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_k+x_j)\otimes (E_{nn}^jE_{n1}^k - E_{n1}^jE_{11}^k)(\mathbf{v}) + \sum_{j=1}^{l} m\widetilde{\mathcal{Y}}_j\otimes E_{n1}^j(\mathbf{v})\\
&= \left(\sum_{j=1}^{l}\widetilde{\mathcal{Y}}_j\otimes E_{n1}^j + \frac{\lambda}{8}\big(X_0^+H_0 + H_0X_0^+ + E_{n1}K_1(H_0) + K_1(H_0)E_{n1}\big)\right)(m\otimes \mathbf{v})
\end{aligned}
$$

Set $\widetilde{J}(X_0^+) = J(X_0^+) - \frac{\lambda}{8}(X_0^+H_0 + H_0X_0^+ + E_{n1}K_1(H_0) + K_1(H_0)E_{n1})$, so

$$
X_{0,1}^+ = \widetilde{J}(X_0^+) - \frac{\lambda}{8}\sum_{\epsilon\in\Delta^+}\big([X_0^+, X_\epsilon^+]X_\epsilon^- + X_\epsilon^-[X_0^+, X_\epsilon^+]\big) + \frac{\lambda}{8}(X_0^+H_0 + H_0X_0^+)
$$

6.2 Action of $H_{0,1}$

We set $\epsilon_0 = -\epsilon_1 - \ldots - \epsilon_{n-1} = -\theta$ and $\epsilon^0 = (\epsilon,\epsilon_0)$ for $\epsilon\in\Delta^+$. From relation (3), we know that $[X_{0,1}^+, X_0^-] = H_{0,1}$, so

$$
\begin{aligned}
H_{0,1} &= [\widetilde{J}(X_0^+), X_0^-] + \frac{\lambda}{4}H_0^2 - \frac{\lambda}{4}(X_0^+X_0^- + X_0^-X_0^+) + \frac{\lambda}{4}(E_{1n}E_{n1} + E_{n1}E_{1n})\\
&\quad -\frac{\lambda}{4}K_1(H_0)K_{-1}(H_0) - \frac{\lambda}{8}\sum_{\substack{\epsilon\in\Delta^+\\ \epsilon\neq-\epsilon_0}}(\epsilon,\epsilon_0)(X_\epsilon^+X_\epsilon^- + X_\epsilon^-X_\epsilon^+)\\
&\quad +\frac{\lambda}{8}\sum_{\substack{\epsilon\in\Delta^+\\ \epsilon\neq-\epsilon_0}}(\epsilon,\epsilon_0)(K_{\epsilon^0}(X_\epsilon^+)K_{-\epsilon_0}(X_\epsilon^-) + K_{-\epsilon^0}(X_\epsilon^-)K_{\epsilon^0}(X_\epsilon^+))\\
&= [\widetilde{J}(X_0^+), X_0^-] - \lambda\widetilde{\nu}_0
\end{aligned} \tag{67}
$$

where

$$
\begin{aligned}
\widetilde{\nu}_0 &= \frac{1}{8}\sum_{\epsilon\in\Delta^+}(\epsilon,\epsilon_0)\big(X_\epsilon^+X_\epsilon^- + X_\epsilon^-X_\epsilon^+ - K_{\epsilon^0}(X_\epsilon^+)K_{-\epsilon^0}(X_\epsilon^-) - K_{-\epsilon^0}(X_\epsilon^-)K_{\epsilon^0}(X_\epsilon^+)\big)\\
&\quad -\frac{1}{4}H_0^2 + \frac{1}{4}K_1(H_0)K_{-1}(H_0)
\end{aligned}
$$

We can use this to find how $H_{0,1}$ acts on $\mathcal{F}(M)$, so let us see explicitly how $[\widetilde{J}(X_0^+), X_0^-]$ acts on $\mathcal{F}(M)$:

$$
\begin{aligned}
[\widetilde{J}(X_0^+), X_0^-](m\otimes \mathbf{v}) &= \sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m[x_k^{-1}, \widetilde{\mathcal{Y}}_j]\otimes E_{n1}^jE_{1n}^k(\mathbf{v})\\
&\quad + \sum_{j=1}^{l} m x_j^{-1}\widetilde{\mathcal{Y}}_j\otimes E_{n1}^jE_{1n}^j(\mathbf{v}) - \sum_{j=1}^{l} m\widetilde{\mathcal{Y}}_j x_j^{-1}\otimes E_{1n}^jE_{n1}^j(\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
[\widetilde{J}(X_0^+), X_0^-](m\otimes\mathbf{v}) &= -\frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j+x_k)(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{n1}^{j}E_{1n}^{k}(\mathbf{v})\\
&\quad +\frac{1}{2}\sum_{j=1}^{l} m(\mathcal{Y}_j + x_j^{-1}\mathcal{Y}_j x_j)\otimes E_{nn}^{j}(\mathbf{v}) - \frac{1}{2}\sum_{j=1}^{l} m(x_j\mathcal{Y}_j x_j^{-1}+\mathcal{Y}_j)\otimes E_{11}^{j}(\mathbf{v})\\
&= -\frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(2+x_j^{-1}x_k + x_k^{-1}x_j)\otimes E_{11}^{j}E_{nn}^{k}(\mathbf{v}) + \sum_{j=1}^{l} m\mathcal{Y}_j\otimes H_0(\mathbf{v})\\
&\quad +\frac{1}{2}\sum_{j=1}^{l} m[x_j^{-1},\mathcal{Y}_j]x_j\otimes E_{nn}^{j}(\mathbf{v}) - \frac{1}{2}\sum_{j=1}^{l} mx_j[\mathcal{Y}_j, x_j^{-1}]\otimes E_{11}^{j}(\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(2+x_j^{-1}x_k + x_k^{-1}x_j)\otimes E_{11}^{j}E_{nn}^{k}(\mathbf{v}) + \sum_{j=1}^{l} m\mathcal{Y}_j\otimes H_0(\mathbf{v})\\
&\quad +\frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} mx_k(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{nn}^{j}(\mathbf{v})\\
&\quad +\frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} mx_j(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{11}^{j}(\mathbf{v}) + \frac{t}{2}(E_{nn}+E_{11})(m\otimes\mathbf{v})\\
&= -\lambda\big(\frac{1}{2}E_{11}E_{nn} + \frac{1}{4}K_{-1}(E_{11})K_1(E_{nn}) + \frac{1}{4}K_{-1}(E_{nn})K_1(E_{11})\big)(m\otimes\mathbf{v})\\
&\quad +\sum_{j=1}^{l} m\mathcal{Y}_j\otimes H_0(\mathbf{v}) - \frac{\lambda n}{4}E_{nn}(m\otimes\mathbf{v}) - \frac{\lambda}{2}\sum_{d=1}^{n}E_{dd}(m\otimes\mathbf{v}) - \frac{\lambda n}{4}E_{11}(m\otimes\mathbf{v})\\
&\quad +\frac{\lambda}{8}\sum_{d=1}^{n}\big(E_{nd}E_{dn}+E_{dn}E_{nd}+K_1(E_{nd})K_{-1}(E_{dn})+K_{-1}(E_{dn})K_1(E_{nd})\big)(m\otimes\mathbf{v})\\
&\quad +\frac{\lambda}{8}\sum_{d=1}^{n}\big(E_{1d}E_{d1}+E_{d1}E_{1d}+K_{-1}(E_{1d})K_1(E_{d1})+K_{-1}(E_{d1})K_{-1}(E_{1d})\big)(m\otimes\mathbf{v})\\
&\quad +\frac{t}{2}(E_{nn}+E_{11})(m\otimes\mathbf{v}) \qquad (68)
\end{aligned}
$$

Putting together equations (67) and (68), we conclude that $H_{0,1}$ acts on $\mathcal{F}(M)$ in the following way:

$$
\begin{aligned}
H_{0,1}(m\otimes\mathbf{v}) &= J(H_0)(m\otimes\mathbf{v}) - \frac{\lambda}{2}\sum_{d=1}^{n}E_{dd}(m\otimes\mathbf{v}) + \frac{\lambda}{2}(E_{n1}E_{1n}+E_{1n}E_{n1})(m\otimes\mathbf{v})\\
&\quad +\frac{\lambda}{4}\sum_{d=2}^{n-1}(E_{nd}E_{dn}+E_{dn}E_{nd}+E_{1d}E_{d1}+E_{d1}E_{1d})(m\otimes\mathbf{v})\\
&\quad +\frac{\lambda}{2}H_0^2(m\otimes\mathbf{v}) + (\beta-\frac{\lambda}{2})(E_{11}+E_{nn})(m\otimes\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
H_{0,1}(m\otimes \mathbf{v}) &= J(H_0)(m\otimes \mathbf{v}) - \lambda\nu_0(m\otimes \mathbf{v}) - l\left(\frac{2t}{n}-\lambda\right)(m\otimes \mathbf{v})\\
&\quad +(\beta-\frac{\lambda}{2})\sum_{j=1}^{n}(\frac{E_{11}-E_{jj}}{n}+\frac{E_{nn}-E_{jj}}{n})(m\otimes \mathbf{v})
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_0 &= -\frac{1}{4}\sum_{d=2}^{n-1}(E_{nd}E_{dn}+E_{dn}E_{nd}+E_{1d}E_{d1}+E_{d1}E_{1d}) - \frac{1}{2}(E_{n1}E_{1n}+E_{1n}E_{n1}) - \frac{1}{2}H_0^2\\
&= \frac{1}{4}\sum_{\epsilon\in\Delta^+}(\epsilon,\epsilon_0)(X_\epsilon^+X_\epsilon^- + X_\epsilon^+X_\epsilon^-) - \frac{1}{2}H_0^2.
\end{aligned}
$$

6.3 Action of $X_{0,1}^-$

Our goal now is to find elements that act on $\mathcal{F}(M)$ by $m\otimes \mathbf{v}\mapsto \sum_{j=1}^{l} my_j\otimes E^j(\mathbf{v}), E\in\mathfrak{sl}_n$. Set $Y_0^+ = \frac{1}{2}[X_0^-,\widetilde{H}_{0,1}]$ where $\widetilde{H}_{0,1} = J(H_0)+\lambda\xi_0$ and

$$
\xi_0 = \frac{1}{4}\sum_{d=2}^{n-1}(E_{1d}E_{d1}+E_{d1}E_{1d}+E_{nd}E_{dn}+E_{dn}E_{nd}) + \frac{1}{2}(E_{n1}E_{1n}+E_{1n}E_{n1})
$$

$$
\begin{aligned}
Y_0^+(m\otimes \mathbf{v}) &= \frac{1}{2}[X_0^-,J(H_0)](m\otimes \mathbf{v}) + \frac{\lambda}{2}[X_0^-,\xi_0](m\otimes \mathbf{v}) = \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m[\mathcal{Y}_j,x_k^{-1}]\otimes E_{1n}^kH_0^j(\mathbf{v})\\
&\quad +\frac{1}{2}\sum_{j=1}^{l} m(\mathcal{Y}_jx_j^{-1}+x_j^{-1}\mathcal{Y}_j)\otimes E_{1n}^j(\mathbf{v}) + \frac{\lambda}{2}[X_0^-,\xi_0](m\otimes \mathbf{v})\\
&= \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{1n}^kH_0^j(\mathbf{v}) + \frac{1}{2}\sum_{j=1}^{l} my_j\otimes E_{1n}^j(\mathbf{v})\\
&\quad +\frac{1}{4}\sum_{j=1}^{l} m(x_jy_jx_j^{-1}+x_j^{-1}y_jx_j)\otimes E_{1n}^j + \frac{\lambda}{2}[X_0^-,\xi_0](m\otimes \mathbf{v})\\
&= \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{1n}^kH_0^j(\mathbf{v}) + \sum_{j=1}^{l} my_j\otimes E_{1n}^j(\mathbf{v})\\
&\quad +\frac{1}{4}\sum_{j=1}^{l} m(x_j[y_j,x_j^{-1}]+[x_j^{-1},y_j]x_j)\otimes E_{1n}^j(\mathbf{v}) + \frac{\lambda}{2}[X_0^-,\xi_0](m\otimes \mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{1n}^{k}H_0^{j}(\mathbf{v}) + \sum_{j=1}^{l} my_j\otimes E_{1n}^{j}(\mathbf{v}) + \frac{\lambda}{2}[X_0^-,\xi_0](m\otimes\mathbf{v}) \\
& +\frac{1}{4}\sum_{j=1}^{l} m\Big(x_j\big(-tx_j^{-2}-\lambda\sum_{k\neq j}x_j^{-1}x_k^{-1}s_{jk}\big)+\big(tx_j^{-2}+\lambda\sum_{k\neq j}x_j^{-1}x_k^{-1}s_{jk}\big)x_j\Big)\otimes E_{1n}^{j}(\mathbf{v}) \\
= \;& \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j^{-1}+x_k^{-1})s_{jk}\otimes E_{1n}^{k}H_0^{j}(\mathbf{v}) + \sum_{j=1}^{l} my_j\otimes E_{1n}^{j}(\mathbf{v}) \\
& +\frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j^{-1}-x_k^{-1})s_{jk}\otimes E_{1n}^{j}(\mathbf{v}) + \frac{\lambda}{2}[X_0^-,\xi_0](m\otimes\mathbf{v}) \\
= \;& \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_j^{-1}+x_k^{-1})\otimes(E_{nn}^{k}E_{1n}^{j}-E_{1n}^{k}E_{11}^{j})(\mathbf{v}) \\
& +\sum_{j=1}^{l} my_j\otimes E_{1n}^{j}(\mathbf{v}) + \frac{\lambda}{2}[X_0^{-1},\xi_0](m\otimes\mathbf{v}) + \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l}\sum_{d=1}^{n} m(x_j^{-1}-x_k^{-1})\otimes E_{1d}^{k}E_{dn}^{j} \\
= \;& \frac{\lambda}{8}\big(E_{nn}K_{-1}(E_{1n})+K_{-1}(E_{1n})E_{nn}+K_{-1}(E_{nn})E_{1n}+E_{1n}K_{-1}(E_{nn})\big)(m\otimes\mathbf{v}) \\
& -\frac{\lambda}{8}\big(E_{1n}K_{-1}(E_{11})+K_{-1}(E_{11})E_{1n}+K_{-1}(E_{1n})E_{11}+E_{11}K_{-1}(E_{1n})\big)(m\otimes\mathbf{v}) \\
& +\sum_{j=1}^{l} my_j\otimes E_{1n}^{j}(\mathbf{v}) + \frac{\lambda}{2}[X_0^-,\xi_0] \\
& +\frac{\lambda}{8}\sum_{d=1}^{n}\big(E_{1d}K_{-1}(E_{dn})+K_{-1}(E_{dn})E_{1d}\big)(m\otimes\mathbf{v}) - \frac{n\lambda}{8}K_{-1}(E_{1n})(m\otimes\mathbf{v}) \\
& -\frac{\lambda}{8}\sum_{d=1}^{n}\big(K_{-1}(E_{1d})E_{dn}+E_{dn}K_{-1}(E_{1d})\big)(m\otimes\mathbf{v}) + \frac{n\lambda}{8}K_{-1}(E_{1n})(m\otimes\mathbf{v}) \\
= \;& \sum_{j=1}^{l} my_j\otimes E_{1n}^{j}(\mathbf{v})
\end{aligned}
$$

It is actually easy to express Y_0^+ in terms of the generators of LY given in definition 3.2. Since $H_0 = -H_1 - \ldots - H_{n-1}$, we can write

$$
\begin{aligned}
\widetilde{H}_{0,1} &= J(H_0)+\lambda\xi_0 = -J(H_1)-\ldots-J(H_{n-1})+\lambda\xi_0 \\
&= -(H_{1,1}+\ldots H_{n-1,1})-\lambda(\nu_1+\ldots+\nu_{n-1})+\lambda\xi_0 \\
&= -(H_{1,1}+\ldots H_{n-1,1})+\frac{\lambda}{2}(H_1^2+\ldots+H_{n-1}^2)
\end{aligned}
$$

Therefore, using relations (6) and (9), we obtain

$$\begin{aligned}
Y_0^+ &= \frac{1}{2}[X_0^-, \widetilde{H}_{0,1}] \\
&= -\frac{1}{2}[X_0^-, H_{1,1}] - \frac{1}{2}[X_0^-, H_{n-1,1}] - \frac{\lambda}{4}(X_0^- H_1 + H_1 X_0^-) - \frac{\lambda}{4}(X_0^- H_{n-1} + H_{n-1} X_0^-) \\
&= \frac{1}{2}X_{0,1}^- + \frac{1}{2}(\beta H_1 X_0^- + (\lambda-\beta) X_0^- H_1) + \frac{1}{2}X_{0,1}^- + \frac{1}{2}((\lambda-\beta) H_{n-1} X_0^- + \beta X_0^- H_{n-1}) \\
&\quad -\frac{\lambda}{4}(X_0^- H_1 + H_1 X_0^-) - \frac{\lambda}{4}(X_0^- H_{n-1} + H_{n-1} X_0^-) \\
&= X_{0,1}^-
\end{aligned}$$

The element $X_{0,1}^-$ will become important in the next section.

It can be seen from the relations in LY that the subalgebra of LY generated by Y_0^+ and $\mathfrak{sl}_n$ is isomorphic to $\mathfrak{U}(\mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u])$. (See also proposition 7.1 below.) We introduce the notation $Q_r(z), r \in \mathbb{Z}_{\geq 0}$, to denote $z \otimes u^r$ as an element of this subalgebra; in particular, $Q_1(E_{1n}) = Y_0^+$. There are three types of operators in $\mathrm{End}_{\mathbb{C}}(M \otimes_{\mathbb{C}[S_l]} V^{\otimes l})$ which are of particular interest to us: those coming from the action of $J(z), K_r(z)$ and $Q_r(z)$. They are related to each other in the following way.

Proposition 6.1 (See also [BHW]). *Suppose that $a \neq b$ and $c \neq d$. Then we have the equality $[Q_1(E_{ab}), K_1(E_{cd})] + [K_1(E_{ab}), Q_1(E_{cd})] = 2\big(\delta_{bc} J(E_{ad}) - \delta_{da} J(E_{cb})\big)$.*

Proof. First, we will prove the equality

$$[Q_1(E_{1n}), K_1(H_0)] + [K_1(E_{1n}), Q_1(H_0)] = 4J(E_{1n}) \tag{69}$$

$$\begin{aligned}
[Q_1(E_{1n}), K_1(H_0)] + [K_1(E_{1n}), Q_1(H_0)] &= \big[X_{0,1}^-, [X_0^+, E_{1n}]\big] + \frac{1}{2}\Big[\big[E_{1n}, [X_0^+, E_{1n}]\big], [E_{n1}, X_{0,1}^-]\Big] \\
&= -[H_{0,1}, E_{1n}] - \frac{1}{2}\Big[\big[H_0, [X_0^+, E_{1n}]\big], X_{0,1}^-\Big] \\
&\quad -\frac{1}{2}\Big[\big[E_{1n}, [X_0^+, H_0]\big], X_{0,1}^-\Big] \\
&\quad +\frac{1}{2}\Big[E_{n1}, \big[E_{1n}, [[X_0^+, X_{0,1}^-], E_{1n}]\big]\Big] \\
&= -[H_{0,1}, E_{1n}] + \big[[E_{1n}, X_0^+], X_{0,1}^-\big] \\
&\quad +\frac{1}{2}\Big[E_{n1}, \big[E_{1n}, [H_{0,1}, E_{1n}]\big]\Big] \\
&= 2[E_{1n}, H_{0,1}] + \frac{1}{2}\Big[E_{n1}, \big[E_{1n}, [H_{0,1}, E_{1n}]\big]\Big] \qquad (70)
\end{aligned}$$

$$\begin{aligned}
[H_{0,1}, E_{1n}] &= \big[[H_{0,1}, E_{12}], E_{2n}\big] + \Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, [H_{0,1}, E_{n-1,n}] \cdots\Big] \\
&= [-X_{1,1}^+ - (\beta H_0 X_1^+ + (\lambda-\beta) X_1^+ H_0), E_{2n}] \\
&\quad +\Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, -X_{n-1,1}^+ - ((\lambda-\beta) H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)] \cdots\Big]
\end{aligned}$$

$$
\begin{aligned}
[H_{0,1}, E_{1n}] &= [-J(X_1^+) + \lambda\omega_1^+ - (\beta H_0 X_1^+ + (\lambda-\beta)X_1^+ H_0), E_{2n}] + \\
&\quad \Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, -J(X_{n-1}^+) + \lambda\omega_{n-1}^+ - ((\lambda-\beta)H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)] \cdot \Big]
\end{aligned}
$$

$$
\begin{aligned}
[H_{0,1}, E_{1n}] &= -J(E_{1n}) - (\beta H_0 E_{1n} + (\lambda-\beta)E_{1n}H_0) + (\beta E_{2n}E_{12} + (\lambda-\beta)E_{12}E_{2n}) \\
&\quad +\lambda[\omega_1^+, E_{2n}] - J(E_{1n}) + \lambda[E_{1,n-1}, \omega_{n-1}^+] \\
&\quad -\Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, ((\lambda-\beta)H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)\big] \cdots \Big]
\end{aligned} \tag{71}
$$

The expression $\Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, ((\lambda-\beta)H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)\big] \cdots \Big]$ is equal to

$$
[E_{12}, (\lambda-\beta)H_0E_{2n} + \beta E_{2n}H_0] = ((\lambda-\beta)E_{12}E_{2n} + \beta E_{2n}E_{12}) + ((\lambda-\beta)H_0E_{1n} + \beta E_{1n}H_0) \tag{72}
$$

$$
\begin{aligned}
[E_{1,n-1}, \omega_{n-1}^+] &= -\frac{1}{4}\sum_{j=2}^{n-2}(E_{jn}E_{1j} + E_{1j}E_{jn}) - \frac{1}{4}\big(E_{1n}(E_{11} - E_{n-1,n-1}) + (E_{11} - E_{n-1,n-1})E_{1n}\big) \\
&\quad -\frac{1}{4}(E_{1n}H_{n-1} + H_{n-1}E_{1n}) - \frac{1}{4}(E_{n-1,n}E_{1,n-1} + E_{1,n-1}E_{n-1,n}) \\
&= -\frac{1}{4}\sum_{j=2}^{n-1}(E_{jn}E_{1j} + E_{1j}E_{jn}) + \frac{1}{4}(E_{1n}H_0 + H_0E_{1n})
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
[\omega_1^+, E_{2n}] &= \frac{1}{4}\sum_{j=3}^{n-1}(E_{jn}E_{1j} + E_{1j}E_{jn}) - \frac{1}{4}\big((E_{22} - E_{nn})E_{1n} + E_{1n}(E_{22} - E_{nn})\big) \\
&\quad -\frac{1}{4}(E_{1n}H_1 + H_1E_{1n}) + \frac{1}{4}(E_{12}E_{2n} + E_{2n}E_{12}) \\
&= \frac{1}{4}\sum_{j=2}^{n-1}(E_{jn}E_{1j} + E_{1j}E_{jn}) + \frac{1}{4}(E_{1n}H_0 + H_0E_{1n})
\end{aligned} \tag{74}
$$

Therefore, combining equations (71),(73),(74) and (72), we obtain the following simple expression for $[H_{0,1}, E_{1n}]$:

$$
[H_{0,1}, E_{1n}] = -2J(E_{1n}) - \frac{\lambda}{2}(H_0E_{1n} + E_{1n}H_0) \tag{75}
$$

Putting together equations (70) and (75) yields equality (69):

$$
\begin{aligned}
[Q(E_{1n}), K_1(H_0)] + [K_1(E_{1n}), Q(H_0)] &= 4J(E_{1n}) + \lambda(H_0E_{1n} + E_{1n}H_0) \\
&\quad -\frac{1}{2}[E_{n1}, [E_{1n}, 2J(E_{1n}) + \frac{\lambda}{2}(H_0E_{1n} + E_{1n}H_0)]] \\
&= 4J(E_{1n}) + \lambda(H_0E_{1n} + E_{1n}H_0) - \lambda[E_{n1}, E_{1n}^2] \\
&= 4J(E_{1n})
\end{aligned}
$$

Applying $[E_{n1},\cdot]$ to both sides of equation (69) yields

$$[K_1(E_{n1}),Q_1(E_{1n})]+[Q_1(E_{n1}),K_1(E_{1n})]=2J(H_0) \tag{76}$$

This proves proposition 6.1 when $a=n,b=1,c=1,d=n$. If $a=n,b=c=1,d\neq 1,n$, we apply $[\cdot,E_{nd}]$ to the previous equation to obtain $[K_1(E_{n1}),Q_1(E_{1d})]+[Q_1(E_{n1}),K_1(E_{1d})]=2J(E_{nd})$. To obtain equation (6.1) for $a=n,b=1,c\neq 1,d,d\neq n,1$, we use $[E_{c1},\cdot]$, which yields $[K_1(E_{n1}),Q_1(E_{cd})]+[Q_1(E_{n1}),K_1(E_{cd})]=0$. Under the assumption $c\neq 1$, we apply $[\cdot,E_{d1}]$ to the previous equation and get $[K_1(E_{n1}),Q_1(E_{c1})]+[Q_1(E_{n1}),K_1(E_{c1})]=0$. Apply $[E_{c1},\cdot]$ if $c\neq 1,n$ to (76) give $[K_1(E_{n1}),Q_1(E_{cn})]+[Q_1(E_{n1}),K_1(E_{cn})]=-2J(E_{c1})$. We have covered all the cases with $a=n,b=1$.

Now suppose that $a=n$ but $\tilde{b}\neq 1$. Equation (6.1) in the case $a=n,b=c=1,,b\neq n,d\neq 1$ along with $[\cdot,E_{1\tilde{b}}]$ yields $[K_1(E_{n\tilde{b}}),Q_1(E_{1d})]+[Q_1(E_{n\tilde{b}}),K_1(E_{1d})]=-2\delta_{ad}J(E_{1\tilde{b}})$. To recover the case when $c\neq 1,\tilde{b}$, we apply $[E_{c1},\cdot]$ to the previous equation, so we have shown the equality

$$[K_1(E_{n\tilde{b}}),Q_1(E_{cd})]+[Q_1(E_{n\tilde{b}}),K_1(E_{cd})]=-2\delta_{ad}J(E_{c\tilde{b}}). \tag{77}$$

Using equation (77) along with $[E_{\tilde{b}c},\cdot]$ yields equation (6.1) in the case $a=n$ and $b=c,d$ are arbitrary. The remaining cases can be obtained using similar computations. □

7 Schur-Weyl dual of the rational Cherednik algebra

Our goal in this section is to establish an equivalence of categories for the rational Cherednik algebra similar to the one given in theorem 4.2 and to identify the Schur-Weyl dual of H with a subalgebra of LY.

7.1 Case of type $\mathfrak{gl}_l$

Definition 7.1. *The subalgebra of LY generated by $X_i^{\pm},1\leq i\leq n-1,X_0^+$ and Y_0^+ is denoted by $\mathbb{L}_{\beta,\lambda}$ and called a Yangian deformed double-loop algebra, as suggested in [BHW]. The polynomial loop algebra generated by $X_i^{\pm},1\leq i\leq n-1$ and X_0^+ (resp. Y_0^+) is denoted L_X (resp. L_Y).*

Remark 7.1. *The algebra $LY_{\beta,\lambda}$ is the same as the subalgebra generated by $z,K_1(z),Q_1(z),\forall z\in\mathfrak{sl}_n$. Furthermore, proposition 6.1 implies that $\mathbb{L}_{\beta,\lambda}$ contains all the elements $X_{i,r}^{\pm},H_{i,r}$ for $1\leq i\leq n,r\geq 0$ and relation (12) shows that it also contains $X_{0,r}^+,\forall r\geq 0$ and $X_{0,r}^-,\forall r\geq 1$. We will abbreviate $\mathbb{L}_{\beta,\lambda}$ by $\mathbb{L}$.*

The computations for the action of $X_{0,1}^-$ on $M\otimes_{\mathbb{C}[S_l]}V^l$ and the anti-symmetric role of $\mathfrak{h}$ and $\mathfrak{h}^*$ in the definition of $\mathsf{H}_{t,c}$, along with the last proposition of the previous section, suggest that the following result is true.

Proposition 7.1. *There exists an anti-involution ι of $\mathbb{L}$ which interchanges L_X and L_Y and which is given on the generators by the formulas*

$$\iota(X_{i,r}^{\pm})=X_{i,r}^{\mp}\ \textit{if}\ i\neq 0,\ \ \iota(H_{i,r})=H_{i,r}$$

$$\iota(X^+_{0,r}) = X^-_{0,r+1} \text{ for } r \geq 0, \quad \iota(X^-_{0,r}) = X^+_{0,r-1} \text{ for } r \geq 1$$

Proof. This can be checked using the relations given in definition 3.2. ☐

Theorem 7.1. *Suppose that $l \geq 1, n \geq 3$. Set $\lambda = c$ and $\beta = \frac{t}{2} - \frac{cn}{4} + \frac{c}{2}$. The functor $M \mapsto M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ sends a right H-module to an integrable left $\mathbb{L}$-module of level l. Furthermore, if $l+2 < n$, this functor is an equivalence.*

Proof. As for theorem 4.2, the proof is in two parts. First, it is enough to take $M = \mathsf{H}$ and show that $\mathcal{F}(M)$ is a module over $\mathbb{L}$. We can view $\mathsf{H} \otimes_{\mathbb{C}[S_l]} \mathbb{C}^{\otimes l}$ as a subspace of $\mathbf{H} \otimes_{\mathbb{C}[S_l]} \mathbb{C}^{\otimes l}$; the later is a module over $\mathbb{L}$ since it is even a module over LY. The subspace $\mathcal{F}(\mathsf{H})$ is stable under the action of the subalgebras L_X and L_Y, so it is a module over the subalgebra of LY generated by L_X and L_Y, which is exactly $\mathbb{L}$. The fact that $\mathcal{F}(M)$ is integrable of level l follows from the same argument as in the proof of theorem 4.2.

Now let N be an integrable module of level l over $\mathbb{L}$ and suppose that $l+2 < n$. We have to show that there exists a module M over H such that $\mathcal{F}(M) = N$. We can argue as for the trigonometric case to conclude that there exists an S_l-module M, which is also a $\mathbb{C}[\mathfrak{h}] \rtimes W$- and a $\mathbb{C}[\mathfrak{h}^*] \rtimes W$-module, such that $\mathcal{F}(M) \cong N$. As before, we must show that M is actually a module over H.

Fix $1 \leq j, k \leq l, j \neq k$. Choose $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ such that $i_k = 2, i_j = n-1$, $i_r = r+2$ if $r < j, r \neq k$, $i_r = r+1$ if $r > j, r \neq k$. Set $\widetilde{\mathbf{v}} = E^k_{n2} E^j_{1,n-1}(\mathbf{v})$.

On one hand,

$$\begin{aligned} &\big(Q_1(E_{1,n-1})K_1(E_{n2}) - K_1(E_{n2})Q_1(E_{1,n-1})\big)(m \otimes \mathbf{v}) = \\ &\sum_{s=1}^{l}\sum_{r=1}^{l} m x_r y_s \otimes E^s_{1,n-1}E^r_{n2}(\mathbf{v}) - \sum_{s=1}^{l}\sum_{r=1}^{l} m y_s x_r \otimes E^r_{n2}E^s_{1,n-1}(\mathbf{v}) \quad = \quad m(x_k y_j - y_j x_k) \otimes \widetilde{\mathbf{v}} \end{aligned} \tag{78}$$

On the other hand, $Q_1(E_{1,n-1}) = [Y_0^+, E_{n,n-1}]$ and $K_1(E_{n2}) = [X_0^+, E_{12}]$, so:

$$\begin{aligned} [Q_1(E_{1,n-1}), K_1(E_{n2})] &= \big[[Y_0^+, E_{n,n-1}], [X_0^+, E_{12}]\big] = \big[[X^-_{0,1}, [X_0^+, E_{12}]], E_{n,n-1}\big] \\ &= \Big[\big[[X^-_{0,1}, X_0^+], X_1^+\big], E_{n,n-1}\Big] = -\big[[H_{0,1}, X_1^+], E_{n,n-1}\big] \\ &= -[-X^+_{1,1} - (\beta H_0 X_1^+ + (\lambda-\beta)X_1^+ H_0), X^-_{n-1}] \\ &= [\beta H_0 X_1^+ + (\lambda-\beta)X_1^+ H_0, X^-_{n-1}] \\ &= \beta E_{n,n-1}E_{12} + (\lambda-\beta)E_{12}E_{n,n-1} \end{aligned}$$

Therefore,

$$\begin{aligned} [Q_1(E_{1,n-1}), K_1(E_{n2})](m \otimes \mathbf{v}) &= m \otimes (\beta E_{n,n-1}E_{12} + (\lambda-\beta)E_{12}E_{n,n-1})(\mathbf{v}) \\ &= \lambda m \otimes E^k_{12}E^j_{n,n-1}(\mathbf{v}) = \lambda m s_{jk} \otimes \widetilde{\mathbf{v}} \end{aligned} \tag{79}$$

Equations (78) and (79) imply that $m(x_k y_j - y_j x_k - \lambda s_{jk}) \otimes \widetilde{\mathbf{v}} = 0$. From lemma 5.2 and our assumption that $\lambda = c$, we conclude that

$$m(x_k y_j - y_j x_k - c s_{jk}) = 0. \tag{80}$$

Now let $\mathbf{v}$ be determined by $i_k = n-1$, $i_j = j+1$ if $j \neq k$. Set $\widehat{\mathbf{v}} = E^k_{n,n-1}(\mathbf{v})$. On one hand,

$$[K_1(E_{n1}), Q_1(E_{1,n-1})](m \otimes \mathbf{v}) = my_kx_k \otimes E^k_{n1}E^k_{1,n-1}(\mathbf{v}) = my_kx_k \otimes \widehat{\mathbf{v}} \tag{81}$$

On the other hand,

$$\begin{aligned}
[K_1(E_{n1}), Q_1(E_{1,n-1})] &= [X_0^+, [Y_0^-, E_{n,n-1}]] = [X_0^+, [X_{0,1}^-, X_{n-1}^-]] \\
&= [H_{0,1}, X_{n-1}^-] = X_{n-1,1}^- + (\beta H_0 X_{n-1}^- + (\lambda-\beta)X_{n-1}^- H_0) \\
&= J(X_{n-1}^-) - \lambda\omega_{n-1}^- + (\beta H_0 X_{n-1}^- + (\lambda-\beta)X_{n-1}^- H_0)
\end{aligned}$$

where

$$\omega_{n-1}^- = -\frac{1}{4}\sum_{d=1}^{n-2}(E_{nd}E_{d,n-1} + E_{d,n-1}E_{nd}) - \frac{1}{4}(X_{n-1}^- H_{n-1} + H_{n-1}X_{n-1}^-).$$

Therefore, we also have:

$$\begin{aligned}
[K_1(E_{n1}), Q_1(E_{1,n-1})](m \otimes \mathbf{v}) &= \big(J(X_{n-1}^-) - \lambda\omega_{n-1}^- + (\beta H_0 X_{n-1}^- + (\lambda-\beta)X_{n-1}^- H_0)\big)(m \otimes \mathbf{v}) \\
&= m\mathcal{Y}_k \otimes E^k_{n,n-1}(\mathbf{v}) - \lambda m \otimes (\omega_{n-1}^-(\mathbf{v})) + \beta m \otimes H_0^k E^k_{n,n-1}(\mathbf{v}) \\
&= m\mathcal{Y}_k \otimes \widehat{\mathbf{v}} + \frac{\lambda}{2}\sum_{d=1}^{n-2}\sum_{r=1}^{l}\sum_{\substack{s=1\\ s\neq d}}^{l} m \otimes (E^s_{nd}E^r_{d,n-1})(\mathbf{v}) \\
&\quad +\lambda\left(\frac{n-2}{4}\right) m \otimes E_{n,n-1}(\mathbf{v}) + \frac{\lambda}{4} m \otimes (X_{n-1}^- H_{n-1} \\
&\quad +H_{n-1}X_{n-1}^-)(\mathbf{v}) + \beta m \otimes H_0^k E^k_{n,n-1}(\mathbf{v}) \\
&= m\mathcal{Y}_k \otimes \widehat{\mathbf{v}} + \frac{\lambda}{2}\sum_{d=1}^{n-2}\sum_{\substack{j=1\\ j\neq k}}^{l} m \otimes (E^j_{nd}E^k_{d,n-1})(\mathbf{v}) + \lambda\left(\frac{n-2}{4}\right) m \otimes \widehat{\mathbf{v}} \\
&\quad +\frac{\lambda}{4} m(E^k_{n,n-1}E^k_{n-1,n-1} - E^k_{nn}E^k_{n,n-1})(\mathbf{v}) + \beta m \otimes E^k_{n,n-1}(\mathbf{v}) \\
&= \frac{1}{2}m(x_ky_k + y_kx_k) \otimes \widehat{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{j=1\\ j\neq k}}^{l} ms_{jk} \otimes \widehat{\mathbf{v}} \\
&\quad +(\frac{\lambda n}{4} - \frac{\lambda}{2} + \beta) m \otimes \widehat{\mathbf{v}}
\end{aligned} \tag{82}$$

From the equations (81) and (82) and our hypothesis that $\beta = \frac{t}{2} - \frac{\lambda n}{4} + \frac{\lambda}{2}$, we deduce the following equality:

$$my_kx_k \otimes \widehat{\mathbf{v}} = \frac{1}{2}m(x_ky_k + y_kx_k) \otimes \widehat{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{j=1\\ j\neq k}}^{l} ms_{jk} \otimes \widehat{\mathbf{v}} + \frac{t}{2}m \otimes \widehat{\mathbf{v}}$$

which implies that $m\big(y_kx_k - x_ky_k - t - \lambda\sum_{\substack{j=1\\ j\neq k}}^{l} s_{jk}\big) \otimes \widehat{\mathbf{v}} = 0$. Since $\lambda = c$ by assumption and $\widehat{\mathbf{v}}$ is a generator of $V^{\otimes l}$ as $\mathfrak{U}\mathfrak{sl}_n$-module, we conclude, using again lemma 5.2, that the equality

$$m\big(y_kx_k - x_ky_k - t - c\sum_{\substack{j=1\\ j\neq k}}^{l} s_{jk}\big) = 0 \tag{83}$$

must be satisfied. Equations (80) and (83) show that M is a right module over H. Finally, that $\mathcal{F}$ is bijective on the set of morphisms follows from an argument similar to the one used in the trigonometric case. $\square$

7.2 Category $\mathcal{O}$

One important category of modules over $\mathsf{H}_{t,c}$ (when $t \neq 0$) is the category $\mathcal{O}$ studied in [GGOR].

Definition 7.2. *We define $\mathcal{O}_{t,c}$ for $t \neq 0$ to be the category of right modules over $\mathsf{H}_{t,c}$ which are finitely generated over $\mathsf{H}_{t,c}$ and locally nilpotent over $\mathbb{C}[\mathfrak{h}^*]$. We set $\mathcal{O} = \mathcal{O}_{t,c}$.*

We see from the definition of the $\mathbb{L}$-module structure on $\mathcal{F}(M)$ that if $M \in \mathcal{O}$ then $\mathcal{F}(M)$ is locally nilpotent over the subalgebra A of $\mathbb{L}$ generated by $Q_r(z), \forall z \in \mathfrak{sl}_n, \forall r \geq 1$. This leads us to our last theorem.

Theorem 7.2. *Assume that $l + 2 < n$, $\lambda = c$ and $\beta = \frac{t}{2} - \frac{cn}{4} + \frac{c}{2}$. The functor $\mathcal{F}$ establishes an equivalence between the category $\mathcal{O}$ and the category of finitely generated left modules over $\mathbb{L}$ which are locally nilpotent over the subalgebra A and integrable of level l.*

Proof. We prove this theorem for H. If $m_1, \dots, m_k$ are generators of M, then $\{m_i \otimes \mathbf{v}, 1 \leq i \leq k, \mathbf{v} \in V^l\}$ is a finite set of generators for $\mathcal{F}(M)$. To see this, we can assume that M is generated over $\mathbb{C}[\mathfrak{h}]$ by $m_1, \dots, m_k$. Take an element $m \otimes \mathbf{v} \in \mathcal{F}(M)$ with $m = m_1 x_1^{a_1} \cdots x_l^{a_l}$. We suppose first that $\mathbf{v} = v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_l$ and set $\mathbf{v}' = v_1 \otimes v_3 \otimes v_4 \otimes \cdots \otimes v_{l+1}$. Then

$$m \otimes \mathbf{v} = K_{a_l}(E_{l,l+1}) \cdots K_{a_2}(E_{23}) K_{a_1}(H_1)(m_1 \otimes \mathbf{v}').$$

Now we can apply elements of $\mathfrak{Usl}_n$ to $v_1 \otimes v_2 \otimes \cdots \otimes v_l$ to obtain any other element of V^l. The general case when $m = \sum_{j=1}^k m_j x_1^{a_{1,j}} \cdots x_l^{a_{l,j}} \otimes \mathbf{v}_j$ follows from this. Conversely, suppose that N is a finitely generated integrable module over $\mathbb{L}$ of level l and $N = \mathcal{F}(M)$. Let $\{n_1, \dots, n_k\}$ be a set of generators of N and write $n_i = \sum_{j=1}^{k_i} m_{ij} \otimes \mathbf{v}_{ij}$ for some $m_{ij} \in M$ and some $\mathbf{v}_{ij} \in V^l$. Then $\{m_{ij} | 1 \leq i \leq k, 1 \leq j \leq k_i\}$ is a set of generators of M.

Now suppose that N is an integrable left module over $\mathbb{L}$ of level l which is locally nilpotent over A. By theorem 7.1, we know that $N = \mathcal{F}(M)$ for a right module M over H. Pick $m \in M$. It is enough to show that $m y_i^s = 0$ for some s. Set $\mathbf{v} = v_1 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_{l+1}$ and choose s so that $Q_1(H_i)^s(m \otimes \mathbf{v}) = 0$. Then $Q_1(H_i)(m \otimes \mathbf{v}) = m y_i \otimes \mathbf{v}$, so $m y_i^s \otimes \mathbf{v} = Q_1(H_i)^s(m \otimes \mathbf{v}) = 0$ and lemma 5.2 implies that $m y_i^s = 0$. $\square$

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