ALGEBRAS OF PRIME DEGREE ON FUNCTION FIELDS OF SURFACES

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Abstract. Let X be a smooth projective surface over an algebraically closed field k of characteristic 0. Let D be a central division algebra of prime index over $K = k(X)$. We prove that D is cyclic, thus proving, in this very special case, a conjecture that Albert probably never made [Sa].

0. Introduction

A central division algebra of degree n over a field K is said to be cyclic if it admits a cyclic splitting field of degree n over K. It is an open question whether central division algebras of prime degree l over any field are cyclic. It is a well-known result of Brauer-Hasse-Noether that all division algebras over number fields are cyclic. Let k be an algebraically closed field of characteristic zero, S a normal surface over k and R the completion or henselization of the local ring at a closed point of S. Then, over the quotient field K of R, Ford and Salman showed that all division algebras are cyclic. Over the quotient field of a 2-dimensional excellent henselian local domain with algebraically closed residue field, it was shown by Colliot-Thélène et al. [CTOP] that all division algebras of index coprime to residue field characteristic are cyclic, thus extending the result of Ford-Saltman. The method of approach here is to kill the ramification of the division algebra on a regular model over a henselian domain in a cyclic extension of K; one then uses the fact that the Brauer group of the model (or its l-primary part) is trivial.

In the case of the function field K of a surface X over an algebraically closed field k, to split a division algebra it is not sufficient to split its ramification, because the Brauer group of X may not be trivial. A theorem of de Jong [dJ] asserts that for every central division algebra D over K with index prime to the characteristic of K, the index coincides with the order of D in $\mathrm{Br}(K)$. In this paper we adapt the techniques of de Jong to prove that if the characteristic of K is zero, then every division algebra of prime degree over K is cyclic.

Given a smooth projective surface X over k and an Azumaya algebra $\mathcal{A}$ on X, we construct a finite flat morphism $Y \to X$ with Y smooth projective which splits $\mathcal{A}$. Such a construction is announced by de Jong and Artin [dJ] under the more general setting in which the characteristic of k is coprime to $\mathrm{index}(\mathcal{A}_{k(X)})$. Following de Jong, we connect Y and X by a family W fibred over $\mathbb{A}^1$ and use this family to prove cyclicity when $\mathcal{A}$ is unramified over X. The ramified case reduces to the unramified case, following de Jong's techniques closely.

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1. Representing Brauer classes

Let X be a noetherian regular integral 2-dimensional scheme, K its field of rational functions and $\mathcal{A}$ an Azumaya algebra over X.

Theorem 1.1. *Suppose that the generic fiber $\mathcal{A}_K$ of $\mathcal{A}$ is of the form $M_n(D)$, where D is a division algebra over K. Then the Brauer class of $\mathcal{A}$ can be represented by an Azumaya algebra Δ over X such that $\Delta_K = D$.*

We first recall the definition and some properties of maximal orders. Let R be an integral domain with field of fractions K and A an associative finite dimensional K-algebra. An *R-order* of A is an R-subalgebra Λ of A which is finitely generated and projective as an R-module and generates A over K: $\Lambda K = A$. It is easy to see that R-orders always exist. (A general reference for the theory of orders is [AG].)

If A is a central simple algebra over K and that R is integrally closed and noetherian, then every R-order is contained in a maximal one. If Λ is an R-Azumaya algebra such that $A = \Lambda_K$, then Λ is a maximal order of A.

Assume now that R is a discrete valuation ring. Then the following holds.

Theorem 1.2. *Let R be a discrete valuation ring, K its field of fractions and A a central simple K-algebra. Any two maximal R-orders Λ and Γ in A are conjugate: $\Lambda = u\Gamma u^{-1}$ for some $u \in A^*$. In particular, if one of the maximal orders is an Azumaya R-algebra, then so is the other one.*

Proof. See for instance [AG], Proposition 3.5.

Proof of Theorem 1.1. Since the canonical map $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is injective, any algebra Δ such that $\Delta_K = D$ will represent the Brauer class of $\mathcal{A}$. Let $x \in X$ be a point of codimension 1. Then $\mathcal{O}_{X,x}$ is a discrete valuation ring and $\mathcal{A}_x$, being an Azumaya algebra over $\mathcal{O}_{X,x}$, is a maximal order in $\mathcal{A}_K = M_n(D)$. Choose now a maximal order $\Delta(x)$ in D. It is easy to check that $M_n(\Delta(x))$ is a maximal order in $M_n(D)$. By Theorem 1.2, $\mathcal{A}_x$ and $M_n(\Delta(x))$ are isomorphic, hence $\Delta(x)$ is Morita equivalent to $\mathcal{A}_x$ and is, therefore, an Azumaya algebra.

We have, for any codimension 1 point $x \in X$, an $\mathcal{O}_{X,x}$-Azumaya algebra $\Delta(x) \subset D$ representing the class of $\mathcal{A}_K$ in $\mathrm{Br}(K)$. In other words, the K-algebra D is unramified over X. We patch all these algebras to get a global representative Δ of $\mathcal{A}_K$, with generic fiber D.

To do this we can invoke (as in [CTS], Proposition 2.4, page 111 for the case of quadratic spaces) some general results on projective limits of schemes or prove a little lemma:

Lemma 1.3. *Let X be a noetherian scheme of finite dimension, U a dense open subset of X, $y \in X \setminus U$ a codimension 1 point, V an open neighbourhood of y and $W \subset U \cap V$ a dense open subset of X. There exists an open neighbourhood V'' of y such that $V'' \cap U \subset W$.*

Proof. The closed set $Z = X \setminus W$ is of dimension $n-1$ and contains y, hence Z has an irredundant decomposition

$$Z = \overline{\{y\}} \cup \overline{\{y_1\}} \cdots \cup \overline{\{y_r\}} \cup \overline{\{x_1\}} \cup \cdots \cup \overline{\{x_s\}}$$

into closed irreducible sets, where $y, y_1, \ldots, y_r \notin U$ and $x_1, \ldots, x_s \in U$. Let $F = \overline{\{x_1\}} \cup \cdots \cup \overline{\{x_s\}}$ and $V' = X \setminus F \supset W$. We have $V' \cap U = \emptyset$ and therefore $U \cap F = U \cap Z$. This shows that $V' \cap U = U \setminus (U \cap Z) = U \cap (X \setminus Z) = U \cap W = W$. Thus we can take $V'' = V \cap V'$.

The next proposition shows that the local algebras $\Delta(x)$ can be patched, thus proving Theorem 1.1.

Proposition 1.4. *Let X be a noetherian integral regular scheme of dimension 2, K its field of rational functions and A a central simple K-algebra. Suppose that A is unramified over X. There exists an Azumaya algebra Λ over X such that $\Lambda_K = A$.*

Proof. The K-algebra A extends over some open set U as an Azumaya algebra Λ^U and we may assume that U is maximal with this property. Suppose that $X \setminus U$ contains a point y of codimension 1. By assumption A is unramified at y, hence there exists an Azumaya $\mathcal{O}_{X,y}$-algebra $\Gamma^y \subset A$ such that $\Gamma^y_K = A$. We extend Γ^y to an Azumaya algebra Γ^V over some suitable open neighbourhood V of y. Since $\Gamma^V_K = A = \Lambda^U_K$, there exists an open set $W \subset U \cap V$ over which $\Gamma^V_W \simeq \Lambda^U_W$ We now choose $V'' \subset W$ as in 1.3 and patch Λ^U with $\Gamma^V_{V''}$ over $U \cap V''$ to get an Azumaya algebra over $U \cup V''$. This shows that if U is maximal, then $X \setminus U$ consists of finitely many closed points. By [CTS], Th. 6.13 applied to the group PGL_n, Λ^U extends to an Azumaya algebra on X.

2. Splitting extensions

We show how to split an Azumaya algebra over a surface X by a finite map $Y \to X$.

Here and in the rest of the article we suppose that k is an algebraically closed field. We denote by $\mathrm{Sing}(X)$ the singular locus of a given scheme X.

Let

$$A_n = \frac{k[X_{11}, X_{12}, \ldots, X_{nn}][T]}{\big(P(T)\big)}$$

where $P(T)$ is the characteristic polynomial of the generic matrix (X_{ij}) with $1 \leq i, j \leq n$. Let $Y_n = \mathrm{Spec}(A_n)$. We study the singular locus of Y_n.

Lemma 2.1. *Let $\beta = \operatorname{diag}(B_1, \dots, B_m)$ be a matrix consisting of m cyclic Jordan blocks*

$$B_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \lambda_i & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \lambda_i & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_i & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \lambda_i \end{pmatrix}$$

with distinct eigenvalues λ_i. Then, for any i, the scheme Y_n is smooth at (β, λ_i).

Proof. We denote by I_n the identity matrix of size n. Developing the determinant of $(X_{ij}) - T \cdot \mathrm{I}_n$ along the first column we get

$$\pm P(T) = (X_{11} - T)P_1(T) + X_{2,1}P_2(T) + \dots + X_{n,1}P_n(T)$$

where the polynomials P_i are the cofactors of the first column. Let k_i be the size of B_i. We see that $P_{k_1}(T)(B, \lambda_1)$ is (up to sign) the determinant of a matrix of the form $\operatorname{diag}(\mathrm{I}_{k_1-1}, B_2 - \lambda_1 \mathrm{I}_{k_2}, \dots, B_m - \lambda_1 \mathrm{I}_{k_m})$, it being understood that the first block is missing if $k_1 = 1$. Since $\lambda_1 \neq \lambda_i$, this shows that $\partial P(T)/\partial X_{k_1,1} = P_{k_1}(T)$ is not zero at (B, λ_1). Thus Y_n is smooth at (β, λ_1) and the same clearly holds for any other λ_i.

Lemma 2.2. *Every neighbourhood of a matrix α with an eigenvalue $\lambda \neq 0$ contains an invertible semisimple matrix with eigenvalue λ.*

Proof. We may assume that α is in Jordan form. The given neighbourhood of α contains an open set defined by the non-vanishing of a polynomial g in the coordinates of the generic matrix (X_{ij}). We may assume that the diagonal entries of α are $(\lambda, \lambda_2, \dots, \lambda_n)$. Since $g(\alpha) \neq 0$ we may find values $\lambda'_2, \dots, \lambda'_n$ all distinct and different from λ and different from 0, such that when we replace λ_i by λ'_i in α we obtain an α' for which $g(\alpha') \neq 0$. This new α' is in the given neighbourhood and is semisimple.

Let Y_n be as before and consider the finite map $\pi : Y_n \to \mathbb{A}^{n^2}$ induced by the injection $k[X_{11}, X_{12}, \dots, X_{nn}] \to A_n$. The projection $C = \pi(\operatorname{Sing}(Y_n))$ is a closed subscheme of $\mathbb{A}^{n^2}$ and is contained in the ramification locus of π, which is the closed subscheme of $\mathbb{A}^{n^2}$ whose closed points correspond to matrices with at least two equal eigenvalues.

Lemma 2.3. *Let $V \subset \mathbb{A}^{n^2}$ be the set of semisimple invertible matrices with at least two coincident eigenvalues. Then $V \subseteq C$.*

Proof. It suffices to check that any matrix of the form $\beta = \operatorname{diag}(\mu_1, \dots, \mu_{n-2}, \lambda, \lambda)$ is in C. We show that (β, λ) belongs to $\operatorname{Sing}(Y_n)$. Writing $X_{ii} = \mu_i + X_i$ for $i \leq n-2$, $X_{ii} = \lambda + X_i$ for $i \geq n-1$, $T = \lambda + t$ and $\nu_i = \mu_i - \lambda$ we see that $P(T)$ is the determinant of the matrix

$$\begin{pmatrix} \nu_1 + X_1 & X_{12} & \cdots & X_{1n} \\ X_{2,1} & \nu_2 + X_2 & \cdots & X_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & X_{n-1} - t & X_{n-1,n} \\ \cdots & \cdots & X_{n,n-1} & X_n - t \end{pmatrix}$$

and it is clear that it does not contain any linear term in X_i,X_{ij} or t. Thus the variety it defines is singular at the origin, which corresponds to the point (β,λ) in the previous coordinates.

Lemma 2.4. *Let $W \subset M_n(k)$ be the set of all semisimple invertible matrices with at least $n-1$ distinct eigenvalues. Then W is open and dense in $M_n(k)$.*

Proof. The set of all semisimple invertible matrices is open and dense in $M_n(k)$. We claim that matrices having at least $n-1$ distinct eigenvalues is open in $M_n(k)$. In fact this set is the inverse image under the eigenvalue map $M_n \to \mathbb{A}^n/\mathcal{S}_n$ of the complement of the closed set of points with three equal coordinates. Hence W is open and clearly non empty.

By 2.4 the set $U = W \cap C$ of all semisimple invertible matrices with exactly two equal eigenvalues is open in C.

Lemma 2.5. *The set U is dense in C.*

Proof. Let (β,λ) be a point of $\mathrm{Sing}(Y_n)$. By 2.1, β, which we may assume to be in Jordan canonical form, contains at least two cyclic Jordan blocks with the same eigenvalue. We write $\beta = \mathrm{diag}(\beta_1,\beta_2,\dots,\beta_r)$ with the β_i's cyclic Jordan blocks of size s_i and β_1, β_2 having the same eigenvalue λ. Suppose that β is in the open set defined by $f \neq 0$ for some polynomial function f in the entries X_{ij} of the generic $n\times n$ matrix. Let $\widetilde{\beta} = \mathrm{diag}(\widetilde{\beta}_1,\widetilde{\beta}_2,\dots,\widetilde{\beta}_r)$ be a matrix where each $\widetilde{\beta}_i$ has the same size as β_i and the same off-diagonal entries. Suppose further that $\widetilde{\beta}$ has $n-1$ distinct eigenvalues, with $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ retaining the eigenvalue λ. Then $\widetilde{\beta}$ is semisimple and, for a general $\widetilde{\beta}$, $f(\widetilde{\beta}) \neq 0$.
For example, if

$$\beta = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

then

$$\widetilde{\beta} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

with λ, λ_1, λ_2, λ_3 distinct.

Corollary 2.6. *The dimension of C is equal to the dimension of U.*

Lemma 2.7. *The dimension of U is n^2-3.*

Proof. Let $\Sigma_{n-1} \subset (k^*)^{n-1}/\mathcal{S}_{n-1}$ be the set of all $\{\lambda,\lambda_3,\dots,\lambda_n\}$ consisting of $n-1$ distinct elements of k^*. Clearly Σ_{n-1} has dimension $n-1$. Mapping each matrix in U to the set of its eigenvalues we obtain a surjective map $p: U \to \Sigma_{n-1}$. The linear group $GL_n(k)$ acts transitively on each fiber of p and the stabilizer of the matrix $\mathrm{diag}(\lambda,\lambda,\lambda_3,\dots,\lambda_n)$ is $GL_2(k)\times(k^*)^{n-2}$. Hence the dimension of U is $\dim\big(GL_n(k)\big) - \dim\big(GL_2(k)\times(k^*)^{n-2}\big) + \dim(\Sigma_{n-1}) = n^2-(4+n-2)+n-1 = n^2-3$.

Corollary 2.8. *The closed set* $\mathrm{Sing}(Y_n)$ *is of codimension* 3.

Proof. The closure of U is $C = \pi(\mathrm{Sing}(Y_n))$ and π is a finite map.

We now show how to use the reduced characteristic polynomial to split an Azumaya algebra over a surface.

If $\mathcal{L}$ is a line bundle over some scheme and n a positive integer, we denote by $\mathcal{L}^{\otimes n}$ the n-fold tensor product of $\mathcal{L}$ with itself and by $\mathcal{L}^{-n}$ the inverse of $\mathcal{L}^{\otimes n}$.

Let X be a smooth projective surface over an algebraically closed field k and $\mathcal{A}$ an Azumaya algebra of rank n^2 over X. Let $\mathcal{L}$ be a line bundle over X such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$ is generated by global sections $s_1, \dots, s_N$ and let s be any global section of $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$. Choose an arbitrary affine nonempty open set $U \subset X$ over which $\mathcal{L}$ is principal: $\mathcal{L}_{|U} = \mathcal{O}_U f$ for some $f \in \mathcal{L}(U)$. Then $sf^{-1} \in \mathcal{A}(U)$, which is an Azumaya algebra over $\mathcal{O}_X(U)$. Let

$$P_{f,U}(T) = T^n + b_1 T^{n-1} + \cdots + b_n$$

with $b_1, \dots, b_n \in k[U]$ be the characteristic polynomial of sf^{-1}. We define $J_{f,U}$ as the ideal of

$$Sym(\mathcal{L}^{-1}|_U) = \mathcal{O}_U \oplus \mathcal{L}^{-1}|_U \oplus \mathcal{L}^{-2}|_U \oplus \cdots = \mathcal{O}_U \oplus \mathcal{O}_U f^{-1} \oplus \mathcal{O}_U f^{-2} \oplus \cdots$$

generated by $f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n$.

Lemma 2.9. *Let Λ be an Azumaya algebra of rank n^2 over a ring R. For any $\alpha \in \Lambda$ and any $c \in R$, the characteristic polynomial $P_\alpha(T)$ of α satisfies the relation $c^n P_\alpha(T) = P_{c\alpha}(cT)$.*

Proof. It immediately follows from the split case $\Lambda = M_n(R)$.

Lemma 2.10. *The ideal $J_{f,U}$ does not depend on the choice of f.*

Proof. We apply 2.9 with $f = ug$ for some other generator g of $\mathcal{L}|_U$ and u invertible on U. (We note that the suffixes f or g stand for the elements s/f, s/g in the algebra). We have

$$P_{g,U}(T) = P_{u^{-1}f,U}(T) = u^n P_{f,U}(u^{-1}T) = T^n + ub_1 T^{n-1} + \cdots + u^n b_n \ .$$

Thus the ideal $J_{g,U}$ is generated by

$$g^{-n} \oplus b_1 u g^{-(n-1)} \oplus \cdots \oplus u^n b_n = u^n (f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n) \ .$$

and coincides therefore with $J_{f,U}$.

Patching the ideals $J_{f,U}$ over a suitable affine covering of X yields a global ideal J_s of $Sym(\mathcal{L}^{-1})$ that only depends on the section s. We call J_s *the characteristic ideal of* s.

We define a projective k-scheme Y_s with a finite map to X as the closed subscheme of $\mathrm{Spec}\,\big(Sym(\mathcal{L}^{-1})\big)$ defined by the ideal J_s.

To simplify notation, if $s = \lambda_1 s_1 + \cdots + \lambda_N s_N$ we put $\lambda = (\lambda_1, \dots, \lambda_N) \in k^N$, $J_s = J_\lambda$ and $Y_s = Y_\lambda$

Theorem 2.11. *Assume that k is of characteristic zero. There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Y_λ is a projective smooth surface.*

Proof. We extend the base to $\widetilde{X} = X \times \mathbb{A}^N$ where $\mathbb{A}^N = \operatorname{Spec}(k[t_1, \ldots, t_N])$. Let $\widetilde{A}$ and $\widetilde{\mathcal{L}}$ be the inverse images of A and $\mathcal{L}$ under the projection $\pi : \widetilde{X} \to X$. Put $\widetilde{s} = t_1 s_1 + \cdots + t_n s_n$ and let $\widetilde{J}_t(T)$ be the characteristic ideal of $\widetilde{s}$ and $\widetilde{Y}$ the closed subscheme of $\operatorname{Spec}\big(\operatorname{Sym}(\widetilde{\mathcal{L}}^{-1})\big)$ defined by $\widetilde{J}_t(T)$. Look at the diagram

$$\begin{tikzcd} & \widetilde{Y} \arrow[dl, "p"'] \arrow[d, "\pi"] \arrow[dr, "q"] & \\ X & X \times \mathbb{A}^N \arrow[l] \arrow[r] & \mathbb{A}^N \end{tikzcd}$$

The map π is clearly finite and flat and the two projections from $X \times \mathbb{A}^N$ are flat, hence p is flat. We try to determine the singularities of $\widetilde{Y}$ using the following lemma.

Lemma 2.12. *Let $f : Z \to X$ be a flat map of schemes. Suppose that X is regular. If $z \in Z$ is a singular point of Z, then z is a singularity of its fiber $f^{-1}(f(z))$.*

Proof. Let C be the local ring of Z at z and A be the local ring of $f(z)$. By assumption the maximal ideal of A is generated by a regular sequence $(x_1, \ldots, x_m)$. Since f is flat, C is faithfully flat over A and this sequence is still regular as a sequence in C. If z is not a singular point of its fiber, then $C/(x_1, \ldots, x_m)$ is regular and hence its maximal ideal is generated by a regular sequence $(\overline{y}_1, \ldots, \overline{y}_r)$. This implies that the maximal ideal of C is generated by the regular sequence $(x_1, \ldots, x_m, y_1, \ldots, y_r)$, hence C is regular.

By 2.12 the singularities of $\widetilde{Y}$ are contained in the union of the singularities of the fibers of p.

Lemma 2.13. *The singular locus of every fiber $p^{-1}(x)$ of p has codimension 3 in $p^{-1}(x)$.*

Proof. Let $k(x)$ be the residue field of $x \in X$, Ω its algebraic closure and F_x the fiber of p at x. The geometric fibre $\mathcal{A}(\overline{x})$ of $\mathcal{A}$ at x is a matrix algebra $M_n(\Omega)$ and

$$F_{\overline{x}} = \operatorname{Spec}\left(\Omega[t_1, \ldots, t_N][T]/(P_x(T))\right) ,$$

where $P_x(T)$ is the characteristic polynomial of $\overline{s} = \big(t_1 s_1(x) + \cdots + t_N s_N(x)\big)/f(x)$ for some generator f of $\mathcal{L}|_U$, U a neighbourhood of x. Since the sections $s_i(x)/f(x)$ generate $M_n(\Omega)$ over Ω, by a linear change of coordinates we may assume that $\overline{s} = t_1 e_1 + \cdots + t_m e_m$ where $m = n^2$ and $\{e_1, \ldots, e_m\}$ form a basis of $M_n(\Omega)$. Then

$$F_{\overline{x}} = Y_n \times \operatorname{Spec}\left(\Omega[t_{m+1}, \ldots, t_N]\right) .$$

We proved that the singular locus of Y_n has codimension 3, hence the same holds for the singular locus of $F_{\overline{x}}$. For every $x \in X$ the fiber F_x is a finite cover of $\mathbb{A}^N$ and hence the dimension of F_x is N. Let $\operatorname{Sing}(\widetilde{Y})$ be the singular locus of $\widetilde{Y}$. By 2.10, for every $x \in X$, the fiber at x of $p|_{\operatorname{Sing}(\widetilde{Y})} : \operatorname{Sing}(\widetilde{Y}) \to X$ is contained in the singular locus of F_x and has therefore dimension at most $N-3$. Since X is 2-dimensional, the dimension of $\mathrm{Sing}(\widetilde{Y})$ is at most $N-1$.

We now look at $q : \widetilde{Y} \to \mathbb{A}^N$. Since $\mathrm{Sing}(\widetilde{Y})$ is at most $(N-1)$-dimensional, its image $q(\mathrm{Sing}(\widetilde{Y}))$ is contained in a proper closed subset of $\mathbb{A}^N$. Choose an open set $W \subset \mathbb{A}^N$ which does not intersect $q(\mathrm{Sing}(\widetilde{Y}))$ and let $\widetilde{V} = q^{-1}(W)$. We now have a map $q : \widetilde{V} \to W$ of smooth varieties. This map is clearly flat and surjective and therefore, k being of characteristic zero, it is generically smooth (see [Ha$_1$], Ch. III, Corollary 10.7). This means that there exists a dense open set $U \subset \mathbb{A}^N$ such that $q^{-1}(U) \to U$ is smooth. For any $\lambda \in U$ the fiber $Y_\lambda = q^{-1}(\lambda)$ is smooth.

Let us denote by $\pi_\lambda : Y_\lambda \to X$ the map induced by π on the fibre Y_λ.

Proposition 2.14. *There exists a dense open set $U \subset k^N$ such that for any $\lambda \in U$ and for all but finitely many closed points $x \in X$ the fibre $\pi_\lambda^{-1}(x)$ contains at least $n-1$ points.*

Proof. For $n = 2$ we may take $U = k^N$. We now assume that $n \geq 3$. Let J be a Jordan block of size $m \times m$. A direct computation shows that its stabilizer in $M_m(k)$ is the group $G(m)$ that consists of all matrices of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ 0 & a_1 & a_2 & \cdots & a_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}$$

and is therefore of dimension m. Thus the stabilizer of $\mathrm{diag}(J_1, \ldots, J_r) \in M_n(k)$ where J_i denotes a Jordan block of size m_i, contains a product of the form $G(m_1) \times \cdots \times G(m_r)$ and is therefore of dimension at least n. It follows that the stabilizer of any matrix $\alpha \in M_n(k)$ is of dimension at least n.
If α is a matrix with less than $n-1$ eigenvalues, the eigenvalues may be chosen in at most $n-2$ different ways. Thus the set of matrices with at most $n-2$ distinct eigenvalues has dimension at most $n^2 - 2$.

Consider now the closed set

$$V = \{(\lambda, x) \mid P_\lambda(x, T) \text{ has at most } n-2 \text{ distinct roots}\} \subset \mathbb{A}^N \times X$$

and its projection $V \to X$. It follows that the fiber of any x has dimension at most $(N - n^2) + (n^2 - 2) = N - 2$ and hence $\dim(V) \leq N$. If the image of the projection $p : V \to \mathbb{A}^N$ is not dense, then there is an open set $U \subset \mathbb{A}^N$ such that for $\lambda \in U$ the characteristic polynomial P_λ has no triple root. If $p(V)$ is dense in $\mathbb{A}^N$, then for an open set $U \subset \mathbb{A}^N$ the fiber of $\lambda \in U$ is finite ([Ha$_1$], Ch. II, Exercise 3.22, page 95). In this case, for each $\lambda \in U$, $P_\lambda(x, T)$ has at most one double root except for a finite number of points $x \in X$.

Proposition 2.15. *Suppose that the algebra $\mathcal{A}_{k(X)}$ is a division ring of index n. Then there exists an open set U of k^N such that, for each $\lambda \in U$, the variety Y_λ is integral.*

Proof. We extend the scalars to $k(X)$. There exists an open set U of k^N such that for each $\lambda \in U$, s_λ/f generates a maximal subfield of degree n in $\mathcal{A}_{k(X)}$. Since $P(T)$ is of degree n and vanishes at s_λ/f, it must be irreducible over $k(X)$, hence $\left(Y_\lambda\right)_{k(X)}$ is integral. This, together with the smoothness of Y, implies that Y is integral.

Summing up, we have proved the following result.

Theorem 2.16. *Let X be a smooth projective surface over an algebraically closed field k of characteristic zero. Let $\mathcal{A}$ be an Azumaya algebra of rank n^2 over X with $\mathcal{A}_{k(X)}$ a division ring. Then there exists an open set U of k^N such that for each $\lambda \in U$,*

(1) Y_λ is a smooth, integral projective surface,

(2) the map $\pi_\lambda : Y_\lambda \to X$ is finite and flat,

(3) $\pi_\lambda^*(\mathcal{A})$ is trivial in $\mathrm{Br}(Y_\lambda)$,

(4) there are only finitely closed point on X which have fewer that $n-1$ preimages in Y.

3. A splitting criterion

Proposition 3.1. *Let X be a smooth projective surface over an algebraically closed field k and $\mathcal{A}$ an Azumaya algebra over X, of rank n^2. Assume that the characteristic of k is zero or a prime that does not divide n. Fix an element $\eta \in H^2(X, \mu_n)$ which maps to $[\mathcal{A}] \in {}_n\mathrm{Br}(X) \subset H^2(X, \mathbb{G}_m)$. Suppose that there exists a diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "f"'] & X \\ \mathbb{A}^1 & \end{tikzcd}$$

with $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ and such that

(1) W is a 3-dimensional integral scheme with $W_{\overline{k(t)}}$ integral,

(2) the map f is proper,

(3) $W_1 = f^{-1}(1)$ has n irreducible components V_i, each with multiplicity 1 and such that $g|_{V_i} : V_i \to X$ is a birational isomorphism for every i,

(4) W is normal at the generic point of each V_i,

(5) $g^*(\eta)|_{W_0} = 0$ in $H^2(W_0, \mu_n)$.

Then $\mathcal{A}_{k(X)}$ is a matrix algebra over $k(X)$.

Proof. Let R be the local ring of $\mathbb{A}^1$ at $t = 0$ and R^h its henselization. Let $g_h : W \times_{\mathbb{A}^1} \mathrm{Spec}(R^h) \to X$ be the composite map $W \times_{\mathbb{A}^1} \mathrm{Spec}(R^h) \to W \xrightarrow{g} X$. The element $g_h^*(\eta) \in H^2(W \times_{\mathbb{A}^1} \mathrm{Spec}(R^h))$ maps to zero in $H^2(W_0, \mu_n)$. By proper base change ([Mi], Ch. VI, 2.7), $g_h^*(\eta) = 0$, hence there exists a finite étale map $C_0 \xrightarrow{\alpha} \mathbb{A}^1$ of a curve onto a neighbourhood of 0, such that if $g_{C_0} : W \times_{\mathbb{A}^1} C_0 \to X$ denotes the restriction of g_h, then $g^*_{C_0}(\eta) = 0$. We extend $\alpha : C_0 \to \mathbb{A}^1$ to an $\alpha : C_1 \to \mathbb{A}^1$ such that the point $t = 1$ is the image of a rational point of C_1. Such a point exists, since k is algebraically closed. Since $W_{\overline{k(t)}}$ is integral, the scheme $W \times_{\mathbb{A}^1} C_1$ is integral, with generic point $\mathrm{Spec}(k(W \times_{\mathbb{A}^1} C_0))$. The class $g^*_{C_1}(\eta) \in H^2(W \times_{\mathbb{A}^1} C_1, \mu_n)$ is generically zero. Since by (3) each V_i occurs with multiplicity 1 in the fibre of 1 and by (4) W is normal at the generic point of V_i, $t-1$ generates the maximal ideal of the discrete valuation ring $\mathcal{O}_{W,V_i}$. Let $1' \in C_1$ be a rational point such that $\alpha(1') = 1$. Then $V_i \times 1' \simeq V_i$ is an irreducible component of the fibre of $1'$. Let S be its local ring in $k(W \times_{\mathbb{A}^1} C_1)$. The maximal ideal of S is generated by a local parameter of C_1 at $1'$, hence S is a discrete valuation ring with quotient field $k(W \times_{\mathbb{A}^1} C_1)$ and the map $H^2(S, \mu_n) \to H^2(k(W \times_{\mathbb{A}^1} C_1), \mu_n)$ is injective. Thus $g^*_{C_1}(\eta)$ restricts to zero in $H^2(S, \mu_n)$ and specializes to zero in

$$H^2(\kappa(V_i \times \{1'\}), \mu_n) = H^2(\kappa(V_i), \mu_n) = H^2(k(X), \mu_n)$$

under the map g. The composite map $k(X) \to \kappa(V_i) \xrightarrow{g} k(X)$ being the identity, we have $\eta_{k(X)} = 0$.

4. Construction of families

We shall first construct a variety W satisfying the assumptions (1) to (4) of 3.1. Let X be a smooth projective surface over an algebraically closed field k and $\mathcal{A}$ an Azumaya algebra of dimension n^2 on X. Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$ is generated by global sections $e_1, \ldots, e_N$. Let $s = \lambda_1 e_1 + \cdots + \lambda_N e_N$ with $\lambda = (\lambda_1, \ldots, \lambda_N) \in k^N$ and $J_s \in Sym(\mathcal{L}^{-1})$ the characteristic ideal of s defined in §2. Recall that Y_λ is the subscheme of $\mathrm{Spec}\big(Sym(\mathcal{L}^{-1})\big)$ defined by J_s and that locally on any affine open set $U \subset X$ over which $\mathcal{L}|_U$ is generated by a section f, $J_s|_U$ is generated by $P_{f,U}(f^{-1}) = f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n$ where $P_{f,U}(T) = T^n + b_1 T^{(n-1)} + \cdots + b_n$ is the characteristic polynomial of $s/f \in H^0(U, \mathcal{A})$. We choose λ such that Y_λ satisfies (1), (2) and (3) of Theorem 2.16. Let $\widehat{X}$ be the scheme $X \times \mathbb{A}^1$, $p : \widehat{X} \to X$ its first projection and t the coordinate on $\mathbb{A}^1$. We put $\widehat{\mathcal{L}} = p^*(\mathcal{L})$ and define an ideal in $Sym(\widehat{\mathcal{L}}^{-1})$ as follows. Let $w_1, \ldots, w_n$ be n distinct global sections of $\mathcal{L}$. We choose them in such a way that no function w_i/f over U is a zero of $P_{f,U}(T)$. We denote by $\widehat{U}$ the inverse image of U. For simplicity, we still denote by the same letter a function (or a section of a bundle, or a polynomial, ...) on an open set of X and its extension to $\widehat{X}$. Let $\widehat{I}_{f,U}$ be the ideal of $Sym(\widehat{\mathcal{L}}^{-1}|_{\widehat{U}})$ generated by $Q_{f,U}(t, f^{-1})$ where

$$Q_{f,U}(t,T) = (1-t)P_{f,U} + t(T - w_1/f)\ldots(T - w_n/f)\ .$$

If we replace f by another generator g such that $g = uf$ for some invertible function u on U, then, as in 2.10, we see that $\widehat{I}_{f,U} = \widehat{I}_{g,U}$. Therefore these ideals patch over X and give rise to an ideal $\widehat{I}_s$ of $Sym(\widehat{\mathcal{L}}^{-1})$. We define W as the closed subscheme of $\mathrm{Spec}\big(Sym(\widehat{\mathcal{L}}^{-1})\big)$ defined by $\widehat{I}_s$.

The composite

$$W \to \mathrm{Spec}\big(Sym(\widehat{\mathcal{L}}^{-1})\big) \to X$$

defines a map $g : W \to X$ and the second projection defines a map $f : W \to \mathbb{A}^1$.

Proposition 4.1. *The triple (W, f, g) satisfies the conditions (2), (3) and (4) of Proposition 3.1. Furthermore $W_0 = f^{-1}(0)$ is a smooth projective surface.*

Proof. Property (2) follows from the fact that W is finite, hence proper over $\widehat{X}$ which is proper over $\mathbb{A}^1$. The fibre W_1 is locally the spectrum of $R[T]/((T - w_1/f)\dots(T - w_n/f))$ whose irreducible components $\operatorname{Spec}\big(R[T]/(T - w_i/f)\big)$ have multiplicity 1 and map isomorphically onto $\operatorname{Spec}(R)$ under g. This proves (3). To show (4) let $\mathfrak{p}_i$ be the generic point of V_i and $U = \operatorname{Spec}(R)$ a suitable affine open set such that its inverse image in W contains $\mathfrak{p}_i$. Then, locally at $\mathfrak{p}_i$, W is the spectrum of

$$S = \big(R[T,t]/\big((1-t)P_{f,U}(T) + t(T - h_1)\dots(T - h_n)\big)\big)_{\mathfrak{p}_i}$$

with $h_i = w_i/f$. Since $T - h_i$ and $1 - t$ are in $\mathfrak{p}_i$ we have $\mathfrak{p}_i = (T - h_i, 1 - t)$. We assumed that $P(h_i) \neq 0$ in $K = S/\mathfrak{p}_i$, hence $\mathfrak{p}_i S$ is generated by $T - h_i$. This proves that W is normal at the generic point of V_i.

It is clear from the construction that W_0 is smooth and projective. It remains to prove property (1).

The proof of irreduciblility will be completed in §6. We begin with the following lemma.

Lemma 4.2. *Let*

$$\Phi_{f,U}(T) = T^n P_{f,U}(T^{-1}) = 1 + b_1 T + \dots + b_n T^n \ ,$$

$$\Psi_{f,U}(T) = (1 - (w_1/f)T)\cdots(1 - (w_n/f)T) = 1 + c_1 T + \dots + c_n T$$

and

$$R_{f,U}(t,T) = (1-t)\Phi_{f,U}(T) + t\Psi_{f,U}(T) = T^n Q_{f,U}(t, T^{-1}) \ .$$

Then $\Phi_{f,U}(f)$, $\Psi_{f,U}(f)$ and $R_{f,U}(t,f)$ do not depend on f and can be patched to yield global sections of

$$\mathcal{O}_{\widehat{X}} \oplus \widehat{\mathcal{L}} \oplus \dots \oplus \widehat{\mathcal{L}}^{\otimes n} \ .$$

Proof. Since these polynomials are determined by their restrictions to the generic fibre, to show that they patch it suffices to show that, for a fixed U, they do not depend on the choice of f. In fact we check that each $b_i f^i$ patches to yield a sections of $\widehat{L}^{\otimes i}$. Let $f = ug$ for a section g of $\mathcal{L}|_U$, u an invertible function on U. By 2.9 we have

$$P_{g,U}(T) = T^n + ub_1T^{n-1} + \dots + u^n b_n$$

and therefore

$$\Phi_{g,U}(T) = 1 + b_1 u T + \dots + b_n u^n T^n \ .$$

The assertion for $\Phi_{g,U}$ follows from $ug = f$ and the same proof holds for $\Psi_{g,U}$ and $R_{g,U}$.

Proposition 4.3. *The varieties W, $W \otimes_{k[t]} k(t)$ and $W \otimes_{k[t]} \overline{k(t)}$ are integral.*

We first recall some elementary well-known facts about integral schemes. A scheme is integral if it is irreducible and reduced. The condition that it is reduced is a local one and can be checked on each set of an affine open covering. The condition of irreducibility is a priori a global one. For quasi-projective schemes, though, it can be checked on affine open sets.

Lemma 4.4. *Let X be a scheme such that any two points of X are contained in an irreducible open affine set. Then X is irredubible.*

Proof. Suppose that $X = Y \cup Z$ with Y and Z closed in X and both different fron X. Then there exist points $y \in Y \backslash Z$ and $z \in Z \backslash Y$. Let U be an irreducible affine open set containing y and z. Then, since $z \notin U \cap Y$ and $y \notin U \cap Z$, $U = (U \cap Y) \cup (U \cap Z)$ is a decomposition of U into two proper subset, which are closed in U, leading to a contradiction.

Lemma 4.5. *Let $f : Y \to X$ be a flat morphism of quasi projective varieties, with X integral. Denote by $k(X)$ the field of rational functions on X. The following two conditions are equivalent:*
(1) Y is integral,
(2) $Y \times_X \operatorname{Spec}(k(X))$ is integral.

Proof. Suppose that (2) holds. Choose any two points y and z in Y and an affine open set $U \subset X$ that contains $f(y)$ and $f(z)$. The preimage $W = f^{-1}(U)$ of U is a quasi-projective variety, hence it contains an affine open set V that contains y and z. We prove that V is integral. The map $f : V \to U$ is flat and since the coordinate ring $k[U]$ of U injects into $k(U)$, $k[V]$ injects into $k[V] \otimes_{k[U]} k(U) = k[V \times_X \operatorname{Spec}(k(X))]$. But if $Y \times_X \operatorname{Spec}(k(X))$ is integral, then its open set $V \times_X \operatorname{Spec}(k(X))$ is integral as well and therefore $k[V] \subset \operatorname{Spec}\big(k[V \times_X \operatorname{Spec}(k(X))]\big)$ is integral. We have found an integral affine open set V containing y and z. By 4.4, Y is irreducible. It is obviously reduced because it can be covered by affine open sets like V.

Suppose now that (1) holds. Then, for any open affine set V of Y, the ring $k[V] \otimes_{k[U]} k(X)$, as a localization of $k[V]$, is integral. Hence every open affine set of $Y \times_X \operatorname{Spec}(k(X)$ is integral and, by 4.4, $Y \times_X \operatorname{Spec}(k(X))$ is integral.

5. Irreducibility

For proving 4.3 we need a result of [AHS] in a slightly different form. For convenience of the readers we give the proof, even if it is the same as that of [AHS].

Let A be a noetherian normal affine domain over an algebraically closed field k of characterisitic zero, and K its field of fractions. Suppose that $F(T)$ and $G(T)$ are two polynomials in $A[T]$ satisfying the following hypotheses:
(1) $F(T)$ is of degree $n \geq 1$ and generates a prime ideal of $A[T]$,
(2) $F(T)$ and $G(T)$ are coprime in $K[T]$ with $\deg(G(T)) \leq \deg(F(T))$.

We want to study the irreducibility of the polynomials

$$F(T) + \lambda G(T)\ , \quad \lambda \in k\ .$$

Note that, since A is normal, irreducibility over A and over K are the same by Gauss' lemma.

We define the set $\Pi \subset A[T]$ as the set of polynomials $f(T) \in A[T]$ for which there exists a $\lambda \in k$ and a proper factorization

$$F(T) + \lambda G(T) = f(T)g(T) ,$$

i.e., $f(T)$ and $g(T)$ are not in A.

Let B be the affine domain

$$\big(A[T]/(F(T))\big)$$

and L its field of fractions. For any $f \in A[T]$ we denote by $\overline{f}$ its image in B.

Lemma 5.1. *Let B be an affine domain over k and L its field of fractions. There exists a finite set of discrete valuations $\{v_1, \dots, v_q\}$ such that the only elements $b \in B$ for which $v_1(b) = \cdots = v_m(b) = 0$ are the elements of k.*

Proof. See [AHS], Lemma, page 55.

Let $\varpi : L^* \to \mathbb{Z}^q$ be defined by $\varpi(x) = (v_1(x), \dots, v_q(x))$. For elements $h \in A[T]$ we write $\varpi(h)$, $v_i(h)$ for $\varpi(\overline{h})$, $v_i(\overline{h})$.

Lemma 5.2. *Assume that F and G satisfy (1) and (2). If $F + \lambda G = fg$ is a proper factorization, then λ is uniquely determined by $\varpi(f)$. Thus $\varpi(\Pi)$ can be mapped bijectively onto the set of λ's for which $F + \lambda G$ is reducible.*

Proof. Suppose that

$$f_1 g_1 = F + \lambda_1 G$$

and

$$f_2 g_2 = F + \lambda_2 G ,$$

and that $\varpi(f_1) = \varpi(f_2)$ and $\lambda_1 \neq \lambda_2$. Then $\overline{f}_1/\overline{f}_2 = \mu$ for some $\mu \in k^*$. This means that $f_1 - \mu f_2 \in (F)$, but F is of higher degree than $f_1 - \mu f_2$, hence $f_1 = \mu f_2$. Substituting in the two equations above and subtracting one from the other we get $f_2(\mu g_1 - g_2) = (\lambda_1 - \lambda_2)G$. Then f_2 divides G and since it divides $F + \lambda G$ it also divides F, which contradicts the assumption that F and G are coprime. Thus $\lambda_1 = \lambda_2$.

Corollary 5.3. *Under the assumptions (1) and (2), if $(1 - \lambda)F + \lambda G = fg$ is a proper factorization, then λ is uniquely determined by $\varpi(g)$. Thus, if Λ is the set of proper factors of $(1 - \lambda)F + \lambda G$, then $\varpi(\Lambda)$ can be mapped bijectively onto the set of λ's for which $(1 - \lambda)F + \lambda G$ is reducible.*

Proof. This is just a restatement of Lemma 5.2.

6. Proof of Proposition 4.3

Consider the diagram

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[d, "\pi"] \arrow[dr, "f"] & \\ X & X \times_k \mathbb{A}^1 \arrow[l] \arrow[r] & \mathbb{A}^1 . \end{tikzcd}$$

The map π is finite and flat because for any f and U the polynomial

$$Q_{f,U}(t,T) = (1-t)P_{f,U}(T) + t(T - w_1/f)\cdots(T - w_n/f))$$

is monic in T. The horizontal maps are clearly flat, hence f and g are flat. We first check that W is integral. We denote by $Q_\xi(t,T)$ the restriction of some $Q_{f,U}(t,T)$ to the generic point ξ of X. By 4.5 it suffices to show that $W \times_X \operatorname{Spec}(k(X)) = \operatorname{Spec}\big(k(X)[T,t]/(Q_\xi(t,T))\big)$ is integral. Since $P_{f,U}(T)$ and $(T-w_1/f)\cdots(T-w_n/f))$ are coprime in $k(X)[T]$, $Q_\xi(t,T)$ is irreducible.

Since W is flat over $\mathbb{A}^1$, 4.3 implies that $W \times_{\mathbb{A}^1} \operatorname{Spec}(k(t))$ is also integral.

Consider now $W \otimes_{k[t]} \overline{k(t)}$ and suppose that it is not integral. This means that $Q_\xi(t,T)$ factors in $k(X) \otimes_k \overline{k(t)}[T]$: $Q_\xi(t,T) = Q_1(T)\cdots Q_m(T)$ with Q_i defined and irreducible over $k(X) \otimes_k \overline{k(t)}$. In this case $Q_\xi(t,T)$ already factors in the same way when we replace $\overline{k(t)}$ by a finite extension $k(t)[w]/(\Phi(t,w))$ of $k(t)$, where $\Phi(t,w) = c_0(t)w^m + \cdots + c_m(t)$ with $c_i(t) \in k[t]$.
There exists $g(t) \in k[t]$ such that $Q_i(T) \in \big(k[t,w,1/g]/(\Phi(t,w))\big)[T]$. If $\lambda \in k$ is different from the roots of $g(t)c_0(t)$, we can choose a $\mu \in k$ such that $\varphi(\lambda,\mu) = 0$. Then, specializing w to μ and t to λ yields a map $\big(k[t,w,1/g]/(\Phi(t,w))\big) \to k$. Denoting the images under specialization by "bar" we obtain $Q_\xi(\lambda,T) = \overline{Q}_1(T)\cdots\overline{Q}_m(T)$. This shows that $Q_\xi(t,T)$ decomposes for all but a finite number of values of t in k. We want to show that this fact leads to a contradiction.

It suffices to show (by Gauss' lemma) that, for some U and some f, $Q_{f,U}(\lambda,T)$ is irreducible for almost all $\lambda \in k$. We now follow the notation of §4. The irreduciblility of $Q_{f,U}(\lambda,T)$ is equivalent to the irreduciblility of $R_{f,U}(\lambda,T)$, and the factorization of $Q_{f,U}(\lambda,T)$ in irreducible monic factors yields a similar factorization $R_{f,U}(\lambda,T) = R_1(T)\cdots R_m(T)$ into irreducible factors with constant term 1.

We shall use the result of the preceding section with $A = H^0(U,\mathcal{O}_X)$, $F = \Phi_{f,U}$ and $G = \Psi_{f,U}$.

We denote by R_ξ the restriction of $R_{f,U}(\lambda,T)$ to the generic point ξ of X and by $R_{\xi,i}$ the restriction to ξ of R_i. It follows from 4.2 that $R_{f,U}(\lambda,T)$ is the restriction of a global section R of $\mathcal{O}_X \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes n}$ and the sections of this bundle form a finite dimensional vector space over k. Since the factorization of $R_{f,U}(\lambda,T)$ over $k[U]$ into factors with constant term 1 is unique, each factor $R_{\xi,i}$ of R_ξ extends to a unique factor of $R_{f,U}$ for any U over which $\mathcal{L}|_U$ is generated by a global section f. By patching, we see that every $R_{\xi,i}$ is the restriction of a global section of

$$\mathcal{O}_X \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{\otimes n} .$$

This shows that the set Λ of proper factors φ of the polynomials $R_{f,U}(\lambda, T)$ when λ varies over k, are contained in a finite dimensional vector space over k. In other words, we can write any $\varphi \in \Lambda$ as $\varphi = \mu_1 h_1 + \cdots + \mu_r h_r$ for some fixed polynomials $h_1, \ldots, h_r$. Let L be the field of fractions of $B = k[U][T]/(\Phi)$ where $\Phi = T^n P(\lambda, T^{-1})$. The image of $R_{f,U}(\lambda, T)$ in B is the class of $\lambda\Psi_{f,U}(T)$ where Ψ is as defined in 4.2. For any discrete k-valuation v of L we have $v(\varphi) \geq \min\{v(h_1), \ldots, v(h_r)\}$, which shows that $v(\Lambda)$ is bounded below. But if $\varphi\psi = R_{f,U}(\lambda, T)$ (necessarily with $\lambda \neq 0$ because $\Phi_{f,U}$ is irreducible) then $-v(\varphi) = v(\psi) - v(R_{f,U}(\lambda, T)) = v(\psi) - v(\Psi_{f,U}(T))$. Since ψ is also in Λ, the value of $v(\varphi)$ is as well bounded above. This proves that if $v_1, \ldots, v_q$ are the valuations described in 5.1 (for $B = k[U][T]/(\Phi)$ and $F(T) = \Phi(T)$), then the image of Λ in $\mathbb{Z}^q$ obtained by mapping $\varphi \in \Lambda$ to $(v_1(\varphi), \ldots, v_q(\varphi))$ is a finite set. We conclude, using 5.3, that $R_{f,U}(\lambda, T)$ is reducible only for finitely many $\lambda \in k$. Thus $Q_\xi(\lambda, T)$ is irreducible for almost all $\lambda \in k$, leading to a contradiction.

This finishes the proof of Proposition 4.3.

7. Galois splittings

We now construct, for any $\lambda \in k^N$, a Galois covering Z_λ of X with group G, such that $X = Z_\lambda/G$. Notice that, in general, even if Y_λ is smooth and $Y_\lambda \to X$ is a projective map, the Galois closure of Y_λ is not smooth. Therefore, in order to have Y and Z smooth, we must construct both at the same time.

We proceed as in the construction of Y_λ. Let $U \subset X$ be an affine open set for which $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U f$ for some section f on U . Let $\mathcal{L}$, $s_1, \ldots, s_N \in H^0(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L})$ and $s = \lambda_1 s_1 + \cdots \lambda_n s_n$ be as before. Let $P_{f,U}(T) = T^n + b_1 T^{n-1} + \cdots + b_n$ be the characteristic polynomial of $s/f \in \mathcal{A}(U)$. We choose n isomorphic copies $\mathcal{L}_1, \ldots, \mathcal{L}_n$ of $\mathcal{L}$ and, for each i, $f_i = f$ the generator of $\mathcal{L}_i|_U$. Consider

$$\mathcal{T} = Sym\big(\mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_n^{-1}\big) .$$

Writing $f_i^{-1} f_j^{-1}$ instead of $f_i^{-1} \otimes_{\mathcal{O}_U} f_j^{-1}$ we shall write the restriction of $\mathcal{T}$ to U simply as

$$\bigoplus \mathcal{O}_U f_1^{-i_1} \cdots f_n^{-i_n} .$$

Note that $\mathcal{T}|_U$ is isomorphic to $\mathcal{O}_U[T_1, \ldots, T_n]$.

For $1 \leq i \leq n$ let σ_i be the i-th elementary symmetric function in the n variables $T_1, \ldots, T_n$. We define $\mathcal{J}_{f,U} \subset \mathcal{T}|_U$ as the ideal generated by

$$\sigma_i(f_1^{-1}, \ldots, f_n^{-1}) - (-1)^i b_i \ 1 \leq i \leq n .$$

It corresponds in the polynomial algebra to the ideal generated by

$$\sigma_i(T_1, \ldots, T_n) - (-1)^i b_i \ 1 \leq i \leq n .$$

As in the preceding section, it is easy to check that these ideals do not depend on the choice of f and can therefore be patched over the various U's to obtain a global ideal $\mathcal{J}_\lambda \subset \mathcal{T}$.

Let Z_λ be the closed subscheme of $\mathrm{Spec}(\mathcal{T})$ defined by $\mathcal{J}_\lambda$.

Proposition 7.1. *The symmetric group $\mathcal{S}_n$ acts on Z_λ via its obvious action on $\mathcal{T}$. The quotient $Z_\lambda/\mathcal{S}_n$ coincides with X and Y_λ coincides with the quotient $Z_\lambda/\mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ is the isotropy group of* 1.

Proof. It suffices to deal with the affine case. Let $P(T) = T^n + b_1T^{n-1} + \cdots + b_n$ be a monic polynomial with coefficients in a ring R and assume (it is the case we are ultimately interested in) that 2 is invertible in R. Let B be the quotient of $R[T_1, \dots, T_n]$ by the ideal I generated by all polynomials $\sigma_i(T_1, \dots, T_n) - (-1)^i b_i$. Denote by $\sigma'_i(T_1, \dots, T_{n-1})$ the i-th elementary symmetric function in $n-1$ indeterminates. To the quotient of Z_λ by $\mathcal{S}_n$ corresponds the ring of invariants $B^{\mathcal{S}_n}$. Since $\mathcal{S}_n$ acts trivially on I and 2 is invertible in R we have

$$H^1(\mathcal{S}_n, I) = Hom(\mathcal{S}_n, I) = Hom(\mu_2, I) = 0 \ .$$

Hence

$$\big(R[T_1, \dots, T_n]/I\big)^{\mathcal{S}_n} = R[T_1, \dots, T_n]^{\mathcal{S}_n}/I^{\mathcal{S}_n} = R[\sigma_1, \dots, \sigma_n]/I = R \ .$$

Similarly we obtain

$$\big(R[T_1, \dots, T_n]/I\big)^{\mathcal{S}_{n-1}} = R[\sigma'_1, \dots, \sigma'_{n-1}, T_n]/I \ .$$

The relations

$$\sigma_1 = \sigma'_1 + T_n \ , \ \sigma_i = \sigma'_i + T_n\sigma'_{i-1} \text{ for } 2 \le i \le n-1 \text{ and } \sigma_n = T_n\sigma'_{n-1}$$

immediately give that $R[\sigma'_1, \dots, \sigma'_{n-1}, T_n]/I = R[T_n]/(P(T_n))$.

Theorem 7.2. *Assume that k is of characteristic zero. There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Z_λ is a projective smooth surface.*

The proof requires some preliminaries. Let X_{ij} with i, j running from 1 to n be indeterminates and write $P(T) = T^n + a_1T^{n-1} + \cdots + a_n$ for the characteristic polynomial of the generic matrix (X_{ij}). Let A be the polynomial k-algebra in the X_{ij}. Consider another set $T_1, \dots, T_n$ of indeterminates and put

$$B_n = A[T_1, \dots, T_n]/I$$

where I is the ideal generated by all the polynomials $\sigma_i(T_1, \dots, T_n) - (-1)^i a_i$ for $1 \le i \le n$. Let $Z_n = \mathrm{Spec}(B_n)$. We want to determine $\mathrm{Sing}(Z_n)$.

A k-point of Z_n is a pair (α, t) with $\alpha \in M_n(k)$ and $t = (t_1, \dots, t_n) \in k^n$ such that $t_1, \dots, t_n$ are the eigenvalues of α, i.e. the roots of the characteristic polynomial of α, which we write as

$$P(\alpha)(T) = T^n + a_1(\alpha)T^{n-1} + \cdots + a_n(\alpha) \ .$$

Let $\pi : Z_n \to \mathrm{Spec}(A)$ be the first projection and let $S = \pi(\mathrm{Sing}(Z_n))$. We want to compute the dimension of S.

Let (α, t) be a singularity of Z_n. Since no $\sigma_i(T_1, \ldots, T_n)$ involves the X_{ij} and no a_j involves the T_i, if we order the X_{ij} lexicographically, the Jacobian matrix of the equations $\sigma_i(T_1, \ldots, T_n) - (-1)^i a_i = 0$ is of size $(n^2 + n) \times n$ and looks as follows:

$$J = \begin{pmatrix} \frac{\partial \sigma_1}{\partial T_1} & \cdots & \frac{\partial \sigma_n}{\partial T_1} \\ \vdots & & \vdots \\ \frac{\partial \sigma_1}{\partial T_n} & \cdots & \frac{\partial \sigma_n}{\partial T_n} \\ \frac{\partial a_1}{\partial X_{11}} & \cdots & \frac{\partial a_n}{\partial X_{11}} \\ \vdots & & \vdots \\ \frac{\partial a_1}{\partial X_{nn}} & \cdots & \frac{\partial a_n}{\partial X_{nn}} \end{pmatrix}.$$

By 7.1, π is a finite map and the dimension of Z_n is n^2. The point (α, t) being a singularity of Z_n, the Jacobian criterion implies that the rank of J at (α, t) is at most $n - 1$. Thus, in particular, the determinant δ of the top $n \times n$ block of J must vanish at (α, t). It is well-known (and can be proved by an easy induction on n) that $\delta = \pm \prod_{i<j}(T_i - T_j)$. This shows that α has at least two equal eigenvalues. In other words, denoting by $V(-)$ the vanishing locus of a given set of polynomials, (α, t) belongs to the vanishing locus $V(\delta^2)$ of the discriminant δ^2 of $P(T)$.

Consider now $\operatorname{Sing}(Z_n) \cap V(a_1, \ldots, a_n)$. Since $\operatorname{Sing}(Z_n) \subset V(\delta^2)$ we have $\operatorname{Sing}(Z_n \cap V(a_1, \ldots, a_n) = \operatorname{Sing}(Z_n \cap V(\delta^2, a_1, \ldots, a_n)$. But the vanishing of $a_1, \ldots, a_{n-1}$ and δ^2 already implies the vanishing of a_n; in fact, if $T^n - a_n$ has a multiple root, then $a_n = 0$ (we are in characteristic 0). Thus

$$\operatorname{Sing}(Z_n) \cap V(a_1, \ldots, a_{n-1}) = \operatorname{Sing}(Z_n) \cap V(a_1, \ldots, a_n)$$

and therefore $\dim(\operatorname{Sing}(Z_n)) \leq \dim(\operatorname{Sing}(Z_n) \cap V(a_1, \ldots, a_n)) + n - 1$. The set $V(a_1, \ldots, a_n)$ is the set $\mathcal{N}$ of nilpotent matrices. On the other hand, the bottom block of the Jacobian matrix must have rank at most $n - 1$, which means that α is a singular point of $\mathcal{N}$. This shows that $\operatorname{Sing}(Z_n) \cap \mathcal{N} \subseteq \operatorname{Sing}(\mathcal{N})$ and from the previous inequality we obtain the next result.

Lemma 7.3. *The dimension of* $\operatorname{Sing}(Z_n)$ *is at most* $\dim(\operatorname{Sing}(\mathcal{N})) + n - 1$.

We now compute the dimension of $\operatorname{Sing}(\mathcal{N})$. We begin with the computation of the dimension of $\mathcal{N}$.

Proposition 7.4. *Let* $\mathcal{N} \subset M_n$ *denote the variety of nilpotent matrices. Then the dimension of* $\mathcal{N}$ *is* $n^2 - n$.

Proof. Since $\mathcal{N}$ is defined by the ideal $(a_1, \ldots, a_n)$ of $A = k[X_{11}, X_{12}, \ldots, X_{nn}]$, it suffices to show that this ideal has height n. Let I be the ideal generated by $(a_1, \ldots a_n, X_{ij} \mid i \neq j)$. We claim that this ideal has height n^2. The ring A/I is isomorphic to $k[X_{11}, X_{2,2}, \ldots, X_{nn}]/J$ where J is the ideal generated by elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ in X_{ii}. Since $k[X_{11}, \ldots, X_{nn}]$ is finite over $k[\sigma_1, \ldots, \sigma_n]$, the ideal J has height n in $k[X_{11}, \ldots, X_{nn}]$. Hence I is supported only at closed points. Since the a_i are homogeneous, it follows that the ideal $(a_1, \ldots, a_n)$ has height n.

Lemma 7.5. *A nilpotent matrix α whose Jordan form consists of only one cyclic block is not a singularity of $\mathcal{N}$. More precisely, the determinant of $\left(\frac{\partial a_i}{\partial X_{j1}}\right)$ is not zero at α.*

Proof. Let A be as before and $P(T) = T^n + a_1 T^{n-1} + \cdots + a_n$ the characteristic polynomial of the generic matrix (X_{ij}). The variety of nilpotent matrices is $\mathcal{N} = V(a_1, \ldots, a_n)$. We show that at

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

the jacobian matrix $\left(\frac{\partial a_i}{\partial X_{ij}}\right)$ has rank n. We compute the $n \times n$ matrix $\left(\frac{\partial a_i}{\partial X_{j1}}\right)$. The derivative of a_i by X_{j1} is the coefficient of T^{n-i} in $\frac{\partial P(T)}{\partial X_{j1}}$. Developing the determinant of $(X_{ij}) - T\mathrm{I}_n$ along the first column we find

$$\pm P(T) = (X_{11} - T)P_1(T) + X_{2,1}P_2(T) + \cdots + X_{n,1}P_n(T)$$

where $P_i(T)$ is the determinant of an $(n-1) \times (n-1)$ matrix M_i. At $(X_{ij}) = \alpha$ we find

$$M_i(\alpha) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

with

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -T & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -T & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -T & 1 \end{pmatrix}$$

of size $j-1$ and

$$B_2 = \begin{pmatrix} -T & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -T & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & -T & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -T & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -T \end{pmatrix}$$

of size $n - j$. Thus $P_j(T) = \pm T^{n-j}$ and $\frac{\partial a_i}{\partial X_{j1}}(\alpha)$ is ± 1 for $j = i$ and zero otherwise. This proves the lemma.

Lemma 7.6. *The set $\mathcal{N}_2$ of nilpotent matrices whose Jordan form has exactly two cyclic blocks are dense in the set of nilpotent matrices whose Jordan form has two or more blocks.*

Proof. Let $\alpha = \mathrm{diag}(B_1, B_2, \ldots, B_m)$ be a nilpotent matrix which we can assume to be in Jordan form with blocks B_1, ... ,B_m, $m \geq 3$. Let $g \neq 0$ with $g \in A$ define a neighbourhood of α. We can find constants $\epsilon_2, \ldots, \epsilon_{m-1}$ such that replacing the zeros between the superdiagonals of B_2 and B_3, between the superdiagonals B_3 and B_4 and so on, by the ϵ_i we obtain a matrix α' such that $g(\alpha') \neq 0$. Clearly α' has two cyclic blocks.

Lemma 7.7. *If $\alpha \in \mathcal{N}$ has a Jordan form with two or more cyclic blocks, then α is a singularity of $\mathcal{N}$.*

Proof. We may assume that α is in Jordan form and can be written as $\alpha = \mathrm{diag}(B_1, B_2, \ldots, B_m)$ where $m \geq 2$, each B_i is a cyclic Jordan block, B_1 is of size p and B_2 of size q. We can write the generic matrix as $(X_{ij}) = (\alpha + Y_{ij})$. Then $\frac{\partial a_i}{\partial X_{ij}}(\alpha) = \frac{\partial a_i}{\partial Y_{ij}}(0)$. But in the matrix $\alpha+(Y_{ij})$ the p-th line and the $(p+q)$-th line are linear homogeneous in the Y_{ij}, hence developing the determinant of $\alpha + (Y_{ij})$ along these two lines we see that $a_n(Y_{ij} \mid 1 \leq i, j \leq n)$ has no constant and no linear term. This shows that all the derivatives $\frac{\partial a_n}{\partial Y_{ij}}$ vanish at the origin and therefore the Jacobian matrix $\frac{\partial a_i}{\partial Y_{ij}}$ cannot be of rank n.

Corollary 7.8. *The set $\mathcal{N}_2$ is dense in* $\mathrm{Sing}(\mathcal{N})$

The set $\mathcal{N}_2$ is the union of the $GL_n(k)$-orbits $S_{p,q}$ of all the matrices of the form $\beta = \mathrm{diag}(B_p, B_q)$ where B_p is the nilpotent cyclic Jordan block of size p and B_q the nilpotent cyclic Jordan block of size $q = n - p$. In particular, it is the finite union of the constructible sets $S_{p,q}$. The dimension of $S_{p,q}$ is $n^2 - s$ where s is the dimension of the isotropy group of β

Lemma 7.9. *For $n \geq 3$ the dimension of the isotropy group of* $\mathrm{diag}(B_p, B_q)$ *is $n + 2\min(p, q)$. In particular it is always at least $n + 2$.*

Proof. Let $\Gamma \subset GL_n(K)$ be the isotropy group of $\beta = \mathrm{diag}(B_p, B_q)$. Let

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an element of Γ, written with blocks A, B, C, D of suitable sizes. The condition $\gamma\beta\gamma^{-1} = \beta$ is equivalent to the conditions

$$AB_p = B_pA\ ,\ DB_q = B_qD\ ,\ BB_q = B_pB\ ,\ CB_p = B_qC\ .$$

We compute the dimension of the linear subspace Γ_0 of $M_n(K)$ consisting of matrices that satisfy the four conditions above.

An explicit matrix computation shows that the first condition gives

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_{p-1} & a_p \\ 0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{p-2} & a_{p-1} \\ 0 & 0 & a_1 & \cdot & \cdot & \cdot & a_{p-3} & a_{p-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_1 & a_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & a_1 \end{pmatrix}$$

A similar result holds for D, hence the matrices $\mathrm{diag}(A, D)$ in Γ_0 span a linear space of dimension $p + q = n$.

Assume now that $p \leq q$. An explicit computation shows that the third condition gives

$$B = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & b_{p-1} & b_p \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & b_1 & b_2 & \cdot & \cdot & \cdot & b_{p-2} & b_{p-1} \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & b_1 & \cdot & \cdot & \cdot & b_{p-3} & b_{p-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & b_1 \end{pmatrix}$$

A similar result holds for C, hence, when $p \leq q$ the dimension of Γ_0 is $n + p + p = n + 2\min(p, q)$ and clearly this is also the dimension (as a variety) of Γ.

Proposition 7.10. *For $n \geq 3$ the dimension of* $\mathrm{Sing}(\mathcal{N})$ *is* $n^2 - n - 2$.

Proof. By 7.8 and 7.9, $\dim(\mathrm{Sing}(\mathcal{N})) = \dim(\mathcal{N}_2) = n^2 - \min_{p,q}(\dim(S_{p,q}))$. The isotropy group of minimal dimension is $S_{1,n-1}$ which has dimension $n + 2$. Thus $\dim(\mathcal{N}_2) = n^2 - (n + 2)$.

Theorem 7.11. *For $n \geq 3$ the dimension of* $\mathrm{Sing}(Z_n)$ *is at most* $n^2 - 3$

Proof. This immediately follows from 7.3 and 7.10.

Proof of Theorem 7.2. If $n = 2$ then $U = k^N$ and for any $\lambda \in k^N$, $Z_\lambda = Y_\lambda$. We therefore assume that $n \geq 3$. In this case the proof is on similar lines as the proof of Theorem 2.11. We extend the base to $\widetilde{X} = X \times \mathbb{A}^N$ where $\mathbb{A}^N = \mathrm{Spec}\,(k[t_1, \ldots, t_N])$ and define $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_i$ for $1 \leq i \leq n$ as the inverse images of $\mathcal{A}$, $\mathcal{L}$ and the $\mathcal{L}_i$'s under the projection $\pi : \widetilde{X} \to X$. Repeating the construction of $\mathcal{J}_\lambda$ we obtain an ideal $\mathcal{J}_t$, where $t = (t_1, \ldots, t_N)$, which specializes to $\mathcal{J}_\lambda$ when we specialize t to λ. The scheme $\widetilde{Z}$ is the closed subscheme of

$$\mathrm{Spec}(\widetilde{\mathcal{T}}) = \mathrm{Spec}\big(Sym\big(\widetilde{\mathcal{L}_1}^{-1} \oplus \cdots \oplus \widetilde{\mathcal{L}_n}^{-1}\big)\big)$$

defined by $\mathcal{J}_t$.

Look at the diagram

$$\begin{tikzcd} & \widetilde{Z} \arrow[dl, "p"'] \arrow[d, "\pi"] \arrow[dr, "q"] & \\ X & X \times \mathbb{A}^N \arrow[l] \arrow[r] & \mathbb{A}^N \end{tikzcd}$$

The map π is clearly finite and flat and the two projections from $X \times \mathbb{A}^N$ are flat, hence p is flat. By 2.12 the singularities of $\widetilde{Z}$ are contained in the union of the singularities of the fibers of p. Since, by Theorem 7.11, the singularities of the closed fibres of p are at worst in codimension 3, we can argue exactly as in the proof of Theorem 2.11 and conclude that q is generically smooth, from which the assertion of Theorem 7.2 immediately follows.

In general the fibre Z_λ is not integral. This happens, for instance, if Y is already a Galois covering of X. But if we choose Z_λ smooth, then it will be a disjoint union of irreducible smooth varieties and picking one of them (which we call Z) we obtain a Galois covering $\pi : Z \to X$. Let G be the Galois group of $k(Z)/k(X)$, $|G|$ its order and $H \subset G$ the Galois group of $k(Z)/k(Y)$. Clearly H is of index n in G.

Proposition 7.12. *Suppose that $n \geq 3$ and let $x \in X$ be a closed point. Then*
(a) if x has n preimages in Y, then $\pi : Z \to X$ is étale at x;
(b) if x has $n-1$ preimages $y_1, \dots, y_{n-1}$ corresponding to the roots $\tau_1, \dots, \tau_{n-1}$ of $P(x,T)$ and τ_1 is the unique double root of $P(x,T)$, then $\pi_Z : Z \to Y$ is étale at y_1;
(c) the fibre $\pi^{-1}(x)$ in Z consists of $|G|/2$ points and each of them has multiplicity 2.

Proof. If the point x has n distinct preimages $y_1, \dots, y_n$ in Y, then clearly π_Z is étale at y. We can identify each y_i with a root τ_i of $P(x,T)$ and we can identify every point z of $\pi^{-1}(x)$ with a permutation $(\sigma(\tau_1), \dots, \sigma(\tau_n))$ where $\sigma \in G$ and π_Z maps z to its first coordinate $\sigma(\tau_1)$. Since locally at y_1 the map $Z \to Y$ is Galois with group H, the fibre of y_1 consists of $|H| = |G|/n$ points $(\tau_1, \sigma(\tau_2), \dots, \sigma(\tau_n))$ where $\sigma \in H$. When $\tau_1 = \tau_2$ and $\tau_i \neq \tau_1$ for $i \neq 1, 2$, the fibre of $y_1 = y_2$ still consists of $|H|$ distinct points $(\tau_1, \sigma(\tau_2), \dots, \sigma(\tau_n))$, hence π_Z is étale at y_1. This proves (b) and shows that every $z \in \pi_Z^{-1}(y_1)$ has multiplicity 2 in $\pi^{-1}(x)$. Since G operates transitively on $\pi^{-1}(x)$, this is true of all the points in the fibre, whence (c).

Theorem 7.13. *Let X be a projective smooth surface over an algebraically closed field k of characteristic zero and $\mathcal{A}$ an Azumaya algebra over X. Suppose that $\mathcal{A}_{k(X)}$ is a division algebra of prime degree l. Then there exist smooth integral projective surfaces Y and Z and finite flat maps*

$$Z \xrightarrow{\pi_Z} Y \xrightarrow{\pi_Y} X$$

such that

(1) $\pi_Y^* \mathcal{A}$ is trivial in the Brauer group of Y;

(2) the composite $Z \xrightarrow{\pi_Z} Y \xrightarrow{\pi_Y} X$ is a (ramified) Galois covering;

(3) the degree of π_Y is l and the degree of π_Z is prime to l;

(4) there are only finitely closed point on X which have fewer that $l-1$ preimages in Y;

(5) if π_Y is ramified at $y \in Y$ and $\pi_Y(y)$ has $l-1$ preimages in Y, then π_Z is étale at y.

8. General sections of a very ample line bundle.

For a finite map $f : V \to W$ of varieties we shall always denote by $\Delta(V/W)$ the ramification locus of f in V and by $\delta(V/W) = f(\Delta(Y/W))$ the ramification locus of f in W, both understood with their reduced structure.

Let $Y \xrightarrow{\pi_Y} X$ be a map of smooth projective surfaces satisfying (1), (2), and (4) of 2.16. We put $\Delta = \pi_Y^{-1}(\delta)$ the set-theoretical preimage of $\delta(Y/X)$ in Y. Note that $\Delta(Y/X) \subset \Delta$.

We recall the local structure of π_Y. The construction of Y in §2 shows that, locally on X, Y is isomorphic to the spectrum of an extension $S = R[T]/(P(T))$, where $R = \mathcal{O}_{X,x}$ is the local ring of the closed point $x = \pi_Y(y)$ of X and $P(T) = T^n + b_1 T^{n-1} + \cdots + b_n$ is the characteristic polynomial of a generic element of an Azumaya algebra over R. We denote by m_x the maximal ideal of R and by m_y the maximal ideal of S corresponding to y.

Let L be a very ample line bundle on Y, generated by N global sections $s_1, \dots, s_N$. We say that a general section s of L satisfies a certain property if there exists a Zariski open dense set $U \subset k^N$ such that, for every $\lambda = (\lambda_1, \dots, \lambda_N) \in U$, $s = \sum_i \lambda_i s_i$ satisfies the property.

We want to show that a general global section s of L has a smooth vanishing locus $V(s)$ whose image $\pi_Y(V(s))$ has no singularity on $\delta(Y/X)$. We begin with a result of Colliot-Thélène ([CT], Lemma 3.1) . We set $V(s) = D$ and $\pi_Y(V(s)) = C$.

Proposition 8.1. *If s is a general section of L, then every point of $C \cap \delta(Y/X)$ has at most one preimage on D*

Proof. We essentially reproduce the proof given in [CT]. Consider the open set of $\Delta \times_X \Delta$

$$U = \big\{(y_1, y_2) \in \Delta \times_k \Delta \mid y_1 \neq y_2,\ \pi_Y(y_1) = \pi_Y(y_2)\big\}\ .$$

Let $H = H^0(Y, L)$ and Z a closed set in $U \times_k H$ defined by

$$Z = \big\{(y_1, y_2, s) \mid s(y_1) = s(y_2) = 0\big\}\ .$$

Since U is a one-dimensional scheme and the fibres of the projection $Z \to U$ have dimension $\dim(H)-2$, the dimension of Z is $\dim(H)-1$. Hence the projection of Z into H is contained in a proper closed subset and any section in the complementary open set has the required property.

A general section s of L has smooth vanishing locus D which crosses Δ transversally. With such a choice of s, our next step is to insure that, if y is in $D \cap \Delta(Y/X)$, then for some open neighbourhood U of y the restriction of π_Y to $D \cap U$ has a smooth image. Let $\varphi = 0$ be a local equation of D in a neighbourhood of y. The point y corresponds to a (multiple) root $\tau \in k$ of $P(x, T)$ and the function φ can be lifted to a polynomial $\Phi \in R[T]$.

Proposition 8.2. *Suppose that y is a closed point of $\Delta(Y/X) \cap D$, $\pi_Y(y) = x$ and $\pi_Y^{-1}(x) = \{y, y_2, \ldots, y_r\}$ where $y_2, \ldots, y_r$ are not on D. Suppose further that*

$$(\star) \qquad \frac{\partial \Phi(x, T)}{\partial T}(\tau) \neq 0\ .$$

Then π_Y induces an isomorphism $\mathcal{O}_{C,x} \to \mathcal{O}_{D,y}$. Thus, D being smooth at y, C is smooth at x.

Proof. Note that by 8.1 every $x \in \delta(Y/X) \cap C$ has exactly one preimage in D. Let R be the local ring $\mathcal{O}_{X,x}$ of X at x. Let $y, y_2, \ldots, y_r$ be the preimages of x in Y. Since $x \in C \cap \delta(Y/X)$ has exactly one preimage y on D, the points $y_2, \ldots, y_r$ do not lie on D.The homomorphism $\pi_Y^* : \mathcal{O}_{C,x} \to \mathcal{O}_{D,y}$ is the finite extension

$$\chi : R\big/(P, \Phi) \cap R \longrightarrow R[T]\big/(P, \Phi)\ .$$

We may assume, by a change of variable, that $y \in \operatorname{Spec}(R[T])$ corresponds to the maximal ideal $m_y = (m_x, T)$. Denoting by "bar" the reduction modulo m_x, we have $\overline{P}(T) = T^e P_1(T)$ with $e \geq 2$, $P_1(0) \neq 0$, and $\overline{\Phi}(T) = T\Phi_1(T)$. By $(\star)$ we have $\Phi_1(0) \neq 0$. Hence $R[T]\big/(P, \Phi, m_x) = k$. By Nakayama's lemma this implies that χ is an isomorphism.

It remains to show that the condition $(\star)$ can be realized at every point of $\Delta(Y/X) \cap D$. We can translate it in terms of sections in the following way. Let $s \in H^0(Y, L)$. Let $y \in \Delta$

be a closed point at which the section s vanishes, $x = \pi_Y(y)$ and $L(y)$ the fibre of L at y. We define $(\partial s/\partial T)(y) \in L(y)$ as follows. Let $R = \mathcal{O}_{X,x}$, $O_{Y,\pi_Y^{-1}(x)} = R[T]/(P(T))$ and let e be a generator of L at $O_{Y,\pi_Y^{-1}(x)}$. Let $s = \varphi e$ with $\varphi \in O_{Y,\pi_Y^{-1}(x)}$. Then, $\varphi(y) = 0$. Set $(\partial s/\partial T)(y) = (\partial \Phi/\partial T)(y) \cdot e(y)$ where $\Phi \in R[T]$ represents φ and $e(y)$ the image of e in $L(y)$. This definition is independent of the choice of e and Φ since $P'(T)$ vanishes at y. The condition $(\star)$ is equivalent to $(\partial s/\partial T)(y) \neq 0$.

Lemma 8.3. *For every closed point $y \in \Delta(Y/X)$ there exists $s \in H^0(Y, L)$ such that if $s(y) = 0$, then $(\partial s/\partial T)(y) \neq 0$.*

Proof. Let $H^0(Y, L)_y$ be the vector space of all sections vanishing at y. By Bertini's theorem ([Ha$_1$], Ch. II, 8.18) the map $H^0(Y, L)_y \to m_y/m_y^2$ sending $s = \varphi e$ to the class of φ is surjective. The maximal ideal m_y at y of $R[T]/(P(T))$ is generated by $(m_x, (T - \alpha))$, where m_x is the maximal ideal of R and α a repeated root of $\overline{P}(T)$ in k. By the surjectivity assertion in the theorem of Bertini stated above, there exists $s \in H^0(Y, L)_y$ such that $s = \varphi e$ with $\varphi - (T - \alpha) \in m_y^2$. For this choice of s, $(\partial s/\partial T)(y) = e(y) \neq 0$.

Proposition 8.4. *For a general $s \in H^0(Y, L)$, $V(s)$ is smooth, irreducible, intersects Δ transversally and avoids the ramification locus of $\pi_Y|_\Delta : \Delta \to \delta(Y/X)$; further, for each intersection y of $V(s)$ with $\Delta(Y/X)$, $\pi_Y(V(s))$ is smooth at $\pi_Y(y)$.*

Proof. The set of singular points of Δ and the ramification locus of $\pi_Y|\Delta : \Delta \to \delta(Y/X)$ form a finite set Σ of closed points of Δ. By Bertini's theorem, for a general $s \in H^0(Y, L)$, $s(y) \neq 0$ for $y \in \Sigma$ and $V(s)$ is smooth, irreducible and intersects Δ transversally. We show that for a general $s \in H^0(Y, L)$, $s(y) = 0$ implies that $(\partial s/\partial T)(y) \neq 0$. Let $s_1, s_2, \ldots, s_N$ be a set of generators of the k-vector space $H^0(Y, L)$. Let $s = \sum X_i s_i$ where X_i are indeterminates and let $\overline{s}$ be the restricition of s to $\Delta \times \mathbb{A}^n$. We look at the closed set $V(\overline{s}, \partial s/\partial T) \subset \Delta \times A^N$. We claim that the dimension of $V(\overline{s}, \partial s/\partial T)$ is at most $N - 1$. We check this at every smooth point $y \in \Delta$. Let B be the local ring of Δ at y. Let e a local generator of s at y and $s_i = f_i e$, $\partial f_i/\partial T = g_i$. We need to show that the dimension of $B[X_1, \ldots, X_N]/(\sum f_i X_i, \sum g_i X_i)$ is at most $N - 1$. Suppose this dimension is greater than $N - 1$. In this case the height of the ideal $(\sum f_i X_i, \sum g_i X_i)$ is at most 1. Since B is a discrete valuation ring, there exists $a, b, h \in B[X_1, \ldots, X_N]$ such that $\sum f_i X_i = ah, \sum g_i X_i = bh$. Suppose that h does not belong to B. Degree considerations show that a belongs to B and, if a is not a unit in B, it divides each f_i. Thus the s_i's vanish at y, contradicting the fact that they generate the sections of L globally. Therefore, a is a unit in B and $\sum g_i X_i = ba^{-1} \sum f_i X_i$. There exists by 8.3, $\lambda = (\lambda_1, \ldots, \lambda_N)$ such that $s_\lambda(y) = 0$ and $\partial s_\lambda/\partial T(y) \neq 0$ where $s_\lambda = \sum \lambda_i s_i$. This contradicts the equality $\sum g_i X_i = ba^{-1} \sum f_i X_i$. The case $h \in B$ is dealt with in the same way. Thus the dimension of $V(s, \partial s/\partial T)$ is at most $N - 1$. Under the projection $\Delta \times \mathbb{A}^N \to \mathbb{A}^N$, the set $V(s, \partial s/\partial T)$ maps into a proper closed subset of $\mathbb{A}^N$. Thus for an open set of $\lambda \in \mathbb{A}^N$, s_λ has the property that if $s_\lambda(y) = 0$ for $y \in \Delta_Y$, then $\frac{\partial s_\lambda}{\partial T} \neq 0$. By 8.2 $\pi_Y(V(s))$ is smooth at $\pi(y)$.

Proposition 8.5. *For a general section s of L each point of $\pi_Y(V(s))$ has at least $l - 1$ distinct preimages in Y.*

Proof. Since under $\pi_Y : Y \to X$ there only are finitely many points of X with fewer that

$l-1$ preimages (condition (4) of 2.16), if s is chosen such that $V(s)$ avoids the preimages of these finitely many points, then s satisfies the condition of the proposition.

Theorem 8.6. *Let $Z \xrightarrow{\pi_Z} Y \xrightarrow{\pi_Y} X$ be maps of smooth projective surfaces as in Theorem 7.13. Let L be a very ample line bundle on Y, generated by global sections $s_1, \dots, s_N$. Assume that $\pi_Z^* L$ is very ample. For any $\lambda = (\lambda_1, \dots, \lambda_N)$ set $s_\lambda = \lambda_1 s_1 + \cdots + \lambda_N s_N$. Let D_λ be the vanishing locus of s_λ and $C_\lambda = \pi_Y(D_\lambda)$ be the image of D_λ in X. Let $D_{Z,\lambda}$ be the vanishing locus of $\pi_Z^*(s_\lambda)$ on Z.*
There exists a dense open subset $U \subset k^N$ such that for every $\lambda \in U$ the following five conditions are satisfied:

(1) D_λ is irreducible, smooth and intersects Δ transversally;

(2) D_λ avoids the ramification locus of $\pi_Y|_\Delta : \Delta \to \delta(Y/X)$;

(3) every point of C_λ has at least $n-1$ preimages in Y;

(4) no singularitiy of C_λ lies on $\delta(Y/X)$;

(5) $D_{Z,\lambda}$ is smooth and irreducible.

Proof. Conditions (1) and (2) follow from 8.4. Condition (3) is a consequence of 8.5. Let x be in $C_\lambda \cap \delta(Y/X)$ and let y be its unique preimage in D. If $y \in \Delta(Y/X)$, then C_λ is smooth at x by 8.4. If y belongs to $\Delta \setminus \Delta(Y/X)$ then C_λ is smooth at x because D_λ is smooth, π_Y is étale at y and y is the unique preimage of x in D. Thus (4) is satisfied. To see condition (5) note that $\pi_Z^{-1}(D_\lambda)$ is the vanishing locus of the section $\sum_i \lambda_i \pi_Z^* s_i$ of $\pi_Z^* L$; since $\pi_Z^* L$ is very ample we can apply Bertini's theorem ([Ha$_1$], Ch. II, 8.18).

9. Extracting a root from a bundle

The main result of this section is the following theorem.

Theorem 9.1. *Let $Z \xrightarrow{\pi_Z} Y \xrightarrow{\pi_Y} X$ be maps of smooth projective surfaces satisfying (2) to (5) of Theorem 7.13. Suppose that n is equal to a prime l. Given a line bundle L_Y on Y with its class in*

$$\mathrm{Pic}(Y)/l \cdot \mathrm{Pic}(Y) \subset H^2(Y, \mu_l)$$

a pull-back of a class $\zeta \in H^2(X, \mu_l)$, there exists a map $h_X : \widetilde{X} \to X$ which is proper, generically cyclic of degree l and such that

(1) the normalization $\widetilde{Y}$ of $Y \times_X \widetilde{X}$ is smooth,

(2) if $h_Y : \widetilde{Y} \to Y$ is the natural map, then $h_Y^*(L_Y)$ vanishes in

$$\mathrm{Pic}(\widetilde{Y})/l\mathrm{Pic}(\widetilde{Y}) + \pi_{\widetilde{Y}}^*(\mathrm{Pic}(\widetilde{X})) \, .$$

This section is entirely dedicated to the proof of this theorem.

For any pair of morphisms $W \to V$ and $\mathrm{Spec}(A) \to V$ we denote the fibre product $W \times_V \mathrm{Spec}(A)$ by W_A.

Let $\mathcal{Z}^1(V)$ denote the free abelian group on codimension 1 cycles of any variety V. For a codimension 1 closed set $W \subset V$ we denote by $\{W\}$ the cycle in $\mathcal{Z}^1(V)$ corresponding to W and by $[W]$ the class of W in the divisor class group of V.

By modifying L_Y by the l^n-th power of an ample line bundle L_0 we may assume that L_Y is very ample. Since the pull-back of an ample line bundle by the finite map π_Z is ample ([Ha$_2$], I, 4.4) we may further assume that $\pi_z^* L$ is very ample. By the previous section there exists a smooth connencted curve D_Y on Y such that $L_Y = \mathcal{O}_Y(D_Y)$ and D_Y satisfies conditions (1) to (5) of Theorem 8.6.

Let $\pi_Y(D_Y) = D_X$. We have $(\pi_Y)_*([D_Y]) = r[D_X]$. Note that r is the degree of $k(D_Y)$ over $k(D_X)$ and thus $r \leq l$. If $r = l$, D_Y is the only divisor on Y lying over D_X, it is not ramified at its generic point, and $\pi_Y^*[D_X]$ is $[D_Y]$. Therefore $L_Y = (\pi_Y)^*(\mathcal{O}_X(D_X))$. In this case we can take $\widetilde{X} = X$ and $\widetilde{Y} = Y$.

Assume now that $r < l$. By assumption we have $[L_Y] = (\pi_Y)^*(\zeta) \in \mathrm{Pic}(Y)/l \subseteq H^2(Y, \mu_l)$. Then $r[D_X] = (\pi_Y)_*(\pi_Y)^*(\zeta) = l\zeta = 0$ in $H^2(X, \mu_l)$. Since $r < l$ this implies $[D_X] = 0$ in $\mathrm{Pic}(X)/l$. Let L_1 be a line bundle on X for which there is an isomorphism $\varphi : \mathcal{O}_X(D_X) \to L_1^{\otimes l}$. Choosing a section $s : \mathcal{O}_X \to L_1^{\otimes l}$ vanishing precisely along D_X we can define, using the dual map $\left(L_1^{-1}\right)^{\otimes l} \to \mathcal{O}_X$ (see [BPV], I, §17), an $\mathcal{O}_X$-algebra structure on

$$\mathcal{S} = \mathcal{O}_X \oplus L_1^{-1} \oplus \cdots \oplus \left(L_1^{-1}\right)^{\otimes(l-1)} .$$

Let $X' = \mathrm{Spec}(\mathcal{S})$ and $g_X : X' \to X$ the map giving the $\mathcal{O}_X$-algebra structure on $\mathcal{S}$. Note that locally on X the scheme X' looks like the spectrum of $R[t]\big/(t^l - f)$, f a local equation of D_X. This map is a cyclic cover of degree l, ramified precisely along D_X. In particular $(g_X)^*\{D_X\} = l\{D'\}$, where D' is an effective Cartier divisor on X'. We put $Y' = Y \times_X X'$, $Z' = Z \times_X X'$ and denote by $\pi_{Y'} : Y' \to X'$, $\pi_{Z'} : Z' \to Y'$ the obvious projections. We have a commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, "\pi_{Z'}"] \arrow[d, "g_Z"'] & Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "g_Y"] & X' \arrow[d, "g_X"] \\
Z \arrow[r, "\pi_Z"] & Y \arrow[r, "\pi_Y"] & X
\end{tikzcd}$$

Let $\pi = \pi_Y \circ \pi_Z$ and $\pi' = \pi_{Y'} \circ \pi_{Z'}$.

Proposition 9.2. *The schemes Z', Y' and X' are irreducible and normal. In particular, they have a finite number of singular points. Further,* $\mathrm{Sing}(X') \cap \delta(Y'/X') = \emptyset$ *and* $\mathrm{Sing}(Y') \cap \Delta(Y'/X')$ *maps under* $\pi_Y g_Y$ *to* $\delta(Y/X) \cap D_X$.

Proof. Since the ramifications $\delta(Y/X)$ and $\delta(X'/X) = D_X$ have no common component, $k(Z)$ and $k(Y)$ are disjoint from $k(X')$ over $k(X)$. The maps π and π_Y are flat, hence by 4.5, Z' and Y' are integral. By construction X', Y' and Z' are locally of the form $\mathrm{Spec}(S)$ where $S = R[t]\big/(t^l - f)$ and R is a regular ring. Thus X', Y' and Z' are locally complete intersections and to show that they are normal it suffices to show that they are nonsingular in codimension 1 ([Ha$_1$], Ch. II, 8.23). The singularities of X' are at most over the singularities of D_X (see [BPV], Ch. I, §17), which by condition (4) of 8.6 are away from $\delta(Y/X)$. Thus $\mathrm{Sing}(X')$ is finite and $\mathrm{Sing}(X') \cap \delta(Y'/X') = \emptyset$.

We now prove that $\mathrm{Sing}(Y')$ is finite. The proof for Z' is similar. Let $(a, b) \in \mathrm{Sing}(Y') \subset Y \times X'$ with $c = \pi_Y(a) = g_X(b) \in X$. If $c \notin \delta(Y/X)$, then π_Y is étale at a and therefore

$b \in \mathrm{Sing}(X')$ which is finite. If $c \notin D_X$, then g_X is étale at b and hence g_Y is étale at (a,b). Since Y is smooth, Y' is smooth at (a,b) and therefore the remaining singularities can only be among the finitely many points of Y' that lie over $\delta(Y/X) \cap D_X$. Thus $\mathrm{Sing}(Y')$ is the disjoint union of $\mathrm{Sing}(Y') \cap \Delta(Y'/X')$ and $\mathrm{Sing}(Y') \cap \pi_{Y'}^{-1}(\mathrm{Sing}(X'))$ and $\mathrm{Sing}(Y') \cap \Delta(Y'/X')$ maps to $\delta(Y/X) \cap D_X$.

We have

$$g_X^*\{D_X\} = l\{D'\}$$

for some curve D' and therefore $\pi_{Y'}^* g_X^*\{D_X\} = l\{D_1\}$ for some $\{D_1\} \in \mathcal{Z}^1(Y')$. On the other hand, $\pi_Y^*\{D_X\} = \{D_Y\} + \{D_2\}$ for some $\{D_2\} \in \mathcal{Z}^1(Y)$. Since D_Y has no common component with the discriminant, $\{D_2\}$ has no component equal to $\{D_Y\}$ and therefore the same is true of $g_Y^*\{D_Y\}$ and $g_Y^*\{D_2\}$. Thus

$$(\dagger) \qquad \pi_{Y'}^* g_X^*\{D_X\} = l\{D_1\} = g_Y^* \pi_Y^*\{D_X\} = g_Y^*\{D_Y\} + g_Y^*\{D_2\}\ .$$

We conclude that $g_Y^*\{D_Y\} = l\{D_{Y'}\}$ for some $\{D_{Y'}\} \in \mathcal{Z}^1(Y')$. The difficulty is that $\{D_{Y'}\}$ is a Weil divisor but not necessarily a Cartier divisor. The rest of the proof consists in trying to replace it by a Cartier divisor.

We start by replacing X' by a smooth surface $\widehat{X}$ and discuss the singularities of $\widehat{Y} = Y \times_X \widehat{X}$. Let $\psi_X : \widehat{X} \to X'$ be a resolution of singularities of X'; ψ_X is a proper birational morphism which is an isomorphism outside $\mathrm{Sing}(X')$ and with $\widehat{X}$ smooth. We set $\widehat{Y} = Y \times_X \widehat{X}$, $\widehat{Z} = Z \times_X \widehat{X}$ and consider the commutative diagram

$$\begin{tikzcd}
\widehat{Z} \arrow[r, "\pi_{\widehat{Z}}"] \arrow[d, "\psi_Z"'] & \widehat{Y} \arrow[r, "\pi_{\widehat{Y}}"] \arrow[d, "\psi_Y"] & \widehat{X} \arrow[d, "\psi_X"] \\
Z' \arrow[r, "\pi_{Z'}"] & Y' \arrow[r, "\pi_{Y'}"] & X'\ .
\end{tikzcd}$$

We set $\widehat{\pi} = \pi_{\widehat{Y}} \pi_{\widehat{Z}}$. Since $\widehat{Y}$ is étale over $\widehat{X}$ away from $\delta(\widehat{Y}/\widehat{X})$ and $\widehat{X}$ is smooth, $\widehat{Y}$ is smooth away from $\delta(\widehat{Y}/\widehat{X})$. Since ψ_X is an isomorphism outside $\mathrm{Sing}(X')$, and $\mathrm{Sing}(X') \cap \delta(Y'/X') = \emptyset$, ψ_X and ψ_Y are isomorhisms along $\delta(Y'/X')$ and $\Delta(Y'/X')$ respectively. Thus ψ_Y maps $\Sigma = \mathrm{Sing}(\widehat{Y})$ bijectively onto $\mathrm{Sing}(Y') \cap \Delta(Y'/X')$ which is finite and lies over to $\delta(Y/X) \cap D_X$ in X by 9.2. Similar arguments lead to the fact that $\mathrm{Sing}(\widehat{Z})$ lies over $\delta(Y/X) \cap D_X$. We set $\Sigma_0 = \pi_{\widehat{Y}}(\Sigma)$ and $\Sigma_1 = \widehat{\pi}^{-1}(\Sigma_0)$. We denote by
$A = \mathcal{O}_{\widehat{X},\Sigma_0}$ the semilocal ring at Σ_0,
A^h the henselization of A at its radical,
and by
$A^h_x = \mathcal{O}^h_{\widehat{X},x}$ the factor of A^h corresponding to a point $x \in \Sigma_0$.
Similarly, we define the rings B, C, and so on, by
$\mathrm{Spec}(B) = \widehat{Y} \times_{\widehat{X}} \mathrm{Spec}(A)$,
$\mathrm{Spec}(C) = \widehat{Z} \times_{\widehat{X}} \mathrm{Spec}(A)$,
$\mathrm{Spec}(B^h) = \widehat{Y} \times_{\widehat{X}} \mathrm{Spec}(A^h)$,

$\mathrm{Spec}(C^h) = \widehat{Z} \times_{\widehat{X}} \mathrm{Spec}(A^h)$
and further, for any $x \in \Sigma_0$,
$B_x = B \otimes_A A_x$,
$C_x = C \otimes_A A_x$,
$B^h_x = B \otimes_A A^h_x$,
$C^h_x = C \otimes_A A^h_x$,
and similarly B_y, C_z and B^h_y, C^h_z when $y \in \Sigma$ and $z \in \Sigma_1$.

We further denote by R the semilocal ring of X at the image of Σ_0, by R_x the local ring of X at the image of an $x \in \Sigma_0$, and by R^h and R^h_x their respective henselizations.

Lemma 9.3. *Let z be a point of Σ_1, $y = \pi_{\widehat{Z}}(z)$ and $x = \widehat{\pi}(z)$.*

(a) The singular locus of $\widehat{Z}$ is Σ_1.

(b) If y is in Σ then $A^h_x \to B^h_y$ is a quadratic extension, $B^h_y \to C^h_z$ is an isomorphism and $B^h_x \simeq B^h_y \times (A^h_x)^{l-2}$.

(c) If y is not in Σ then $A^h_x \to B^h_y$ is an isomorphism, $B^h_y \to C^h_z$ is a quadratic extension and $B^h_x \simeq (A^h_x)^l$.

(d) The A^h_x-algebra C^h_x is a product of $|G|/2$ quadratic extensions of A^h_x.

(e) Denoting by $Cl(-)$ the class group, if $y \in \Sigma$ we have

$$Cl(B) = Cl(B_x) = Cl(B_y) = Cl(C_z) \subsetneq Cl(C^h_z) = \mathbb{Z}/l\ .$$

In particular $Cl(B)$ is either zero or $\mathbb{Z}/l$.

Proof. In a suitable neighbourhood of Σ the map $\pi_{\widehat{Z}} : \widehat{Z} \to \widehat{Y}$ coincides with $\pi_{Z'} : Z' \to Y'$ which, by 7.12, is étale at every $y \in \Sigma$. Hence over every $x \in \Sigma_0$ there is a singularity z of $\widehat{Z}$ which clearly is in Σ_1. Since G acts transitively over Σ_1, we have $\mathrm{Sing}(\widehat{Z}) = \Sigma_1$. This proves (a).

Let u be a semilocal equation of D_X in R. Since ψ_X and ψ_Y are isomorphisms along Σ_0 we have

$$B = \frac{R[T,t]}{(P(T), t^l - u)} = \frac{A[T]}{(P(T))}\ .$$

We know by 8.6 that $P[T]$ has one double root and $l-2$ simple roots at every $x \in \Sigma_0$, hence we may assume that

$$\frac{R^h[T]}{(P(T))} \simeq \frac{R^h[T]}{(T^2-\theta)} \times \prod_{x\in\Sigma_0} \left(R^h_x\right)^{l-2}$$

where θ is a semilocal equation of $\delta(Y/X)$ at the image of Σ_0. Note that locally θ must be a regular parameter because $R^h[T]\big/(P(T))$ is regular. Extending scalars to A^h_x we find

$$B^h_x = \frac{A^h_x[T]}{(T^2-\theta)} \times \prod_{x\in\Sigma_0} \left(A^h_x\right)^{l-2}$$

from which we see that if y corresponds to the double root, then $B_y^h = A_x^h[T]\big/(T^2-\theta)$ is a quadratic extension of A_x^h. In this case we know that π_Z is étale at z, hence the extension $B_y^h \to C_z^h$ is an isomorphism. This proves (b).

If $y \notin \Sigma$ then π_Y is étale at y and therefore the extension $A_x^h \to B_y^h$ is an isomorphism. On the other hand, because of the Galois action, all the extensions $A_x^h \to C_{\sigma(z)}^h$ are isomorphic to each other, therefore they are all quadratic. This proves (c).

It follows from (b) and (c) that the Galois extension $A_x^h \to C_x^h = \prod_z C_z^h$ is a product of quadratic extensions and since C_x^h is of rank $|G|$ over A_x^h, the assertion (d) follows.

Let $f \in B$ be a semilocal equation of D_Y at Σ. By the construction of D_Y made in §8 (see Lemma 8.3), in $A^h[T]\big/(T^2-\theta)$ we can write $f = aT - \pi$ where a is a unit of A^h and $\pi \in A^h$ is a semilocal parameter (that is a parameter of A_x^h for every $x \in \Sigma_0$). Further, $u = N_{Y/X}(f) = \pi^2 - a^2T^2 = \pi^2 - a^2\theta = fg$ for $g = -aT - \pi$. Then, at any $y \in \Sigma$, we have

$$B_y^h \simeq \frac{R_x^h[T]}{(T^2-\theta, t^l - a^2T^2 + \pi^2)} = \frac{S[t]}{(t^l - fg)}$$

where x is the image of y in Σ_0 and $S = A_x[T]\big/(T^2-\theta)$ is a regular 2-dimensional henselian ring. Its maximal ideal is (f, g) because D_Y and $\Delta(Y/X)$ meet transversally and they are defined, respectively, by f and T.

The class group of B_y^h is $\mathbb{Z}/l$ generated by the prime $\mathfrak{p}(y) = (t, aT-\pi) = (t, f)$ or by the prime $\mathfrak{q}(y) = (t, g)$ whose class is the inverse of the class of $\mathfrak{p}(y)$. Since $\mathfrak{p}(y)$ comes from B_y, it follows that $Cl(B_y)$ is generated by the ideal $\mathfrak{p}(y) \cap B_y$, which we still denote by $\mathfrak{p}(y)$). Since $Cl(B) \to \prod_{y\in\Sigma} Cl(B_y)$ is injective, inverting t we obtain a factorial ring $B[1/t]$ and therefore $Cl(B)$ is generated by those prime ideals of height 1 that contain t. Two of them are $\mathfrak{p} = (t, f)$ and $\mathfrak{q} = (t, g)$. Any other one must coincide with $\mathfrak{p}$ or $\mathfrak{q}$ at a given $y \in \Sigma$ and therefore it must coincide with $\mathfrak{p}$ or $\mathfrak{q}$ because B is a domain. At every $y \in \Sigma$ the class of $\mathfrak{p}$ is the inverse of the class of $\mathfrak{q}$, the divisor class group of B is generated by each of them and is, therefore, zero or $\mathbb{Z}/l$. The rest of the proposition follows from the fact that the maps induced on divisor class groups are surjective for localizations and injective for henselizations of local rings.

We now look at the pull-back under ψ_Y and ψ_X of the Cartier divisors occurring in (†). We have

$$\psi_Y^* g_Y^*\{D_Y\} + \psi_Y^* g_Y^*\{D_2\} = l \cdot \psi_Y^* \pi_{Y'}^*\{D'\}\ .$$

Since the centres of the blow-up for ψ_Y are away from $\Delta(Y'/X')$ and since $g_Y^*\{D_Y\}$ and $g_Y^*\{D_2\}$ have no common composnents in Y' and meet only along $\Delta(Y'/X')$, it follows that $\psi_Y^* g_Y^*\{D_Y\}$ and $\psi_Y^* g_Y^*\{D_2\}$ have no common component in $\widehat{Y}$. Therefore

$$\psi_Y^* g_Y^*\{D_Y\} = l\{D_{\widehat{Y}}\}$$

for some $D_{\widehat{Y}}$ on $\widehat{Y}$ and

$$\psi_Z^* g_Z^*\{D_Z\} = l\{D_{\widehat{Z}}\}$$

where $D_{\widehat{Z}} = \pi_{\widehat{Z}}^*(D_{\widehat{Y}})$. We shall now construct maps $\widetilde{X} \to \widehat{X}$ and $\widetilde{Y} \to \widehat{Y}$ satisfying the conditions of Theorem 9.1.

We recall a result of Abhyankar and Manish Kumar:

Lemma 9.4. *Let R be a 2-dimensional complete local domain which is regular, with quotient field K. Let L be a quadratic extension of K and S the normalization of R in L. Suppose that the extension L/K is defined by $z^2 = ux^iy^j$ where $(x, y) = m$ is the maximal ideal of R, u is a unit and $0 \leq i, j \leq 1$. If $(i, j) \neq (1, 1)$ then S is regular. Suppose $(i, j) = (1, 1)$ and let $X \to \mathrm{Spec}(R)$ be the blow-up of $\mathrm{Spec}(R)$ at the closed point. Then the normalization of X in L is regular.*

Proof. See [AMK], §2.

Let y be a singular point of $\widehat{Y}$ mapping to $x \in \widehat{X}$, to x_0 in X and to y_0 in Y. We note that $x_0 \in \delta(Y/X) \cap D_X$.
Let $\widehat{\mathcal{O}}_{\widehat{Y},y}$ and $\widehat{\mathcal{O}}_{\widehat{X},x}$ be the completions of $\mathcal{O}_{\widehat{Y},y}$ and $\mathcal{O}_{\widehat{X},x}$ respectively. By condition (3) of Theorem 8.6 the image x_0 of y in X has $l-1$ preimages in Y, hence $Y \times_X \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x_0})$ splits as a disjoint union of $l - 2$ copies of $\mathrm{Spec}(\widehat{\mathcal{O}}_{X,x_0})$ and one degree 2 extension $\mathrm{Spec}(\widehat{\mathcal{O}}_{Y,y_0})$ of $\mathrm{Spec}(\widehat{\mathcal{O}}_{X,x_0})$. It follows by base change that $\widehat{\mathcal{O}}_{\widehat{Y},y}$ is a degree 2 extension of $\widehat{\mathcal{O}}_{\widehat{X},x}$.

Let Σ_0 be the image of Σ in $\widehat{X}$. Clearly $\Sigma_0 \subset \delta(\widehat{Y}/\widehat{X})$. Let $\varphi_1 : \widetilde{X}_1 \to \widehat{X}$ be a proper birational morphism which is an isomorphism outside Σ_0 and such that the total transform of $\delta(\widehat{Y}/\widehat{X})$ has normal crossings over Σ_0. Let $\widetilde{\Sigma}_0 \subset \widetilde{X}_1$ be the image of the set of singular points of the normalization of $Y \times \widetilde{X}_1$. Let $\varphi_2 : \widetilde{X} \to \widetilde{X}_1$ be the blow-up at $\widetilde{\Sigma}_0$ and $\varphi_X = \varphi_2\varphi_1$. We have a commutative diagram

$$\begin{tikzcd}
Z \times_X \widetilde{X} \arrow[r] \arrow[d, "\varphi_Z"'] & Y \times_X \widetilde{X} \arrow[r] \arrow[d, "\varphi_Y"] & \widetilde{X} \arrow[d, "\varphi_X"] \\
\widehat{Z} \arrow[r, "\pi_Z"] & \widehat{Y} \arrow[r, "\pi_Y"] & \widehat{X}
\end{tikzcd}$$

Let $n_Y : \widetilde{Y} \to Y \times_X \widetilde{X}$, $n_Z : \widetilde{Z} \to Z \times_X \widetilde{X}$ be the normalization maps. We claim that $\widetilde{Y}$ and $\widetilde{Z}$ are smooth. Outside Σ_0, φ_X is an isomorphism and hence φ_Y is also an isomorphism outside $\pi_{\widehat{Y}}^{-1}(\Sigma_0) \supset \Sigma = \mathrm{Sing}(\widehat{Y})$. Let y be a singular point of $Y \times_X \widetilde{X}$. The image of y in $\widehat{Y}$ is contained in $\pi_{\widehat{Y}}^{-1}(\Sigma_0)$ which maps to $\delta(Y/X) \cap D_X$ in X. Thus the image x_0 of y in X belongs to $\delta(Y/X) \cap D_X$. The point x_0 has $l - 1$ preimages in Y in view of condition (3) of 8.6. Let x be the image of y in $\widetilde{X}$.
Let $\mathcal{O}_{\widetilde{Y},x}$ be the semilocal ring of $\widetilde{Y}$ at $\pi_{\widetilde{Y}}^{-1}(x)$ and $\overline{\mathcal{O}}_{\widetilde{Y},x}$ be its integral closure in $k(\widetilde{Y})$. Let $\widehat{\mathcal{O}}_{\widetilde{X},x}$ and $\widehat{\mathcal{O}}_{\widetilde{Y},y}$ denote respectively the completion of $\mathcal{O}_{\widetilde{X},x}$ and $\mathcal{O}_{\widetilde{Y},y}$. Then $\mathcal{O}_{\widetilde{Y},x} \otimes_{\mathcal{O}_{\widetilde{X},x}} \widehat{\mathcal{O}}_{\widetilde{X},x}$ splits as a product

$$\widehat{\mathcal{O}}_{\widetilde{Y},y} \times \widehat{\mathcal{O}}_{\widetilde{X},x} \times \cdots \times \widehat{\mathcal{O}}_{\widetilde{X},x} \,.$$

Since $\overline{\mathcal{O}}_{\widetilde{Y},x}$ is finite over $\mathcal{O}_{\widetilde{X},x}$, the tensor product $\overline{\mathcal{O}}_{\widetilde{Y},x} \otimes_{\mathcal{O}_{\widetilde{X},x}} \widehat{\mathcal{O}}_{\widetilde{X},x}$ is the completion of $\overline{\mathcal{O}}_{\widetilde{Y},x}$ with respect to its radical and hence, by [ZS], Ch. VIII, §30, Theorem 32, it is normal. Clearly it is the normalization of

$$\mathcal{O}_{\widetilde{Y},x} \otimes_{\mathcal{O}_{\widetilde{X},x}} \widehat{\mathcal{O}}_{\widetilde{X},x} = \widehat{\mathcal{O}}_{\widetilde{Y},y} \times \widehat{\mathcal{O}}_{\widetilde{X},x} \times \cdots \times \widehat{\mathcal{O}}_{\widetilde{X},x} \,.$$

Let $\overline{\widehat{\mathcal{O}}}_{\widetilde{Y},y}$ denote the normalization of $\widehat{\mathcal{O}}_{\widetilde{Y},y}$. By construction the ramification divisor on $\widehat{\mathcal{O}}_{\widetilde{X},x}$ for the extension $\widehat{\mathcal{O}}_{\widetilde{X},x} \to \widehat{\mathcal{O}}_{\widetilde{Y},y}$ has normal crossings and further $\overline{\widehat{\mathcal{O}}}_{\widetilde{Y},y}$ is regular by 9.4. It follows that $\overline{\mathcal{O}}_{\widetilde{Y},x} \otimes_{\mathcal{O}_{\widetilde{X},x}} \widehat{\mathcal{O}}_{\widetilde{X},x}$ is regular and hence $\overline{\mathcal{O}}_{\widetilde{Y},x}$ itself is regular. This shows that $\widetilde{Y}$ is smooth. Exactly the same argument shows that $\widetilde{Z}$ is smooth because, by 8.3, $Z \to Y$ is étale at the image of y in Y.

We thus have a commutative diagram

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\pi_{\widetilde{Z}}"'] \arrow[d, "f_Z"'] \arrow[rr, bend left, "\widetilde{\pi}"] \arrow[dd, bend right=60, "h_Z"'] & \widetilde{Y} \arrow[r, "\pi_{\widetilde{Y}}"'] \arrow[d, "f_Y"] & \widetilde{X} \arrow[d, "f_X"] \arrow[dd, bend left=60, "h_X"] \\
\widehat{Z} \arrow[r, "\pi_{\widehat{Z}}"'] \arrow[d, "g_Z\psi_Z"'] & \widehat{Y} \arrow[r, "\pi_{\widehat{Y}}"'] \arrow[d, "g_Y\psi_Y"] & \widehat{X} \arrow[d, "g_X\psi_X"] \\
Z \arrow[r, "\pi_Z"] \arrow[rr, bend right, "\pi"'] & Y \arrow[r, "\pi_Y"] & X
\end{tikzcd}$$

where the undefined maps are the obvious ones, $f_Z = \varphi_Z n_Z$, $f_Y = \varphi_Y n_Y$ and $f_X = \varphi_X$. The surfaces $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ are smooth, $\widetilde{\pi}$ is finite, and $\widetilde{Z}$ is Galois over $\widetilde{X}$ with group $G = Gal\big(k(Z)/k(X)\big)$.

Proposition 9.5. *The line bundle $(h_Z)^*(\mathcal{O}_Z(D_Z))$ belongs to*

$$l \cdot \operatorname{Pic}(\widetilde{Z}) + \widetilde{\pi}^*(\operatorname{Pic}(\widetilde{X}))$$

To prove 9.5 we need several preliminaries. By construction φ_X is an isomorphism outside Σ_0, hence φ_Z is an isomorphism outside $Z \times_X \Sigma_0$. It follows that $Z \times_X \widetilde{X}$ is smooth —in particular normal— outside $Z \times_X \Sigma_0$, hence the normalization map n_Z is an isomorphism outside Σ_1. It follows that $f_z = \varphi_Z n_Z$ is an isomorphism outside Σ_1.
We denote by
$F_1 = f_Z^{-1}(\Sigma_1)$ the exceptional fibre for f_Z,
$F = f_Y^{-1}(\pi_{\widehat{Y}}^{-1}(\Sigma_0))$ the exceptional fibre for f_Y
$F_0 = f_X^{-1}(\Sigma_0)$ the exceptional fibre for f_X.
To simplify notation we write F, F_1 instead of $F_B, (F_1)_C$ etc.
Let $\widetilde{L}$ be the line bundle $h_Z^*(\mathcal{O}_Z(D_Z))$. Since $\widehat{Z} \setminus \Sigma_1$ is smooth, the restriction $D_{\widehat{Z}}\big|_{\widehat{Z}\setminus\Sigma_1}$ is a Cartier divisor and defines a line bundle L_0 on $\widetilde{Z} \setminus F_1 \simeq \widehat{Z} \setminus \Sigma_1$. We have an exact sequence

$$0 \to \operatorname{Pic}_{F_1}(\widetilde{Z}) \to \operatorname{Pic}(\widetilde{Z}) \to \operatorname{Pic}(\widetilde{Z} \setminus F_1) \to 0$$

where $\operatorname{Pic}_{F_1}(\widetilde{Z})$ is the subgroup generated by the irreducible components of F_1. Thus there exists a line bundle $\widetilde{L}_0$ on $\widetilde{Z}$ whose restriction to $\widetilde{Z} \setminus F_1$ is isomorphic to $f_Z^*(L_0)$. The line bundle $\widetilde{L}(\widetilde{L}_0^{-1})^{\otimes l}$ restricted to $\widetilde{Z} \setminus F_1$ is trivial and hence its class belongs to $\mathrm{Pic}_{F_1}(\widetilde{Z})$. We fix a point $x \in \Sigma_0$ and let $z_1, \ldots, z_m \in \Sigma_1$ be the preimages of x in $\widehat{Z}$. Let F_x be the fibre of x in $\widetilde{X}$ and, for $1 \leq j \leq m$, let F_j be the fibre of z_j in $\widetilde{Z}$. By Zariski's main theorem ([Ha$_1$], Ch. III, Cor. 11.4) the curves F_x and $F_1, \ldots, F_m$ are connected. Let E be an irreducible component of F_x and D_j^k, $k = 1 \ldots, m_j$ the irreducible components of F_j mapping to E. The class $\left[\widetilde{L}\left(\widetilde{L}_0^{-1}\right)^{\otimes l}\right]$ is the class of a divisor

$$\sum_{x \in \Sigma_0} \sum_E \sum_j r_j^k D_j^k$$

as E runs over the irreducible components of F_x. Denoting $C_{z_j}^h$ by C_j^h and, as usual, $\widetilde{Z} \times_{\widehat{Z}} C_j^h$ by $\widetilde{Z}_{C_j^h}$ and so on, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}\left(\widetilde{Z}_{C_j^h}\right)/l \arrow[r] \arrow[d, "\alpha_Z"] & H^2\left(\widetilde{Z}_{C_j^h}, \mu_l\right) \arrow[d, "\beta"] \\
0 \arrow[r] & \mathrm{Pic}(F_j)/l \arrow[r] & H^2(F_j, \mu_l).
\end{tikzcd}$$

By the proper base change theorem ([Mi], Ch. VI, 2.7) β is an isomorphism and by [CTOP], Th. 1.7, (c) α_Z is surjective, hence α_Z is an isomorphism. Further, for any connected curve $\mathcal{C}$ the inclusion $\mathcal{C}_{red} \hookrightarrow \mathcal{C}$ induces an isomorphism $\mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{C}_{red})$. We thus have isomorphisms

$$\alpha_Z : \mathrm{Pic}\left(\widetilde{Z}_{C_j^h}\right)/l \simeq \mathrm{Pic}((F_j)_{red}))/l$$

and, similarly,

$$\alpha_X : \mathrm{Pic}\left(\widetilde{X}_{A_x^h}\right)/l \simeq \mathrm{Pic}((F_x)_{red}))/l\ .$$

By a result of Artin ([Ar], §1) the degree map yields identifications

$$\mathrm{Pic}((F_x)_{red}))/l = \bigoplus_E \mathbb{Z}/l \quad \text{and} \quad \mathrm{Pic}((F_j)_{red}))/l = \bigoplus_E \bigoplus_{D_j^k} \mathbb{Z}/l$$

where E runs over the irreducible components of F_x and D_j^k runs over the irreducible components of F_j mapping to E. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}\left(\widetilde{X}_{A_x^h}\right)/l \arrow[r, "\widetilde{\pi}^*"] \arrow[d, "\alpha_X"'] & \bigoplus_j \mathrm{Pic}\left(\widetilde{Z}_{C_j^h}\right)/l \arrow[d, "\alpha_Z"] \\
\bigoplus_E \mathbb{Z}/l \arrow[r, "\theta"] & \bigoplus_E \bigoplus_{D_j^k} \mathbb{Z}/l
\end{tikzcd}$$

where the vertical isomorphisms are obtained by associating to a divisor its intersection multiplicity with each E and D_j^k respectively and $\theta = \left(\theta_{E,D_j^k}\right)$ where θ_{E,D_j^k} is the degree of $k(D_j^k)$ over $k(E)$. Since the Galois action transitively permutes the D_j^k 's lying over E, these degrees are all equal to some f_E. Since, for $x \in \Sigma_0$, $g_X(x)$ is in $D_X \subset X$, by 8.6 x has at least $l-1 \geq 2$ preimages in $\widehat{Y}$ and therefore the preimage of E in $\widetilde{Y}$ has at least two components. Since the degree of $\widetilde{Y} \to \widetilde{X}$ is l and the degree of $\widetilde{Z} \to \widetilde{Y}$ is prime to l, this implies that f_E is prime to l. The Galois action on $\operatorname{Pic}(\widetilde{Z})$ transitively permutes the components in $\bigoplus_{D_j^k} \mathbb{Z}/l$. Hence θ maps $\bigoplus_E \mathbb{Z}/l$ isomorphically onto the subgroup of $\bigoplus_E \bigoplus_{D_j^k} \mathbb{Z}/l$ consisting of all the elements fixed by the Galois action and the same is true of $\widetilde{\pi}^*$. Since $[D_Z] \in \operatorname{Pic}(Z)/l$ is the pull-back of an element in $H^2(X, \mu_l)$ under π, it is Galois invariant; its pull-back in $\operatorname{Pic}(\widetilde{Z})/l$ and its restriction to $\operatorname{Pic}(\widetilde{Z}_{A^h})/l$ are again Galois invariant. Thus $\left[\sum_E \sum_j r_j^k D_j^k\right]$ belongs to the image of $\widetilde{\pi}^*$. We claim that $\left[\sum_E \sum_j r_j^k D_j^k\right]$ is of the form $\widetilde{\pi}^*(\sum r_E[E])$, where E varies in the exceptional fibre of $\widetilde{X} \to \widehat{X}$.

Since $\widehat{X}$ is smooth at x, A_x^h is regular and we have an exact sequence

$$0 \to \operatorname{Pic}_{F_x}\left(\widetilde{X}_{A_x^h}\right) \to \operatorname{Pic}\left(\widetilde{X}_{A_x^h}\right) \to \operatorname{Pic}\left(\widetilde{X}_{A_x^h} \setminus F_x\right) \to 0$$

with $\operatorname{Pic}\left(\widetilde{X}_{A_x^h} \setminus F_x\right) = \operatorname{Pic}\left(\operatorname{Spec}(A_x^h) \setminus \{x\}\right) = 0$. The claim follows.

Let M be the line bundle on $\widetilde{X}$ representing the class of $\sum_{x \in \Sigma_0} \sum_E r_E[E]$. We claim that $N = \widetilde{L}\left(\widetilde{L}_0^{-1}\right)^{\otimes l} \widetilde{\pi}^*(M^{-1})$, which belongs to $\operatorname{Pic}_{F_1}(\widetilde{Z})$, is an l-th power in $\operatorname{Pic}(\widetilde{Z})$. We note that, by construction, it maps to zero in $\operatorname{Pic}\left(\widetilde{Z}_{A_x^h}\right)/l$ for all $x \in \Sigma_0$.

Lemma 9.6. *The map*

$$\operatorname{Pic}_{F_1}\left(\widetilde{Z}_A\right) \to \operatorname{Pic}_{F_1}\left(\widetilde{Z}_{A^h}\right)$$

is an isomorphism.

Proof. We have a commutative diagram

$$\begin{array}{ccccccc}
H^0(\widetilde{Z}_A, \mathbb{G}_m) & \xrightarrow{\gamma_0} & H^0(\widetilde{Z}_A \setminus F_A, \mathbb{G}_m) & \xrightarrow{\delta_0} & H^1_{F_1}(\widetilde{Z}_A, \mathbb{G}_m) & \xrightarrow{\varphi_0} & \operatorname{Pic}_{F_1}(\widetilde{Z}_A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\widetilde{Z}_{A^h}, \mathbb{G}_m) & \xrightarrow{\gamma} & H^0(\widetilde{Z}_{A^h} \setminus F_{A^h}, \mathbb{G}_m) & \xrightarrow{\delta} & H^1_{F_1}(\widetilde{Z}_{A^h}, \mathbb{G}_m) & \xrightarrow{\varphi} & \operatorname{Pic}_{F_1}(\widetilde{Z}_{A^h})
\end{array}$$

with exact rows (see [Mi], III, 1.27 and the correction in [CTO], Proposition 4.4) and where φ_0 and φ are surjective. Since $\widetilde{Z}_A \to \widehat{Z}_A$ is proper and birational we have $\mathbb{G}_m(\widetilde{Z}_A) = \mathbb{G}_m(\widehat{Z}_A)$. For the same reason, $\mathbb{G}_m(\widetilde{Z}_{A^h}) = \mathbb{G}_m(\widehat{Z}_{A^h})$. Since $\widetilde{Z}_A \setminus F_1 \simeq \widehat{Z}_A \setminus \Sigma_1$ and $\widetilde{Z}_{A^h} \setminus F_1 \simeq \widehat{Z}_{A^h} \setminus \Sigma_1$ we have

$$\mathbb{G}_m(\widetilde{Z}_A \setminus F_1) = \mathbb{G}_m(\widehat{Z}_A \setminus \Sigma_1) \quad \text{and} \quad \mathbb{G}_m(\widetilde{Z}_{A^h} \setminus F_1) = \mathbb{G}_m(\widehat{Z}_{A^h} \setminus \Sigma_1).$$

Since $\widehat{Z}_A$ is a 2-dimensional connected scheme and Σ_1 a finite set of closed points of $\widehat{Z}$,

$$\mathbb{G}_m(\widehat{Z}_A \setminus \Sigma_1) = \mathbb{G}_m(\widehat{Z}_A) \quad \text{and} \quad \mathbb{G}_m(\widehat{Z}_{A^h} \setminus \Sigma_1) = \mathbb{G}_m(\widehat{Z}_{A^h}).$$

Thus γ and γ_0 are isomorphisms. and therefore φ and φ_0 are isomorphisms.
Let $h : \widetilde{Z}_{A^h} \to \widetilde{Z}_A$ be the projection. We note that $h^*\mathbb{G}_m = \mathbb{G}_m$, hence by étale excision using a limit argument [Mi] the map $H^1_{F_1}(\widetilde{Z}_A, \mathbb{G}_m) \to H^1_{F_1}(\widetilde{Z}_{A^h}, \mathbb{G}_m)$ is an isomorphism and the lemma is proved.

To complete the proof of Theorem 9.1 we may assume that $Cl(B) = \mathbb{Z}/l$, because if B is factorial then Weil divisors coincide with Cartier divisor and there is nothing to prove.

Proposition 9.7. *Identifying $Cl(B)$ with a subgroup of $Cl(C)$ under the pull-back map, $Cl(C)$ is generated by the Galois conjugates of $\pi^*_{\widehat{Z}}(Cl(B))$ in $Cl(C)$.*

Proof. We know by 8.6 (5) that the inverse image of D_Y in Z is an irreducible smooth curve D_Z. Let h be a local equation of D_Z on Σ_1. Then for every $z \in \Sigma_1$ we can choose a $\sigma \in G$ for which $\sigma(z)$ lies over a singular $y \in \Sigma$. From $C^h_z \simeq B^h_y$ it follows that $Cl(C^h_z)$ is generated by the ideal $(t, \sigma(h))$. This shows that $Cl(C)$ is generated by the primes containing some $\sigma(h)$ and the argument used in the proof of 9.3 shows that $Cl(C)$ is generated by the Galois conjugates of $\pi^*_{\widehat{Z}}(Cl(B))$.

Lemma 9.8. *The group $\mathrm{Pic}(\widetilde{Z}_{C^h})$ has no l-torsion.*

Proof. Using the fact that $\mathrm{Pic}(\widetilde{Z}_{C^h} \setminus F_1) = Cl(C^h)$, we have the exact sequence

$$(\dagger) \qquad 0 \to \mathrm{Pic}_{F_1}(\widetilde{Z}_{C^h}) \to \mathrm{Pic}(\widetilde{Z}_{C^h}) \to Cl(C^h) \to 0$$

where $\mathrm{Pic}_{F_1}(\widetilde{Z}_{C^h})$ is the subgroup of $\mathrm{Pic}(\widetilde{Z}_{C^h})$ generated by the irreducible components of F_1. Suppose that their number is r. Then they generate a free abelian group isomorphic to $\mathbb{Z}^r$ because $\mathrm{Pic}_{F_1}(\widetilde{Z}_{C^h})$ carries a nondegenerate quadratic form given by intersection multiplicities ([Mu], §1). It follows that $\mathrm{Pic}(\widetilde{Z}_{C^h})$ is a finitely generated group of rank r, of the form $\mathbb{Z}^r \oplus T$, where T is its torsion subgroup. Since $Cl(C^h)$ is l-torsion, T can only be l-torsion, but we have seen that

$$\mathrm{Pic}_{F_1}(\widetilde{Z}_{C^h})/l \simeq (\mathbb{Z}/l)^r$$

hence T must be zero.

Corollary 9.9. *The group $\mathrm{Pic}(\widetilde{Z}_C)$ has no l-torsion.*

Proof. The corollary follows from the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}_{F_1}(\widetilde{Z}_C) \arrow[r] \arrow[d, "\alpha"', "\simeq"] & \mathrm{Pic}(\widetilde{Z}_C) \arrow[r] \arrow[d] & Cl(C) \arrow[r] \arrow[d, tail, "\beta"] & 0 \\
0 \arrow[r] & \mathrm{Pic}_{F_1}(\widetilde{Z}_{C^h}) \arrow[r] & \mathrm{Pic}(\widetilde{Z}_{C^h}) \arrow[r] & Cl(C^h) \arrow[r] & 0
\end{tikzcd}$$

using that α is an isomorphism by 9.6 and that β is injective.

Lemma 9.10. *We have*

$$\big(Cl(C)\big)^G = \big(Cl(C^h)\big)^G = 0.$$

Proof. It suffices to show that $\big(Cl(C^h)\big)^G = 0$. We have

$$C^h = \prod_{z\in\Sigma_1} C_z^h$$

and each factor on the right is isomorphic to

$$\frac{R_x^h[T,t]}{(T^2-\theta, t^l - a^2T^2+\pi^2)}\ .$$

The isotropy group $G_z \subset G$ of z acts by changing the sign of T and therefore maps the class of the prime $t, aT-\pi$ to the class of $(t, -at-\pi)$ which is its inverse and is distinct from the class of $(t, at-\pi)$ because l is at least 3. Hence $\big(Cl(S_z^h)\big)^{G_z} = 0$. From this the claim follows.

Proposition 9.11. *The map*

$$\mathrm{Pic}_{F_1}\big(\widetilde{Z}_{C^h}\big) \hookrightarrow \mathrm{Pic}\big(\widetilde{Z}_{C^h}\big)$$

induces an injection

$$\big(\mathrm{Pic}_{F_1}\big(\widetilde{Z}_{C^h}\big)/l\big)^G \hookrightarrow \big(\mathrm{Pic}\big(\widetilde{Z}_{C^h}\big)/l\big)^G$$

Proof. From the exact sequence (†) used for proving 9.8, using multiplication by l and the snake lemma we get a long exact sequence

$${}_l\mathrm{Pic}\big(\widetilde{Z}_{C^h}\big) \to Cl(C^h) \to \mathrm{Pic}_{F_1}\big(\widetilde{Z}_{C^h}\big)/l \to \mathrm{Pic}\big(\widetilde{Z}_{C^h}\big)/l.$$

By Lemma 9.8, $\mathrm{Pic}\big(\widetilde{Z}_{C^h}\big)$ has no l-torsion, hence we have an exact sequence

$$0 \to Cl(C^h) \to \mathrm{Pic}_{F_1}\big(\widetilde{Z}_{C^h}\big)/l \to \mathrm{Pic}\big(\widetilde{Z}_{C^h}\big)/l$$

from which the assertion immediately follows, since, by Lemma 9.10, $\big(Cl(C^h)\big)^G = 0$.

Proof of Proposition 9.5. Since $\widetilde{Z}_C \setminus (\widetilde{Z}_C \setminus F_1) \to F_1$ is an isomorphism, we have a patching diagram [Jo]

$$\begin{array}{ccc}
\widetilde{Z}_C \setminus F_1 & \longrightarrow & \widetilde{Z}_C \\
\downarrow & & \downarrow \\
\widetilde{Z} \setminus F_1 & \longrightarrow & \widetilde{Z}
\end{array}$$

and an exact sequence

$$(\star\star) \qquad 0 \to \mathrm{Pic}(\widetilde{Z}) \to \mathrm{Pic}(\widetilde{Z}_C) \oplus \mathrm{Pic}(\widetilde{Z} \setminus F_1) \to Cl(C) \to 0.$$

Surjectivity on the right is due to the the fact that $\mathrm{Pic}(\widetilde{Z} \setminus F_1)$ surjects onto $Cl(C)$. From this sequence, multiplying by l, using the snake lemma and remembering that ${}_l\mathrm{Pic}(\widetilde{Z}_C) = 0$ and that $Cl(C)$ is l-torsion, we obtain a long exact sequence

$$(\ddagger) \qquad \begin{aligned} 0 \to {}_l\mathrm{Pic}(\widetilde{Z}) \to {}_l\mathrm{Pic}(\widetilde{Z} \setminus F_1) \to Cl(C) \to \\ \to \mathrm{Pic}(\widetilde{Z})/l \to \mathrm{Pic}(\widetilde{Z}_C)/l \to \mathrm{Pic}(\widetilde{Z} \setminus F_1)/l \end{aligned}$$

We now distinguish two cases:

I. Suppose that ${}_l\mathrm{Pic}(\widetilde{Y} \setminus F) \to Cl(B)$ is surjective. Consider the commutative diagram

$$\begin{tikzcd} & & \mathrm{Pic}(\widetilde{Y} \setminus F) \arrow[d, "(f_Y)_*"'] \arrow[dr, "i"] & & \\ 0 \arrow[r] & \mathrm{Pic}(\widehat{Y}) \arrow[r] & Cl(\widehat{Y}) \arrow[r, "j"] & Cl(B) \arrow[r] & 0 \end{tikzcd}$$

Recall that $g_Y^*\{D_Y\} = l\{D_{\widehat{Y}}\}$ for some Weil divisor $D_{\widehat{Y}}$. By assumption $j\{D_{\widehat{Y}}\} = i(\gamma)$ for some $\gamma \in {}_l\mathrm{Pic}(\widetilde{Y} \setminus F)$. Then, since $j(\{D_{\widehat{Y}}\} - (f_Y)_*(\gamma)) = 0$, there exists a $\gamma' \in \mathrm{Pic}(\widehat{Y})$ such that $\{D_{\widehat{Y}}\} = (f_Y)_*(\gamma)) + \gamma'$. We conclude that $\psi_Y^* g_Y^*\{D_Y\} = l\{D_{\widehat{Y}}\} = l\gamma'$ because $l\gamma = 0$. From this it follows that $h_Z^*\{D_Z\}$ is divisible by l in $\mathrm{Pic}(\widetilde{Z})$.

II. Suppose that ${}_l\mathrm{Pic}(\widetilde{Y} \setminus F) \to Cl(B)$ is not surjective, in which case, since $Cl(B) = \mathbb{Z}/l$, it is the zero map. By Proposition 9.7, ${}_l\mathrm{Pic}(\widetilde{Z} \setminus F_1) \to Cl(C)$ is also zero, hence from the exact sequence ($\ddagger$) we conclude that the map ${}_l\mathrm{Pic}(\widetilde{Z}) \to {}_l\mathrm{Pic}(\widetilde{Z} \setminus F_1)$ is an isomorphism. In this case the exact sequence

$$0 \to \mathrm{Pic}_{F_1}(\widetilde{Z}) \to \mathrm{Pic}(\widetilde{Z}) \to \mathrm{Pic}(\widetilde{Z} \setminus F_1) \to 0$$

yields that $\mathrm{Pic}_{F_1}(\widetilde{Z})/l \hookrightarrow \mathrm{Pic}(\widetilde{Z})/l$ is injective. The element $[N] \in \mathrm{Pic}_{F_1}(\widetilde{Z})$ is Galois invariant in $\mathrm{Pic}(\widetilde{Z})/l$ and hence, because of this injection, it is in $\left(\mathrm{Pic}_{F_1}(\widetilde{Z})/l\right)^G$. From the exact sequence

$$0 \to \mathrm{Pic}_{F_1}(\widetilde{Z}_C) \to \mathrm{Pic}(\widetilde{Z}_C) \to Cl(C) \to 0$$

multiplying by l, using the snake lemma and remembering that ${}_l\mathrm{Pic}(\widetilde{Z}_C) = 0$ we get the exact sequence

$$0 \to Cl(C) \to \mathrm{Pic}_{F_1}(\widetilde{Z})/l \to \mathrm{Pic}(\widetilde{Z}_C)/l.$$

Taking invariants under G and using the fact that $Cl(C)^G = 0$ (Proposition 9.10) we see that $[N] \in \left(\mathrm{Pic}_{F_1}(\widetilde{Z})/l\right)^G$. On the other hand, we know that $[N]$ is zero in $\mathrm{Pic}(\widetilde{Z}_{C^h})/l$, hence by Proposition 9.11 it is zero in $\mathrm{Pic}_{F_1}(\widetilde{Z}_{C^h})/l$. Noting that $\widetilde{Z}_C = \widetilde{Z} \times_{\widehat{Z}} \mathrm{Spec}(C) =$

$\widetilde{Z} \times_{hx} \mathrm{Spec}(A) = \widetilde{Z}_A$, 9.6 implies that $[N]$ vanishes in $\mathrm{Pic}_{F_1}(\widetilde{Z}_C)/l$. Since the map $\mathrm{Pic}_{F_1}(\widetilde{Z}) \to \mathrm{Pic}_{F_1}(\widetilde{Z}_C)$ is surjective, there exists $N_1 \in \mathrm{Pic}_{F_1}(\widetilde{Z})$ such that $NN_1^{\otimes l}$ maps to zero in $\mathrm{Pic}(\widetilde{Z}_C)$. The element $NN_1^{\otimes l}$ also maps to zero in $\mathrm{Pic}(\widetilde{Z} \setminus F_1)$. By the exact sequence $(\star\star)$, $NN_1^{\otimes l}$ is zero in $\mathrm{Pic}(\widetilde{Z})$, completing the proof of the proposition.

We now complete the proof of Theorem 9.1 which says that the class of $h_Y^*(\mathcal{O}_Y(D_Y))$ in $\mathrm{Pic}(\widetilde{Y})$ belongs to

$$l \cdot \mathrm{Pic}(\widetilde{Y}) + \pi_{\widetilde{Y}}^* \mathrm{Pic}(\widetilde{X}).$$

This immediately follows from 9.4 and the fact that $\pi_{\widetilde{Z}}$ is a finite map of degree dividing $(l-1)!$ which is coprime to l.

10. Proof of the main theorem

Theorem 10.1. *Let X be a smooth projective surface over an algebraically closed field k of characteristic zero. Let D be a central division algebra of prime degree l over $k(X)$. Then D is cyclic.*

Proof. If $l = 2$ then D is a quaternion algebra and every quadratic extension in D is cyclic. If $l = 3$ then D is cyclic by a result of Wedderburn [W]. However we show for all primes $l \geq 3$ that the cyclic extension $k(\widetilde{X})$ of $k(X)$ of degree l splits D.

Suppose first that D is unramified over X. By §1 the class of D is equivalent to an Azumaya algebra of rank l^2 over X. Let $\eta \in H^2(X, \mu_l)$ be an element mapping to the class of D in $H^2(X, \mathbb{G}_m)$. Let $\pi_Y : Y \to X$ be a generic splitting of D as in Theorem 7.13. The class $\pi^*(\eta) \in H^2(Y, \mu_l)$ maps to zero in $H^2(Y, \mathbb{G}_m)$ since the map $H^2(Y, \mathbb{G}_m) \to H^2(k(Y), \mathbb{G}_m)$ is injective, Y being smooth. In view of the Kummer exact sequence

$$1 \to \mu_l \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

we have the exact sequence

$$0 \to \mathrm{Pic}(Y)/l \to H^2(Y, \mu_l) \to H^2(k(Y), \mathbb{G}_m)$$

from which we see that there exists a line bundle L on Y such that $\pi^*(\eta) = [L]$ in $\mathrm{Pic}(Y)/l$. By Theorem 9.1 there exists a map $h_X : \widetilde{X} \to X$ which is proper, generically cyclic of degree l and such that the normalization $\widetilde{Y}$ of $Y \times_X \widetilde{X}$ is smooth; further if $h_Y : \widetilde{Y} \to Y$ is the natural map, then $h_Y^*(L_Y)$ vanishes in

$$\mathrm{Pic}(\widetilde{Y})/l\mathrm{Pic}(\widetilde{Y}) + \pi_{\widetilde{Y}}^*(\mathrm{Pic}(\widetilde{X}))\ .$$

We show that $k(\widetilde{X})$ splits D. Let $h_Y^*(L) = L_1^{\otimes l} \otimes_{\mathcal{O}_{\widetilde{Y}}} \pi_{\widetilde{Y}}^*(M)$ where M is a line bundle on $\widetilde{X}$. The element $\xi = \eta + [M^{-1}] \in H^2(\widetilde{X}, \mu_l)$ maps to the class of A in $H^2(\widetilde{X}, \mathbb{G}_m)$. Further $\pi_{\widetilde{Y}}^*(\zeta)$ is zero in $H^2(\widetilde{Y}, \mu_l)$.

Let W and the maps $f : W \to \mathbb{A}^1$, $g : W \to X$ be as constructed in §4. Let $P(T)$ be the characteristic polynomial of a general element of $D_{k(X)}$ defining $k(Y)$. We have already seen in 9.2 that $P(T)$ is irreducible over $k(\widetilde{X}) = k(X')$. Repeating the proof of the irreducibility of W over $\overline{k(t)}$ we see that $W' = W \times_X \widetilde{X}$ is integral and remains irreducible over $\overline{k(t)}$. Let f', g' denote the base changes of f, g to $\widetilde{X}$. Then $W'_0 = (f')^{-1}(0) = Y \times_X \widetilde{X}$ and $g'|_{W'} : W'_0 \to \widetilde{X}$ is finite. Further, $W'_1 = (f')^{-1}(1)$ has l irreducible components V'_i, and $g'|_{V'_i} : V'_i \to \widetilde{X}$ is a birational isomorphism (in fact an isomorphism). It is also clear that W' is normal at the generic point of V'_i. Let I be the ideal of $\mathcal{O}_{Y \times_X \widetilde{X}}$ such that the normalization map $\widetilde{Y} \to Y \times_X \widetilde{X}$ is the blow up at I ([Ha$_1$], Ch. II, Th. 7.17). Let $\widetilde{W}$ be the blow up of W' at the ideal (I, t) and $\widetilde{f}$, $\widetilde{g}$ the obvious maps from $\widetilde{W}$ to $\mathbb{A}^1$, $\widetilde{X}$. The fibre $\widetilde{W_0} = \widetilde{Y}$ and $\widetilde{W}$ is birational to W' away from $\widetilde{W_0}$. We have $\widetilde{g}^*(\zeta)|_{\widetilde{W_0}} = 0$ in $H^2(\widetilde{W_0}, \mu_l)$. Thus conditions (1) to (5) of 3.1 are satisfied by $\widetilde{W}, \widetilde{f}, \widetilde{g}$. Hence the class of B in $\mathrm{Br}(k(\widetilde{X}))$, which is the same as the class of D, is zero.

Suppose now that D is ramified on X. The proof in this case consists in a reduction to the unramified case, which is described in de Jong's paper [dJ], §1. We follow de Jong's notation.

Let $\mathcal{A}$ be an Azumaya algebra over $k(X)$. Let $h = (g, f) : W \to X \times \mathbb{P}^1$ be the map constructed in [dJ], satisfying

(1) $W_{\overline{\xi}}$ is smooth, where $\overline{\xi}$ is the geometric generic point of $\mathbb{P}^1$.

(2) The extension of $\mathcal{A}$ to the function field $L_{\overline{\xi}}$ of $W_{\overline{\xi}}$ is unramified on $W_{\overline{\xi}}$.

(3) The fiber of f at infinity has l components of multiplicity 1, each birationally isomorphic to X under g.

By the unramified case, there exists an element $a \in L_{\overline{\xi}}$ such that $D \otimes L_{\overline{\xi}}(a^{1/l})$ is split. There is a finite map $p : C \to \mathbb{P}^1$ with C a smooth affine curve such that a belongs to the fraction field L_C of $W \times_{\mathbb{P}^1} C$ and $D \otimes L_C$ is split by $L_C(a^{1/l})$ and hence is cyclic. The map p extends to $\widetilde{p} : \widetilde{C} \to \mathbb{P}^1$ where $\widetilde{C}$ is a smooth projective curve containing C as an open set. Let ∞' be a point of $\widetilde{C}$ mapping to the point ∞ of $\mathbb{P}^1$. The discrete valuation of $k(X \times \mathbb{P}^1)$ corresponding to a the generic point of $X \times \infty$ extends to a discrete valuation of $M = k(W \times_{\mathbb{P}^1} \widetilde{C})$, with the same residue field. Let R be the corresponding discrete valuation ring. The algebra D extended to M being cyclic, is given by a symbol $(u, v)_\zeta$, where ζ is a primitive l-th root of 1 in k. Since it is unramified at R, we may assume, by the following lemma that u and v are units of $\widehat{R}$, the completion of R. Its specialization to the residue field of R is therefore cyclic and coincides with the class of D.

Lemma 10.2. *Let $\widehat{R}$ be a complete discrete valuation ring with quotient field K. Assume that K contains a primitive $2l$-th root of 1. Then a cyclic algebra of index l over K and unramified at $\widehat{R}$ is represented by a symbol $(u, v)_\zeta$ with u and v units of $\widehat{R}$.*

Proof. We fix ζ and write $(-,-)$ for $(-,-)_\zeta$. Let the κ be the residue field of R. Let $\mathcal{A}$ be represented by the symbol $(ut^m, vt^n) \in H^2(K, \mu_l)$. Since $\mathcal{A}$ is unramified the residue of (ut^m, vt^n) is zero. Since κ contains μ_l this residue is $1 = (u^n/v^m) \in \kappa^*/(\kappa^*)^l$. Thus we have

$$(ut^m, vt^n) = (u, v) + (u, t^n) + (t^m, v) + (t^m, t^n)\ .$$