Zero cycles on twisted Cayley plane

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Abstract

In the present paper we compute the group of zero-cycles modulo rational equivalence of a twisted form X of a Cayley plane. More precisely, we show that the degree map $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$ is injective.

1 Introduction

Let G be a split exceptional group of type E_6 and G/P a projective G-homogeneous variety with a stabilizer given by a maximal parabolic subgroup P corresponding to the first (or last) root of the respective Dynkin diagram. This variety is called *Cayley plane* and is denoted by $\mathbb{OP}^2$. Its geometry is a subject of intensive investigations (see [IM05]).

Let X be a twisted form of the Cayley plane, i.e., a smooth projective variety over a field k which becomes isomorphic to $\mathbb{OP}^2$ over the separable closure of k. In this case the variety X can have no rational points and its geometry turns to be very complicated.

In the present paper we compute the group of zero-cycles $\mathrm{CH}_0(X)$ which is an important geometric invariant of a variety. Namely, we prove:

Theorem. *Let k be a field of characteristic different from 2 and 3. Let X be a twisted form of a Cayley plane corresponding to a group of inner type E_6 over k with trivial Tits algebras. Then the degree map $\mathrm{CH}_0(X) \to \mathbb{Z}$ is injective.*

The history of the question starts with the work of I. Panin [Pa84] where he proved the injectivity of the degree map for Severi-Brauer varieties. For quadrics this was proved by R. Swan in [Sw89]. The case of involutive varieties was considered by A. Merkurjev in [Me95]. For varieties of type F_4 it was announced by M. Rost.

Our work was mostly motivated by the paper of D. Krashen [Kr05] where he reformulated the question above in terms of R-triviality of certain symmetric powers and proved the injectivity for a wide class of generalized Severi-Brauer varieties and some involutive varieties, hence, generalizing the previously known results by Panin and Merkurjev. Another motivating point was the result of V. Popov [Po05] where he gave a full classification of generically n-transitive actions of a split linear algebraic group G on a projective homogeneous variety G/P. The case of $\mathbb{OP}^2 = E_6/P_6$ which is the main point of our investigations provides an example of such an action for $n = 3$. As a consequence, one can identify the open orbit X^o of this action with a homogeneous variety $E_6/T \cdot \mathrm{Spin}_8$, where T is the torus which is complementary to Spin_8. Then the result of Krashen reduces the question of injectivity to the question of R-triviality of the twisted form of X^o/S_3.

Apart from the main result concerning exceptional varieties we give shortened proofs for injectivity of the degree map for quadrics and Severi-Brauer varieties as well.

As it was mentioned to us by V. Chernousov and M. Rost, it is possible to prove the same results using Rost invariant and Chain Lemma. Our proof doesn't use these tools but only the geometry and several classical results concerning exceptional groups.

2 Zero cycles and symmetric powers

In this section we collect some facts from [Kr05] on the relations between zero cycles on a projective variety X and R-equivalence classes on symmetric powers of X.

2.1. We systematically use the Galois descent language, i.e., identify a (quasi-projective) variety X over k with the variety $X_{k_{\mathrm{sep}}}$ over the separable closure k_{sep} equipped with the action (by semiautomorphisms of $X_{k_{\mathrm{sep}}}$) of the absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$. It means that for any $\sigma \in \Gamma$ we are given by an automorphism φ_σ of $X_{k_{\text{sep}}}$ over k such that the diagram

$$\begin{array}{ccc} X_{k_{\text{sep}}} & \xrightarrow{\varphi_\sigma} & X_{k_{\text{sep}}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k_{\text{sep}} & \xrightarrow{\sigma^\sharp} & \operatorname{Spec} k_{\text{sep}} \end{array}$$

commutes, and $\varphi_{\sigma\tau} = \varphi_\sigma\varphi_\tau$. The set of k-rational points of X is precisely the set of k_{sep}-rational points of $X_{k_{\text{sep}}}$ stable under the action of Γ.

2.2. Let X be a variety over k. Two rational points $p, q \in X(k)$ are called *elementary linked* if there exists a rational morphism $\varphi\colon \mathbb{P}^1_k \dashrightarrow X$ such that $p, q \in \operatorname{Im}(\varphi(k))$. The R-equivalence is the equivalence relation generated by this relation. A variety X is called *R-trivial* if there exists exactly one class of R-equivalence on X, and *algebraically R-trivial* if $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is R-trivial for any finite field extension K/k.

The n-th symmetric power of X is by definition the variety $S^nX = X^n/S_n$ where S_n is the symmetric group acting on X^n via permutations.

Let p be a prime number. A field k is called *prime to p closed* if there is no finite field extension K/k of degree prime to p. For any field k we will denote by k_p its *prime to p closure* that is the algebraic extension of k which is prime to p closed.

Let X be a projective variety over k. By $\widetilde{\mathrm{CH}}_0(X)$ we denote the kernel of the degree map:

$$\widetilde{\mathrm{CH}}_0(X) = \operatorname{Ker}(\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}).$$

The crucial role in the sequel will play the following results:

2.3 Proposition. *([Kr05, Lemma 1.3]) If $\widetilde{\mathrm{CH}}_0(X_{k_p}) = 0$ for each prime p then $\widetilde{\mathrm{CH}}_0(X) = 0$.*

2.4 Proposition. *([Kr05, Theorem 1.4]) Suppose that k is prime to p closed and the following conditions are satisfied:*

1. *$S^{p^n}X$ is algebraically R-trivial for some $n \geq 0$,*
2. *For any field K/k such that $X(K) \neq \emptyset$ X_K is R-trivial.*

Then $\widetilde{\mathrm{CH}}_0(X) = 0$.

2.5. As an easy application we give a sketch of a proof of the well-known result (first appeared in [Pa84, Theorem 2.3.7]) that $\widetilde{\mathrm{CH}}_0(\mathrm{SB}(A)) = 0$ where A is a central simple algebra (a more general case of flag varieties is settled in [Kr05]). For simplicity we assume that $\deg A = p$ is prime.

We may assume that k is prime to p closed by Proposition 2.3 (for primes q different from p A splits over k_q). By Proposition 2.4 it is enough to show that $S^p\,\mathrm{SB}(A_K)$ is R-trivial for every finite extension K/k (the second hypothesis holds for any twisted flag variety). Making base change we may assume $K = k$. If A splits, the assertion is trivial; so we assume that A is not split.

According to our conventions (2.1) the variety $\mathrm{SB}(A)$ is the variety of all parabolic subgroups P of type P_1 in $\mathrm{PGL}_1(A \otimes_k k_{\mathrm{sep}})$ with the action of Γ coming from the action on k_{sep}. Therefore S^pX is the variety of all unordered p-tuples $[P^{(1)}, \dots, P^{(p)}]$ of parabolic subgroups of type P_1 in $\mathrm{PGL}_1(A \otimes_k k_{\mathrm{sep}})$. Let U be the open subvariety of S^pX defined by the condition that the intersection $P^{(1)} \cap \ldots \cap P^{(p)}$ is a maximal torus in $\mathrm{PGL}_1(A \otimes_k k_{\mathrm{sep}})$. Every maximal torus T in $\mathrm{PGL}_1(A \otimes_k k_{\mathrm{sep}})$ is contained in precisely p parabolic subgroups of type P_1, whose intersection is T. Therefore U is isomorphic to the variety of all maximal tori in $\mathrm{PGL}_1(A)$. This variety is known to be rational (and therefore R-trivial since it is homogeneous). Moreover, one can check that if A is not split then the embedding $U \to S^pX$ is surjective on k-points. So S^pX is R-trivial, and we are done.

2.6. The same method can be applied to prove that $\widetilde{\mathrm{CH}}_0(Q) = 0$ for a nonsingular projective quadric Q over a field of characteristic not 2 (the result of Swan [Sw89], see [Me95] and [Kr05] for generalizations).

As above, we may assume that $p = 2$, Q is anisotropic, and we have to prove that S^2Q is R-trivial. Let q be a corresponding quadratic form on a vector space V. Q is the variety of all lines $\langle v \rangle$ where $v \in V \otimes_k k_{\mathrm{sep}}$ satisfies $q(v) = 0$, with the obvious action of Γ. So S^2Q can be identified with the variety of pairs $[\langle v_1 \rangle, \langle v_2 \rangle]$ of lines of this kind, with induced action of Γ. Consider the open subvariety U defined by the condition $b_q(v_1, v_2) \neq 0$ (b_q stands for the polarization of q). Clearly the embedding $U \to S^2Q$ is surjective on k-points (otherwise the subspace $\langle v_1, v_2 \rangle$ defines a totally isotropic subspace of V defined over k). So it is enough to check that U is R-trivial.

Consider the open subvariety W of $\mathrm{Gr}(2, V)$ consisting of planes $H \leq V \otimes_k k_{\mathrm{sep}}$ such that $q|_H$ is nonsingular. For every such a plane there exists up to scalar factors exactly one hyperbolic base $\{v_1, v_2\}$ over k_{sep}. Therefore the map from U to W sending $[\langle v_1\rangle, \langle v_2\rangle]$ to $\langle v_1, v_2\rangle$ is an isomorphism. But any open subvariety of $\mathrm{Gr}(2, V)$ is R-trivial, and we are done.

We will use in the sequel the following observation.

2.7 Lemma. *Let $H \leq K \leq G$ be algebraic groups over k. Suppose that the map $\mathrm{H}^1(k, H) \to \mathrm{H}^1(k, K)$ is surjective. Then the morphism $G/H \to G/K$ is surjective on k-points.*

Proof. An element x of $G/K(k)$ is presented by an element $g \in G(k_{\mathrm{sep}})$ satisfying the condition $\gamma(\sigma) = g^{-1} \cdot {}^{\sigma}g$ lies in $K(k_{\mathrm{sep}})$ for all $\sigma \in \Gamma$. But γ is clearly a 1-cocycle with coefficients in K. Therefore by the assumption there exists some $h \in K$ such that $h^{-1}\gamma(\sigma) \cdot {}^{\sigma}h = (gh)^{-1} \cdot {}^{\sigma}(gh)$ is a 1-cocycle with coefficients in H. But then gh presents an element of $G/H(k)$ which goes to x under the morphism $G/H \to G/K$, and the lemma is proved. □

3 The twisted Cayley Plane

From now on we assume that $\operatorname{char} k \neq 2, 3$. In the present section we prove the theorem of the introduction.

3.1. Let J be a simple exceptional 27-dimensional Jordan algebra over k, N_J be its norm (which is a cubic form on J). An invertible linear map $f\colon J \to J$ is called a *similitude* if there exists some $\alpha \in k^*$ (called the *multiplier* of f) such that $N_J(f(v)) = \alpha N_J(v)$ for all $v \in J$. The group $G = \mathrm{Sim}(J)$ of all similitudes is a reductive group of inner type E_6, and every group of inner type E_6 with trivial Tits algebras can be obtained in this way up to isogeny (see [Ga01, Theorem 1.4]).

The *(twisted) Cayley plane* $\mathbb{OP}^2(J)$ is the variety obtained by the Galois descent from the variety of all parabolic subgroups of type P_1 (enumeration of roots follows Bourbaki) in $\mathrm{Sim}(J \otimes_k k_{\mathrm{sep}})$ (the action of $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ comes from the action on k_{sep}). This variety can be identified with the variety of all lines $\langle e\rangle$ spanned by elements $e \in J \otimes k_{\mathrm{sep}}$ satisfying the condition $e \times e = 0$ (see [Ga01, Theorem 7.2]).

Theorem. $\widetilde{\mathrm{CH}}_0(\mathbb{OP}^2(J)) = 0$.

Proof.

3.2. Note that if $\mathbb{OP}^2(J)(K) \neq \emptyset$ for some extension K/k then $\mathbb{OP}^2(J \otimes_k K)$ is rational and, since it is homogeneous, is R-trivial.

Let p be a prime number. If $p \neq 3$ then $\mathbb{OP}^2(J)(k_p) \neq \emptyset$. Indeed, choose any cubic étale subalgebra L of J ([Inv, Proposition 39.20]). It splits over k_p, and therefore $L \otimes_k k_p$ contains a primitive idempotent e. As an element of $J \otimes_k k_p$ it satisfies the condition $e \times e = 0$ (see [SV, Lemma 5.2.1(i)]).

So we assume $p = 3$. We have to prove that $S^3\mathbb{OP}^2(J \otimes_k K)$ is R-trivial for all finite field extensions K/k. Making the base change, it suffices to prove it for $K = k$. Moreover, we assume that J is not reduced (otherwise $\mathbb{OP}^2(J)$ is rational).

3.3. $S^3(\mathbb{OP}^2(J))$ is the variety of all unordered triples $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$ where e_i are elements of $J \otimes_k k_{\mathrm{sep}}$ satisfying the conditions $e_i \times e_i = 0$, with the natural action of Γ. Denote by U the open subvariety of $\mathbb{OP}^2(J)$ defined by the condition

$$N_{J\otimes_k k_{\mathrm{sep}}}(e_1, e_2, e_3) \neq 0$$

(we denote by N the polarization of the norm as well).

The embedding $U \to S^3(\mathbb{OP}^2(J))$ is surjective on k-points. Indeed, if $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$ is stable under the action of Γ and $N_{J\otimes_k k_{\mathrm{sep}}}(e_1, e_2, e_3) = 0$, then $\langle e_1, e_2, e_3\rangle$ gives by descent a k-defined subspace V of J such that $N|_V = 0$. But then J is reduced ([SV, Theorem 5.5.2]), which leads to a contradiction. So it is enough to show that U is R-trivial.

3.4. Choose a cubic étale subalgebra L in J. Over the separable closure we have

$$L \otimes_k k_{\mathrm{sep}} = k_{\mathrm{sep}}e_1 \oplus k_{\mathrm{sep}}e_2 \oplus k_{\mathrm{sep}}e_3$$

where e_1, e_2, e_3 are primitive idempotents. Then $e_i \times e_i = 0$ in $J \otimes_k k_{\mathrm{sep}}$,

$$N_{J\otimes_k k_{\mathrm{sep}}}(e_1, e_2, e_3) = N_{L\otimes_k k_{\mathrm{sep}}}(e_1, e_2, e_3) = 1,$$

and the triple $[e_1, e_2, e_3]$ is stable under Γ since L is so. Therefore $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$ is a k-rational point of U.

By [SV68, Proposition 3.12] G acts transitively on U (in geometric sense, that is over a separable closure). Therefore

$$U \simeq G/\operatorname{Stab}_G([\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]).$$

This stabilizer is defined over k (since it is stable under Γ) and coincides with $\operatorname{Stab}_G(L)$ (one inclusion is obvious, and the other one follows from the fact that e_1, e_2, e_3 are the only elements e of $L \otimes_k k_{\mathrm{sep}}$ satisfying the condition $e \times e = 0$, up to scalar factors (see [SV, Theorem 5.5.1]).

3.5. Consider the Springer decomposition of J respect to L: $J = L \oplus V$. Then the pair (L, V) has a natural structure of a twisted composition, and there is a monomorphism $\mathrm{Aut}(L, V) \to \mathrm{Aut}(J)$ sending a pair (φ, t) (where $\varphi\colon L \to L$, $t\colon V \to V$) to $\varphi \oplus t\colon J \to J$ (see [Inv, § 38.A]). Note that $\mathrm{Aut}(L, V)$ coincides with the stabilizer of L in $\mathrm{Aut}(J)$.

Show that there exists an exact sequence of algebraic groups:

$$1 \longrightarrow \mathrm{Aut}(L,V) \longrightarrow \mathrm{Stab}_G(L) \longrightarrow R_{L/k}(\mathbb{G}_m) \longrightarrow 1,$$
$$f \mapsto f(1)$$

where $R_{L/k}$ stands for the Weil restriction. Exactness at the middle term follows from the considerations above and the fact that the stabilizer of 1 in G coincides with $\mathrm{Aut}(J)$ ([SV, Proposition 5.9.4]).

Prove the exactness at the last term. A k_{sep}-point of $R_{L/k}(\mathbb{G}_m)$ is a triple of scalars $(\alpha_0, \alpha_1, \alpha_2) \in k^*_{\mathrm{sep}} \times k^*_{\mathrm{sep}} \times k^*_{\mathrm{sep}}$. Multiplying f by the scalar transformation with the coefficient $(\alpha_0\alpha_1\alpha_2)^{\frac{1}{3}}$ (which is an element of $\mathrm{Stab}_G(L)(k_{\mathrm{sep}})$) we may assume that $\alpha_0\alpha_1\alpha_2 = 1$. Choose a *related triple* (t_0, t_1, t_2) of elements of $\mathrm{GO}^+(\mathbb{O}_d, N_{\mathbb{O}_d})$ ($\mathbb{O}_d$ is the split Cayley algebra) such that $\mu(t_i) = \alpha_i$, $i = 0, 1, 2$ (see [Inv, Corollary 35.5]). Now the transformation f of J defined by

$$\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \alpha_1\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \alpha_2\varepsilon_2 \end{pmatrix}$$

lies in $\mathrm{Sim}(J)$ by [Ga01, (7.3)], stabilizes $L_{k_{\mathrm{sep}}} = \mathrm{diag}(k_{\mathrm{sep}}, k_{\mathrm{sep}}, k_{\mathrm{sep}})$, and sends $1 \in J_{k_{\mathrm{sep}}}$ to $\mathrm{diag}(\alpha_0, \alpha_1, \alpha_2)$.

Since $\mathrm{H}^1(k, L^*) = 1$ by Hilbert '90, the map $\mathrm{H}^1(k, \mathrm{Aut}(L, V)) \to \mathrm{H}^1(k, \mathrm{Stab}_G(L))$ is surjective. By Lemma 2.7 the morphism

$$G/\mathrm{Aut}(L, V) \to G/\mathrm{Stab}_G(L) \simeq U$$

is surjective on k-points. Therefore it suffices to show that $G/\mathrm{Aut}(L, V)$ is R-trivial.

3.6. Consider the morphism

$$\psi\colon G/\mathrm{Aut}(L, V) \to G/\mathrm{Aut}(J).$$

By [Kr05, Corollary 3.14] it suffices to show that

1. ψ is surjective on k-points;
2. $G/\operatorname{Aut}(J)$ is R-trivial;
3. The fibers of ψ (which are isomorphic to $\operatorname{Aut}(J)/\operatorname{Aut}(L,V)$) are unirational and R-trivial.

3.7. In order to prove surjectivity of ψ on k-points it is enough by Lemma 2.7 to prove surjectivity of the map $\mathrm{H}^1(k,\operatorname{Aut}(L,V)) \to \mathrm{H}^1(k,\operatorname{Aut}(J))$. Now $\mathrm{H}^1(k,\operatorname{Aut}(L,V))$ classifies all twisted compositions (L',V') which become isomorphic to (L,V) over k_{sep}, and $\mathrm{H}^1(k,\operatorname{Aut}(J))$ classifies all (exceptional 27-dimensional) Jordan algebras J'. It is easy to verify that the morphism sends (L',V') to the Jordan algebra $L'\oplus V'$, and hence the surjectivity follows from the fact that any Jordan algebra admits a Springer decomposition (cf. [Inv, Proposition 38.7]).

3.8. Let W be the open subvariety of J consisting of elements v with $N_J(v) \neq 0$. Then G acts transitively (in geometric sense) on W (see [SV, Proposition 5.9.3]), and the stabilizer of the point 1 coincides with $\operatorname{Aut}(J)$. So $G/\operatorname{Aut}(J) \simeq W$ is clearly R-trivial.

3.9. Consider the variety Y of all étale cubic subalgebras of J. By [Inv, Proposition 39.20(1)] there is a map from an open subvariety J_0 of regular elements in J to Y (sending a to $k[a]$), surjective on k-points. Therefore Y is unirational and R-trivial 24-dimensional irreducible variety.

$\operatorname{Aut}(J)$ acts on Y naturally. Let L' be any k-point of Y. The stabilizer of L' in $\operatorname{Aut}(J)$ obviously equals $\operatorname{Aut}(L',V')$ ($J = L' \oplus V'$ is the Springer decomposition). So the orbit of L' is isomorphic to $\operatorname{Aut}(J)/\operatorname{Aut}(L',V')$ and, in particular, has dimension 24. Therefore it is open, and since L' is arbitrary, the action is transitive. So we have $\operatorname{Aut}(J)/\operatorname{Aut}(L,V) \simeq Y$ is unirational and R-trivial, and the proof is completed.

□

3.10 Remark. The main result of the paper also implies the injectivity of the degree map for the twisted form Y of F_4/P_4. To see this one observes that (if the splitting field has degree $p = 3$) both varieties $\mathbb{OP}^2(J)$ and Y split completely over the function fields of each other. This together with [Me, Cor. 3.6] implies that the groups of zero-cycles are isomorphic.

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