Quadratic and differential forms over function fields of Pfister quadrics in characteristic two*

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Abstract

Let F be a field of characteristic 2. Let Ω_F^n be the F-space of differential forms over F. There is a homomorphism $\wp : \Omega_F^n \longrightarrow \Omega_F^n / \operatorname{d}\Omega_F^{n-1}$ given by $\wp \quad x\frac{\operatorname{d}x_1}{x_1} \wedge \cdots \wedge \frac{\operatorname{d}x_n}{x_n} \quad = (x^2 - x)\frac{\operatorname{d}x_1}{x_1} \wedge \cdots \wedge \frac{\operatorname{d}x_n}{x_n} \bmod \operatorname{d}\Omega_F^{n-1}$. Let $H^{n+1}(F) = \operatorname{coker}(\wp)$. If $p = \ll a_1, \ldots, a_n; b]]$ is an anisotropic quadratic Pfister form over F and $F(p)$ the function field of the Pfister quadric $\{p = 0\}$, we compute the kernel $H^{n+1}(F(p)/F) = \ker \quad H^{n+1}(F) \to H^{n+1}(F(p))$ for all m. Using Kato's correspondence between differential and quadratic forms we compute the kernels $I^m W_q(F(p)/F) = \ker\,[I^m W_q(F) \to I^m W_q(F(p))]$, where $W_q(F)$ denotes the Witt group of quadratic forms over F and I_F is the maximal ideal of the Witt ring $W(F)$ of symmetric bilinear forms over F.

Keywords: Quadratic Forms, Bilinear forms, Pfister forms, Witt ring, Differential forms.

1 Introduction

We continue in this paper our previous work [Ar-Ba$_1$] on the behavior of quadratic and differential forms under function field of Pfister quadrics over fields with $2 = 0$. In [Ar-Ba$_1$] we considered bilinear Pfister quadrics, and in this paper we will treat the case of quadratic Pfister quadrics. If F is a field with $2 = 0$,

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we will denote by $W(F)$ the Witt ring of non singular symmetric bilinear forms over F and by $W_q(F)$ the $W(F)$- module of non singular quadratic forms over F. Let $I_F \subset W(F)$ be the maximal ideal of even dimensional forms (see [Sa], [Ba], [Mi] for details). For $a_i \in F^* = F\backslash\{0\}$, $1 \leq i \leq n$, we will denote by $< a_1, \ldots, a_n >$ the bilinear form with diagonal Gram-matrix and entries $a_1, \ldots, a_n$ on the diagonal. The quadratic form $x^2 + xy + ay^2$ will be denoted by $[1, a]$, $a \in F$. The maximal ideal $I_F \subset W(F)$ is additively generated by the forms $\ll a \gg = < 1, a >$, $a \in F^*$, so that the powers I_F^n are generated by the n-fold bilinear Pfister forms $\ll a_1, \ldots, a_n \gg = \ll a_1 \gg \cdots \ll a_n \gg$, $a_i \in F^*$. The submodules $I^n W_q(F)$ of $W_q(F)$ are generated by the n-fold quadratic Pfister forms $\ll a_1, \ldots, a_n; a]] = \ll a_1, \ldots, a_n \gg \cdot [1, a]$. The graded objects I_F^n / I_F^{n+1}, resp. $I^n W_q(F) / I^{n+1} W_q(F)$ will be denoted by $\overline{I_F^n}$, resp. $\overline{I^n W_q(F)}$.

Let p be an anisotropic bilinear or quadratic n-fold Pfister form. Let $F(p)$ denote the function field of the quadric $\{p = 0\}$ over F. In [Ar-Ba$_1$] we computed the kernels $\ker \left[\overline{I^m W_q(F)} \longrightarrow \overline{I^m W_q(F(p))}\right]$ if p is a bilinear n-fold Pfister form for all $m \geq 0$. In this paper we will compute these kernels if p is a quadratic n-fold Pfister form. The bilinear case implies the quadratic case for $m \leq n$, as it is shown in [Ar-Ba$_1$]. Although the methods are similar, the arguments we need in the latter case are much more delicate. Actually we will compute the absolute kernels $\ker [I^m W_q(F) \longrightarrow I^m W_q(F(p))]$ by a trick used in [Ar-Ba$_2$] (see the statements below).

As in [Ar-Ba$_1$] we will use Kato's correspondence between differential forms and quadratic forms. We will work with differential forms and then translate the results into the language of quadratic forms. Let us first introduce briefly some notations and results. If Ω_F^1 denotes the F-vector space generated by the symbols $\mathrm{d}\, a$, $a \in F$, with the relations $\mathrm{d}(a + b) = \mathrm{d}\, a + \mathrm{d}\, b$, $\mathrm{d}(ab) = b\, \mathrm{d}\, a + a\, \mathrm{d}\, b$, then let $\Omega_F^n = \bigwedge^n \Omega_F^1$ and $\mathrm{d} : \Omega_F^n \longrightarrow \Omega_F^{n+1}$ be the differential operator $\mathrm{d}(x\, \mathrm{d}\, x_1 \wedge \cdots \wedge \mathrm{d}\, x_n) = \mathrm{d}\, x \wedge \mathrm{d}\, x_1 \wedge \cdots \wedge \mathrm{d}\, x_n$. Then $\mathrm{d}^2 = 0$ and d extends the derivation $\mathrm{d} : F \longrightarrow \Omega_F^1$. The space $\Omega_F^* = \bigoplus_{n=0}^{\infty} \Omega_F^n$ $(\Omega_F^0 = F)$ is a $\mathbb{Z}$-graded algebra with the exterior multiplication $\wedge : \Omega_F^n \times \Omega_F^m \longrightarrow \Omega_F^{n+m}$. The usual Artin-Schreier operator $\wp : F \longrightarrow F$, $\wp(x) = x^2 - x$ extends to a well defined homomorphism

$$\wp : \Omega_F^n \longrightarrow \Omega_F^n / \mathrm{d}\, \Omega_F^{n-1}$$

through

$$\wp\left(x \frac{\mathrm{d}\, x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\, x_n}{x_n}\right) = \overline{(x^2 - x) \frac{\mathrm{d}\, x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\, x_n}{x_n}}.$$

Let $\nu_F(m) = \ker(\wp)$, $H^{m+1}(F) = \operatorname{coker}(\wp)$, so that

$$0 \longrightarrow \nu_F(m) \longrightarrow \Omega_F^m \longrightarrow \Omega_F^m / \mathrm{d}\, \Omega_F^{m-1} \longrightarrow H^{m+1}(F) \longrightarrow 0$$

is exact. The groups $\nu_F(n)$ act on the groups $H^{m+1}(F)$ by exterior multiplication of forms

$$\bigwedge : \nu_F(n) \times H^{m+1} \longrightarrow H^{n+m+1}(F).$$

These groups behave functorially with respect to field extensions. If $F \hookrightarrow L$ is a field extension we will denote by $\nu_{L/F}(m)$, resp. $H^{m+1}(L/F)$, the kernels $\ker(\nu_F(m) \longrightarrow \nu_L(m))$, resp. $\ker(H^{m+1}(F) \longrightarrow H^{m+1}(L))$. A basic lemma due to Kato and stated below shows that $\nu_F(m)$ is additively generated by the pure logarithmic differential forms $\frac{\mathrm{d}\, x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}\, x_n}{x_n}$, $x_i \in F^*$. Recall that a 2-basis of F is a subset $\mathcal{B} = \{a_i,\, i \in I\}$ such that the elements $\{a^{\varepsilon} = \prod\limits_{i \in I} a^{\varepsilon_i},\, \varepsilon = (\varepsilon_i)_{i \in I},\, \varepsilon_i \in \{0,1\},\, \varepsilon_i = 0$ for almost all $i \in I\}$ form a F^2-basis of F. Then ordering I, it is easy to see that for any $n \geq 1$, $\{\frac{\mathrm{d}\, a_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{i_n}}{a_{i_n}},\, i_1 < \cdots < i_n\}$ is a F-basis of Ω_F^n. Using such a basis we can define a $\mathcal{B}$-depending Artin-Schreier operator $\wp : \Omega_F^n \longrightarrow \Omega_F^n$, $\wp\left(\sum_{i_1 < \cdots < i_n} c_{i_1, \cdots, i_n} \frac{\mathrm{d}\, a_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{i_n}}{a_{i_n}}\right) = \sum_{i_1 < \cdots < i_n} \wp\left(c_{i_1, \cdots, i_n}\right) \frac{\mathrm{d}\, a_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{i_n}}{a_{i_n}}$. This operator $\wp$ is modulo $\mathrm{d}\, \Omega_F^{n-1}$, well defined, and we will use it some times when the 2-basis $\mathcal{B}$ is given. Moreover if we set for $\eta = \sum_{i_1 < \cdots < i_n} c_{i_1, \cdots, i_n} \frac{\mathrm{d}\, a_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{i_n}}{a_{i_n}}$, $\eta^{[2]} = \sum_{i_1 < \cdots < i_n} c^2_{i_1, \cdots, i_n} \frac{\mathrm{d}\, a_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{i_n}}{a_{i_n}}$, we get $\wp(\eta) = \eta^{[2]} - \eta$. In fact the square operation $\eta^{[2]}$ is well defined modulo $\mathrm{d}\, \Omega_F^{n-1}$. Using this notation we can write $H^{m+1}(F) = \Omega_F^m / \left(\wp \Omega_F^m + \mathrm{d}\, \Omega_F^{m-1}\right)$. Another important operator on differential forms is the Cartier operator. Given a 2-basis $\mathcal{B} = \{a_i, i \in I\}$ of F we define

$$[\Omega_F^n]^2 = \{\eta^{[2]} \,|\, \eta \in \Omega_F^n\}$$

which is the subgroup of squares of Ω_F^n (with respect to $\mathcal{B}$). In [Ca] it is shown that the space $Z_F^n = \ker(\mathrm{d}) \subset \Omega_F^n$ of closed n-forms satisfies $Z_F^n = [\Omega_F^n]^2 \bigoplus \mathrm{d}\, \Omega_F^{n-1}$. Then we define the Cartier operator

$$C : Z_F^n \longrightarrow \Omega_F^n$$

by

$$C(\eta^{[2]} + \mathrm{d}\, \omega) = \eta$$

C is well defined, and independent of the 2-basis $\mathcal{B}$. This operator satisfies the following rules:

- $\mathrm{C}\left(a^2 \dfrac{\mathrm{d}\, b}{b}\right) = a \dfrac{\mathrm{d}\, b}{b}$ for all $a, b \in F^*$
- $\mathrm{C}\left(\mathrm{d}\, \eta\right) = 0$ for all $\eta \in \Omega_F^{n-1}$
- $\mathrm{C}\left(a^2 w\right) = aw$ for all $w \in Z_F^n$
- $\mathrm{C}\left(w \wedge \lambda\right) = \mathrm{C}\left(w\right) \wedge \mathrm{C}\left(\lambda\right)$ for all $w \in Z_F^n,\ \lambda \in Z_F^m$.

The elements of $\nu_F(n)$ are characterized by the following properties: for any $\eta \in \Omega_F^n$ it holds

$$\eta \in \nu_F(n) \quad \text{if and only if} \quad \eta \in Z_F^n \ \text{ and } \ C(\eta) = \eta.$$

Moreover for $\eta \in \Omega_F^n$

$$\eta \in \mathrm{d}\, \Omega_F^{n-1} \quad \text{if and only if} \quad \eta \in Z_F^n \ \text{ and } \ C(\eta) = 0.$$

We will use these properties thoroughly in this paper without mentioning any source. Let us state now Kato's result (see [Ka]). To this end we need some notation. Let $\mathcal{B}$ be a fixed 2-basis and take any fixed ordering on I. For any $n \geq 1$, let $\Sigma_{n,F}$ denote the set of functions $\sigma : \{1, \dots, n\} \longrightarrow I$ with $\sigma(i) < \sigma(j)$ whenever $i < j$. We order $\Sigma_{n,F}$ lexicographically. For any $\alpha \in \Sigma_{n,F}$ we denote by $\frac{\mathrm{d}\, a_\alpha}{a_\alpha}$ the form $\frac{\mathrm{d}\, a_{\alpha(1)}}{a_{\alpha(1)}} \wedge \dots \wedge \frac{\mathrm{d}\, a_{\alpha(n)}}{a_{\alpha(n)}}$. Thus $\{\frac{\mathrm{d}\, a_\alpha}{a_\alpha},\, \alpha \in \Sigma_{n,F}\}$ is a F-basis of Ω^n_F. Let $\Omega^n_{F,\alpha}$ be the subspace generated by the forms $\frac{\mathrm{d}\, a_\beta}{a_\beta}$, $\beta \leq \alpha$ and $\Omega^n_{F,<\alpha}$ the subspace generated by $\frac{\mathrm{d}\, a_\beta}{a_\beta}$ with $\beta < \alpha$. For any $i \in I$ we will also denote by F_i the subfield $F^2(a_j | \, j \leq i)$ and by $F_{<i}$ the subfield $F^2(a_j | \, j < i)$. Thus $F_i = F_{<i}(a_i)$. (If i_0 is the first element in I, then $F_{<i_0} = F^2$). Kato's lemma (see [Ka]) can be stated as follows:

Lemma (Kato [Ka]) *Fix $\alpha \in \Sigma_{n,F}$, $y \in F$ and assume $\wp(y\frac{\mathrm{d}\, a_\alpha}{a_\alpha}) \in \Omega^n_{F,<\alpha} + \mathrm{d}\,\Omega^{n-1}_F$. Then there exist $v \in \Omega^n_{F,<\alpha}$ and $c_i \in F^*_{\alpha(i)}$, $1 \leq i \leq n$, such that*

$$y\frac{\mathrm{d}\, a_\alpha}{a_\alpha} = v + \frac{\mathrm{d}\, c_1}{c_1} \wedge \dots \wedge \frac{\mathrm{d}\, c_n}{c_n}.$$

This implies in particular, that any $\eta \in \nu_F(n)$ can be written as

$$\eta = \sum_{\beta \leq \alpha} \varepsilon_\beta \frac{\mathrm{d}\, f_\beta}{f_\beta}$$

for some $\alpha \in \Sigma_{n,F}$, $f_{\beta(i)} \in F^*_{\beta(i)}$, $\varepsilon_\beta \in \{0, 1\}$ and $\frac{\mathrm{d}\, f_\beta}{f_\beta} = \frac{\mathrm{d}\, f_{\beta(1)}}{f_{\beta(1)}} \wedge \dots \wedge \frac{\mathrm{d}\, f_{\beta(n)}}{f_{\beta(n)}}$.

In [Ar-Ba$_2$] we have shown the following result, which we quote here for the sake of completeness, since we will use it several times in the future.

Lemma 1.1 *Let $\mathcal{B} = \{a_i,\, i \in I\}$ be a 2-basis of F with a given ordering on I. Let $\alpha \in \Sigma_{n,F}$ and let $\omega = \sum_{\gamma \leq \alpha} c_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} \in \Omega^n_F$ be a differential form with $c_\alpha \neq 0$. If $\omega \in \mathrm{d}\,\Omega^{n-1}_F$, then there exist elements $M_i \in F_{<\alpha(i)}$, $1 \leq i \leq n$ such that*

$$c_\alpha = a_{\alpha(1)} M_1 + \dots + a_{\alpha(n)} M_n.$$

After these preliminaries we proceed now to formulate our main result. Let $p = \ll a_1, \dots, a_n; b]]$ be an anisotropic quadratic n-fold Pfister form over F. Since we can alter b modulo $\wp(F)$ we may assume $b \in F^2$. Moreover since $\ll a_1, \dots, a_n \gg$ is anisotropic, $a_1, \dots, a_n$ are part of a 2-basis of F. Let $F(p)$ be the function field of the quadric $\{p = 0\}$ over F. Then the main result in this paper is

Theorem 1.2 *For all $m \geq 0$ it holds*

$$H^{m+1}(F(p)/F) = \begin{cases} 0 & \text{if } m < n \\ \nu_F(m-n) \wedge \overline{b\frac{\mathrm{d}\, a_1}{a_1} \wedge \dots \wedge \frac{\mathrm{d}\, a_n}{a_n}} & \text{if } m \geq n. \end{cases}$$

This result follows easily from (4.1) (5.4) in [Ar-Ba$_1$] for $n = 0$ and for $m \leq n$, $n \geq 1$. Thus we can assume $n \geq 1$ and $m \geq n$. Using Kato's correspondence (1.2) reads

Theorem 1.3 *For all* $m, n \geq 0$

$$\overline{I^m W_q(F(p)/F)} = \begin{cases} 0 & \text{if } m < n \\ \\ \overline{I_F^{m-n} \cdot p} & \text{if } m \geq n. \end{cases}$$

We give now the full graded version of this result

Theorem 1.4 *Let $p = \ll a_1, \ldots, a_n; b]]$ be an anisotropic quadratic n-fold Pfister form over F. Then for all $m \geq 0$*

$$I^m W_q(F(p)/F) = \begin{cases} W(F) \cdot p & \text{if } m < n \\ \\ I_F^{m-n} \cdot p & \text{if } m \geq n \end{cases}$$

(here we write $W(F)$ instead of I_F^0).

Remark 1.5 *In [Ar-Ba$_1$] we computed $\overline{I^m W_q(F(\phi)/F)}$ for $\phi = \ll a_1, \ldots, a_n \gg$. The corresponding full graded version of the main result of [Ar-Ba$_1$] is now*

$$I^m W_q(F(\phi)/F) = \begin{cases} \phi \cdot W_q(F) & \text{if } m < n \\ \\ \phi \cdot I^{m-n} W_q(F) & \text{if } m \geq n \end{cases} \tag{1.6}$$

Since the proofs of (1.4) and (1.6) are very similar and rely on the same principle, we will derive only (1.4) from (1.3).

Proof of (1.4). Let $p = \ll a_1, \ldots, a_n; b]]$ be anisotropic. Let us first assume that the field F has a finite 2-basis $\mathcal{B} = \{a_1, \ldots, a_N\}$. Then (see [Mi]) $I_F^{N+1} = 0$. Take $q \in \overline{I^m W_q(F(p)/F)}$ i.e. $q \in I^m W_q(F)$ and $q \otimes F(p) \sim 0$. Assume $q \neq 0$. Then $\overline{q} \in \overline{I^m W_q(F(p)/F)}$ and hence if $m \geq n$, $\overline{q} \in \overline{I^{m-n} \cdot p}$, i.e. $q = \phi_1 p + q_1$ with $\phi_1 \in I_F^{m-n}$ and $q_1 \in I^{m+1} W_q(F)$. Since $q \otimes F(p) = 0$, $p \otimes F(p) = 0$ it follows $q_1 \otimes F(p) = 0$. Thus $q_1 = \phi_2 p + q_2$ with $\phi_2 \in I_F^{m+1-n}$ and $q_2 \in I^{m+2} W_q(F)$. Iterating this procedure $M \geq N + 1 - m$ times and using $I^{N+1} = 0$ we arrive at an equation $q = (\phi_1 + \phi_2 + \cdots + \phi_M) p \in I_F^{m-n} \cdot p$. This proves the claim in this case. In the general case we have the relation $q \otimes F(p) = 0$ and this relation involves finitely many elements $a_1, \ldots, a_N \in \mathcal{B}$. Let $\mathcal{B}_0 = \{a_1, \ldots, a_N\} \subset \mathcal{B}$ and set $F_0 = F^2(a_1, \ldots, a_N)$. Then there exist forms q_0, p_0 over F_0, $p_0 = \ll a_1, \ldots, a_n; b]]$, such that $q = q_0 \otimes F$, $p = p_0 \otimes F$ and $q_0 \otimes F(p_0) = 0$. From the first part we conclude $q_0 \in I_{F_0}^{m-n} \cdot p_0$, and extending scalars to F we obtain $q \in I_F^{m-n} \cdot p$. This proves the claim in the case $m \geq n$. If $m < n$, then in the first case we would obtain $q \in I^{m+1} W_q(F)$ and iterating, we may assume $m = n$, i.e. $q \in W(F) \cdot p$. The rest of the proof is the same. This concludes the proof of (1.4) and also of (1.6). ■

We proceed now to describe briefly the plan of the paper. In section 2 we reduce the computation of $H^{m+1}(F(p)/F)$ to study for a form $w \in \Omega_F^m$ the equation $w = \wp u + \mathrm{d}\, v + T\eta$, where $u \in \Omega_F^m \otimes M$, $v \in \Omega_F^{m-1} \otimes M$, $\eta \in \nu_M(m)$ and T is the polynomial (2.1). Here M denotes the field $F\left(X_\mu^2, Y_\mu^2, \mu \in S_n\right)$, X_μ, Y_μ are variables for $\mu \in S_n =$ set of maps $\mu : \{1, \ldots, n\} \to \{0, 1\}$, $\mu \neq 0$ (see (2.8)).

The main idea is now to reduce this equation to one without variables, so that during this process the form η changes and successively it gets factors of the form $\dfrac{\mathrm{d}\, a_i}{a_i}$, thereby at the end of the reduction process we get an equation

$$w = \wp u_0 + \mathrm{d}\, v_0 + b\frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \xi$$

with $\xi \in \nu_F(m-n)$, $u_0, v_0 \in \Omega_F$, which is the content of theorem (1.2). This reduction process is achieved in section 4 and 6. Section 3 is of preparatory character (see also [Ar-Ba$_1$]), and we analyze here the decomposition in partial fractions (= p-components) of forms contained in $\nu_{N(Y^2)}(m)$ for some field N, where Y is a variable over N (see (3.6)). We show that the $p-$components are also forms contained in groups $\nu_\bullet(m)$. In section 5, we characterize the forms $\eta \in \nu_{N(Y^2)}(m)$ with the property that $a^\sigma \eta \in \mathrm{d}\, \Omega_{M(Y^2)}^{m-1}$ for all $\sigma \leq \tau \in S_n$, where $a_1, \ldots, a_n \in M$ are part of a 2-basis of N (see (5.1)). This property, called *good shape*, plays a crucial role during the reduction process done in section 6 (see main lemma (6.1)).

2 First reduction step in the proof of (1.2)

Let F be a field with $2 = 0$ and let $p = \ll a_1, \ldots, a_n; b]]$ be an anisotropic quadratic n-fold Pfister form over F with function field $F(p)$. We may assume without restriction $b \in F^2$. Let S_n be the set of maps $\mu : \{1, \ldots, n\} \longrightarrow \{0, 1\}$, $\mu \neq 0$, and let $\mathcal{B}$ be a 2-basis of F, which we may assume contains $a_1, \ldots, a_n$ as the first elements in some ordering of the index set I of $\mathcal{B}$. Let $\Sigma_{n,F}$ be the set of maps $\sigma : \{1, \ldots, n\} \longrightarrow I$ with $\sigma(i) < \sigma(j)$ whenever $i < j$. For any $\mu \in S_n$ we take a pair of independent variables X_μ, Y_μ and we form the field $L = F(X_\mu, Y_\mu, \mu \in S_n)$. For $m < n$ we will identify $S_m \subset S_n$ in the obvious way. Let $T_0 = b + \sum_{\mu \in S_n} a^\mu Y_\mu^2 (b + X_\mu + X_\mu^2)$ be the generic polynomial representing the pure part of p. Here $a^\mu = a_1^{\mu(1)} \cdots a_n^{\mu(n)}$. Then $F(p) = L(z_0)$ with $z_0^2 + z_0 = T_0$. We can change T_0 modulo $\wp F[X_\mu, Y_\mu, \mu \in S_n]$, so that we will work in this paper with the polynomial

$$\begin{aligned} T &= b + \sum_{\mu \in S_n} a^\mu Y_\mu^2 (b + X_\mu^2) + \sum_{\mu \in S_n} \left(a^\mu Y_\mu^2\right)^2 X_\mu^2 \\ &= \prod_{\mu \in S_n} \left(1 + a^\mu Y_\mu^2\right)\left(b + a^\mu Y_\mu^2 X_\mu^2\right) \end{aligned} \tag{2.1}$$

and $F(p) = L(z)$, $z^2 + z = T$.

The 2-basis $\mathcal{B}$ of F extends to the 2-basis $\mathcal{B}_L = \mathcal{B} \cup\{X_\mu,\, Y_\mu,\, \mu \in S_n\}$ of L, where we order the X's, Y's lexicographically. Since $F(p)/L$ is a separable quadratic extension then $\mathcal{B}_L$ remains a 2-basis of $F(p)$. This is consequence of the following

Lemma 2.2 *If E/F is a finite separable extension, then any 2-basis of F is also a 2-basis of E. Thus $\Omega^*_E = \Omega^*_F \otimes_F E$.*

In particular we obtain $\Omega^n_{F(p)} = \Omega^n_L \otimes_L F(p)$ for all $n \geq 1$, and $\mathcal{B}_F \cup\{X_\mu,\, Y_\mu,\, \mu \in S_n\}$ is 2-basis of $F(p)$.

Take now $w \in H^{m+1}(F(p)/F)$, i.e. $w \in \Omega^m_F$ satisfies

$$w = \wp(u) + \mathrm{d}\, v \tag{2.3}$$

with $u \in \Omega^m_{F(p)}$, $v \in \Omega^{m-1}_{F(p)}$. Thus u, v are forms generated over $L(z) = L + Lz$ by the differentials $\mathrm{d}\, a_i$, $i \in I$, $\mathrm{d}\, X_\mu$, $\mathrm{d}\, Y_\mu$, $\mu \in S_n$.

Lemma 2.4 *The forms u, v in (2.3) can be chosen in $\Omega^m_F \otimes F(X^2_\mu,\, Y^2_\mu,\, \mu \in S_n)(z)$.*

Proof. Let us represent w, u and $\mathrm{d}\, v$ in the 2-basis $\mathcal{B}_L = \{b_j, j \in J\}$ of $F(p)$ introduced above, i.e. they are generated by the differentials $\frac{\mathrm{d}\, a_i}{a_i}$, $a_i \in \mathcal{B}_F$, $\frac{\mathrm{d}\, X_\mu}{X_\mu}$, $\frac{\mathrm{d}\, Y_\mu}{Y_\mu}$ over $F(p)$. Let $\gamma \in \Sigma_{m,F}$ be the maximal multi-index occurring in w, and $\delta \in \Sigma_{m,L}$ the maximal one occurring in u. Then the maximal multi-index of $\mathrm{d}\, v$ is $\leq \max\{\gamma, \delta\}$. If $\gamma < \delta$ then we have from (2.3)

$$\wp\left(u_\delta \frac{\mathrm{d}\, b_\delta}{b_\delta}\right) \equiv 0 \mod \left(\mathrm{d}\, \Omega^{m-1}_{F(p)} + \Omega^m_{F(p), < \delta}\right) \tag{2.5}$$

where $u_\delta \frac{\mathrm{d}\, b_\delta}{b_\delta}$ is the maximal ($\neq 0$) element in the expansion of u. Using Kato's lemma we can write

$$u_\delta \frac{\mathrm{d}\, b_\delta}{b_\delta} = \frac{\mathrm{d}\, f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}\, f_m}{f_m} + u'$$

with $f_i \in F(p)_{\delta(i)}$, $1 \leq i \leq m$, and $u' \in \Omega^m_{F(p), < \delta}$. Thus we get

$$w = \wp\overline{u} + \mathrm{d}\, \overline{v}$$

with $\overline{u} \in \Omega^m_{F(p), < \delta}$. Hence we can assume $\gamma = \delta$. In particular u belongs to $\Omega^m_F \otimes F(p)$ and let $w_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma}$ be the maximal term in w as well as $u_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma}$ the maximal term of u. Using lemma (3.3) of [Ar-Ba$_2$] we conclude

$$w_\gamma = \wp u_\gamma + \sum_{i=1}^{m} a_{\gamma(i)} M_{\gamma(i)} \tag{2.6}$$

with $M_{\gamma(i)} \in F(p)_{< \gamma(i)}$, $1 \leq i \leq m$. In particular $M_{\gamma(i)}$ is contained in $F(X^2_\mu, Y^2_\mu)(z)$ for all $1 \leq i \leq m$. Then (2.6) implies $\wp u_\gamma \in F(X^2_\mu, Y^2_\mu)(z)$ and hence also

$u_\gamma \in F(X_\mu^2, Y_\mu^2)(z)$, i.e. u_γ does not contain odd powers of X_μ, Y_μ in its 2-basis expansion. Thus $\wp(u) = \wp(u_\gamma)\frac{\mathrm{d}\, a_\gamma}{a_\gamma} + \wp(u_{<\gamma})$ with $u_{<\gamma} \in \Omega^m_{F(p),<\gamma}$ and $\wp(u_\gamma) \in F(X_\mu^2, Y_\mu^2)(z)$. We claim now that there is a decomposition $v = v_\gamma + v_{<\gamma}$, where $\mathrm{d}\, v_\gamma$ is free from $\mathrm{d}\, X_\mu$, $\mathrm{d}\, Y_\mu$'s and its coefficients lie in $F(X_\mu^2, Y_\mu^2)(z)$, and $\mathrm{d}\, v_{<\gamma} \in \Omega^{m-1}_{F(p),<\gamma}$. To see this we set $\mathrm{d}\, v = h_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} + v'$ with $v' \in \Omega^m_{F(p),<\gamma}$. Using lemma (3.2) of [Ar-Ba$_2$] again, we have $h_\gamma = \sum_{i=1}^m a_{\gamma(i)} M_{\gamma(i)}$ with $M_{\gamma(i)} \in F(X_\mu^2, Y_\mu^2)(z)$ for $1 \le i \le m$ (see above remarks). Thus

$$\begin{aligned}
\mathrm{d}\, v &= \left(\sum_{i=1}^m a_{\gamma(i)} M_{\gamma(i)}\right) \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}} + v' \\
&= \sum_{i=1}^m a_{\gamma(i)} M_{\gamma(i)} \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(i)}}{a_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}} + v' \\
&= \sum_{i=1}^m \mathrm{d}\left(a_{\gamma(i)} M_{\gamma(i)} \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \overset{i}{\vee} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}} \right) \\
&\qquad\qquad + a_{\gamma(i)} M_{\gamma(i)} \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, M_{\gamma(i)}}{M_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}} + v'.
\end{aligned}$$

Since $\lambda = a_{\gamma(i)} M_{\gamma(i)} \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, M_{\gamma(i)}}{M_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}} + v' \in \Omega^{m-1}_{F(p),<\gamma}$ and from the last relation we see that λ is exact, we conclude $\lambda = \mathrm{d}(v_{<\gamma})$ for some $v_{<\gamma} \in \Omega^{m-1}_{F(p)}$. Hence we have $\mathrm{d}\, v = \mathrm{d}\, v_\gamma + \mathrm{d}(v_{<\gamma})$, where

$$v_\gamma = \sum_{i=1}^m a_{\gamma(i)} M_{\gamma(i)} \frac{\mathrm{d}\, a_{\gamma(1)}}{a_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(i-1)}}{a_{\gamma(i-1)}} \wedge \frac{\mathrm{d}\, a_{\gamma(i+1)}}{a_{\gamma(i+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}}$$

belongs to $\Omega^{m-1}_F \otimes F(X_\mu^2,\, Y_\mu^2)(z)$. Inserting all of this in (2.3) it follows

$$w + \wp(u_\gamma) + \mathrm{d}\, v_\gamma = \wp(u_{<\gamma}) + \mathrm{d}\,(v_{<\gamma})$$

where the left hand side of this equation is in $\Omega^m_F \otimes F(X_\mu^2,\, Y_\mu^2)(z)$. Repeating the above arguments for this relation we finally arrive at an equation $w = \wp(u) + \mathrm{d}\, v$ in $\Omega^m_F \otimes F(X_\mu^2,\, Y_\mu^2)(z)$ which proves the lemma. ■

Let us now consider the equation (2.3) with $u,\ v \in \Omega^m_F \otimes F(X_\mu^2,\, Y_\mu^2)(z)$. We write $u = u_0 + u_1 z$, $v = v_0 + v_1 z$ with $u_0, u_1 \in \Omega^m_F \otimes F(X_\mu^2, Y_\mu^2)$, $v_0, v_1 \in \Omega^{m-1}_F \otimes F(X_\mu^2,\, Y_\mu^2)$. Then

$$\begin{aligned}
w &= \wp(u_0 + u_1 z) + \mathrm{d}(v_0 + v_1 z) \\
&= \wp u_0 + T u_1^{[2]} + z u_1^{[2]} + z u_1 + \mathrm{d}\, v_0 + z\, \mathrm{d}\, v_1 + v_1 \wedge \mathrm{d}\, T
\end{aligned}$$

i.e.

$$\begin{aligned}
w &= \wp u_0 + \mathrm{d}\, v_0 + T u_1^{[2]} + v_1 \wedge \mathrm{d}\, T \\
0 &= \wp u_1 + \mathrm{d}\, v_1.
\end{aligned}$$

The second equation implies (with $L = F(X_\mu, Y_\mu)$)

$$u_1 \in \nu_L(m) \cap \Omega_F^m \otimes F(X_\mu^2, Y_\mu^2)$$

i.e.

$$u_1 \in \nu_{F(X_\mu^2, Y_\mu^2)}(m)$$

according to lemma (2.9) below. Inserting $u_1^{[2]} = u_1 + \mathrm{d}\, v_1$ in the first equation, we get

$$w = \wp u_0 + \mathrm{d}\, v_0 + T u_1 + T\, \mathrm{d}\, v_1 + v_1 \wedge \mathrm{d}\, T$$

and we finally obtain an equation for $w \in \Omega_F^m$

$$w = \wp u + \mathrm{d}\, v + T\eta \tag{2.7}$$

with $u \in \Omega_F^m \otimes F(X_\mu^2, Y_\mu^2)$, $v \in \Omega_F^{m-1} \otimes F(X_\mu^2, Y_\mu^2)$ and $\eta \in \nu_{F(X_\mu^2, Y_\mu^2)}(m)$. Here we have set $u = u_0$, $v = v_0 + T v_1$, $\eta = u_1$. Equation (2.7) will be the basic relation which will be used in the proof of theorem (1.2). Thus we summarize the above results in

Proposition 2.8 *Let $w \in H^{m+1}(F)$. Then $w \in H^{m+1}(F(p)/F)$ if and only if there exist $u \in \Omega_F^m \otimes F(X_\mu^2, Y_\mu^2)$, $v \in \Omega_F^{m-1} \otimes F(X_\mu^2, Y_\mu^2)$ and $\eta \in \nu_{F(X_\mu^2, Y_\mu^2)}(m)$ such that*

$$w = \wp u + \mathrm{d}\, v + T\eta.$$

In the proof of the above result we have used the following

Lemma 2.9

$$\nu_{F(X_\mu, Y_\mu)}(m) \cap \Omega_F^m \otimes F(X_\mu^2, Y_\mu^2) = \nu_{F(X_\mu^2, Y_\mu^2)}(m).$$

Proof. The contention $\supseteq$ is trivial. Write $L = F(X_\mu, Y_\mu)$ and $M = F(X_\mu^2, Y_\mu^2)$. Take now $\lambda \in \nu_L(m) \cap \Omega_F^m \otimes F(X_\mu^2, Y_\mu^2)$, and write

$$\lambda = c_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} + \lambda_{<\gamma}$$

with $\lambda_{<\gamma} \in \Omega_{F,<\gamma}^m \otimes M$. Since $\wp\lambda \in \mathrm{d}\, \Omega_L$, it follows

$$\wp(c_\gamma) \frac{\mathrm{d}\, a_\gamma}{a_\gamma} \in \mathrm{d}\, \Omega_L^{m-1} + \Omega_{L,<\gamma}^m.$$

Applying now Kato's lemma, we obtain

$$c_\gamma \frac{\mathrm{d}\, a_\gamma}{a_\gamma} = \frac{\mathrm{d}\, f_\gamma}{f_\gamma} + \lambda'_{<\gamma}$$

where $f_{\gamma(i)} \in L^2(a_j,\, j \le \gamma(i))$, $1 \le i \le m$ and $\lambda'_{<\gamma} \in \Omega_{L,<\gamma}^m$. Thus

$$\lambda = \frac{\mathrm{d}\, f_\gamma}{f_\gamma} + \lambda_{<\gamma} + \lambda'_{<\gamma}.$$

In particular, since $\frac{\mathrm{d}\, f_\gamma}{f_\gamma} \in \nu_M(m)$, we get $\lambda' = \lambda_{<\gamma} + \lambda'_{<\gamma} \in \nu_L(m) \cap \Omega_F \otimes M$, but now $\lambda' \in \Omega_{L,<\gamma}$ is in a lower filtration. We proceed with λ' as above and after finitely many steps we obtain $\lambda \in \nu_{F(X_\mu^2, Y_\mu^2)}(m)$. This proves the lemma.

■

Similarly we can prove the following result, which we will need later.

Lemma 2.10 *Let $F \subseteq M$ be fields and Y a variable over M. Assume $\mathrm{d}\, M \subseteq \Omega_F^1 \otimes M$. Then,*

$$\nu_{M(Y^2)}(m) \cap \Omega_F^m M[Y^2] = \nu_M(m)$$

(we will apply this result for $M = F(X_\mu^2, Y_\mu^2)$ for certain variables X_μ, Y_μ, $\mu \in S_n$).

Proof. Since $\mathrm{d}\, M \subseteq \Omega_F^1 \otimes M$, the contention $\supseteq$ is clear. Let us show now $\subseteq$. Let $\eta \in \nu_{M(Y^2)}(m) \cap \Omega_F^m M[Y^2]$ and write

$$\eta = \sum_\gamma \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}} = \sum_\alpha c_\alpha \frac{\mathrm{d}\, a_\alpha}{a_\alpha}$$

with $c_\alpha \in M[Y^2]$, $f_{\gamma(i)} \in M_{\gamma(i)}(Y^2)$, $1 \le i \le m$ (according to Kato's lemma). Choose γ to be the maximal multi-index on the left hand side. Expanding the term $\mathrm{d}\, f_\gamma / f_\gamma$, we see that $\mathrm{d}\, f_\gamma / f_\gamma = \left(\prod_{i=1}^m a_{\gamma(i)} D_{\gamma(i)}(f_{\gamma(i)}) \right) / \left(\prod_{i=1}^m f_{\gamma(i)} \right) \mathrm{d}\, a_\gamma / a_\gamma +$terms of lower filtration. On the right the maximal term is $c_\gamma \, \mathrm{d}\, a_\gamma / a_\gamma$, thus we obtain

$$c_\gamma = \frac{\prod_{i=1}^m a_{\gamma(i)} D_{\gamma(i)}(f_{\gamma(i)})}{\prod_{i=1}^m f_{\gamma(i)}}$$

with $c_\gamma \in M[Y^2]$ and all $f_{\gamma(i)} \in M[Y^2]$. Moreover we can assume that the $f_{\gamma(i)}$ have only irreducible factors of multiplicity one. We want to show $f_{\gamma(i)} \in M$ for all i. Let $k = \max\{i \mid f_{\gamma(i)} \notin M\}$ and consider an irreducible monic factor p of $f_{\gamma(k)}$ (which has without restriction multiplicity one). From the lemma (2.11) below, we know that filtration(p)=$\gamma(k)$ =filtration of $f_{\gamma(k)}$. In particular (loc.cit.) $p \nmid f_{\gamma(i)}$ for all $i < k$. Thus (2.11) implies $p | D_{\gamma(k)}(f_{\gamma(k)})$, because filtration($D_{\gamma(i)} f_{\gamma(i)}$) $\le$ filtration($f_{\gamma(i)}$) and for $i > k$, all $f_{\gamma(i)} \in M$. Let us write $f_{\gamma(k)} = p \cdot h$ with $(p, h) = 1$ in $M[Y^2]$. Then $D_{\gamma(k)}(f_{\gamma(k)}) = p D_{\gamma(k)}(h) + h D_{\gamma(k)}(p)$ and hence $p | D_{\gamma(k)}(p)$ which is a contradiction since p being monic, $\deg(D_{\gamma(k)}(p)) < \deg(p)$. This proves $f_{\gamma(i)} \in M$ for all i and the lemma follows.

■

Lemma 2.11 *Let $f(Y) \in M_l[Y]$ be a polynomial with filtration l and whose irreducible monic factors have multiplicity 1. Assume that all of these factors have filtration $\ge l$. Then all such factors have exactly filtration $= l$.*

Proof. Let $p|f$ be a monic irreducible factor such that filtration$(p) = k$ is maximal among all such factors. Write $p = p_0 + a_k p_1$ with fil(p_0), fil$(p_1) < k$. By assumption $k \geq l$. We have $(p_0 + a_k p_1)h = f$ in $M[Y]$, where fil$(f) = l \leq k$. Then fil$(h) \leq k$, otherwise we get a contradiction. Let $r =$ fil$(h) \leq k$, and set $h = h_0 + a_r h_1$, fil$(h_i) < r$, $i = 1, 2$. Obviously $p_1, h_1 \neq 0$. Assume now $k > l$. We have

$$\begin{aligned} f &= (p_0 + a_k p_1)(h_0 + a_r h_1) \\ &= p_0 h_0 + a_k p_1 h_0 + a_r h_1 (p_0 + a_k p_1). \end{aligned}$$

If $r < k$, then the term $a_k p_1 (h_0 + a_r h_1)$ has filtration $= k$, because fil$(p_1) < k$, and hence fil$(f) = k > l$, contradiction. Then necessarily $r = k$ and we can write

$$f = p_0 h_0 + a_k^2 p_1 h_1 + a_k (p_1 h_0 + h_1 p_0).$$

Since fil$(p_1 h_1) < k$, the last term dominates if it is $\neq 0$. But $l < k$ implies

$$p_1 h_0 + h_1 p_0 = 0$$

and this means, since $(p_0, p_1) = 1$, that $p_1 | h_1$, $p_0 | h_0$ and

$$\frac{h_1}{p_1} = \frac{h_0}{p_0} = t \in M[Y]$$

is a certain polynomial. Thus $h = t \cdot p$ and hence $p^2 | f$, contradicting our assumption on the multiplicity. This shows $k = l$. ■

3 p-components of differential forms

Our basic equation (2.7) holds in $\Omega_F^m \otimes F(X_\mu^2,\, Y_\mu^2,\, \mu \in S_n)$. More generally fix some index $\mu \in S_n$, and let $M = F(X_\nu^2, Y_\lambda^2 | \, \nu > \mu, \lambda \geq \mu)$ and set $X = X_\mu$. Assume that we have a form $u \in \Omega_F^m \otimes M(X^2)$. The p-component of this form u arise from the partial fraction decomposition in $M(X)$ of the coefficients of the form. We will call the subgroup $\Omega_F^m M[X^2]$ the integral forms and for any irreducible monic polynomial $p \in M[X]$ set

$$p^{-\infty} \Omega_F^m M[X^2] = \left\{ \frac{v}{p^s} | \, v \in \Omega_F^m M[X^2],\, s \geq 1, \deg_X v < s \deg_X p \right\} \qquad (3.1)$$

if $p \in M[X^2]$ and

$$p^{-\infty} \Omega_F^m M[X^2] = \left\{ \frac{w}{p^{2s}} | \, w \in \Omega_F^m M[X^2],\, s \geq 1, \deg_X v < 2s \deg_X p \right\}$$

if $p \notin M[X^2]$. Here we set $\deg_X w = 2t$ whenever we have $w = w_0 + w_1 X^2 + \cdots + w_t X^{2t}$, $w_t \neq 0$ and $w_i \in \Omega_F^m$. Then using partial fraction decomposition in $M(X)$ we obtain

$$\Omega_F^m M(X^2) = \Omega_F^m M[X^2] \oplus \bigoplus_p p^{-\infty} \Omega_F^m M[X^2]. \qquad (3.2)$$

It is shown in [Ar-Ba$_1$] that this decomposition is compatible with the operators d and $\wp$. If $u \in \Omega_F^m M(X^2)$, then $u = u_0 + \sum_p u_p$, and the forms u_p (including u_0) are called the p-components of u.

Let us consider the p-decomposition of the form $\eta \in \nu_L(m)$ where $L = F(X_\mu, Y_\mu,\, \mu \in S_n)$. Assume $\eta \in \nu_{M(X^2)}(m)$ where M and X are as above. Then $\eta = \eta_E + \sum_p \eta_p$ is the p-decomposition of η, $\eta_E \in \Omega_F^m M[X^2]$ and $\eta_p \in p^{-\infty}\Omega_F^m M[X^2]$. We claim

Lemma 3.3 *(i)* $\eta_E \in \nu_{M(X^2)}(m) \cap \Omega_F^m M[X^2] = \nu_M(m)$

(ii) $\eta_p \in \nu_{M(X^2)}(m) \cap p^{-\infty}\Omega_F^m M[X^2]$.

Proof. We use $\mathrm{d}\,\eta = 0$ (since $\eta \in \nu(m)$) and the compatibility of the p-decomposition with d to get from $\mathrm{d}\,\eta_E + \sum_p \mathrm{d}\,\eta_p = 0$ necessarily

$$\mathrm{d}\,\eta_E = 0, \qquad \mathrm{d}\,\eta_p = 0 \qquad \text{for all } p.$$

Now we apply the Cartier operator to η to obtain from $\eta = C(\eta)$

$$\eta_E + \sum_p \eta_p = C(\eta_E) + \sum_p C(\eta_p)$$

and hence

$$\begin{aligned} \eta_E &= C(\eta_E) \\ \eta_p &= C(\eta_p) \end{aligned}$$

i.e. $\eta_E \in \nu_M(m)$, $\eta_p \in \nu_{M(X^2)}(m)$. This proves (i) and (ii). ■

The equality

$$\nu_{M(X^2)}(m) \cap \Omega_F^m M[X^2] = \nu_M(m)$$

has been shown in (2.10). We give here a shorter proof of it. The contention $\supseteq$ is clear. Let $\alpha \in \nu_{M(X^2)}(m) \cap \Omega_F^m M[X^2]$ and write $\alpha = \alpha_0 + \alpha_1 X^2 + \cdots + \alpha_s X^{2s}$ with $\alpha_i \in \Omega_F^m M$. From $\mathrm{d}\,\alpha = 0$ it follows $\mathrm{d}\,\alpha_i = 0$ for all $0 \le i \le s$ and the relation $C(\alpha) = \alpha$ implies

$$\alpha_0 + \alpha_1 X^2 + \cdots + \alpha_s X^{2s} = C(\alpha_0) + C(\alpha_1) X + \cdots + C(\alpha_s) X^s$$

which holds only if $s = 0$, $\alpha = \alpha_0 \in \Omega_F^m M$. But $\mathrm{d}\,\alpha_0 = 0$ and $C(\alpha_0) = \alpha_0$ imply $\alpha_0 \in \nu_M(m)$. This shows (2.10) again.

Let us now study the intersection

$$\nu_{M(X^2)}(m) \cap p^{-\infty}\Omega_F^m M[X^2] \tag{3.4}$$

and take η_p contained in it.

We can write $\eta_p = \frac{\xi}{p^{2t}}$ or $\eta_p = \frac{\xi}{p^{2t+1}}$ where $\xi \in \Omega_F^m M[X^2]$, $p \nmid \xi$ and $\deg \xi < 2t \deg p$, resp. $\deg \xi < (2t+1)\deg p$, with $t \ge 1$, resp. $t \ge 0$. We set

$$\xi = \xi_0 + \xi_1 X^2 + \cdots + \xi_s X^{2s}$$

with $\xi_i \in \Omega_F^m M$, $0 \le i \le s$, and we choose s and t in the above representations to be minimal. We have $\mathrm{d}\,\eta_p = 0$, $C(\eta_p) = \eta_p$ because $\eta_p \in \nu_{M(X^2)}$. The first equation implies $\mathrm{d}\,\xi_i = 0$ for all i and the second one means

$$\frac{C(\xi_0) + C(\xi_1)X + \cdots + C(\xi_s)X^s}{p^t} = \frac{\xi_0 + \xi_1 X^2 + \cdots + \xi_s X^{2s}}{p^{2t}}$$

and this implies $s = 0$, $t = 0$ i.e. $\eta_p = 0$. Thus only the second case can occur. We have $\eta_p = \dfrac{p\xi}{p^{2t+2}}$ and from $\mathrm{d}\,\eta_p = 0$, $C(\eta_p) = \eta_p$ it follows $\mathrm{d}\,(p\xi_i) = 0$ for all i and

$$\frac{C(p\xi)}{p^{t+1}} = \frac{\xi}{p^{2t+1}}.$$

Thus $\xi = p^t C(p\xi)$. Since $p \nmid \xi$ we conclude $t = 0$, i.e.

$$\eta_p = \frac{\xi}{p} \tag{3.5}$$

with $\xi \in \Omega_F^m M[X^2]$, $\deg \xi < \deg p$. Thus we have shown that the p-component η_p of a form $\eta \in \nu_{M(X^2)}(m)$ is of the form (3.5). Furthermore we claim that $\eta_p \neq 0$ only if $p \in M[X^2]$. Assume $p \notin M[X^2]$ and set $p = p_0 + Xp_1$ with p_0, $p_1 \in M[X^2]$, $p_1 \neq 0$. Then $\xi = p_0\eta_p + Xp_1\eta_p$. Since $\eta_p \in \Omega_F M(X^2)$, it follows $Xp_1\eta_p \in \Omega_F M(X^2)$ with $p_1 \in M[X^2]$. This is impossible if $p_1 \neq 0$. Therefore $p = p_0 \in M[X^2]$. Summing up these results we have shown:

Lemma 3.6 *Let $\eta \in \nu_{M(X^2)}(m)$. Then the p-decomposition $\eta = \eta_E + \sum \eta_p$ of η satisfies*

(i) $\eta_E \in \nu_M(m)$

(ii) Only for $p \in M[X^2]$ we may have $\eta_p \neq 0$, and then

$$\eta_p = \frac{\xi}{p}$$

with $\xi \in \Omega_F^m M[X^2]$, $\deg \xi < \deg p$. Moreover $\eta_p \in \nu_{M(X^2)}(m)$, $\mathrm{d}(p\xi) = 0$ and $C(p\xi) = \xi$.

Let us return to the basic equation

$$w = \wp u + \mathrm{d}\,v + T\eta$$

in $\Omega_F^m F(X_\mu^2, Y_\mu^2) = \Omega_F^m M(X^2)$, where $X = X_1$ and $M = F(X_\nu^2, Y_\mu^2,\ \mu, \nu \in S_n,\ \nu > 1)$. We insert the p-decomposition of the forms involved in this equation and we obtain as in [Ar-Ba$_1$]

$$w = \wp u_E + \mathrm{d}\,v_E + T\eta_E + \sum_p E_p \tag{3.7}$$

$$E_p = \wp u_p + \mathrm{d}\,v_p + T\eta_p \tag{3.8}$$

where the forms E_p belong to $\Omega_F M[X^2]$ and arise from the multiplication of T with η_p. For all irreducible polynomials $p \in M[X]\backslash M[X^2]$ we have shown $\eta_p = 0$ and hence $E_p = \wp u_p + \mathrm{d}\, v_p$. Since $\wp$, d are compatible with the p-decomposition, we have in this case $E_p = 0$. Thus in (3.7) only polynomials $p \in M[X^2]$ do appear. The main idea is now to get rid of the forms E_p in the equation (3.7), and this will be accomplished in the last section of this paper.

4 Second reduction step in the proof of the main theorem

We keep the same notation as in the last section. Thus in the basic equations (3.7), (3.8) only polynomials $p \in M[X^2]$ occur. Since p is monic and irreducible, we have $p(0) \neq 0$ for all such p's. Therefore we can specialize in these equations $X \longrightarrow 0$ and we obtain

$$w = \wp u_0 + \mathrm{d}\, v_0 + T_1 \eta_0 \tag{4.1}$$

where $u_0,\ v_0 \in \Omega_F M$ and $\eta_0 \in \nu_M(m)$. Here T_1 is now the polynomial $b + a_1 b Y_1^2 + \sum_{\mu>1} a^\mu Y_\mu^2 (b + X_\mu^2 + a^\mu Y_\mu^2 X_\mu^2)$. Notice that the specialization $X \longrightarrow 0$ is compatible with the operators $\wp$ and d. Proceeding with the other variables X_μ in the same way as above, we finally obtain an equation

$$w = \wp u + \mathrm{d}\, v + T\eta \tag{4.2}$$

with $u \in \Omega_F^m F(Y_\mu^2,\ \mu \in S_n)$, $v \in \Omega_F^{m-1} F(Y_\mu^2,\ \mu \in S_n)$, $\eta \in \nu_{F(Y_\mu^2)}(m)$ and

$$T = b \left(1 + \sum_{\mu \in S_n} a^\mu Y_\mu^2 \right) \tag{4.3}$$

This equation will be from now on our basic equation and our goal is now to eliminate in an appropriate way the variables Y_μ. Thus we have shown

Proposition 4.4 *If $w \in H^{m+1}(F(p)/F)$, then there exist $u \in \Omega_F^m F(Y_\mu^2,\ \mu \in S_n)$, $v \in \Omega_F^{m-1} F(Y_\mu^2,\ \mu \in S_n)$ and $\eta \in \nu_{F(Y_\mu^2)}(m)$ such that (4.2) holds, where T is given by (4.3).*

During the process of elimination of the variables Y_μ, we will see that the following type of conditions on the forms $\eta \in \nu(m)$ appear:

For some $\tau \in S_n$ and $E \subset F(Y_\mu^2,\ \mu \in S_n)$ intermediate field (to be later specified) one has $\eta \in \nu_E(m)$ and $a^\sigma \eta \in \mathrm{d}\, \Omega_E^{m-1}$ for all $\sigma \leq \tau$. We will in the next section study this property and characterize these forms for some specific subfields of $F(Y_\mu^2,\ \mu \in S_n)$.

5 Forms in good shape

Let N be some field (with $2 = 0$) and $a_1, \ldots, a_n \in N$ elements contained in a 2-basis of N. Let Y be a variable over N and fix some $\tau \in S_n$. Set $t_\tau = t = \max\{i | \tau(i) = 1\}$. Then

Proposition 5.1 *Let $\eta \in \nu_{N(Y^2)}(m)$ be such that $a^\sigma \eta \in \mathrm{d}\,\Omega^{m-1}_{N(Y^2)}$ for all $\sigma \leq \tau$. Then η has the form*

$$\eta = \frac{\mathrm{d}\,a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\,a_{t-1}}{a_{t-1}} \wedge \sum_\gamma \frac{\mathrm{d}\,(a_t + d_\gamma)}{a_t + d_\gamma} \wedge \frac{\mathrm{d}\,f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}$$

where $d_\gamma = \sum_\alpha a^\alpha c^2_{\alpha,\gamma}$, $c_{\alpha,\gamma} \in N(Y)$, $\alpha \in S_{t-1}$ such that $\alpha + t > \tau$, $f_{\gamma(i)} \in N_{\gamma(i)}[Y^2]$, $i \geq t+1$, $\gamma \in \Sigma_n$ with $\gamma(1) = 1, \ldots, \gamma(t) = t$. (Here $\alpha + t$ means that one adds 1 at the t-th place of α). Conversely, each form of the above type satisfies $a^\sigma \eta \in \mathrm{d}\,\Omega$ for all $\sigma \leq \tau$.

As a special case $\tau = \max\{\sigma | \sigma \in S_n\}$ we get $t = n$ and the set $\alpha \in S_{n-1}$ with $\alpha + n > \tau$ is empty. Thus we get the useful

Corollary 5.2 *If $\eta \in \nu_{N(Y^2)}(m)$ satisfies*

$$a^\sigma \eta \in \mathrm{d}\,\Omega^{m-1}_{N(Y^2)}$$

for all $\sigma \in S_n$, then

$$\eta = \frac{\mathrm{d}\,a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\,a_n}{a_n} \wedge \xi$$

with some $\xi \in \nu(m-n)$.

Proof of (5.1). Let first consider the case $\tau = (1, 0, \ldots, 0)$ = first element in S_n. Then $t = 1$. By assumption $a_1 \eta \in \mathrm{d}\,\Omega$, which implies $\mathrm{d}\,a_1 \wedge \eta = 0$ (since $\mathrm{d}\,\eta = 0$) and hence $\eta = \frac{\mathrm{d}\,a_1}{a_1} \wedge \xi$ with some $\xi \in \Omega^{m-1}$. From Kato's lemma we can write

$$\eta = \sum_\gamma \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}$$

with $\gamma(1) = 1$ for all γ, i.e. $f_{\gamma(1)} = n_\gamma^2 a_1 + m_\gamma^2$ and $n_\gamma \neq 0$ in $N[Y]$. For simplicity replace $f_{\gamma(1)}$ by $n_\gamma^{-2} f_{\gamma(1)}$ so that we can assume $n_\gamma = 1$ and $m_\gamma \in N(Y)$. Then

$$\eta = \sum_\gamma \frac{\mathrm{d}\left(a_1 + m_\gamma^2\right)}{a_1 + m_\gamma^2} \wedge \xi_\gamma$$

where $\xi_\gamma = \frac{\mathrm{d}\,f_{\gamma(2)}}{f_{\gamma(2)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}$. Hence

$$a_1 \eta = \sum_\gamma \frac{\left(a_1^3 + a_1^2 m_\gamma^2\right)}{\left(a_1 + m_\gamma^2\right)^2} \frac{\mathrm{d}\,a_1}{a_1} \wedge \xi_\gamma.$$

The form $\sum_\gamma \frac{a_1^3}{\left(a_1+m_\gamma^2\right)^2} \frac{\mathrm{d}\, a_1}{a_1} \wedge \xi_\gamma$ is exact, so that our hypothesis implies

$$\sum_\gamma \frac{a_1^2 m_\gamma^2}{\left(a_1+m_\gamma^2\right)^2} \frac{\mathrm{d}\, a_1}{a_1} \wedge \xi_\gamma \in \mathrm{d}\, \Omega_{N(Y^2)}^{m-1}.$$

Applying the Cartier operator to this form, we obtain

$$\sum_\gamma \frac{a_1 m_\gamma}{\left(a_1+m_\gamma^2\right)} \frac{\mathrm{d}\, a_1}{a_1} \wedge \xi_\gamma = 0.$$

But the elements $\frac{\mathrm{d}\, a_1}{a_1} \wedge \xi_\gamma$ belong to different filtrations for each γ, thus we obtain

$$m_\gamma = 0 \quad \text{for all } \gamma$$

and hence $\eta = \frac{\mathrm{d}\, a_1}{a_1} \wedge \xi$ with $\xi = \sum_\gamma \xi_\gamma \in \nu(m-1)$. This proves our assertion in the case $\tau = (1, 0, \ldots, 0)$.

Assume now our assertion for all $\sigma \leq \tau$, $\tau \in S_n$. Let δ be the next element in S_n. We want to show the assertion for δ. Assume $t = t_\tau = \max\{i |\, \tau(i) = 1\}$ and let $t_\delta = \max\{i |\, \delta(i) = 1\}$. Then $t_\delta = t$ or $t_\delta = t+1$. First case: $t_\delta = t$. Then $\delta = (\ldots, 1, 0, \ldots)$ with 1 at the t-th place. By induction we have $\eta = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{t-1}}{a_{t-1}} \wedge \sum_\gamma \frac{\mathrm{d}(a_t+d_\gamma)}{a_t+d_\gamma} \wedge \xi_\gamma$ with $d_\gamma = \sum_{\substack{\sigma \in S_{t-1} \\ \sigma+t>\tau}} a^\sigma m_{\sigma,\gamma}^2$, $m_{\sigma,\gamma} \in N(Y)$, and $\xi_\gamma = \frac{\mathrm{d}\, f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$. Thus we can write

$$d_\gamma = \sum_{\substack{\sigma \in S_{t-1} \\ \sigma+t \geq \delta}} a^\sigma m_{\sigma,\gamma}^2 = a^{\delta-t} m_{\delta-t,\gamma}^2 + \sum_{\substack{\sigma \in S_{t-1} \\ \sigma+t>\delta}} a^\sigma m_{\sigma,\gamma}^2$$

where $\delta - t$ is the element $\sigma \in S_{t-1}$ with $\sigma + t = \delta$. Now we use $a^\delta \eta \in \mathrm{d}\, \Omega_{N(Y^2)}^{m-1}$. The form

$$\begin{aligned} a^\delta \eta &= \sum_\gamma \frac{\mathrm{d}\, a}{a} \wedge \frac{a^\delta\, \mathrm{d}\,(a_t + d_\gamma)}{a_t + d_\gamma} \wedge \xi_\gamma \\ &= \sum_\gamma \frac{\mathrm{d}\, a}{a} \wedge \frac{a^\delta\,(a_t+d_\gamma)\, a_t}{\left(a_t+d_\gamma\right)^2} \frac{\mathrm{d}\, a_t}{a_t} \wedge \xi_\gamma \in \mathrm{d}\, \Omega_{N(Y^2)}^{m-1} \end{aligned}$$

because $\mathrm{d}\, a \wedge \mathrm{d}\, a^{\delta-t} = \mathrm{d}\, a \wedge \mathrm{d}\, a^\sigma = 0$ for all $\sigma \in S_{t-1}$. Now the terms

$$\frac{a^\delta a_t^2}{\left(a_t+d_\gamma\right)^2} \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, a_t}{a_t} \wedge \xi_\gamma$$

are all exact, as well as the forms

$$\frac{a^\delta a_t \sum_{\sigma+t>\delta} a^\sigma m_{\sigma,\gamma}^2}{\left(a_t+d_\gamma\right)^2} \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, a_t}{a_t} \wedge \xi_\gamma$$

because each summand of these expressions has the form

$$\frac{a^{\delta+t+\sigma}m_{\sigma,\gamma}^2}{(a_t+d_\gamma)^2}\frac{\mathrm{d}\,a}{a}\wedge\frac{\mathrm{d}\,a_t}{a_t}$$

which is indeed exact, since $\delta+t+\sigma>0$ and $\delta+t+\sigma\in S_{t-1}$. Therefore we obtain

$$\sum_\gamma \frac{a^{2\delta}m_{\delta-t,\gamma}^2}{(a_t+d_\gamma)^2}\frac{\mathrm{d}\,a}{a}\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\xi_\gamma\in\mathrm{d}\,\Omega^{m-1}_{N(Y^2)}.$$

Applying the Cartier operator to this form we obtain

$$\sum_\gamma \frac{a^{\delta}m_{\delta-t,\gamma}}{(a_t+d_\gamma)}\frac{\mathrm{d}\,a}{a}\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\xi_\gamma\in 0.$$

Since the forms $\frac{\mathrm{d}\,a}{a}\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\xi_\gamma$ are independent, we conclude $m_{\delta-t,\gamma}=0$ for all γ and hence $d_\gamma=\sum_{\substack{\sigma\in S_{t-1}\\ \sigma+t>\delta}}a^\sigma m_{\sigma,\gamma}^2$. This concludes the proof in this case. We have to look at the case $t_\delta=t+1$. This case is only possible for $\tau=(1,1,\ldots,1,0,\ldots,0)$ (the last 1 at the t-th place) and $\delta=(0,\ldots,0,1,0,\ldots,0)$ with 1 at the $(t+1)$-th place. In this case there is no $\alpha\in S_{t-1}$ with $\alpha+t>\tau$, hence we have by induction, because of $a^\sigma\eta\in\mathrm{d}\,\Omega$ for $\sigma\leq\tau$,

$$\eta=\sum_\gamma\frac{\mathrm{d}\,a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}\,a_{t-1}}{a_{t-1}}\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\frac{\mathrm{d}\,f_{\gamma(t+1)}}{f_{\gamma(t+1)}}\wedge\cdots\wedge\frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}.$$

Since $a^\delta=a_{t+1}$, we have still the condition $a_{t+1}\eta\in\mathrm{d}\,\Omega$. In particular $\mathrm{d}\,a_{t+1}\wedge\eta=0$. It follows $\eta\in\mathrm{d}\,a_1\wedge\cdots\wedge\mathrm{d}\,a_{t+1}\wedge\Omega^{m-t-1}$. Comparing this with the above expression of η, we conclude $\gamma(t+1)=t+1$ for all γ in the sum. Thus $f_{\gamma(t+1)}\in M_{t+1}[Y^2]$ for all γ, i.e. $f_{\gamma(t+1)}=A_\gamma(a_{t+1}+B_\gamma)$ with $A_\gamma,B_\gamma\in M_t[Y^2]$. Since $\mathrm{d}\,a_1\wedge\cdots\wedge\mathrm{d}\,a_t\wedge\mathrm{d}\,A_\gamma=0$, we may assume $f_{\gamma(t+1)}=a_{t+1}+B_\gamma$. Write now $B_\gamma=n_\gamma^2+\sum a^\alpha n_{\alpha,\gamma}^2$, where α runs over S_t with $\alpha+(t+1)>\delta$. We want to show $n_\gamma=0$ for all γ. The assumption $a_{t+1}\eta\in\mathrm{d}\,\Omega$ means

$$\sum_\gamma\frac{\mathrm{d}\,a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\frac{(a_{t+1}+B_\gamma)\,a_{t+1}^2}{(a_{t+1}+B_\gamma)^2}\frac{\mathrm{d}\,a_{t+1}}{a_{t+1}}\wedge\xi_\gamma\in\mathrm{d}\,\Omega$$

with $\xi_\gamma=\frac{\mathrm{d}\,f_{\gamma(t+2)}}{f_{\gamma(t+2)}}\wedge\cdots\wedge\frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}$. A similar argument as before shows that

$$\sum_\gamma\frac{\mathrm{d}\,a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\frac{a_{t+1}^2n_\gamma^2}{(a_{t+1}+B_\gamma)^2}\frac{\mathrm{d}\,a_{t+1}}{a_{t+1}}\wedge\xi_\gamma$$

is exact. Applying the Cartier operator, we obtain

$$\sum_\gamma\frac{a_{t+1}n_\gamma}{(a_{t+1}+B_\gamma)}\frac{\mathrm{d}\,a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}\,a_t}{a_t}\wedge\frac{\mathrm{d}\,a_{t+1}}{a_{t+1}}\wedge\xi_\gamma=0$$

and again it follows $n_\gamma = 0$ for all γ. This completes the proof of our proposition. ■

The property assumed in the last proposition for forms $\eta \in \nu(m)$ will play a crucial role in the proof of our main theorem. We will state this property in the following definition.

Definition 5.3 *Let E be any field with $2 = 0$ and $a_1, \ldots, a_n \in E$ be elements contained in a 2-basis of E. Fix some $\tau \in S_n$. Let $\eta \in \nu_E(m)$. We say that η is in* good shape *with respect to τ (and $a_1, \ldots, a_n$) if*

$$a^\sigma \eta \in \mathrm{d}\,\Omega_E^{m-1}$$

for all $\sigma \leq \tau$.

Thus proposition (5.1) characterizes forms in good shape for certain fields.

6 Last reduction step in the proof of the main theorem

We fix the lexicographic ordering in the set S_n, and for any $\mu \in S_n$ let μ^+ be the next element in this ordering. Let $T = T_0 = b(1 + \sum_\mu a^\mu Y_\mu^2)$ and we define T_μ inductively by

$$T_\mu = b a^\mu Y_\mu^2 + T_{\mu^+}$$

Let $M = F(Y_\nu^2)_{\nu > \mu}$ (for a given μ) and let $w \in \Omega_F^n$. We consider the following assertions on w and μ:

(A_μ) There exist $u \in \Omega_F^m M(Y_\mu^2)$, $v \in \Omega_F^{m-1} M(Y_\mu^2)$ and $\eta \in \nu_{M(Y_\mu^2)}(m) \cap \Omega_F^m M(Y_\mu^2)$ such that

$$w = \wp(u) + \mathrm{d}\,v + T_\mu \eta$$

and $a^\sigma \eta \in \mathrm{d}\,\Omega$ for all $\sigma < \mu$.

(B_μ) There exist $u \in \Omega_F^m M[Y_\mu^2]$, $v \in \Omega_F^{m-1} M[Y_\mu^2]$ and $\eta \in \nu_M(m) \cap \Omega_F^m M$ such that

$$w = \wp(u) + \mathrm{d}\,v + T_\mu \eta$$

and $a^\sigma \eta \in \mathrm{d}\,\Omega$ for all $\sigma < \mu$.

Lemma 6.1 (Main lemma) *For any $\mu \in S_n$*

$$(A_\mu) \Longrightarrow (B_\mu) \Longrightarrow (A_{\mu^+}).$$

Clearly (A_0) is the basic equation (4.2). Thus (6.1) implies that starting with (A_0), i.e. (4.2), we arrive at an equation (for the last element in S_n)

$$w = \wp u + \mathrm{d}\, v + b\eta \tag{6.2}$$

with $u \in \Omega_F^m$, $v \in \Omega_F^{m-1}$ and $\eta \in \nu_F(m)$ such that $a^\sigma \eta \in \mathrm{d}\, \Omega_F$ for all $\sigma \in S_n$. Applying now corollary (5.2) we conclude $\eta = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_n}{a_n} \wedge \xi$ with $\xi \in \nu_F(m-n)$. This implies the main theorem.

The rest of this section is devoted to the proof of (6.1). Let us first show the easy part, namely $(B_\mu) \Longrightarrow (A_{\mu^+})$. Assuming (B_μ) one easily checks that, without restriction, $u = u_0 \in \Omega_F^m M$ and $v = v_0 + v_2 Y_\mu^2$ with $v_0, v_2 \in \Omega_F^{m-1} M$, as well as $\eta \in \nu_M(m) \cap \Omega_F^m M$ with $a^\sigma \eta \in \mathrm{d}\, \Omega$ for all $\sigma < \mu$.

From

$$w = \wp u_0 + \mathrm{d}(v_0 + v_2 Y_\mu^2) + (ba^\mu Y_\mu^2 + T_{\mu^+})\eta$$

we obtain

$$w = \wp u_0 + \mathrm{d}(v_0) + T_{\mu^+}\eta$$

and

$$\mathrm{d}(v_2) = ba^\mu \eta.$$

Since $b \in F^2$, we get $a^\mu \eta = b^{-1}\, \mathrm{d}(v_2) \in \mathrm{d}\, \Omega$, i.e. it holds $a^\sigma \eta \in \mathrm{d}\, \Omega$ for all $\sigma < \mu^+$. This proves (A_{μ^+}).

The hard part of the lemma is the implication $(A_\mu) \Longrightarrow (B_\mu)$. Let us start with the basic equation (i.e. (A_μ))

$$w = \wp u + \mathrm{d}\, v + T\eta$$

with $w \in \Omega_F^m$, $u \in \Omega_F^m M(Y^2)$, $v \in \Omega_F^{m-1} M(Y^2)$, $\eta \in \nu_{M(Y^2)}(m) \cap \Omega_F^m M(Y^2)$, $T = T_\mu$, $Y = Y_\mu$ for some $\mu \in S_n$, where $a^\sigma \eta \in \mathrm{d}\, \Omega$ for all $\sigma < \mu$. For $\mu = 0$, this is the equation (4.2). Let $t = \max\{i \mid \mu^-(i) = 1\}$, where μ^- denotes the antecessor of μ. We consider the p-decomposition (see §3 and [Ar-Ba$_1$]) of the form η

$$\eta = \eta_E + \sum_p \eta_p \tag{6.3}$$

with $\eta_E \in \nu_{M(Y^2)}(m) \cap \Omega_F^m M[Y^2] = \nu_M(m)$, $\eta_p \in \frac{1}{p}\Omega_F M[Y^2] \cap \nu_{M(Y^2)}(m)$ (see (2.10)). Then we have

Lemma 6.4 *If η in (6.3) satisfies $a^\sigma \eta \in \mathrm{d}\, \Omega$ for all $\sigma < \mu$, then η_E, η_p, for all p, have the same property.*

Proof. Since d is compatible with the p-decomposition (see [Ar-Ba$_1$]) we obtain

$$0 = \mathrm{d}(a^\sigma \eta) = \mathrm{d}(a^\sigma \eta_E) + \sum_p \mathrm{d}(a^\sigma \eta_p)$$

and hence $\mathrm{d}(a^\sigma \eta_E) = 0$, $\mathrm{d}(a^\sigma \eta_p) = 0$ for all p and $\sigma < \mu$. Also C is compatible with p-decomposition (loc. cit.), so that from $C(a^\sigma \eta) = 0$ it follows

$$C(a^\sigma \eta_E) = 0$$
$$C(a^\sigma \eta_p) = 0.$$

This implies $a^\sigma \eta_E$, $a^\sigma \eta_p \in \mathrm{d}\,\Omega$ for all p, $\sigma < \mu$.

Let us now insert the p-decompositions $u = u_E + \sum u_p$, $v = v_E + \sum v_p$, $\eta = \eta_E + \sum \eta_p$ into the equation (A_μ). Here we have $u_p \in \frac{1}{p}\Omega_F^m M[Y^2]$, $v_p \in \frac{1}{p^2}\Omega_F^{m-1} M[Y^2]$, $\eta_E \in \nu_{M(Y^2)}(m) \cap \Omega_F^m M[Y^2] = \nu_M(m)$ and $\eta_p \in \frac{1}{p}\Omega_F M[Y^2] \cap \nu_{M(Y^2)}(m)$. We obtain

$$w = \wp u_E + \mathrm{d}\, v_E + T\eta_E + \sum_p E_p \tag{6.5}$$

$$E_p = \wp u_p + \mathrm{d}\, v_p + T\eta_p \tag{6.6}$$

with $E_p \in \Omega_F^m M[Y^2]$ for all p (see [Ar-Ba$_1$]). We have seen in (3.6) that the irreducible polynomials appearing in the p-decomposition of η are in $M[Y^2]$. Thus for the others polynomials p we have $E_p = \wp u_p + \mathrm{d}\, v_p$ and it follows immediately that $E_p = 0$ since E_p is integral and $\wp u_p$, $\mathrm{d}\, v_p$ are in the p-component so these terms can be discarded in the above expression (6.5). Thus we assume in (6.5) and (6.6) that all irreducible polynomials p are in $M[Y^2]$. Moreover, according to (6.4) we know that the forms η_E, η_p satisfy $a^\sigma \eta_E$, $a^\sigma \eta_p \in \mathrm{d}\,\Omega$ for all $\sigma < \mu$. ■

In the next lemma we will use the following notation. Given a polynomial $f \in M[Y^2]$, we denote by $\mathrm{fil}(f)$ the maximal filtration of the coefficients of f (with respect to the given 2-basis). For $f, g \in M[Y^2]$ we write $f < g$ if $\deg f < \deg g$ or $\deg f = \deg g$ and $\mathrm{fil}\, f < \mathrm{fil}\, g$.

Lemma 6.7 *For any p and η_p as above it holds*

$$\eta_p = \eta_0 + \eta_1 \wedge \frac{\mathrm{d}\, p}{p}$$

with η_0, $\eta_1 \in \nu_{M(Y^2)} \cap \bigoplus_{q<p} \frac{1}{q}\Omega_F M[Y^2]$ and $a^\sigma \eta_0$, $a^\sigma \eta_1 \in \mathrm{d}\,\Omega$ for all $\sigma < \mu$.

Proof. From the characterization of forms in good shape given in (5.1) we get

$$\eta_p = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{t-1}}{a_{t-1}} \wedge \sum_\gamma \frac{\mathrm{d}\left(n_\gamma^2 a_t + m_\gamma\right)}{n_\gamma^2 a_t + m_\gamma} \wedge \frac{\mathrm{d}\, f_\gamma}{f_\gamma}$$

where $m_\gamma = \sum_\sigma a^\sigma m_{\sigma,\gamma}^2$, $m_{\sigma,\gamma}$, $n_\gamma \in M[Y]$ and σ runs over those $\sigma \in S_{t-1}$ with $\sigma + t \geq \mu$, and $\frac{\mathrm{d}\, f_\gamma}{f_\gamma} = \frac{\mathrm{d}\, f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$ with $t = \max\{i \mid \mu^-(i) = 1\}$. Moreover for all $\gamma \in \Sigma_m$ in this sum we can write $\gamma(i) = i$ for $i \leq t$, $f_{\gamma(i)} = a_i$,

$i \leq t-1$ and $f_{\gamma(t)} = n_\gamma^2 a_t + m_\gamma$ and $f_{\gamma(j)} \in M_{\gamma(j)}[Y^2]$ for all j. Let γ be the maximal multi-index in the above sum, and set

$$\eta_{p,\gamma} = \frac{\mathrm{d}\, f_\gamma}{f_\gamma} = \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{t-1}}{a_1} \wedge \frac{\mathrm{d}\left(n_\gamma^2 a_t + m_\gamma\right)}{n_\gamma^2 a_t + m_\gamma} \wedge \frac{\mathrm{d}\, f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}} \tag{6.8}$$

and

$$\eta_{p,<\gamma} = \frac{\mathrm{d}\, a}{a} \wedge \sum_{\lambda<\gamma} \frac{\mathrm{d}\left(n_\lambda^2 a_t + m_\lambda\right)}{n_\lambda^2 a_t + m_\lambda} \wedge \frac{\mathrm{d}\, f_{\lambda(t+1)}}{f_{\lambda(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\lambda(m)}}{f_{\lambda(m)}}.$$

Thus $\eta_p = \eta_{p,\gamma} + \eta_{p,<\gamma}$. We use now the fact that $\eta_p \in \frac{1}{p}\Omega_F M[Y^2]$ to obtain the $2-$basis expansion

$$\eta_p = \sum_{\lambda \leq \gamma} c_\lambda \frac{\mathrm{d}\, a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{t-1}}{a_{t-1}} \wedge \frac{\mathrm{d}\, a_t}{a_t} \wedge \frac{\mathrm{d}\, a_{\lambda(t+1)}}{a_{\lambda(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, a_{\lambda(m)}}{a_{\lambda(m)}}$$

with $c_\lambda \in \frac{1}{p} M[Y^2]$. Let $c_\lambda = \frac{h_\lambda}{p}$, where $h_\lambda \in M[Y^2]$, $\deg_Y h_\lambda < \deg_Y p$. On the other hand only the term $\gamma_{p,\gamma}$ contributes to the maximal term in this expansion, and its contribution is (see the proof of (2.10))

$$\begin{aligned}
&\frac{\mathrm{d}\, a}{a} \wedge \frac{a_t n_\gamma^2}{n_\gamma^2 a_t + m_\gamma} \frac{\mathrm{d}\, a_t}{a_t} \wedge \frac{a_{\gamma(t+1)} D_{\gamma(t+1)}\left(f_{\gamma(t+1)}\right)}{f_{\gamma(t+1)}} \frac{\mathrm{d}\, a_{\gamma(t+1)}}{a_{\gamma(t+1)}} \wedge \\
&\qquad \cdots \wedge \frac{a_{\gamma(m)} D_{\gamma(m)}\left(f_{\gamma(m)}\right)}{f_{\gamma(m)}} \frac{\mathrm{d}\, a_{\gamma(m)}}{a_{\gamma(m)}} \\
&= \prod_{j=t}^{m} \left(\frac{a_{\gamma(j)} D_{\gamma(j)}\left(f_{\gamma(j)}\right)}{f_{\gamma(j)}} \right) \frac{\mathrm{d}\, a_\gamma}{a_\gamma}.
\end{aligned}$$

Thus we obtain

$$\frac{h_\lambda}{p} = \prod_{j=t}^{m} \left(\frac{a_{\gamma(j)} D_{\gamma(j)}\left(f_{\gamma(j)}\right)}{f_{\gamma(j)}} \right)$$

where $\deg h_\gamma < \deg p$, $\deg\left(D_{\gamma(i)}\left(f_{\gamma(i)}\right)\right) \leq \deg f_{\gamma(i)}$, $\operatorname{fil}\left(D_{\gamma(i)}\left(f_{\gamma(i)}\right)\right) < \gamma(i) = \operatorname{fil}\left(f_{\gamma(i)}\right)$. This last remark follows from the definition of the operator $D_{\gamma(i)}$. Therefore

$$h_\lambda \prod_{j=t}^{m} f_{\gamma(j)} = ap \prod_{j=t}^{m} D_{\gamma(j)}\left(f_{\gamma(j)}\right) \tag{6.9}$$

with $a = \prod_{j=t}^{m} a_{\gamma(j)} \in F$. In particular $p | \prod_{j=t}^{m} f_{\gamma(j)}$ and there is some $t \leq k \leq m$ with $p | f_{\gamma(k)}$. This k is unique and it follows, since p is irreducible and monic (see (2.11)), that $\operatorname{fil}(p) = \gamma(k)$. We can assume without restriction that $f_{\gamma(j)}$ are products of irreducible polynomials q with multiplicity 1 and filtration $\gamma(j)$.

Now we distinguish three cases:

(a) $j > k$. Then $f_{\gamma(j)} \in M$ is constant.

To see this, we choose $j > k$ maximal with $f_{\gamma(j)} \in M[Y^2]\backslash M$. Let $q|f_{\gamma(j)}$ be a monic irreducible factor. Then $\mathrm{fil}(q) = \gamma(j)$. From (6.7) we get $q|\prod_{i=t}^{m} D_{\gamma(i)}\left(f_{\gamma(i)}\right)$, because it can not be p since $\mathrm{fil}(p) = \gamma(k) \neq \gamma(j)$. Hence $q|D_{\gamma(i)}\left(f_{\gamma(i)}\right)$ for some $i \geq t$. If $l > j$, by the choice of j, $f_{\gamma(l)} \in M$. Hence $i \leq j$. But $i \leq j$ implies $\mathrm{fil}\left(D_{\gamma(i)}\left(f_{\gamma(i)}\right)\right) < \gamma(i) \leq \gamma(j)$ and this contradicts $q|D_{\gamma(i)}\left(f_{\gamma(i)}\right)$, since $\mathrm{fil}(q) = \gamma(j)$. This implies $f_{\gamma(j)} \in M$ for all $j > k$.

(b) $j = k$. We claim $f_{\gamma(k)} = cp$ with $c \in M_{\gamma(k)}$.

Assume $q|f_{\gamma(k)}$ is an irreducible monic factor different from p. We have $\mathrm{fil}(q) = \gamma(k)$. But from (6.9) it follows $q|D_{\gamma(i)}\left(f_{\gamma(i)}\right)$ for some $i \geq t$, and this implies $\mathrm{fil}(q) < \gamma(i)$. Since $f_{\gamma(l)} \in M$ for $l > k$, it follows $i \leq k$, and hence $\mathrm{fil}(q) < \gamma(i) \leq \gamma(k)$, which is a contradiction. Thus $q = p$ and since its multiplicity is one, we have $f_{\gamma(k)} = cp$. Since $\mathrm{fil}(f_{\gamma(k)}) = \gamma(k)$, it follows $c \in M_{\gamma(k)}$.

(c) $j < k$. We claim $\deg(f_{\gamma(j)}) \leq \deg(p)$ for all $j < k$.

Since $\mathrm{fil}(f_{\gamma(j)}) = \gamma(j) < \gamma(k) = \mathrm{fil}(p)$, we see that any monic irreducible factor of $f_{\gamma(j)}$, say q, has $\mathrm{fil}(q) = \gamma(j) < \gamma(k)$ and hence $q \neq p$. Inserting (a) and (b) in the relation (6.9) we obtain an equation of the form

$$dh_\gamma f_{\gamma(t)} \cdots f_{\gamma(k-1)} = \prod_{t \leq j \leq k-1} D_{\gamma(j)}\left(f_{\gamma(j)}\right) \cdot D_{\gamma(k)}\left(cp\right) \tag{6.10}$$

with some $d \in M$. If $q|f_{\gamma(k-1)}$ is a monic irreducible factor, we have $\mathrm{fil}(q) = \gamma(k-1)$ and since $\mathrm{fil}(D_{\gamma(j)}\left(f_{\gamma(j)}\right)) < \gamma(j)$ for all $j \leq k-1$ and therefore $q|D_{\gamma(k)}\left(cp\right)$. Since all irreducible factors of $f_{\gamma(k-1)}$ have multiplicity 1 we obtain $f_{\gamma(k-1)}|D_{\gamma(k)}\left(cp\right)$ and hence $\deg(f_{\gamma(k-1)}) \leq \deg(p)$. We set $D_{\gamma(k)}\left(cp\right) = l_{\gamma(k)} f_{\gamma(k-1)}$ with $\mathrm{fil}(l_{\gamma(k)}) < \gamma(k)$ and $\deg(l_{\gamma(k)}) = \deg(p) - \deg(f_{\gamma(k-1)})$. From (6.10) it follows

$$dh_\gamma f_{\gamma(t)} \cdots f_{\gamma(k-2)} = l_{\gamma(k)} \prod_{t \leq j \leq k-1} D_{\gamma(j)}\left(f_{\gamma(j)}\right). \tag{6.11}$$

Let now q be a monic irreducible factor of $f_{\gamma(k-2)}$. Then $\mathrm{fil}(q) = \gamma(k-2)$ and from (6.10) we conclude $q|l_{\gamma(k)} D_{\gamma(k-1)}\left(f_{\gamma(k-1)}\right)$. Since all factors of $f_{\gamma(k-2)}$ have multiplicity 1, it follows $f_{\gamma(k-2)}|l_{\gamma(k)} D_{\gamma(k-1)}\left(f_{\gamma(k-1)}\right)$. Hence

$$\begin{aligned}
\deg(f_{\gamma(k-2)}) &\leq \deg(l_{\gamma(k)}) + \deg(D_{\gamma(k-1)}\left(f_{\gamma(k-1)}\right)) \\
&\leq \deg(p) - \deg(f_{\gamma(k-1)}) + \deg(D_{\gamma(k-1)}\left(f_{\gamma(k-1)}\right)) \\
&\leq \deg(p).
\end{aligned}$$

Iterating this argument we obtain our claim.

Summing up, we have in (6.8) the following situation. There is an integer k, $t \leq k \leq m$, such that $f_{\gamma(k)} = cp$ with $c \in M_{\gamma(k)}$, $f_{\gamma(j)} \in M$ for all $j > k$ and if $j < k$, $\deg f_{\gamma(j)} \leq \deg p$.

If $\gamma(k) > t$, then

$$\begin{aligned}\eta_{p,\gamma} = & \frac{\mathrm{d}\,a}{a} \wedge \frac{\mathrm{d}(n_t^2 a_t + m_t)}{n_t^2 a_t + m_t} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(k-1)}}{f_{\gamma(k-1)}} \wedge \frac{\mathrm{d}\,c}{c} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}} \\ & + \frac{\mathrm{d}\,a}{a} \wedge \frac{\mathrm{d}(n_t^2 a_t + m_t)}{n_t^2 a_t + m_t} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(k-1)}}{f_{\gamma(k-1)}} \wedge \frac{\mathrm{d}\,p}{p} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}\end{aligned}$$

where both summands are in good shape with respect to μ^- and satisfy the requirements of the lemma, since all $f_{\gamma(j)}$ $(j \neq k)$, have irreducible factors $< p$.

If $\gamma(k) = t$, then $f_{\gamma(k)} = f_t = cp$ is the polynomial $n_t^2 a_t + m_t$, and obviously $\eta_{p,\gamma}$ has the decomposition stated in the lemma, and is in good shape with respect to μ.

Let us now consider $\eta_{p,<\gamma}$. We have

$$\eta_{p,<\gamma} = \frac{\mathrm{d}\,a}{a} \wedge \sum_{\alpha<\gamma} \frac{\mathrm{d}(n_\alpha^2 a_t + m_\alpha)}{n_\alpha^2 a_t + m_\alpha} \wedge \frac{\mathrm{d}\,f_\alpha}{f_\alpha}$$

which is in good shape with respect to μ^-. Decomposing $\eta_{p,<\gamma}$ in p-components we get

$$\eta_{p,<\gamma} = \eta_{p,\gamma,p} + \eta_{p,\gamma,0},$$

where $\eta_{p,\gamma,p}$ belongs to $\frac{1}{p}\Omega_F M[Y^2] \cap \nu_{M(Y^2)}(m)$ and $\eta_{p,\gamma,0} \in \bigoplus_{q<p} \frac{1}{q}\Omega_F M[Y^2] \cap \nu_{M(Y^2)}(m)$. From (6.4) we know that all these summands are in good shape with respect to μ^-. Thus $\eta_{p,\gamma,0}$ contributes to h_0. We apply now the above procedure to $\eta_{p,\gamma,p}$ considering the highest multi-index $\lambda < \gamma$ appearing in $\eta_{p,\gamma,p}$. We continue with this procedure to finally get the desired decomposition of η_p. This proves the lemma. ■

Lemma 6.12 *For any p appearing in (6.6) it holds*

$$u_p = u_0 + u_1 \wedge \frac{\mathrm{d}\,p}{p}$$
$$v_p = v_0 + v_1 \wedge \frac{\mathrm{d}\,p}{p}$$

with u_0, u_1, v_0, $v_1 \in \Omega_F M[Y^2]$, $\deg u_i$, $\deg v_i < \deg(p)$.

Proof. Set $u_p = \frac{u'}{p}$, $v_p = \frac{v'}{p^2}$ with u', $v' \in \Omega_F M[Y^2]$, $\deg u' < \deg p$, $\deg v' < 2 \deg p$. Inserting in (6.6) we get

$$p^2 E_p = u'^{[2]} + pu' + \mathrm{d}\,v' + pT(p\eta_p)$$

with $p\eta_p \in \Omega_F M[Y^2]$. Set $M(p) = M[Y]/ < p >$ and consider the rest class homomorphism $\Omega_F M[Y^2] \longrightarrow \Omega_{M(p)}$. The last equation induces the relation

$$u'^{[2]} + \mathrm{d}\, v' = 0$$

in $\Omega_{M(p)}$. Using the Cartier-operator we conclude from $\mathrm{d}\, v' = u'^{[2]}$ that $u' = 0$ and $\mathrm{d}\, v' = 0$ in $\Omega_{M(p)}$. From lemma (3.2) in [Ar-Ba$_1$] we get

$$u' = pu_0 + u_1 \wedge \mathrm{d}\, p$$
$$\mathrm{d}\, v' = p^2\, \mathrm{d}\, v_0 + p\, \mathrm{d}\, v_1 \wedge \mathrm{d}\, p$$

with $u_0,\ u_1,\ v_0,\ v_1 \in \Omega_F M[Y^2]$, and $\deg u_i, v_i < \deg p$. Thus

$$u_p = u_0 + u_1 \wedge \frac{\mathrm{d}\, p}{p}$$

and

$$v_p = v_0 + v_1 \wedge \frac{\mathrm{d}\, p}{p}.$$

This proves the lemma. ■

We insert now the above relations in (6.6). It follows

$$E_p + \wp u_0 + \mathrm{d}\, v_0 + T\eta_0 = (\wp u_1 + \mathrm{d}\, v_1 + T\eta_1) \wedge \frac{\mathrm{d}\, p}{p}. \tag{6.13}$$

Here η_0 is in good shape with respect to μ^- as well as $\eta_1 \wedge \frac{\mathrm{d}\, p}{p}$. Actually if $\mathrm{fil}(p) > t$, then η_1 is in good shape with respect to μ^-, and if $\mathrm{fil}(p) = t$, then $\eta_1 \wedge \frac{\mathrm{d}\, cp}{cp}$ with some $c \in M_t$, is in good shape with respect to μ^-.

In what follows we will need the following remark concerning 2-basis of $M(p)$. Assume $p = p_0 + a_k p_1$ with $p_0,\ p_1 \in M_{<k}[Y^2]$, $\mathrm{fil}(p_0),\ \mathrm{fil}(p_1) < t$, and $t = \mathrm{fil}(p)$. Then we obtain the following 2-basis of $M(p) = M[y]$ where $y \equiv Y \mod (p)$: $\{a_1, \ldots, a_{t-1}, a_{t+1}, \ldots\} = \mathcal{B} - \{a_t\} \cup \{y\}$. We keep the same ordering in $\mathcal{B} - \{a_t\}$as in $\mathcal{B}$, but we set y as the maximal element in this 2-basis.

If $f \in M[Y^2]$ has filtration $s < t = \mathrm{fil}(p)$, then it follows easily that $\mathrm{fil}(\overline{f}) = s$ in $M(p)$. If $f \in M[Y^2]$ has degree $< \deg p$ and filtration $\mathrm{fil}(f) = s > t$, then $\mathrm{fil}(\overline{f}) \geq s$ in $M(p)$ too (and in fact $\mathrm{fil}(\overline{f}) = s$). To see this, let us assume $\mathrm{fil}(\overline{f}) < s$ and write in $M[Y^2]$

$$f = n_0 + a_t n_1 + a_s(n_2 + a_t n_3)$$

where n_i has degree $< p$ and do not contain a_t or a_s in the 2-basis expansion, $0 \leq i \leq 3$. In particular $\mathrm{fil}(n_i) < s$, $0 \leq i \leq 3$. In $M(p)$ we get

$$\overline{f} = \overline{n}_0 + a_t \overline{n}_1 + a_s(\overline{n}_2 + a_t \overline{n}_3)$$

where $a_t = \dfrac{p_0 p_1}{p_1^2}$ has filtration $< t$ in $M(p)$. If $\mathrm{fil}(\overline{f}) < s$, then we have $\overline{n}_2 + a_t \overline{n}_3 = 0$ in $M(p)$ and this implies $n_2 = a_t n_3 + lp$ with some polynomial

$l \in M[Y^2]$. Since $\deg n_2$, $\deg n_3 < \deg p$, it follows $l = 0$, i.e. $n_2 = a_t n_3$, and this is a contradiction.

From (6.13) we obtain

$$\wp u_1 + \mathrm{d}\, v_1 + T\eta_1 = 0$$

in $\Omega_{M(p)}$ (see [Ar-Ba$_1$]). In this relation we know that η_1 is in good shape with respect to μ^- if $\gamma(k) > t$ or if $\gamma(k) = t$, then $\eta_1 \wedge \frac{\mathrm{d}\, p}{p}$ is in good shape with respect to μ^-. Moreover since for all $\gamma(j) > t$, the elements $f_{\gamma(i)}$ appears in η_1 are constants (see (a) in the proof of (6.7)), we get from the last remark that the multi-indices of η_1 keep the same ordering in $M(p)$ with respect to the 2-basis of $M(p)$ introduced in this remark. Now the equation

$$T\eta_1 = \wp u_1 + \mathrm{d}\, v_1$$

leads to the relation

$$T = \wp c + \sum_{i \neq k} A_{\gamma(i)} f_{\gamma(i)}$$

with $A_{\gamma(i)} \in M(p)_{\gamma(i)}$ and $\deg A_{\gamma(i)} < \deg p$. This follows from lemma (3.3) in [Ar-Ba$_2$], and the fact that we can change the 2-basis of $M(p)$ through the replacement $f_{\gamma(i)} \longleftrightarrow a_{\gamma(i)}$ for $i \neq k$. Here γ is the maximal multi-index in η_1. Thus in $M[Y^2]$ we obtain

$$T = \wp c + \sum_{i \neq k} A_{\gamma(i)} f_{\gamma(i)} + ph \tag{6.14}$$

with $c \in M[Y^2]$, $\deg c < \deg p$, $A_{\gamma(i)} \in M[Y^2]$, $\deg A_{\gamma(i)} < \deg p$ and hence also $\deg h < \deg p$. We also have $\operatorname{fil} A_{\gamma(i)} < \gamma(i)$. If $\gamma(k) = t$, recall that $f_{\gamma(k)} = f_t = cp$ with $c \in M_t$, and in this case we write cp instead of p in the above relation. Changing the notation a little, we will write $f_{\gamma(k)} = p$ if $\gamma(k) > t$ and $f_{\gamma(k)} = cp$ if $\gamma(k) = t$, and $h = A_{\gamma(k)}$ in both cases. Thus we have

$$T = \wp c + \sum_{i} A_{\gamma(i)} f_{\gamma(i)}. \tag{6.15}$$

We will now use this relation to get rid of the term $T\eta_1$ in the equation (6.13).

To this end we distinguish two cases: $\gamma(k) = \operatorname{fil} p$ for $k > t$ and $\gamma(k) = t$ for $k = t$.

First Case : $\gamma(k) = \operatorname{fil} p$, $k > t$. We can write

$$\begin{aligned}\eta_1 &= \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}(n_t^2 a_t + m_t)}{n_t^2 a_t + m_t} \wedge \frac{\mathrm{d}\, f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \overset{k}{} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}} \\ &\quad + \eta_{1,<\gamma} \\ &= \eta_{1,\gamma} + \eta_{1,<\gamma}\end{aligned}$$

for the form η_1 in lemma (6.7). Here γ is the maximal multi-index appearing in η_1. Let us now use (6.15) with $k > t$, to compute $T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\,p}{p}$. Notice that in this case we have written $f_{\gamma(k)} = p$, $A_{\gamma(k)} = h$, where $\deg h < \deg p$, and $f_{\gamma(t)} = f_t = n_t^2 a_t + m_t$. We get

$$T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\,p}{p} = \left(\wp c + \sum_i A_{\gamma(i)} f_{\gamma(i)}\right) \frac{\mathrm{d}\,a}{a} \wedge \frac{\mathrm{d}\,f_{\gamma(t)}}{f_{\gamma(t)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}.$$

It is convenient to write $f_{\gamma(i)} = a_i$, $\gamma(i) = i$, for $i < t$, so that

$$T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\,p}{p} = \left(\wp c + \sum_{i=1}^{m} A_{\gamma(i)} f_{\gamma(i)}\right) \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}.$$

Let $\nu_{<p,\mathrm{gs}}(m)$ denote the subgroup of $\nu_{M(Y^2)}(m)$ of forms in good shape with respect to μ^- and where the generators are logarithmic differential forms $\mathrm{d}\,q/q$ with $q < p$. We will compute the above expression modulo the group $\wp\Omega^m + \mathrm{d}\,\Omega^{m-1} + T\nu_{<p,\mathrm{gs}}(m)$. Since

$$A_{\gamma(i)} f_{\gamma(i)} \frac{\mathrm{d}\,f_{\gamma(i)}}{f_{\gamma(i)}} \equiv A_{\gamma(i)} f_{\gamma(i)} \frac{\mathrm{d}\,A_{\gamma(i)}}{A_{\gamma(i)}}$$

we have

$$\begin{aligned}
T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\,p}{p} &\equiv \sum_{i=1}^{m} A_{\gamma(i)} f_{\gamma(i)} \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}} \\
&\equiv \sum_{i=1}^{m} \left(T + \sum_{j\neq i} A_{\gamma(j)} f_{\gamma(j)}\right) \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}} \\
&\equiv T \sum_{i=1}^{m} \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}} \\
&\quad + \sum_{i=1}^{m} \left(\sum_{j\neq i} A_{\gamma(j)} f_{\gamma(j)}\right) \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\,f_{\gamma(m)}}{f_{\gamma(m)}}.
\end{aligned}$$

Since $\mathrm{fil}(A_{\gamma(i)}) < \gamma(i)$, it follows that $\mathrm{d}\,a \wedge \mathrm{d}\,A_{\gamma(i)} = 0$ for all $i < t$. Hence the above sums can be taken for $i > t$. Moreover the term $\frac{\mathrm{d}\,a}{a} \wedge \frac{\mathrm{d}\left(n_t^2 a_t + m_t\right)}{n_t^2 a_t + m_t}$ appears in the first summand, so that $T \sum_{i=1}^{m} \frac{\mathrm{d}\,f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\,A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge$

$\frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$ is contained in $T\nu_{<p,\mathrm{gs}}$, and can be dropped. Hence

$$\begin{aligned}
T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\, p}{p} &\equiv \sum_{i=1}^{m} \left(\sum_{j\neq i} A_{\gamma(j)} f_{\gamma(j)} \right) \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}} \\
&\equiv \sum_{1\leq j<i\leq m}^{m} \big(A_{\gamma(j)} f_{\gamma(j)} + A_{\gamma(i)} f_{\gamma(i)} \big) \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(j)}}{A_{\gamma(j)}} \wedge \cdots \\
&\quad \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}
\end{aligned}$$

Replacing $A_{\gamma(j)} f_{\gamma(j)} + A_{\gamma(i)} f_{\gamma(i)}$ by $T + \sum_{l\neq i,j} A_{\gamma(l)} f_{\gamma(l)} \bmod \wp$ we obtain

$$\begin{aligned}
T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\, p}{p} &\equiv \sum_{1\leq j<i\leq m}^{m} \left(T + \sum_{l\neq i,j} A_{\gamma(l)} f_{\gamma(l)} \right) \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(j)}}{A_{\gamma(j)}} \wedge \cdots \\
&\quad \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}} \\
&\equiv \sum_{1\leq j<i\leq m}^{m} \sum_{l\neq i,j} A_{\gamma(l)} f_{\gamma(l)} \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(j)}}{A_{\gamma(j)}} \wedge \cdots \wedge \\
&\quad \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}.
\end{aligned}$$

Iterating this reasoning we obtain

$$\begin{aligned}
T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\, p}{p} &\equiv \sum_{i=1}^{m} A_{\gamma(i)} f_{\gamma(i)} \frac{\mathrm{d}\, A_{\gamma(1)}}{A_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(m)}}{A_{\gamma(m)}} \\
&\equiv T \frac{\mathrm{d}\, A_{\gamma(1)}}{A_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(m)}}{A_{\gamma(m)}} \\
&\equiv 0
\end{aligned}$$

i.e.

$$T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\, p}{p} \in \wp\Omega^m + \mathrm{d}\,\Omega^{m-1} + T\nu_{<p,\mathrm{gs}}(m)$$

Thus we get rid of the term $T\eta_{1,\gamma} \wedge \frac{\mathrm{d}\, p}{p}$ in the equation (6.13). We can now continue with $T\eta_{1,<\gamma}$ which has lower filtration, and this shows that we can assume $\eta_1 = 0$ in (6.13).

Thus we have

$$E_p + \wp u_0 + \mathrm{d}\, v_0 + T\eta_0 = (\wp u_1 + \mathrm{d}\, v_1) \wedge \frac{\mathrm{d}\, p}{p} \tag{6.16}$$

where $\eta_0 \in \nu_{M(Y^2)}(m)$ is in good shape with respect to μ and is in $\nu_{<p,\mathrm{gs}}(m)$, i.e. its q-components are in lower filtration than p. Now a

simple argument, see [Ar-Ba$_1$], shows that we can get rid of the term $(\wp u_1 + \mathrm{d}\, v_1) \wedge \frac{\mathrm{d}\, p}{p}$ and hence

$$E_p = \wp u_0 + \mathrm{d}\, v_0 + T\eta_0$$

and therefore E_p disappears in the basic equation (6.5), because the u_0, v_0, η_0 are in lower components. Thus we are done in the case $k > t$. Let us now consider the

Second case: $k = t$ i.e. $\gamma(k) = k = t = \mathrm{fil}(p) = \mathrm{fil}(f_t)$ where $f_{\gamma(t)} = n_t^2 a_t + m_t = c_\gamma p$ with $c_\gamma \in M_t$, $m_t = \sum_\sigma a^\sigma m_\sigma^2$, $\sigma \in S_{t-1}$, $\sigma + t \geq \mu$ and n_t, $m_t \in M[Y]$. From the proof of (6.7), case (b) we know that $f_{\gamma(i)} \in M_{\gamma(i)}$ for all $i > t$ and $f_t = f_{\gamma(t)} = c_\gamma p$, $\mathrm{fil}(c_\gamma) \leq t$. Let us at this place introduce the following M^2-vector space

$$W = \{x^2 a_t + \sum_\sigma a^\sigma y_\sigma \,|\, x, y_\sigma \in M,\, \sigma \in S_{t-1}, \sigma + t \geq \mu\} \tag{6.17}$$

so that $f_{\gamma(t)} \in W[Y^2]$. We will say that the polynomials in $W[Y^2]$ are in *good shape* with respect to μ^-. In particular $c_\gamma \in W$. We have in this case

$$\eta_p = \frac{\mathrm{d}\, a}{a} \wedge \sum_\gamma \frac{\mathrm{d}\, c_\gamma p}{c_\gamma p} \wedge \frac{\mathrm{d}\, f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$$

and $f_{\gamma(t)} = c_\gamma p$ for all γ. We write $f_{\gamma(i)} = a_i$, $\gamma(i) = i$ for $i < t$, so that we have $\eta_p = \sum_\gamma \frac{\mathrm{d}\, f_\gamma}{f_\gamma}$. Let γ be the maximal multi-index in this summand, set

$$\eta_{\gamma,p} = \frac{\mathrm{d}\, f_\gamma}{f_\gamma} = \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, c_\gamma p}{c_\gamma p} \wedge \frac{\mathrm{d}\, f_{\gamma(t+1)}}{f_{\gamma(t+1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$$

with $f_{\gamma(j)} \in M$ for all $j > t$. Set $\eta_{<\gamma,p} = \sum_{\lambda<\gamma} (\mathrm{d}\, f_\lambda / f_\lambda)$ so that $\eta_p = \eta_{\gamma,p} + \eta_{<\gamma,p}$.

The equation

$$E_p + \wp u_0 + \mathrm{d}\, v_0 + T\eta_0 = (\wp u_1 + \mathrm{d}\, v_1 + T\eta_1) \wedge \frac{\mathrm{d}\, p}{p}$$

implies $\wp u_1 + \mathrm{d}\, v_1 + T\eta_1 = 0$ in $\Omega_{M(p)}$. Notice in this case $\eta_p = \eta_1 \wedge \frac{\mathrm{d}\, p}{p}$, $\eta_0 = 0$, since $k = t$. Since the ordering of the multi-index appearing in η_p remains the same in the 2-basis of $M(p)$, we obtain in $M[Y^2]$ (see remark after the proof of (6.12))

$$T = \wp c + \sum_{i \neq t} A_{\gamma(i)} f_{\gamma(i)} + (c_\gamma p) h \tag{6.18}$$

with $h \in M[Y^2]$. Here all $f_{\gamma(i)}$ are constants, c_γ, $A_{\gamma(i)}$, $h \in M[Y^2]$ with $\deg c_\gamma$, $\deg A_{\gamma(i)}$, $\deg h < \deg p$ and $A_{\gamma(i)} \in M_{<\gamma(i)}[Y^2]$. Moreover $f_{\gamma(t)} = c_\gamma p$ and $A_{\gamma(t)} = h$. Then

$$T = \wp c + \sum_i A_{\gamma(i)} f_{\gamma(i)}$$

Lemma 6.19 *If* $\deg p > 2$, *then* $h \in W[Y^2]$

We postpone the proof of (6.19) to the end of the section. We look at the basic equation for p of degree > 2.

$$E_p + \wp u_0 + \mathrm{d}\, v_0 = (\wp u_1 + \mathrm{d}\, v_1 + T\eta_1) \wedge \frac{\mathrm{d}\, p}{p}.$$

We want to get rid of the term $T\eta_1 \wedge \frac{\mathrm{d}\, p}{p} = T\eta_p$. We have for $\gamma =$ maximal multi-index in η_p

$$T\eta_{\gamma,p} = \left(\wp c + \sum_{i=1}^{m} A_{\gamma(i)} f_{\gamma(i)}\right) \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, c_\gamma p}{c_\gamma p} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$$

and we compute again modulo the subgroup $\wp\Omega^m + \mathrm{d}\,\Omega^{m-1} + T\nu_{<p,\mathrm{gs}}(m)$. Then

$$\begin{aligned}
T\eta_{\gamma,p} &\equiv \left(\sum_{i=1}^{m} A_{\gamma(i)} f_{\gamma(i)}\right) \frac{\mathrm{d}\, f_\gamma}{f_\gamma} \\
&\equiv \sum_{i=1}^{m} A_{\gamma(i)} f_{\gamma(i)} \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}} \\
&\equiv \sum_{i=1}^{m} \left(T + \sum_{j \neq i} A_{\gamma(j)} f_{\gamma(j)}\right) \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}.
\end{aligned}$$

Whenever $i < t$, the terms $\frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$ are zero because $A_{\gamma(i)} \in M_{<\gamma(i)}$, thus only for $i \geq t$ there are contribution to the sum. Moreover, all the terms $T\frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}$ for $i \geq t$ belong to $T\nu_{<p,\mathrm{gs}}(m)$. Only the case $i = t$ in non trivial, and this assertion follows from lemma (6.19) since $A_t = h \in W[Y^2]$. Thus we have

$$T\eta_{\gamma,p} \equiv \sum_{i=1}^{m} \left(\sum_{j \neq i} A_{\gamma(j)} f_{\gamma(j)}\right) \frac{\mathrm{d}\, f_{\gamma(1)}}{f_{\gamma(1)}} \wedge \cdots \wedge \frac{\mathrm{d}\, A_{\gamma(i)}}{A_{\gamma(i)}} \wedge \cdots \wedge \frac{\mathrm{d}\, f_{\gamma(m)}}{f_{\gamma(m)}}.$$

Now we continue as in the case $k > t$ to conclude $T\eta_{\gamma,p} \in \wp\Omega^m + \mathrm{d}\,\Omega^{m-1} + T\nu_{<p,\mathrm{gs}}(m)$. Doing the same with $T\eta_{<\gamma,p}$ we finally obtain the desired result $T\eta_p \in \wp\Omega^m_{<p} + \mathrm{d}\,\Omega^{m-1}_{<p} + T\nu_{<p,\mathrm{gs}}(m)$ and we are done in this case.

Thus we are left with the case $\deg p = 2$, i.e. $f_{\gamma(k)} = f_{\gamma(t)} = c_\gamma p = c_\gamma Y^2 + d_\gamma \in W[Y^2]$, $c_\gamma \in W$, $d_\gamma \in W$. From the equation

$$T = \wp c + \sum_{i \neq t} A_{\gamma(i)} f_{\gamma(i)} + (c_\gamma p) h$$

we see $\deg h = 0$,i.e. $h \in M$, as well as $A_{\gamma(i)} \in M$. Hence comparing coefficients we obtain $ba^\mu = c_\gamma h$, i.e.

$$h = \frac{ba^\sigma a_t}{c_\gamma} = \frac{ba^\sigma a_t}{c_\gamma^2} c_\gamma.$$

But $b \in F^2$ and $\sigma < \mu$, and

$$\frac{\mathrm{d}\, h}{h} = \frac{\mathrm{d}\, a^\sigma}{a^\sigma} + \frac{\mathrm{d}\, a_t}{a_t} + \frac{\mathrm{d}\, c_\gamma}{c_\gamma}$$
$$\frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, h}{h} = \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, a^\sigma}{a^\sigma} + \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, a_t}{a_t} + \frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, c_\gamma}{c_\gamma}$$

and we see immediately that the terms $\frac{\mathrm{d}\, a}{a} \wedge \frac{\mathrm{d}\, h}{h} \wedge \frac{\mathrm{d}\, f_\lambda}{f_\lambda}$ are in good shape with respect to μ^- and belongs to lower degree or filtrations (in fact they are constants). Thus we also get rid of the terms $T\eta_p = T\eta_1 \wedge \frac{\mathrm{d}\, p}{p}$ in the basic equation.

It remains

$$E_p + \wp u_0 + \mathrm{d}\, v_0 + T\eta_0 = (\wp u_1 + \mathrm{d}\, v_1) \wedge \frac{\mathrm{d}\, p}{p}$$

with u_0, v_0, η_0 having at most denominators divisible by q's with $q < p$, and η_0 in good shape respect to μ^-. Again $\wp u_1 + \mathrm{d}\, v_1 = 0$ in $M(p)$ enable us to eliminate E_p from the basic equation (6.5).

Proceeding in the same way we finally arrive at a relation

$$w = \wp u_E + \mathrm{d}\, v_E + T\eta_E$$

with $u_E, v_E, \eta_E \in \Omega_F M[Y^2]$ and $\eta_E \in \nu_{M(Y^2)}(m)$ in good shape with respect to μ, i.e. $a^\sigma \eta_E \in \mathrm{d}\,\Omega$ for all $\sigma < \mu$. But since (see (2.10)) $\Omega_F^m M[Y^2] \cap \nu_{M[Y^2]}(m) = \nu_M(m)$, we get $\eta_E \in \nu_M(m)$. This conclude the proof of the basic lemma: $(A_\mu) \Longrightarrow (B_\mu)$. $\square$

Now we return to the proof of lemma(6.19).

Proof of lemma 6.19. To this end, we look at the relation

$$T = \wp c + \sum_{i \neq t} A_{\gamma(i)} f_{\gamma(i)} + (c_\gamma p)h$$

with $c_\gamma \in W$, $\deg c$, $\deg A_{\gamma(i)}$, $\deg h < \deg p$ and also $\deg f_{\gamma(i)} = 0$ for $i \neq t$, i.e. $f_{\gamma(i)} \in M$ for all $i \neq t$. Here $T = ba^\mu Y^2 + T_{\mu^+}$. We write $c_\gamma p = p_0 + p_1 Y^2 + \cdots + p_s Y^{2s}$ with $p_0, \ldots, p_s \in W$ and $p_s = c_\gamma$. Set also $h = h_0 + h_1 Y^2 + \cdots + h_{s-1} Y^{2s-2}$ and $c = c_0 + c_1 Y^2 + \cdots + c_{s-1} Y^{2s-2}$, $c_i, h_i \in M$. Then the equation above implies $h_{s-1} = 0$ and the other coefficients satisfy the following system of linear equations

$$\begin{pmatrix} p_s & p_{s-1} & p_{s-2} & \cdots & p_3 & p_2 \\ 0 & p_s & p_{s-1} & \cdots & p_4 & p_3 \\ 0 & 0 & p_s & \cdots & & p_4 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & p_s & p_{s-1} \\ 0 & 0 & 0 & \cdots & 0 & p_s \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{s-3} \\ h_{s-2} \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ 0 \\ c_{s-2}^2 \\ 0 \\ c_{s-1}^2 \end{pmatrix}$$

where $* = c_{s/2}$ if s is even and $* = 0$ otherwise. Since the matrix of this system has entries in W, it suffices to show that the inverse matrix also has entries in W. This will imply $h_i \in W$ for all i. Let I_n denote the identity matrix and consider the $n \times n$-matrix $E = (e_{i,j})$ with $e_{i,i+1} = 1$ and $e_{i,j} = 0$ otherwise. Then $E^n = 0$. Fix n elements $q_i \in M$ and define the M^2-vector space $W = M^2 q_1 + \cdots + M^2 q_n$. Assume $q_n \neq 0$. Let $M_{n\times n}(W)$ be the abelian group of $n \times n$-matrices with entries in W and let

$$A = \begin{pmatrix} q_n & q_{n-1} & \cdots & q_1 \\ 0 & q_n & \cdots & q_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n \end{pmatrix}$$

i.e. $A = q_n I_n + q_{n-1}E + \cdots + q_1 E^{n-1} = q_n \left(I_n + \sum_{j=1}^{n-1} \frac{q_{n-j}}{q_n} E^j \right) = q_n (I_n + B)$ where $B^n = 0$. Then $A^{-1} = q_n^{-1} \left(I_n + B + \cdots + B^{n-1} \right)$.

We claim $A^{-1} \in M_{n\times n}(W)$. Obviously this claim proves our lemma (6.19). We have

$$A^{-1} = \frac{q_n}{q_n^2} I_n + \frac{q_n}{q_n^2} B + \cdots + \frac{q_n}{q_n^2} B^{n-1}$$

and it suffices to show $\frac{q_n}{q_n^2} B^i \in M_{n\times n}(W)$.

If $i = 0$ this is clear. Assume $i \geq 1$.

If $i = 2t$ is even, then

$$\frac{q_n}{q_n^2} B^{2t} = \frac{q_n}{q_n^2} \sum_{j=1}^{n-1} \left(\frac{q_{n-j}}{q_n} \right)^{2t} E^{2tj} = q_n \sum_{j=1}^{n-1} \left(\frac{q_{n-j}^t}{q_n^{t+1}} \right)^2 E^{2tj}$$

is clearly in $M_{n\times n}(W)$.

If $i = 2t+1$ is odd, then

$$\frac{q_n}{q_n^2} B^i = \left(\sum_{j=1}^{n-1} \left(\frac{q_{n-j}}{q_n^2} \right) E^j \right) \left(\sum_{j=1}^{n-1} \left(\frac{q_{n-j}}{q_n^2} \right)^{2t} E^{2tj} \right)$$

which also belongs to $M_{n\times n}(W)$ since $q_{n-j}/q_n^2 \in W$.

This proves the claim. ■

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