Harrison's criterion, Witt equivalence and reciprocity equivalence

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Abstract. Harrison's criterion characterizes the isomorphy of the Witt rings of two fields in terms of properties of these fields. In this article, we discuss about the existence of such characterizations for the isomorphism of Witt groups of hermitian forms over certain algebras with involution. In the cases where we consider the Witt group of a quadratic extension with its non-trivial automorphism or the Witt group of a quaternion division algebra with its canonical involution, such criteria are proved. In the framework of global fields, the first of these criteria is reformulated in terms of properties involving real places of the considered fields.

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1 Introduction

One of the basic questions in the algebraic theory of quadratic forms is to give necessary and sufficient conditions for two fields K_1 and K_2 to have isomorphic Witt rings: in this case, K_1 and K_2 are said to be *Witt equivalent*. In [7], Harrison expresses Witt equivalence in the following terms:

Theorem 1.1 (Harrison). *Let K_1 and K_2 be two fields of characteristic different from 2. Then the following are equivalent:*
(1) *K_1 and K_2 are Witt equivalent.*
(2) *There is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ sending -1 to -1 such that the quadratic form $\langle x, y\rangle$ represents 1 over K_1 if and only if the quadratic form $\langle t(x), t(y)\rangle$ represents 1 over K_2 for all $x,\ y \in {K_1}^*$.*

In the literature, the previous theorem is known as "Harrison's criterion". In [4, Theorem 2.3], Cordes shows that the two conditions of Theorem 1.1 are equivalent to the following one, known as "Harrison-Cordes condition":
(3) There is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ sending -1 to -1 such that

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$t(D_{K_1}(\langle 1,a\rangle)) = D_{K_2}(\langle 1,t(a)\rangle)$ (if K is a field of characteristic different from 2 and q is a nondegenerate quadratic form, recall that $D_K(q)$ denotes the set of nonzero square classes represented by q). With this criterion, Cordes classifies Witt rings of order at most 32 up to Witt equivalence by their group structure: see [4, Theorem 7.1].

In [1], Baeza and Moresi study the possibilities to extend Harrison's criterion to fields K_1 and K_2 of characteristic 2. On the one hand, they show that the bilinear Witt rings $W(K_1)$ and $W(K_2)$ of K_1 and K_2 are isomorphic if and only if K_1 and K_2 are isomorphic in the case where $\dim_{{K_1}^2} K_1 = \dim_{{K_2}^2} K_2 > 2$, and they give a complete treatment of the cases where $\dim_{{K_1}^2} K_1 = \dim_{{K_2}^2} K_2 = 1,\ 2$: see [1, Theorem 2.9, Proposition 2.10]. On the other hand, in [1, Theorem 3.1], they characterize the isomorphy of the quadratic Witt modules $W_q(K_1)$ et $W_q(K_2)$ in the following way:

Theorem 1.2 (Baeza-Moresi). *Let K_1 and K_2 be two fields of characteristic 2. Then the following are equivalent:*
(1) *There exist a ring isomorphism $\Phi : W(K_1) \to W(K_2)$ and a group isomorphism $\Psi : W_q(K_1) \to W_q(K_2)$ such that*
$$\Psi(b.q) = \Phi(b).\Psi(q)$$
for all $b \in W(K_1)$ and for all $q \in W_q(K_1)$.
(2) *There exist groups isomorphisms*
$$t_1 : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}, \quad t_2 : K_1/\wp(K_1) \to K_2/\wp(K_2)$$
such that
$$t_1(D_{K_1}(\langle 1,a\rangle)) = D_{K_2}(\langle 1,t_1(a)\rangle), \quad t_2(D_{K_1}[1,b]) = D_{K_2}[1,t_2(b)]$$
for all $a \in {K_1}^*$ and for all $b \in K_1$ (where $\wp(K_i) = \{a + a^2 \mid a \in K_i\},\ i = 1,\ 2$).

Note that the first condition in (2) is similar to "Harrison-Cordes condition".

Such criteria are very useful. For example, Harrison's criterion is used by Mináč and Spira to connect the Witt equivalence of two fields K_1 and K_2 to the isomorphy of some groups G_{K_1} and G_{K_2} (called W-groups), G_{K_i} being the Galois group of a certain field extension ${K_i}^{(3)}$ of K_i for $i = 1,\ 2$: see [14].

In this context, a natural question arises: is it possible to obtain such criteria for the Witt group of a central simple algebra with involution ? After recalling some notations and basic facts in Section 2, we explain how to obtain such criteria in two particular cases in Section 3. First, we obtain a criterion similar to Theorem 1.1 for the Witt group of a quadratic field extension equipped with its nontrivial automorphism:

Theorem 1.3. *Let K_1 and K_2 be two fields of characteristic different from 2. Let $L_1 = K_1(\sqrt{a})$ (resp. $L_2 = K_2(\sqrt{b})$) be a quadratic field extension of K_1 (resp. K_2) equipped with its non trivial automorphism σ_1 (resp. σ_2). Then, the following are equivalent:*
(1) *$W(L_1,\sigma_1) \simeq W(L_2,\sigma_2)$ as rings.*
(2) *There is a group isomorphism $t : {K_1}^*/N_{L_1/K_1}({L_1}^*) \to K_2^*/N_{L_2/K_2}({L_2}^*)$ sending -1 to -1 such that the quadratic form $\langle\langle a,x,y\rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle b,t(x),t(y)\rangle\rangle$ is hyperbolic over K_2 for all $x,\ y \in {K_1}^*$, where $N_{L_i/K_i}({L_i}^*)$ denotes the norm group of the extension L_i/K_i for $i = 1,\ 2$.*

Moreover, in Assertion (2), we show that the condition "$t(-1) = -1$" is not a consequence of the two other conditions: see Example 3.5. Next, in the case of the Witt group of a quaternion division algebra endowed with its canonical involution, two examples point out that its structure of module seems to be the right choice to obtain such a criterion: see Examples 3.6. In this direction, we obtain Theorem 3.9 which is at the same time similar to Theorem 1.1 and to Theorem 1.2 (in some sense). At first sight, Theorem 3.9 seems to be more analogous to Baeza and Moresi's results, but the similarity with Harrison's criterion shows up by taking $K_1 = K_2 = K$:

Corollary 1.4. *Let Q_1 (resp. Q_2) be quaternion division algebras over K endowed with its canonical involution γ_1 (resp. γ_2). Then, the following are equivalent:*
(1) *$W(Q_1, \gamma_1) \simeq W(Q_2, \gamma_2)$ as $W(K)$-modules.*
(2) *There is a group isomorphism $\tilde{t} : K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*) \simeq K^*/\mathrm{Nrd}_{Q_2/K}({Q_2}^*)$ with $\tilde{t}(-1) = -1$ such that the quadratic form $\langle\langle a, b, u, v\rangle\rangle$ is hyperbolic over K if and only if the quadratic form $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ is hyperbolic over K for all $u,\ v \in K^*$.*

In this framework, another interesting problem is to give necessary and sufficient conditions for two global fields to be Witt equivalent. This problem is now entirely solved. The first step is due to Baeza and Moresi who show that two global fields of characteristic 2 are always Witt equivalent: see [1] or [16, §2]. Besides, if two global fields are Witt equivalent then either $\mathrm{char}(K_1),\ \mathrm{char}(K_2) \neq 2$ or $\mathrm{char}(K_1) = \mathrm{char}(K_2) = 2$: see [16, §2]. Lastly, in [16, §3, §4], Perlis, Szymiczek, Conner and Litherland prove that two global fields K_1 and K_2 of characteristic different from 2 are Witt equivalent if and only if they are reciprocity equivalent (i.e. if there exist a group isomorphism t between their square class groups and a bijection T between their non trivial places such that the Hilbert symbols $(x, y)_P$ and $(t(x), t(y))_{T(P)}$ are equal for any $x,\ y \in {K_1}^*/{K_1}^2$ and for any non trivial place P over K_1).

In Section 4, we explain how to obtain such a classification for the Witt group of a quadratic field extension endowed with its nontrivial automorphism. For this purpose, we define the notion of (a, b)-quadratic reciprocity equivalence between two global fields of characteristic different from 2. This notion is a natural adaptation of the reciprocity equivalence when considering norm class groups of quadratic extensions instead of square class groups (see Definition 4.7).

Theorem 1.5. *Let K_1 and K_2 be two global fields of characteristic different from 2. Let $L_1 = K_1(\sqrt{a})$ (resp. $L_2 = K_2(\sqrt{b})$) be a quadratic field extension of K_1 (resp. K_2) equipped with its nontrivial automorphism σ_1 (resp. σ_2). Then, the following are equivalent:*
(1) *$W(L_1, \sigma_1) \simeq W(L_2, \sigma_2)$ as rings.*
(2) *There is an (a, b)-quadratic reciprocity equivalence between K_1 and K_2.*

2 Basic results and notations

In this Section, we fix some notations and recall some basic definitions and results.

2.1 Central simple algebras wth involution

The general reference for the theory of central simple algebras with involution is [10]: see also [17, Chapter 8]

In this Section, K will denote a field of characteristic different from 2, D will denote a finite-dimensional division algebra over K. Then $\dim_K D = n^2$ for some $n \in \mathbb{N}$, and $n = \deg D$ is called the degree of D. Suppose that D is endowed with an involution σ. The map σ restricts to an involution of the center K and we can distinguish two cases: if $\sigma|_K$ is the identity, we say that σ is of *the first kind*, otherwise $\sigma|_K$ is an automorphism of order two of K and we say that σ is of *the second kind*.

A central simple algebra D of degree 2 is called a *quaternion algebra*. As $\mathrm{char}(K) \neq 2$, every quaternion algebra has a quaternion basis $\{1, i, j, k\}$, that is a basis of the K-algebra Q subject to the relations

$$i^2 = a \in K^*,\ j^2 = b \in K^*, ij = k = -ji.$$

This algebra Q is then denoted by $Q = (a, b)_K$. Note also that every quaternion algebra has a canonical involution (usually denoted by γ) which is of the first kind and defined as follows:

$$\gamma(i) = -i,\ \gamma(j) = -j.$$

2.2 Hermitian forms

The standard reference for the theory of hermitian forms is [17]. All the vector spaces considered are supposed to be finite dimensional right vector spaces.

A *hermitian form* over (D, σ) is a pair (V, h) where V is a finite dimensional D-vector space and h is a map $h : V \times V \to D$ which is σ-sesquilinear in the first argument, K-linear in the second argument and which satisfies

$$\sigma\big(h(x, y)\big) = h(y, x) \text{ for any } x,\ y \in V.$$

If $D = K$ and $\sigma = \mathrm{id}_K$ then a hermitian form is a symmetric bilinear form which can be identified with a quadratic form as $\mathrm{char}(K) \neq 2$. The integer $\dim_K V$ is called the dimension (or the rank) of the hermitian form. Since $\mathrm{char}(K) \neq 2$ and D is division, a hermitian form over (D, σ) may be given by a diagonalization $\langle a_1, \cdots, a_n\rangle$.

Note that the dual vector space $V^* := \mathrm{Hom}_D(V, D)$ can be considered as a right D-vector space by twisting the action of D by σ. We define the adjoint map of (V, h) by the following morphism of right D-vector spaces

$$\widehat{h} : \left\{ \begin{array}{ccc} V & \to & V^* \\ x & \mapsto & h_x \end{array} \right.$$

where $h_x(y) = h(x, y)$ for all $y \in M$. If this map is an isomorphism of right D-vector spaces, h is said to be *nondegenerate*. We say that h is *isotropic* if there exists an $x \in V \setminus \{0\}$ such that $h(x, x) = 0$, anisotropic otherwise. If y is an element of D such that $h(x, x) = y$ for a certain $x \in V \setminus \{0\}$, then we say that h *represents* y.

Let (V, h) and (V', h') be two hermitian forms over (D, σ). If these forms are isometric then we write $h \simeq h'$ for short. Their *orthogonal sum* is denoted by $h \perp h'$.

If (V, h) is a hermitian form over (D, σ), one can attach to it a nondegenerate hermitian form over (D, σ) denoted by $(V \oplus V^*, \mathbb{h}_V)$ with

$$\mathbb{h}_V(v \oplus f, v' \oplus g) = f(v') + \sigma\big(g(v)\big)$$

for all $f,\ g \in V^*,\ v,\ v' \in V$. A hermitian form over (D, σ) which is isometric to such a form is called a *hyperbolic form*.

Witt's cancelation theorem says that if (V, h), (V', h'), (V'', h'') are three hermitian forms over (D, σ) and if $h \perp h' \simeq h \perp h''$ then $h' \simeq h''$: see [9, Chapter 5.11].

2.3 The Witt group of a division algebra with involution

We refer to [17] for more details about the Witt group.

Let (V, h) be a nondegenerate hermitian form over (D, σ). The isometry class of this form is denoted by $[V, h]$ or $[h]$ for short and $[0]$ denotes the zero-dimensional hermitian form which is supposed to be nondegenerate by convention. Let $S(D, \sigma)$ denote the set of isometry classes of nondegenerate hermitian forms over (D, σ). This set is endowed with a structure of commutative monoïd by the law

$$[h] + [h'] = [h \perp h'].$$

The *Witt group* of (D, σ) is the quotient group of the Grothendieck group of $S(D, \sigma)$ by the subgroup generated by hyperbolic forms and is denoted by $W(D, \sigma)$. In the case where $D = K$, Kronecker's product can be used to define a structure of ring on $W(K, \sigma)$. If moreover $\sigma = \mathrm{id}_K$, this ring is called the *Witt ring* of K and is denoted by $W(K)$.

If $[V, h] \in W(K, \sigma|_K)$ and $[V', h'] \in W(D, \sigma)$, we define $[V \otimes_K V', h \cdot h'] \in W(D, \sigma)$ as the Witt class of the following nondegenerate hermitian form

$$h \cdot h' : \left\{ \begin{array}{ccl} (V \otimes_K V') \times (V \otimes_K V') & \rightarrow & D \\ (v_1 \otimes v_1', v_2 \otimes v_2') & \mapsto & h(v_1, v_2) h'(v_1', v_2') \end{array} \right. .$$

It is easy to see that this action endows $W(D, \sigma)$ with a structure of $W(K, \sigma|_K)$-module. The submodule generated by nondegenerate hermitian forms of even rank is denoted by $I_1(D, \sigma)$ or by $I(K)$ if $D = K$ and $\sigma = \mathrm{id}_K$. We write $I^n(K)$ for $\big(I(K)\big)^n$. It is easy to see that $I^n(K)$ is additively generated by the so-called Pfister forms

$$\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle := \langle\langle a_1, \cdots, a_n \rangle\rangle,$$

for $a_1, \cdots, a_n \in K^*$.

2.4 The (refined) discriminant of a hermitian form

We refer to [2, §2] and [3, §3] for more general statements about this invariant.

Let (V, h) be a hermitian form over (D, σ) and suppose first that σ is an involution of the first kind. Let $\{e_1, \cdots, e_n\}$ be a D-basis of the right D-vector space V. Let M be the matrix of h with respect to this basis, $E = M_n(D)$ and $m = n \deg(D)$. We define the *signed discriminant* of (V, h) by

$$\mathrm{d}_{\pm}(h) = (-1)^{\frac{m(m-1)}{2}} \mathrm{Nrd}_{E/K}(M) \in K^*,$$

where $\mathrm{Nrd}_{E/K}$ denotes the usual reduced norm map: see [5, §22]. One can show that $\mathrm{d}_{\pm}$ induces a well-defined group homomorphism, again denoted by $\mathrm{d}_{\pm}$

$$\mathrm{d}_{\pm} : I_1(D, \sigma) \rightarrow K^*/K^{*2}.$$

If σ is an involution of the second kind and if F is the fixed field of σ in K, one can define the signed discriminant of (V, h) by the formula above and show that it induces a group homomorphism

$$\mathrm{d}_{\pm} : I_1(D, \sigma) \to F^*/N_{K/F}(K^*),$$

where $N_{K/F}(K^*)$ is the group of norms of K/F. The kernel of the signed discriminant is denoted by $I_2(D, \sigma)$.

In both cases, the signed discriminant can be refined as follows. We will describe it only for σ of the first kind. Denote by $\mathrm{Nrd}_{D/K}(D^*)$ the group of reduced norms from D. With the same formula as above, on can define a group homomorphism

$$\mathrm{Disc} : I_1(D, \sigma) \to \mathrm{Nrd}_{D/K}(D^*)/\mathrm{Nrd}_{D/K}(D^*)^2,$$

which is called the *refined discriminant*.

2.5 Chain equivalence

Throughout this Subsection we will use the notations of [11, Chapter I, §5] and will refer to it for more general statements.

Let D be a division algebra over K endowed with an involution σ (of arbitrary kind). Let $h = \langle a_1, \cdots, a_n\rangle$ and $h' = \langle a'_1, \cdots, a'_n\rangle$ be two hermitian forms over (D, σ). They are said to be *simply equivalent* if there exists indices $i,\ j \in \{1, \cdots, n\}$ such that $\langle a_i, a_j\rangle \simeq \langle b_i, b_j\rangle$ and $a_k = b_k$ for every k different from i and j (note that, if $i = j$, the expression $\langle a_i, a_j\rangle$ is understood to be $\langle a_i\rangle$). Two (diagonalized) hermitian forms h and h' over (D, σ) are *chain equivalent* if there is a sequence of diagonalized hermitian forms $f_0, \cdots, f_m$ over (D, σ) such that $h = f_0$, $h' = f_m$ and such that f_i is simply equivalent to f_{i+1} for $0 \leq i \leq m-1$. We immediately see that two chain equivalent forms are isometric. In fact, the converse is also true by "Witt's Chain equivalence Theorem":

Theorem 2.1 (Witt). *If h and h' are two (diagonalized) hermitian forms over (D, σ) and if h is isometric to h' then h and h' are chain equivalent.*

Proof. The proof can be easily adapted from [11, Chapter I, Theorem 5.2] by replacing usual squares by hermitian squares (that is, elements of the form $\sigma(x)x$). □

Remark 2.2. The above proposition is very useful for showing general properties for quadratic (resp. hermitian) forms by reducing it to properties for two-dimensional quadratic (resp. hermitian) forms. See the proofs of Theorems 1.1, 1.3 and 3.9.

2.6 Further results

Here we want to state two important results that are used several times in this paper.

The following result known as "Arason-Pfister Hauptsatz" gives a sufficient condition for a quadratic form to belong to $I^n(K)$.

Theorem 2.3 (Arason-Pfister). *Let q be a positive-dimensional anisotropic quadratic form over K. If $q \in I^n(K)$, then $\dim q \geq 2^n$.*

Proof. See [11, Chapter X, Hauptsatz 5.1] or [17, Chapter 4, Theorem 5.6]. □

Let L/K, $L = K(\sqrt{a})$, be a quadratic field extension endowed with its non trivial automorphism $-$ and $D = (a,b)_K$ be a quaternion algebra endowed with its canonical involution γ. We define the following usual transfer maps

$$\pi_L : \begin{cases} W(L,-) & \to & W(K) \\ [h] & \mapsto & [x \mapsto h(x,x)] \end{cases}, \quad \pi_D : \begin{cases} W(D,\gamma) & \to & W(K) \\ [h] & \mapsto & [x \mapsto h(x,x)] \end{cases}.$$

Theorem 2.4 (Jacobson). *With the above notations, the maps π_L and π_D are injective.*

Proof. See [17, Chapter 10, 1.1, 1.2, 1.7] or [8]. □

Moreover, $\mathrm{im}(\pi_L) = \langle\langle a \rangle\rangle W(K)$ and for any positive integer n, $\pi_L(I_1(L,-)^n) = \langle\langle a \rangle\rangle I^n(K)$.

3 Analogues of Harrison's criterion

In this Section, we state and prove results for quadratic field extension with their nontrivial automorphism and for quaternion division algebras with their canonical involutions in analogy with Harrison's criterion 1.1.

3.1 The case of quadratic field extensions

First, we rephrase Theorem 1.1 by introducing another equivalent condition and it is this condition we will then generalize to the setting of hermitian forms.

Lemma 3.1. *Let K_1 and K_2 be two fields of characterisic different from 2. Then the following are equivalent:*
(1) K_1 and K_2 are Witt equivalent.
(2) There is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ sending -1 to -1 and such that the quadratic form $\langle\langle x, y \rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle t(x), t(y) \rangle\rangle$ is hyperbolic over K_2 for all $x,\ y \in {K_1}^*$.

Proof. The quadratic form $\langle x, y \rangle$ represents 1 over K_1 if and only if the quadratic form $\langle 1, -x, -y \rangle$ is isotropic over K_1 if and only if the Pfister form $\langle\langle x, y \rangle\rangle$ is hyperbolic over K_1. The equivalence then follows from the second statement of Harrison's criterion 1.1. □

We now come to the proof of Theorem 1.3:

Theorem *Let K_1 and K_2 be two fields of characteristic different from 2. Let $L_1 = K_1(\sqrt{a})$ (resp. $L_2 = K_2(\sqrt{b})$) be a quadratic field extension of K_1 (resp. K_2) equipped with its non trivial automorphism σ_1 (resp. σ_2). Then, the following are equivalent:*
(1) $W(L_1, \sigma_1) \simeq W(L_2, \sigma_2)$ as rings.
(2) There is a group isomorphism $t : {K_1}^*/N_{L_1/K_1}({L_1}^*) \to K_2^*/N_{L_2/K_2}({L_2}^*)$ sending -1 to -1 such that the quadratic form $\langle\langle a, x, y \rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle b, t(x), t(y) \rangle\rangle$ is hyperbolic over K_2 for all $x,\ y \in {K_1}^*$.

In the proof, we will need the following two lemmas.

Lemma 3.2. *For $i = 1,\ 2$, the signed discriminant induces a group isomorphism*

$$\mathrm{d}_{\pm} : I_1(L_i, \sigma_i)/I_2(L_i, \sigma_i) \simeq {K_i}^*/N_{L_i/K_i}({L_i}^*).$$

Proof. We know that

$$\mathrm{d}_{\pm} : I_1(L_i, \sigma_i) \to {K_i}^*/N_{L_i/K_i}({L_i}^*)$$

is a group homomorphism whose kernel is $I_2(L_i, \sigma_i)$. If $b \in {K_i}^*$ then

$$\mathrm{d}_{\pm}(\langle 1, -b\rangle) = (-1)^1 \mathrm{Nrd}_{M_2(L_i)/L_i}(\langle 1, -b\rangle) = -\mathrm{Nrd}_{L_i/K_i}(-b) = b \mod N_{L_i/K_i}({L_i}^*),$$

hence $\mathrm{d}_{\pm}$ is onto. □

We consider $W(L_i, \sigma_i)$ as a ring, $I_1(L_i, \sigma_i)$ and $I_2(L_i, \sigma_i)$ as ideals for $i = 1,\ 2$. As in the case of quadratic forms, these two ideals are related in the following way:

Lemma 3.3. *For $i = 1,\ 2$, $\big(I_1(L_i, \sigma_i)\big)^2 = I_2(L_i, \sigma_i)$.*

Proof. By using the signed discriminant, it is easy to show that $\big(I_1(L_i, \sigma_i)\big)^2 \subseteq I_2(L_i, \sigma_i)$.

Conversely, if $\phi \in I_2(L_i, \sigma_i)$ with $\phi = \langle a_1, \cdots, a_{2s}\rangle$, $a_j \in {K_i}^*,\ j = 1, \cdots, 2s$, then

$$1 = \mathrm{d}_{\pm}(\phi) = (-1)^{\frac{2s(2s-1)}{2}}\big(\prod_{j=1}^{2s} a_j\big) \in {K_i}^*/N_{L_i/K_i}({L_i}^*).$$

We proceed by induction on s.

If $s = 1$ and $\phi \simeq \langle a, b\rangle$ then $-a = b \in {K_i}^*/N_{L_i/K_i}({L_i}^*)$. Now $\phi \simeq \langle a, -a\rangle$ and $\phi \in \big(I_1(L_i, \sigma_i)\big)^2$.

If $s = 2$ and $\phi \simeq \langle a, b, c, d\rangle$ then $d = abc \in {K_i}^*/N_{L_i/K_i}({L_i}^*)$ and

$$\phi \simeq \langle a\rangle \otimes \langle 1, ab, ac, bc\rangle \simeq \langle a\rangle \otimes \langle 1, ab\rangle \otimes \langle 1, ac\rangle.$$

We thus have $\phi \in \big(I_1(L_i, \sigma_i)\big)^2$.

Suppose now $s \geq 3$. One can write $\phi = \langle a, b, c\rangle \perp \phi'$ and

$$\phi = \underbrace{\langle a, b, c, abc\rangle}_{\alpha} \perp \underbrace{(\phi' \perp \langle -abc\rangle)}_{\beta} \in W(K, \sigma_a).$$

By assumption, $\mathrm{d}_{\pm}(\phi) = 1$ and $\mathrm{d}_{\pm}(\alpha) = 1$ and this implies that $\mathrm{d}_{\pm}(\beta) = 1$. By induction, $\beta \in \big(I_1(L_i, \sigma_i)\big)^2$ and $\phi \in \big(I_1(L_i, \sigma_i)\big)^2$, thus finishing the proof. □

Proof of Theorem 1.3: (1) $\Rightarrow$ (2) : Let $\Phi : W(L_1, \sigma_1) \simeq W(L_2, \sigma_2)$ be a ring isomorphism. The rank induces a group homomorphism, rank : $W(L_1, \sigma_1) \to \mathbb{Z}/2\mathbb{Z}$ and this shows that $I_1(L_1, \sigma_1)$ is a maximal ideal of index 2 in $W(L_1, \sigma_1)$. In fact, it can be seen that $I_1(L_1, \sigma_1)$ is the unique ideal of index 2 in $W(L_1, \sigma_1)$. As a consequence, Φ induces the following group isomorphism

$$\Phi|_{I_1(L_1, \sigma_1)} : I_1(L_1, \sigma_1) \to I_1(L_2, \sigma_2).$$

Hence, by Lemma 3.3, $\Phi(I_2(L_1,\sigma_1)) = I_2(L_2,\sigma_2)$ and we can define the following group isomorphism by means of Lemma 3.2:

$$t : \left\{ \begin{array}{ccc} {K_1}^*/N_{L_1/K_1}({L_1}^*) & \to & {K_2}^*/N_{L_2/K_2}({L_2}^*) \\ c & \mapsto & \mathrm{d}_\pm\left(\Phi(\langle 1,-c\rangle)\right) \end{array} \right. .$$

Now, it is easy to see that $t(-1) = -1$.

Φ induces also the following group isomorphism

$$u : I_1(L_1,\sigma_1)^2/I_1(L_1,\sigma_1)^3 \simeq I_1(L_2,\sigma_2)^2/I_1(L_2,\sigma_2)^3,$$

which can be used to obtain the following commutative diagram:

$$\begin{tikzcd} \left({K_1}^*/N_{L_1/K_1}({L_1}^*)\right) \times \left({K_1}^*/N_{L_1/K_1}({L_1}^*)\right) \arrow[r, "\theta_{L_1}"] \arrow[d, "(t,t)"] & \left(I_1(L_1,\sigma_1)\right)^2/\left(I_1(L_1,\sigma_1)\right)^3 \arrow[d, "u"] \\ \left({K_2}^*/N_{L_2/K_2}({L_2}^*)\right) \times \left({K_2}^*/N_{L_2/K_2}({L_2}^*)\right) \arrow[r, "\theta_{L_2}"] & \left(I_1(L_2,\sigma_2)\right)^2/\left(I_1(L_2,\sigma_2)\right)^3 \end{tikzcd}$$

where

$$\theta_{L_i}(x,y) = \langle 1,-x\rangle \otimes \langle 1,-y\rangle \mod I_1(L_i,\sigma_i)^3$$

for all $x,\ y \in {K_i}^*$ and for $i = 1,\ 2$. If the hermitian form $\langle 1,-x,-y,xy\rangle$ is hyperbolic over (L_1,σ_1), then $\langle 1,-x,-y,xy\rangle \in I_1(L_1,\sigma_1)^3$. Conversely, let $\langle 1,-x,-y,xy\rangle \in I_1(L_1,\sigma_1)^3$. Using the notations of Subsection 2.6, we know that the map $\pi_{L_1} : W(L_1,\sigma_1) \to \langle\langle a\rangle\rangle W(K_1)$ is an additive group isomorphism and that

$$\Psi(\langle 1,-x,-y,xy\rangle) \in I^4(K_1).$$

By Theorem 2.3, the quadratic form $\Psi(\langle 1,-x,-y,xy\rangle)$ is hyperbolic over K_1 and it follows that the hermitian form $\langle 1,-x,-y,xy\rangle$ is hyperbolic over (L_1,σ_1).

Lastly, if the quadratic form $\langle\langle a,x,y\rangle\rangle = \pi_{L_1}(\langle 1,-x,-y,xy\rangle)$ is hyperbolic over K_1 then, the hermitian form $\langle 1,-x,-y,xy\rangle$ is hyperbolic over (L_1,σ_1) and $\langle 1,-x,-y,xy\rangle \in I_1(L_1,\sigma_1)^3$. By commutativity of the previous diagram,

$$0 = u(\theta_{L_1}(x,y)) = \theta_{L_2}(t(x),t(y)).$$

We conclude that the hermitian form $\langle 1,-t(x),-t(y),t(xy)\rangle \in I_1(L_2,\sigma_2)^3$ which implies that the quadratic form $\langle\langle b,t(x),t(y)\rangle\rangle = \pi_{L_2}(\langle 1,-t(x),-t(y),t(xy)\rangle)$ is hyperbolic over K_2. The converse is similar.

(2) $\Rightarrow$ (1) : We define a map Φ on diagonal forms by

$$\Phi(\langle a_1,\cdots,a_n\rangle) = \langle t(a_1),\cdots,t(a_n)\rangle.$$

We have to show that this definition does not depend on the chosen diagonalization. If $n = 1$, this is clear. If $n = 2$, suppose that $\langle u,v\rangle \simeq \langle u',v'\rangle$ as hermitian forms over (L_1,σ_1). By taking the signed discriminant on both sides, we obtain that $uv = u'v' \in {K_1}^*/N_{L_1/K_1}({L_1}^*)$.

If we let the one-dimensional hermitian form $\langle u\rangle$ act on both sides, it follows that the hermitian form $\langle 1, -uu', -uv', u'v'\rangle$ is hyperbolic over (L_1, σ_1). As a consequence, the hermitian form $\langle 1, -t(u)t(u'), -t(u)t(v'), t(u')t(v')\rangle$ is hyperbolic over (L_2, σ_2) and, multiplying by the form $\langle t(u)\rangle$, it follows that the hermitian forms $\langle t(u), t(v)\rangle$ and $\langle t(u'), t(v')\rangle$ are isometric over (L_2, σ_2). If $n > 2$, the result comes from Theorem 2.1 and from the fact that the property holds for $n = 2$. As $t(-1) = -1$, Φ preserves hyperbolicity and induces a well-defined map between $W(L_1, \sigma_1)$ and $W(L_2, \sigma_2)$. Besides, Φ is additive and multiplicative (Φ being multiplicative over rank one forms which generate additively $W(L_1, \sigma_1)$) and t^{-1} provides an inverse for Φ which is thus a ring isomorphism. □

Remarks 3.4. (1) The previous proof is similar to the proof of Harrison's criterion: see [7] or [16, §2].
(2) As ${K_i}^{*2} = D_{K_i}(\langle 1\rangle)$ and $N_{L_i/K_i}({L_i}^*) = D_{K_i}(\langle 1, -a\rangle)$ for $i = 1,\ 2$, Theorem 1.3 is a quadratic analogue of Theorem 1.1.

In Theorem 1.3 we can show that the condition $t(-1) = -1$ is not a consequence of the other two conditions of Assertion (2):

Example 3.5. Let $K_1 = \mathbb{Q}_3$ and $K_2 = \mathbb{Q}_5$. Then

$$ {K_1}^*/{K_1}^{*2} = \{1, -1, 3, -3\},\ \ {K_2}^*/{K_2}^{*2} = \{1, 2, 5, 10\},\ \ u(K_1) = u(K_2) = 4, $$

and the unique anisotropic quadratic form of dimension 4 over K_1 (resp. over K_2) is $\langle 1, 1, -3, -3\rangle$ (resp. $\langle 1, -2, -5, 10\rangle$) (see [11, Chapter 6, Theorem 2.2]). Let $L_1 = K_1(\sqrt{3})$ and $L_2 = K_2(\sqrt{2})$. It is easy to show that $|{K_1}^*/D_{K_1}(\langle 1, -3\rangle)| = 2 = |{K_2}^*/D_{K_2}(\langle 1, -2\rangle)|$ and that we have a group isomorphism defined by

$$ \begin{array}{rcl} t : {K_1}^*/D_{K_1}(\langle 1, -3\rangle) & \to & {K_2}^*/D_{K_2}(\langle 1, -2\rangle) \\ 1 & \mapsto & 1 \\ -1 & \mapsto & 5 \end{array} $$

As $u(K_1) = u(K_2) = 4$, the quadratic form $\langle\langle 3, x, y\rangle\rangle$ (resp. $\langle\langle 2, t(x), t(y)\rangle\rangle$) is hyperbolic over K_1 (resp. over K_2) for all $x,\ y \in {K_1}^*$. Finally, $\langle 1, -2\rangle$ clearly represents -1 over K_2 and

$$ t(-1) \neq -1 = 1 \in {K_2}^*/D_{K_2}(\langle 1, -2\rangle). $$

3.2 The case of quaternion division algebras

In this Subsection, $Q_1 = (a, b)_{K_1}$ (resp. $Q_2 = (c, d)_{K_2}$) will denote a quaternion division algebra over K_1 (resp. over K_2) with its canonical involution γ_1 (resp. γ_2).

First, we motivate our choice of the module structure in Theorem 3.9 and Corollary 1.4 by giving two examples showing that, for quadratic forms, the group structure of the Witt ring is not sufficient to classify fields up to Witt equivalence as in Theorem 1.1. In the first example, the cardinality of the Witt ring is infinite and in the second, it is finite.

Examples 3.6. (1) One can find this example in [16, §7]. If $K_1 = \mathbb{Q}(\sqrt[3]{2})$ and $K_2 = \mathbb{Q}$, one can show that $W(K_1) \simeq W(K_2)$ as groups. But, by [16, §4, Corollary 2], $W(K_1)$ and $W(K_2)$ are not isomorphic as rings.
(2) One can find this example in [4, Example 7.2]. The construction is based on [6, §II.1] which was obtained in 1965 by Gross and Fischer. We choose $K_1 = \mathbb{Q}_2(\sqrt{d})$ where $d \in \mathbb{Q}_2^* \backslash \pm \mathbb{Q}_2^{*2}$. Then K_1 is a local field and we have

$$|{K_1}^*/{K_1}^{*2}| = 16, \ s(K_1) = 2, \ u(K_1) = 4$$

(see [11, Chapter VI, Corollary 2.24], [11, Chapter XI, Examples 2.4(7), 6.2(4)]). Let F be a field satisfying

$$|F^*/F^{*2}| = 4, \ s(F) = 2, \ u(F) = 2$$

(there is such a field by Cordes' results in [4]). Then $K_2 := F((X))$ is such that

$$|{K_2}^*/{K_2}^{*2}| = 8, \ s(K_2) = 2, \ u(K_2) = 4.$$

By [4, Theorem 4.5],

$$W(K_1) \simeq W(K_2) \simeq C_4 \times C_4 \times C_2 \times C_2$$

as groups. But $W(K_1)$ and $W(K_2)$ are not isomorphic as rings by Harrison's criterion 1.1 as we have $|{K_1}^*/{K_1}^{*2}| = 16 \neq 8 = |{K_2}^*/{K_2}^{*2}|$.

In order to simplify the statement of Theorem 3.9, we define:

Definition 3.7. Two fields K_1 and K_2 whose characteristic is different from 2 are said to be (Q_1, Q_2)-*equivalent* if there is a group morphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ sending -1 to -1 such that, if the quadratic form $\langle\langle x, y\rangle\rangle$ is hyperbolic over K_1, then the quadratic form $\langle\langle t(x), t(y)\rangle\rangle$ is hyperbolic over K_2 for all $x,\ y \in {K_1}^*$ and which induces a group isomorphism $\tilde{t} : {K_1}^*/D_{K_1}(\langle\langle a, b\rangle\rangle) \simeq {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle)$. The pair $(t, \tilde{t})$ is called a (Q_1, Q_2)-*equivalence*.

Remark 3.8. If A is a central simple algebra over a field K, let

$$\mathrm{SH}_0(A) = K^*/\mathrm{Nrd}_{A/K}(A^*).$$

In the literature, the group $\mathrm{SH}_0(A)$ is called the reduced norm residue group of the algebra A. In particular, if Q is a quaternion division algebra with norm form N_Q, $\mathrm{SH}_0(Q) = K^*/D_K(N_Q)$ which allows us to translate Definition 3.7 in terms of these groups.

Theorem 3.9. *The following are equivalent:*
(1) *There exist a ring homomorphism* $\Phi : W(K_1) \to W(K_2)$ *sending one-dimensional forms to one-dimensional forms and a group isomorphism* $\Psi : W(Q_1, \gamma_1) \to W(Q_2, \gamma_2)$ *such that* $\Psi(\langle 1\rangle) = \langle 1\rangle$ *and*

$$\Psi(q.h) = \Phi(q).\Psi(h),$$

for all $q \in W(K_1)$, $h \in W(Q_1, \gamma_1)$.
(2) *There is a* (Q_1, Q_2)*-equivalence* $(t, \tilde{t})$ *between* K_1 *and* K_2 *such that the hermitian forms* $\langle u, v\rangle$ *and* $\langle u', v'\rangle$ *are isometric over* (Q_1, γ_1) *if and only if the hermitian forms* $\langle \tilde{t}(u), \tilde{t}(v)\rangle$ *and*

$\langle \tilde{t}(u'), \tilde{t}(v')\rangle$ *are isometric over* (Q_2, γ_2) *for all* $u,\ v,\ u',\ v' \in {K_1}^*$.
(3) *There is a* (Q_1, Q_2)*-equivalence* $(t, \tilde{t})$ *between* K_1 *and* K_2 *such that the quadratic form* $\langle\langle a, b, u, v\rangle\rangle$ *is hyperbolic over* K_1 *if and only if the quadratic form* $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ *is hyperbolic over* K_2 *for all* $u,\ v \in {K_1}^*$.

First, we need to prove the following lemma:

Lemma 3.10. *Let* Q *be a quaternion division algebra over a field* K *with norm form* N_Q. *Let* $u,\ v,\ u',\ v' \in K^*$. *Suppose that the quadratic form* $N_Q \otimes \langle u, v, -u', -v'\rangle$ *is hyperbolic over* K. *Then* $uvu'v' \in D_K(N_Q)$.

Proof. As the quadratic forms $q = \langle u, v, -u', -v'\rangle$ and $q' = \langle 1, -uu'vv'\rangle$ have the same signed discriminant, $q \perp (-q')$ belongs to $I^2(K)$. Thus,

$$N_Q \otimes \langle u, v, -u', -v'\rangle \equiv N_Q \otimes \langle 1, -uu'vv'\rangle \mod I^4(K).$$

By assumption and by Theorem 2.3, the quadratic form $N_Q \otimes \langle 1, -uu'vv'\rangle$ is hyperbolic over K and it follows that $uvu'v' \in D_K(N_Q)$. □

Proof of Theorem 3.9: (3) $\Rightarrow$ (2) : Let $u,\ v,\ u',\ v' \in {K_1}^*$ be such that

$$\langle u, v\rangle \simeq \langle u', v'\rangle \tag{1}$$

as hermitian forms over (Q_1, γ_1). We now use the notations of Subsection 2.6. By Theorem 2.4, $\pi_{Q_1} : W(Q_1, \gamma_1) \to W(K_1)$ is injective. The fact (1) is thus equivalent to the fact that

$$\langle\langle a, b\rangle\rangle \otimes \langle u, v\rangle \simeq \langle\langle a, b\rangle\rangle \otimes \langle u', v'\rangle$$

as quadratic forms, which in turn is equivalent to the hyperbolicity of the quadratic form $\langle\langle a, b\rangle\rangle \otimes \langle u, v, -u', -v'\rangle$ over K_1. By Lemma 3.10, $uvu'v' \in D_{K_1}(\langle\langle a, b\rangle\rangle)$ and $t(uvu'v') \in D_{K_2}(\langle\langle c, d\rangle\rangle))$. Now we let the quadratic form $\langle u\rangle$ act on (1) so we have $\langle 1, uv\rangle \simeq \langle uu', uv'\rangle$ as hermitian forms over (Q_1, γ_1) and, since $uvu'v' \in D_{K_1}(\langle\langle a, b\rangle\rangle)$, it follows that $\langle 1, u'v'\rangle \simeq \langle vv', vu'\rangle$ as hermitian forms over (Q_1, γ_1). This is equivalent to the hyperbolicity of the quadratic form $\langle\langle a, b, vv', vu'\rangle\rangle$ over K_1 and by Assertion (3), the quadratic form $\langle\langle c, d, \tilde{t}(vv'), \tilde{t}(vu')\rangle\rangle$ is hyperbolic over K_2. Using the fact that $t(uvu'v') \in D_{K_2}(\langle\langle c, d\rangle\rangle))$, we deduce that the hermitian forms $\langle \tilde{t}(u), \tilde{t}(v)\rangle$ and $\langle \tilde{t}(u'), \tilde{t}(v')\rangle$ are isometric over (Q_2, γ_2). The converse is similar.
(2) $\Rightarrow$ (1) : let $(t, \tilde{t})$ be a (Q_1, Q_2)-equivalence between K_1 and K_2 satisfying the conditions of Assertion (2). Mimicking the first part of the proof of Theorem 1.3, one can define a group homomorphism $\Phi : W(K_1) \to W(K_2)$ sending a one-dimensional form to a one-dimensional form. We define Ψ in the following way

$$\Psi : \begin{cases} W(Q_1, \gamma_1) & \to & W(Q_2, \gamma_2) \\ \langle a_1, \cdots, a_n\rangle & \mapsto & \langle \tilde{t}(a_1), \cdots, \tilde{t}(a_n)\rangle \end{cases}.$$

As in the proof of Theorem 1.3, by using Proposition 2.1, we can show that Ψ is a well-defined map which induces a group homomorphism, and that the inverse of $\tilde{t}$ induces an inverse for Ψ. Finally, the compatibility relation between Φ and Ψ is easily proved.

$(1) \Rightarrow (3)$: Let us suppose the existence of Φ and Ψ as in Assertion (1). As $\Phi(I(K_1)) \subset I(K_2)$, Φ induces the following group homomorphism

$$t : \left\{ \begin{array}{ccc} {K_1}^*/{K_1}^{*2} & \to & {K_2}^*/{K_2}^{*2} \\ a & \mapsto & \mathrm{d}_\pm\left(\Phi(\langle 1, -a\rangle)\right) \end{array} \right.$$

and t satisfies the other properties stated in Definition 3.7 by the implication $(1) \Rightarrow (2)$ in Harrison's criterion 1.1. We are going to show that

$$D_{K_1}(\langle\langle a, b\rangle\rangle)/{K_1}^{*2} = t^{-1}(D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2}). \tag{2}$$

Let $\overline{u} \in D_{K_1}(\langle\langle a, b\rangle\rangle)/{K_1}^{*2}$. Then $\Psi(\langle u\rangle) = \Psi(\langle 1\rangle) = \langle 1\rangle$ on the one hand, and $\Psi(\langle u\rangle) = \Phi(\langle u\rangle).\langle 1\rangle$ on the other hand (note that $\Phi(\langle u\rangle)$ is a quadratic form over K_2 whereas $\Psi(\langle u\rangle)$ is a hermitian form over (Q_2, γ_2)). Letting $\Phi(\langle u\rangle) = \langle x\rangle$, we easily see that $t(\overline{u}) = x$ and that $x \in D_{K_2}(\langle\langle c, d\rangle\rangle)$. This implies that $t(\overline{u}) \in D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2}$ and finally $D_{K_1}(\langle\langle a, b\rangle\rangle)/{K_1}^{*2} \subset t^{-1}(D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2})$.

Let $\overline{v} \in D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2}$ be such that $t(\overline{y}) = \overline{v}$ for a $y \in {K_1}^*$. As $\Phi(\langle y\rangle) = \langle v\rangle$,

$$\Psi(\langle y\rangle) = \Phi(\langle y\rangle).\langle 1\rangle = \langle v\rangle.\langle 1\rangle = \langle 1\rangle.$$

By injectivity of Ψ, it follows that $\langle y\rangle \simeq \langle 1\rangle$ and (2) holds.

Hence, t induces after factorization a unique injective group homomorphism

$$\tilde{t} : \left\{ \begin{array}{ccc} {K_1}^*/D_{K_1}(\langle\langle a, b\rangle\rangle) & \to & {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle) \\ x & \mapsto & t(x) \end{array} \right.$$

Denote by $s_1 : {K_1}^*/{K_1}^{*2} \to {K_1}^*/D_{K_1}(\langle\langle a, b\rangle\rangle)$ and $s_2 : {K_2}^*/{K_2}^{*2} \to {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle)$ the two canonical surjections. Then, the following diagram is commutative

$$\begin{tikzcd} {K_1}^*/{K_1}^{*2} \arrow[r, "t"] \arrow[d, "s_1"] & {K_2}^*/{K_2}^{*2} \arrow[r, "s_2"] & {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle) \\ {K_1}^*/D_{K_1}(\langle\langle a, b\rangle\rangle) \arrow[urr, "\tilde{t}"] & & \end{tikzcd}$$

Now, we show that $s_2 \circ t$ is onto. Let $\overline{w} \in {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle)$. Ψ being surjective, there is a hermitian form h over (Q_1, γ_1) such that $\Psi(h) = \langle w\rangle = \Phi(q).\langle 1\rangle$ where $h = q.\langle 1\rangle$ and q is a quadratic form over K. Without loss of generality, one can suppose that $h = \langle a_1, \cdots, a_n\rangle$ et $\Phi(q) = \langle b_1, \cdots, b_n\rangle$ with $a_1, \cdots, a_n,\ b_1, \cdots, b_n \in {K_1}^*$ (note that n is odd). By taking the refined signed discriminant on both sides of the previous equality, we obtain

$$\prod_{i=1}^{n} {b_i}^2 = w^2 \mod D_{K_2}(\langle\langle c, d\rangle\rangle)^2.$$

Consequently, there is a $\delta \in {Q_2}^*$ such that $\big(\prod_{i=1}^n {b_i}^2).\mathrm{Nrd}_{Q_2/K_2}(\delta)^2 = w^2$ and

$$w = \pm(\prod_{i=1}^n b_i).\mathrm{Nrd}_{Q_2/K_2}(\delta).$$

The element $\pm(\prod_{i=1}^n a_i)$ is an antecedent of w via $s_2 \circ t$. Indeed,

$$\begin{aligned} t(\pm\prod_{i=1}^n a_i) &= \pm\prod_{i=1}^n t(a_i) \\ &= \pm\,\mathrm{d}_\pm\left(n\langle 1\rangle \perp -\Phi(\langle a_1, \cdots, a_n\rangle)\right) \\ &= \pm\,\mathrm{d}_\pm(n\langle 1\rangle \perp \langle -b_1, \cdots, -b_n\rangle) \\ &= \pm\prod_{i=1}^n b_i \in {K_2}^*/{K_2}^{*2}. \end{aligned}$$

So

$$(s_2 \circ t)(\pm\prod_{i=1}^n a_i) = \pm\prod_{i=1}^n b_i \mod D_{K_2}(\langle\langle c, d\rangle\rangle) = \overline{w}.$$

It follows that $s_2 \circ t$ is surjective and that $\tilde{t}$ is a group isomorphism.

Let $u,\ v \in {K_1}^*$ be such that the quadratic form $\langle\langle a, b, u, v\rangle\rangle$ is hyperbolic over K_1. The hermitian form $\langle 1, -u, -v, uv\rangle$ is also hyperbolic over (Q_1, γ_1) by Theorem 2.4. We obtain (in $W(Q_1, \gamma_1)$)

$$\begin{aligned} 0 &= \Psi(\langle 1, -u, -v, uv\rangle) \\ &= \Phi(\langle 1, -u, -v, uv\rangle).\langle 1\rangle \\ &= \big(\Phi(\langle 1, -u\rangle) \otimes \Phi(\langle 1, -v\rangle)\big).\langle 1\rangle. \end{aligned}$$

By definition of t and $\tilde{t}$, we then have

$$\begin{aligned} \Psi(\langle 1, -u, -v, uv\rangle) &= \langle 1, -t(u)\rangle \otimes \langle 1, -t(v)\rangle \\ &= \langle 1, -t(u), -t(v), t(u)t(v)\rangle \\ &= \langle 1, -\tilde{t}(u), -\tilde{t}(v), \tilde{t}(u)\tilde{t}(v)\rangle. \end{aligned}$$

It follows that the quadratic form $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ is hyperbolic over K_2. Conversely, if the quadratic form $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ is hyperbolic over K_2 then the quadratic form $\langle\langle a, b, u, v\rangle\rangle$ is hyperbolic over K_1 by Theorem 2.4 and by injectivity of Ψ. □

In the particular case where $K_1 = K_2 = K$, Theorem 3.9 readily implies:

Corollary *Then, the following are equivalent:*
(1) $W(Q_1, \gamma_1) \simeq W(Q_2, \gamma_2)$ *as* $W(K)$*-modules.*
(2) *There is a group isomorphism* $\tilde{t} : K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*) \simeq K^*/\mathrm{Nrd}_{Q_2/K}({Q_2}^*)$ *with* $\tilde{t}(-1) = -1$ *such that the quadratic form* $\langle\langle a, b, u, v\rangle\rangle$ *is hyperbolic over* K *if and only if the quadratic form* $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ *is hyperbolic over* K *for all* $u,\ v \in K^*$.

Remarks 3.11. (1) The condition on the hyperbolicity of quadratic forms in Theorem 3.9 can be replaced by a condition similar to the "Harrison-Cordes condition" (see Introduction). Theorem 3.9 is thus analogous to Theorem 1.2 proved by Baeza and Moresi. There are also some differences:
− we do not suppose Φ to be bijective in Assertion (1) ;
− in addition, we suppose that Φ respects one-dimensional forms and that $\Psi(\langle 1\rangle) = \langle 1\rangle$ in Assertion (1) ;
(2) As ${K_1}^{*2} = D_{K_1}(\langle 1\rangle)$ and $\mathrm{Nrd}_{Q_1/K_1}(Q_1^*) = D_{K_1}(\langle\langle a, b\rangle\rangle)$, Corollary 1.4 is an analogue of Theorem 1.1 for quaternion division algebras (in the case where $K = L$).
(3) In [12], Leep and Marshall construct a surjective map between $\mathrm{Aut}\left(W(K_1)\right)$ and the set of the so-called "Harrison maps" (i.e. satisfying Assertion (2) of Theorem 1.1) and describe the kernel of this map. To prove their results, they used the fact that every $\rho \in \mathrm{Hom}_{\mathrm{ring}}\left(W(K_1), W(K_2)\right)$ induces an element $\overline{\rho} \in \mathrm{Hom}_{\mathrm{ring}}\left(W(K_1), W(K_2)\right)$ respecting the dimension of quadratic forms and characterized by

$$\overline{\rho}(q) \equiv \rho(q) \mod \left(I(K_2)\right)^2$$

for all $q \in W(K_1)$. It might be interesting to check if such properties hold for hermitian forms over a quaternion division algebra.
(4) In Assertion (1) of Theorem 3.9, the fact that $\Psi(q.h) = \Phi(q).\Psi(h)$, for all $q \in W(K_1)$, $h \in W(Q_1, \gamma_1)$ is equivalent to the commutativity of the following diagram (with exact rows)

$$\begin{tikzcd}
W(K_1) \arrow[r, "\rho_1"] \arrow[d, "\Phi"] & W(Q_1, \gamma_1) \arrow[r] \arrow[d, "\Psi"] & 0 \\
W(K_2) \arrow[r, "\rho_2"] & W(Q_2, \gamma_2) \arrow[r] & 0
\end{tikzcd}$$

where $\rho_i : W(K_i) \to W(Q_i, \gamma_i)$ is the scalar extension map, for $i = 1,\ 2$.

4 Reciprocity equivalence

In this Section we recall some basic results about global fields and reciprocity equivalence: we refer to [15] for more details about quadratic forms over global fields and to [16] for a complete treatment of reciprocity equivalence. Finally, we define the notion of quadratic reciprocity equivalence and prove Theorem 1.5.

4.1 Preliminaries

A *global field* is either an algebraic number field (i.e. a finite field extension of $\mathbb{Q}$) or an algebraic function field (i.e. a finite field extension of a field of the form $\mathbb{F}_q(X)$ where $\mathbb{F}_q$ is the finite field with q elements for some prime power q and X is an indeterminate).

Let K be a global field, P be a nontrivial place of K and K_P be a completion of K at P. If P is non archimedian then P is discrete and K_P is a local field whose residue field is finite: we say that P is a *finite place*. If P is archimedian then there is a topological isomorphism between K_P and $\mathbb{R}$ or $\mathbb{C}$: the place is called *real* in the first case, *complex* in the second case. An archimedian place is also called an *infinite place*. If K is an algebraic function field, every place over K is finite and if K is an algebraic number field, there is a finite number of archimedian places (more precisely $[K:\mathbb{Q}] = 2r + s$ where r is the number of complex places and s is the number of real places). We introduce the following notations:

$$\begin{aligned}
\Omega_K &= \{\text{nontrivial places of } K\},\\
\Omega_K^f &= \{\text{finite places of } K\},\\
\Omega_K^\infty &= \{\text{infinite places of } K\},\\
\Omega_K^r &= \{\text{real places of } K\} = \{P \in \Omega_K^\infty \mid K_P \simeq \mathbb{R}\},\\
\Omega_K^c &= \{\text{complex places of } K\} = \{P \in \Omega_K^\infty \mid K_P \simeq \mathbb{C}\}.
\end{aligned}$$

So we have the following disjoint unions: $\Omega_K = \Omega_K^f \cup \Omega_K^\infty$ and $\Omega_K^\infty = \Omega_K^r \cup \Omega_K^c$.

Let K be a global field and suppose $P \in \Omega_K^r$. Then, there is a topological isomorphism $\phi : K_P \simeq \mathbb{R}$. Via ϕ, K_P is an ordered field, real closed and euclidian, with unique ordering ${K_P}^2$ (see [17, Chapter 3, Theorem 1.1.4]). We thus say that an element $a \in K^*$ is positive (resp. negative) at P if $a \in {K_P^*}^2$ (resp. $a \in -{K_P^*}^2$), and we write $a \underset{\mathrm{P}}{>} 0$ (resp. $a \underset{\mathrm{P}}{<} 0$). If $a \in K^*$, we introduce the notation

$$\Omega_K^a = \{P \in \Omega_K^r \mid a \underset{\mathrm{P}}{<} 0\}$$

to denote the set of real places at which a is negative.

The following is immediate:

Lemma 4.1. *Let $P \in \Omega_K^r$ and q be a quadratic form over K. Then q is hyperbolic over K_P if and only if* $\mathrm{sgn}_P(q) = 0$ *where sgn_P denotes the usual signature at P (see [17, Chapter 2, §4]).*

It is easy to calculate the signature of a 2-fold Pfister form $\langle\langle x, y\rangle\rangle$ at a real place P:

Table 4.2.

x	y	xy	$\mathrm{sgn}_P(\langle\langle x, y\rangle\rangle)$
$+$	$+$	$+$	0
$-$	$+$	$-$	0
$+$	$-$	$-$	0
$-$	$-$	$+$	4

Finally, we get the following result which will be useful in Subsection 4.3:

Lemma 4.3. *Let K be a global field and $P \in \Omega_K^r$. A n-fold Pfister form $q = \langle\langle a_1, \cdots, a_n\rangle\rangle$ is anisotropic over K_P if and only if $a_i \underset{\mathrm{P}}{<} 0$ for $i = 1, \cdots, n$.*

Proof. The fact that $a_i \underset{\mathrm{P}}{<} 0$ for all $i = 1, \cdots, n$ is equivalent to $\mathrm{sgn}_P(q) = 2^n$ by mean of Table 4.2 and an induction on n. This property is also equivalent to the fact that q is anisotropic over K_P by Lemma 4.1, a Pfister form being either anisotropic or hyperbolic. □

4.2 Reciprocity equivalence

Throughout this Subsection, K_1 and K_2 will denote global fields of characteristic different from 2. The notion of reciprocity equivalence between such fields has been defined in [16, §1]:

Definition 4.4. A *reciprocity equivalence* between K_1 and K_2 is a pair of maps (t,T), where t is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ and T is a bijection $T : \Omega_{K_1} \to \Omega_{K_2}$ such that (t,T) respects Hilbert symbols, i.e.

$$(x,y)_P = (tx,ty)_{TP}$$

for all $x,\ y \in {K_1}^*/{K_1}^{*2}$ and for all $P \in \Omega_{K_1}$.

Remark 4.5. As $(x,y)_P = 1$ if and only if the 2-fold Pfister form $\langle\langle x,y\rangle\rangle$ is hyperbolic over $(K_1)_P$, one can replace the condition concerning Hilbert symbols in Definition 4.4 by $\langle\langle x,y\rangle\rangle$ being hyperbolic over $(K_1)_P$ if and only if the quadratic form $\langle\langle t(x),t(y)\rangle\rangle$ is hyperbolic over $(K_2)_{T(P)}$.

The main Theorem of [16] says that

Theorem 4.6. *K_1 and K_2 are Witt equivalent if and only if they are reciprocity equivalent.*

The proof of the "if"-part of this Theorem very much relies on Harrison's criterion 1.1. The proof of the converse is much more difficult and is based on a description of the 2-torsion of the Brauer group by the set of subsets of $\Omega_{K_1}\backslash\Omega^c_{K_1}$ of even order endowed with symmetric difference. We refer to[16, §3, 4] for more details.

4.3 Quadratic Reciprocity equivalence

The purpose of this Subsection is to give a proof of Theorem 1.5. Throughout, K_1 (resp. K_2) will denote a global field of characteristic different from 2, $L_1 = K_1(\sqrt{a})$ (resp. $L_2 = K_2(\sqrt{b})$) will denote a quadratic field extension of K_1 (resp. K_2) and σ_1 (resp. σ_2) will denote the nontrivial automorphism of L_1 (resp. L_2).

Definition 4.7. An (a,b)-*quadratic reciprocity equivalence* between K_1 and K_2 is a pair of maps (t,T) where t is a group isomorphism $t : {K_1}^*/N_{L_1/K_1}({L_1}^*) \to {K_2}^*/N_{L_2/K_2}({L_2}^*)$ with $t(-1) = -1$ and where T is a bijection $T : \Omega^a_{K_1} \to \Omega^b_{K_2}$ such that the quadratic form $\langle\langle a,x,y\rangle\rangle$ is hyperbolic over $(K_1)_P$ if and only if the quadratic form $\langle\langle b,t(x),t(y)\rangle\rangle$ is hyperbolic over $(K_2)_{T(P)}$ for all $x,\ y \in {K_1}^*/N_{L_1/K_1}({L_1}^*)$ and for all $P \in \Omega^a_{K_1}$.

We now prove Theorem 1.5:

Theorem *The following are equivalent:*
(1) $W(L_1,\sigma_1) \simeq W(L_2,\sigma_2)$ *as rings.*
(2) *There is an (a,b)-quadratic reciprocity equivalence between K_1 and K_2.*

Proof. (2) $\Rightarrow$ (1) : by Theorem 1.3, it suffices to show that the quadratic form $\langle\langle a,x,y\rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle b,t(x),t(y)\rangle\rangle$ is hyperbolic over K_2. Note that, for all $P \in \Omega_{K_1}\backslash\Omega^a_{K_1}$ (resp. for all $Q \in \Omega_{K_2}\backslash\Omega^b_{K_2}$) and for all $x,\ y \in {K_1}^*$, the quadratic form $\langle\langle a,x,y\rangle\rangle$ (resp. $\langle\langle b,t(x),t(y)\rangle\rangle$) is hyperbolic over $(K_1)_P$ (resp. over $(K_2)_Q$). This fact is obvious if the place is complex or if $a \underset{P}{>} 0$ (resp. if $b \underset{Q}{>} 0$). If P (resp. Q) is finite, it comes from the fact that $(K_1)_P$ (resp. $(K_2)_Q$) is a local field with $u\big((K_1)_P\big) = 4 = u\big((K_2)_Q\big)$ (see [11, Chapter XI, Example 6.2(4)]). Suppose now that the quadratic form $\langle\langle a, x, y\rangle\rangle$ is hyperbolic over K_1. Then $\phi = \langle\langle b, t(x), t(y)\rangle\rangle$ is hyperbolic over $(K_2)_Q$ for all $Q \in \Omega^b_{K_2}$ hence ϕ is hyperbolic over K_2 by the Hasse-Minkowski-Principle (see [11, Chapter VI, Hasse-Minkowski-Principle 3.1]). The converse is similar.

(1) $\Rightarrow$ (2) : by Theorem 1.3, we only have to find the map T of Definition 4.7.

We first show that K_1 and K_2 have the same number of distinct real places for which a and b are negative. First of all, K_1 has no real place where a is negative if and only if K_2 has no real place where b is negative (this is a consequence of Theorem 1.3, Lemma 4.3 and the Hasse-Minkowski Principle applied to $\langle\langle a, -1, -1\rangle\rangle$ and to $\langle\langle b, t(-1), t(-1)\rangle\rangle = \langle\langle b, -1, -1\rangle\rangle$). In this case, (for example, if K_1 and K_2 are algebraic function fields), there is nothing to prove. Suppose now that $\Omega^a_{K_1} = \{P_1, \cdots, P_n\}$, $n \geq 1$. As K_1 is a global field which has at least one real place, K_1 is an SAP field (see [13, §2]). As a consequence, there exist $x_1, \cdots, x_n \in {K_1}^*$ such that

$$x_i \underset{P_i}{>} 0, \quad x_i \underset{P_j}{<} 0 \text{ for all } j \neq i,$$

for $i = 1, \cdots, n$. On the one hand, it is easy to see that for each i,

$$(-1)^n x_1 \cdots x_{i-1} x_{i+1} \cdots x_n \underset{P_i}{<} 0.$$

This implies, by Lemma 4.3, that the quadratic form $\langle\langle a, -1, (-1)^n x_1 \cdots x_{i-1} x_{i+1} \cdots x_n\rangle\rangle$ is anisotropic over K_1. Consequently, the quadratic form

$$\langle\langle b, -1, (-1)^n t(x_1) \cdots t(x_{i-1}) t(x_{i+1}) \cdots t(x_n)\rangle\rangle$$

is anisotropic over K_2 and there is a place Q_i (which has to be real) such that the quadratic form $\langle\langle b, -1, (-1)^n t(x_1) \cdots t(x_{i-1}) t(x_{i+1}) \cdots t(x_n)\rangle\rangle$ is anisotropic over $(K_2)_{Q_i}$ by the Hasse-Minkowski-Principle. Applying Lemma 4.3, we see that $b \underset{Q_i}{<} 0$ and

$$(-1)^n t(x_1) \cdots t(x_{i-1}) t(x_{i+1}) \cdots t(x_n) \underset{Q_i}{<} 0, \tag{3}$$

for all $i = 1, \cdots, n$. On the other hand, one also has

$$(-1)^n x_1 \cdots x_n \underset{P_i}{<} 0$$

for all $i = 1, \cdots, n$. This implies that the quadratic form $\langle\langle a, -1, (-1)^n x_1 \cdots x_n\rangle\rangle$ is anisotropic over K_1 and that $\langle\langle b, -1, (-1)^n t(x_1) \cdots t(x_n)\rangle\rangle$ is anisotropic over K_2. By the Hasse-Minkowski-Principle,

$$(-1)^n t(x_1) \cdots t(x_n) \underset{Q_i}{<} 0, \tag{4}$$

for all $i = 1, \cdots, n$. By combining (3) and (4), we get that $t(x_i) \underset{Q_i}{>} 0$ for all $i = 1, \cdots, n$. We now show that the n places Q_i are distinct. If conversely, there are $i \neq j$ such that $Q = Q_i = Q_j$ then

$$t(x_i) \underset{Q}{>} 0, \quad t(x_j) \underset{Q}{>} 0$$

and the quadratic form $\langle\langle b, t(-x_i), t(-x_j)\rangle\rangle$ is anisotropic over K_2. This implies that the quadratic form $\langle\langle a, -x_i, -x_j\rangle\rangle$ is anisotropic over K_1. By the Hasse-Minkowski-Principle and Lemma 4.3, there is a place $P \in \Omega^a_{K_1}$ for which

$$x_i \underset{P}{>} 0, \quad x_j \underset{P}{>} 0.$$

That is impossible, by construction of the x_i's. We have thus shown that there is an injection $T : \Omega^a_{K_1} \to \Omega^b_{K_2}$. By Symmetry, this is a bijection.

Finally, suppose that $\langle\langle b, t(x), t(y)\rangle\rangle$ is anisotropic over $(K_2)_{Q_i}$. By Lemma 4.3, $t(x) \underset{Q_i}{<} 0$ and $t(y) \underset{Q_i}{<} 0$. As $t(x_i) \underset{Q_i}{>} 0$ by definition, the quadratic forms $\langle\langle b, t(-x_i), t(x)\rangle\rangle$ and $\langle\langle b, t(-x_i), t(y)\rangle\rangle$ are anisotropic over $(K_2)_{Q_i}$, hence over K_2. As a consequence, the quadratic forms $\langle\langle a, -x_i, x\rangle\rangle$ and $\langle\langle a, -x_i, y\rangle\rangle$ are anisotropic over K_1. By the Hasse-Minkowski-Principle, there are $P,\ R \in \Omega^a_{K_1}$ such that

$$x_i \underset{P}{>} 0,\ x \underset{P}{<} 0 \text{ et } x_i \underset{R}{>} 0,\ y \underset{R}{<} 0.$$

Thus $P = R = P_i$ and the quadratic form $\langle\langle a, x, y\rangle\rangle$ is anisotropic over $(K_1)_{P_i}$. The converse is similar. $\square$

Remarks 4.8. (1) By means of Remark 4.5, one can say that Theorem 4.6 and Theorem 1.5 are analogous. There are also some major differences between the two results. In the proof of Theorem 4.6, the most important places are the finite places whereas in Theorem 1.5, they do not play a part, the completion of a field at such places being local with u-invariant 4. Besides, in Theorem 1.5, it seems that we have to suppose that $t(-1) = -1$ which is not the case in Theorem 4.6. In fact, in this last case, this condition comes from the "Global Square Theorem" (see [11, Chapter VI, Theorem 3.7]) which has no analogue (as far as we know) in our situation.
(2) While proving (1) $\Rightarrow$ (2) in Theorem 1.5, instead of saying that a certain field is SAP, we could have used the Strong Approximation Theorem for global fields: see [15, Chapter II, Theorem 2.1.2].
(3) In the case of algebraic function fields, the ring isomorphism $W(L_1, \sigma_1) \simeq W(L_2, \sigma_2)$ is equivalent to having an isomorphism ${K_1}^*/N_{L_1/K_1}({L_1}^*) \simeq K_2^*/N_{L_2/K_2}({L_2}^*)$ sending -1 to -1.

Similarly, we can prove:

Theorem 4.9. *Let K be a global field of characteristic different from 2. Let $Q_1 = (a, b)_K$ (resp. $Q_2 = (c, d)_K$) be a quaternion algebra over K endowed with its canonical involution γ_1 (resp. γ_2). For $\alpha,\ \beta \in K^*$, denote by $\Omega^{(\alpha,\beta)}_K$ the set of real places at which α and β are negative. Then, the following are equivalent:*
(1) $W(Q_1, \gamma_1) \simeq W(Q_2, \gamma_2)$ as $W(K)$-modules.
(2) There exists a pair of maps (t, T) where t is a group isomorphism $t : K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*) \simeq K^*/\mathrm{Nrd}_{Q_2/K}({Q_2}^*)$ with $t(-1) = -1$ and where T is a bijection $T : \Omega^{(a,b)}_K \to \Omega^{(c,d)}_K$ such that the quadratic form $\langle\langle a, b, x, y\rangle\rangle$ is hyperbolic over K_P if and only if the quadratic form $\langle\langle c, d, t(x), t(y)\rangle\rangle$ is hyperbolic over $K_{T(P)}$ for all $x,\ y \in K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*)$ and for all $P \in \Omega^{(a,b)}_K$.

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