NONTRIVIALITY OF $\mathrm{NK}_1(D)$ FOR DIVISION ALGEBRAS

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Abstract. A field E is said to be NKNT if for any noncommutative division algebra D finite dimensional over $E \subseteq Z(D) = F$ with index $\mathrm{ind}(D)$, $\mathrm{Nrd}(D^*)/F^{*\mathrm{ind}(D)}$ is nontrivial. It is proved that if E is a field finitely generated but not algebraic over some subfield then E is NKNT. As a consequence, if $F = Z(D)$ is finitely generated over its prime subfield or over an algebraically closed field, then $\mathrm{CK}_1(D) = \mathrm{Coker}(K_1F \to K_1D)$ is nontrivial.

Let D be a division algebra over its center F of index n. Denote by D^* and F^* the multiplicative group of D and F respectively. Let $\mathrm{Nrd}_D\colon D^* \to F^*$ be the reduced norm map, $D^{(1)}$ the kernel of this map and D' the commutator subgroup of D^*. The inclusion map $F \hookrightarrow D$ induces a homomorphism $K_1(F) = F^* \to K_1(D) = D^*/D'$. Consider the group

$$\mathrm{CK}_1(D) = \mathrm{Coker}(K_1F \to K_1D) \cong D^*/F^*D'\,.$$

Since $x^{-n}\mathrm{Nrd}(x) \in D^{(1)}$ and the reduced Whitehead group $\mathrm{SK}_1(D) = D^{(1)}/D'$ is n-torsion (by [3], p. 157, Lemma 2), it follows that $\mathrm{CK}_1(D)$ is an abelian group of bounded exponent n^2 (In fact one can show that the bound is n, see the proof of Lemma 4, p. 154 in [3] or pp. 579–580 in [5]). Thus, by the Prüfer-Baer theorem $\mathrm{CK}_1(D) \cong \bigoplus \mathbb{Z}_{k_i}$ where each $k_i \mid n$ (see [11], p. 105). Therefore if $\mathrm{CK}_1(D)$ is nontrivial then D^* has a (normal) maximal subgroup. The question of whether D^* has a maximal subgroup seems to remain open and thus by the above observation is limited to the case when $\mathrm{CK}_1(D)$ is trivial. In [4], Th. 2.12 it was proved that if D is a tensor product of cyclic algebras then $\mathrm{CK}_1(D)$ is trivial if and only if D is a quaternion division algebra $(\frac{-1,-1}{F})$ where F is a real Pythagorean field (see also [8]). It has been conjectured in [6] that if $\mathrm{CK}_1(D)$ is trivial then D is a quaternion division algebra.

The group CK_1 has been computed in [5] for certain division algebras, and its connection with SK_1 was also studied. But, $\mathrm{CK}_1(D)$ is often difficult to work with. We will focus here on a related invariant, $\mathrm{NK}_1(D)$, which is sometimes more tractable, and can yield information about $\mathrm{CK}_1(D)$. Define

$$\mathrm{NK}_1(D) \;=\; D^*/F^*D^{(1)} \;\cong\; \mathrm{Nrd}(D^*)/F^{*\mathrm{ind}(D)}$$

(with the isomorphism given by the reduced norm map). Observe that $\mathrm{NK}_1(D)$ is a homomorphic image of $\mathrm{CK}_1(D)$ and that whenever $\mathrm{SK}_1(D) = 1$, we have $\mathrm{CK}_1(D) = \mathrm{NK}_1(D)$. (Recall that $\mathrm{SK}_1(D) = 1$ whenever $\mathrm{ind}(D)$ is square-free, or the center F of D is a local or a

Date: 31 jan 06.

global field, by [3], p. 164, Cor. 4, Th. 3, p. 165, (17), p. 166, (18).) For example, if $Q = (\frac{a,b}{F})$ is a quaternion division algebra with $\mathrm{char}(F) \neq 2$, we have

$$\mathrm{CK}_1(Q) = \mathrm{NK}_1(Q) \cong \big(\{r^2 - as^2 - bt^2 + abu^2 \mid r,s,t,u \in F\}\backslash\{0\}\big)/F^{*2}.$$

From this formula, it is immediate that $\mathrm{CK}_1(Q)$ is trivial iff F is a real Pythagorean field and $Q \cong (\frac{-1,-1}{F})$.

Observe that the condition that $\mathrm{NK}_1(D)$ be trivial for a noncommutative division ring D is an extremely strong one. Indeed, if $\mathrm{ind}(D) = d$ then $\mathrm{NK}_1(D) = 1$ iff $\mathrm{Nrd}_D(D^*) = F^{*d} = \mathrm{Nrd}_D(F^*)$, which holds iff for every maximal subfield L of F, $N_{L/F}(L^*) = F^{*d} = N_{L/F}(F^*)$. It was shown in [4] that if C is a noncommutative cyclic algebra with $\mathrm{NK}_1(C) = 1$, then $C \cong (\frac{-1,-1}{F})$ with F a Pythagorean field. We will show here that $\mathrm{NK}_1(D)$ is nontrivial for a great many other noncommutative division algebras D. Of course, whenever $\mathrm{NK}_1(D) \neq 1$, we also have $\mathrm{CK}_1(D) \neq 1$.

Definition. A field E is said to be NKNT (for NK_1 nontrivial) if for any noncommutative division algebra D finite dimensional over E (and not necessarily central over E), $\mathrm{NK}_1(D)$ is nontrivial.

It is clear from the definition that if a field E is NKNT then so is every finite degree field extension of E. Here are some examples of NKNT fields: Clearly a finite field is NKNT; so is any algebraically closed field; so also is any field of transcendence degree 1 over an algebraically closed field, by Tsen's Theorem. Since every division algebra over a global field is a cyclic algebra, the result quoted above shows that every global field is NKNT. Likewise, every nonreal local field is NKNT. However the field of real numbers $\mathbb{R}$ is not NKNT, since $\mathbb{R}$ is a real Pythagorean and thus $\mathrm{CK}_1(\mathbb{H}_\mathbb{R}) = 1$, but the rational function field $\mathbb{R}(t)$ is NKNT. This is a consequence of our main theorem below. But we can see it directly as follows: If L is a finite degree extension of $\mathbb{R}(t)$ and D is an L-central noncommutative division algebra, then by Tsen's Theorem, D is split by $L(\sqrt{-1})$, so D is a quaternion algebra; but $\mathrm{NK}_1(D)$ is then nontrivial because L is not Pythagorean.

In this note our main result is:

Theorem. *Let F be a field which is finitely generated but not algebraic over some subfield F_0. Then, F is* NKNT.

Let D be a division algebra with center F. In showing that $\mathrm{NK}_1(D)$ is nontrivial, Lemma 1 below allows us to reduce to the case where $\mathrm{ind}(D)$ is a prime power. Our arguments then divide into two cases depending on whether $\mathrm{ind}(D)$ is a power of $\mathrm{char}(F)$. Lemma 2 and its Corollary handle the first case:

Lemma 1. *Let $D_1, \dots, D_k$ be division algebras with center F such that gcd*$(\mathrm{ind}(D_i), \mathrm{ind}(D_j)) = 1$ *whenever $i \neq j$. Then,*

$$\mathrm{NK}_1(D_1 \otimes_F D_2 \otimes_F \dots \otimes_F D_k) \;\cong\; \mathrm{NK}_1(D_1) \times \dots \times \mathrm{NK}_1(D_k)\,.$$

Proof. It suffices by induction to prove the result for $k = 2$. This can be done the same way as the corresponding result for CK_1 was proved in [5], Th. 2.8. □

Lemma 2. *Let D be a noncommutative division algebra similar to a cyclic algebra A. In each of the following cases $\mathrm{NK}_1(D)$ is nontrivial:*

(1) *F contains a square root of -1;*
(2) *The characteristic of F is 2;*
(3) *The degree of A is odd.*

Proof. Since the primary components of D are similar to tensor powers of A which are similar to cyclic algebras, it suffices by Lemma 1 to consider the case when $\mathrm{ind}(D)$ is a power of a prime. Thus, assume $\mathrm{ind}(D) = p^c$, where p is a prime number and $c \geq 1$ and D is similar to a cyclic algebra $A = (E/F, \sigma, a)$. Choose A of minimal degree. Then, $\deg(A) = p^e$ and $a \notin F^{*p}$; for, if $a = b^p$, then A is Brauer equivalent to $(E_0/F, \sigma, b)$, where $[E : E_0] = p$, contradicting the minimality of $\deg(A)$.

Let $d = p^e = \deg(A)$. Let α be the standard generator of A with $\alpha^d = a$. Since the powers of α up to the d-th are part of a base of A over F, they are F-linearly independent. Therefore, the minimal polynomial of α over F is $x^d - a$. If M is any splitting field of A, then for $\alpha \otimes 1 \in A \otimes_F M$, the minimal polynomial of $\alpha \otimes 1$ over M is again $x^d - a$. Since the characteristic polynomial of $\alpha \otimes 1$ has degree d, this polynomial is also $x^d - a$. Hence, $\mathrm{Nrd}_A(\alpha) = \det(\alpha \otimes 1) = (-1)^{d-1} a$. So, if p is odd, or $\mathrm{char}(F) = 2$, or F contains a square root of -1, then $\mathrm{Nrd}_A(\alpha) \notin F^{*p}$. But thanks to the Dieudonné determinant, $\mathrm{Nrd}_A(A^*) = \mathrm{Nrd}_D(D^*)$. Thus $\mathrm{Nrd}_A(\alpha) \in \mathrm{Nrd}_D(D^*) \backslash F^{*p^e}$, so $\mathrm{NK}_1(D)$ is nontrivial. □

Recall that a p-algebra is a central simple algebra of degree a power of the prime p over a field of characteristic p. Albert's main theorem in the theory of p-algebras states that every p-algebra is similar to a cyclic p-algebra (see [1], p. 109, Th. 31). Combining this with the Lemma above, we obtain:

Corollary 3. *Let D be a noncommutative p-division algebra. Then $\mathrm{NK}_1(D)$ is nontrivial.*

Remark. Let $G(D) = D^* \big/ \mathrm{Nrd}(D^*) D'$, which is a bigger group than $\mathrm{CK}_1(D)$ in general. It is much easier to see that $G(D) \neq 1$ for every noncommutative p-division algebra D. Indeed, if $G(D) = 1$ then $\mathrm{Nrd}(D^*) = \mathrm{Nrd}(D)^{*p^n}$ where $\mathrm{ind}(D) = p^n$. So, for $F = Z(D)$,

$$F^{*p^n} \subseteq \mathrm{Nrd}(D^*) = \mathrm{Nrd}(D)^{*p^n} = \mathrm{Nrd}(D)^{*p^{2n}} \subseteq F^{*p^{2n}}.$$

Hence, $F^{*p^n} = F^{*p^{2n}}$. Since $\mathrm{char}(F) = p$, this implies $F^* = F^{*p}$, i.e., F has no proper purely inseparable extensions. But one knows by [1], p. 104, Th. 21, that any p-algebra has a purely inseparable splitting field; hence, $D = F$, a contradiction. (Compare this argument with [8], Th. 2).

In order to prove the main Theorem, we need two propositions.

Proposition 4. *Let $F \subseteq L$ be fields with $[L:F] = d < \infty$ such that $N_{L/F}(L^*) = F^{*d}$. If V is a discrete valuation ring of F with residue field $\overline{V}$, and if the integral closure of V in L is a finite V-module, then V has a unique extension to a DVR W of L, and $[\overline{W}:\overline{V}] = [L:F]$.*

Proof. Let v be the normalized discrete valuation on F corresponding to the valuation ring V. "Normalized" means that the value group $v(F^*) = \mathbb{Z}$. Let $v_1, \ldots, v_s$ be all the (inequivalent) extensions of v to L. Since v is discrete and the integral closure of V in L is a finite V-module, we have $\sum_{i=1}^{s} e_i f_i = [L:F]$, where e_i is the ramification index of v_i/v and f_i is the residue degree of v_i/v ([2], VI, §8.3, Cor. 3). Since v_i extends v, the value group of v_i is $\frac{1}{e_i}\mathbb{Z}$. For any $x \in L$, we have

$$v(N_{L/F}(x)) = \sum_{i=1}^{s} e_i f_i v_i(x), \tag{1}$$

by, [2], VI, §8.5, Cor. 2. Now by the Approximation Theorem ([2], VI, §7.2, Cor. 1) one can choose $x \in L$ such that $v_1(x) = 1/e_1$ and $v_i(x) = 0$ for all $i > 1$. Thus by (1), $v(N_{L/F}(x)) = f_1$. But since $N_{L/F}(x) \in F^{*d}$, we must have $d \mid v(N_{L/F}(x)) = f_1 \leq d$. This, combined with $\sum_{i=1}^{s} e_i f_i = d$ with all $e_i \geq 1$ and $f_i \geq 1$, forces $f_1 = d = [L:F]$ and $s = 1$, as desired. □

Using Proposition 4, we obtain the following Theorem which provides a further class of fields with the NKNT property which is not covered by the main Theorem 7. For example, it shows that if F is NKNT, then so is the Laurent power series field $F((x))$.

Theorem 5. *Let F be a discrete valued field with residue field $\overline{F}$ such that* $\operatorname{char}(F) = \operatorname{char}(\overline{F})$. *If $\overline{F}$ is* NKNT, *then so is F.*

Proof. Suppose there is a noncommutative division algebra D finite dimensional over F with center K such that $\mathrm{NK}_1(D) = 1$. We can assume $K = F$. Since by Lemma 1, NK_1 respects the primary decomposition of D, it is enough to consider the case when $\operatorname{ind}(D) = p^k$, where p is prime and $k \geq 1$. If $\operatorname{char}(F) = p$, then D is a p-algebra and by Corollary 3, $\mathrm{NK}_1(D)$ is nontrivial. Thus we may assume that $\operatorname{char}(F) \neq p$. Hence, every subfield of D containing F is separable over F.

Let $d = p^k = \operatorname{ind}(D)$. Since $\mathrm{NK}_1(D) = 1$, we have $N_{L/F}(L^*) = F^{*d}$ for every maximal subfield L of D. Since L is separable over F, the integral closure in L of the discrete valuation ring V_F of v on F is a finitely generated V_F-module. Thus by Proposition 4, v extends uniquely to any maximal subfield of D, with no ramification. So, v extends uniquely to any subfield of D. By the theorem of Ershov-Wadsworth (see [13], Th. 2.1 or [12]), it follows that v extends to a valuation on D, which is denoted again by v. Furthermore D is not ramified over F, i.e. the value group Γ_D of D coincides with the value group Γ_F of F. Let $\overline{D}$ and $\overline{F}$ be the residue division algebra and the residue field of the valuations on D and F. Since $\operatorname{char}(\overline{F}) = \operatorname{char}(F)$ does not divide $\operatorname{ind}(D)$, the Ostrowski theorem for valued division algebras, [9] Th. 3, yields $[D:F] = [\overline{D}:\overline{F}]\,\big|\Gamma_D : \Gamma_F\big| = [\overline{D}:\overline{F}]$. Note also that $[Z(\overline{D}):\overline{F}] \,\big|\, [D:F]$; hence, $Z(\overline{D})$ is separable over $\overline{F}$. Thus, the surjectivity of the fundamental homomorphism $\Gamma_D/\Gamma_F \to \mathrm{Gal}(Z(\overline{D})/\overline{F})$, together with the fact that $Z(\overline{D})$ is normal and separable over $\overline{F}$ and $\Gamma_D/\Gamma_F = 1$ force that $Z(\overline{D}) = \overline{F}$ (see [13], Prop. 2.5 or [7], Prop. 1.7). Hence, $\mathrm{ind}(\overline{D}) = \mathrm{ind}(D)$.

Since $\mathrm{NK}_1(\overline{D}) \neq 1$ and $\mathrm{ind}(\overline{D}) = \mathrm{ind}(D) = d$, there is $\overline{a} \in \overline{D}^*$ with $\mathrm{Nrd}_{\overline{D}}(\overline{a}) \notin \overline{F}^d$. Let a be any inverse image of $\overline{a}$ in the valuation ring V_D of D, and let L be any maximal subfield of D containing a. Let V_L be the valuation ring of the restriction of v to L, and let $\overline{L}$ be the residue field of V_L. Because V_L is the unique extension of V_F to L, V_L is the integral closure of V_F in L; hence, it is a finitely-generated V_F-module. Since $[\overline{D}:\overline{F}] = [D:F]$, we must have $[\overline{L}:\overline{F}] = [L:F]$, showing that $\overline{L}$ is a maximal subfield of $\overline{D}$. If $\overline{b}_1, \ldots, \overline{b}_d$ are any $\overline{F}$-vector space base of $\overline{L}$, then any inverse images $b_1, \ldots, b_d$ of the $\overline{b}_i$ in the valuation ring V_L form a base of V_L as a free V_F-module. (The b_i generate V_L over V_F by Nakayama's Lemma, and they are V_F-independent because V_F is a valuation ring and the $\overline{b}_i$ are $\overline{F}$-independent.) By computing the norm $N_{L/F}(a)$ as the determinant of the F-linear map multiplication by a using the base $b_1, \ldots, b_d$, we obtain $N_{L/F}(a) \in V_F$ and

$$\overline{N_{L/F}(a)} = N_{\overline{L}/\overline{F}}(\overline{a}) \text{ in } \overline{F}\,.$$

Because we have assumed $\mathrm{NK}_1(D) = 1$, we have

$$N_{L/F}(a) = \mathrm{Nrd}_D(a) \in F^{*d} \cap V_F = V_F^d\,.$$

Hence,

$$\mathrm{Nrd}_{\overline{D}}(\overline{a}) = N_{\overline{L}/\overline{F}}(\overline{a}) = \overline{N_{L/F}(a)} \in \overline{V_F^d} = \overline{F}^d\,,$$

contradicting the choice of $\overline{a}$. So, $\mathrm{NK}_1(D) \neq 1$, contradicting the choice of D. Thus, F is NKNT. □

Proposition 6. *Let $F \subseteq F(t) \subseteq L$ be fields with t transcendental over F and $[L:F(t)] < \infty$. If $L = F(t)(\alpha)$ for some α, then there is a discrete valuation ring V of $F(t)$ with $F \subseteq V$ such that V has an extension to a DVR W of L such that $\overline{W} = \overline{V}$. (In fact, there are infinitely many such V.)*

Proof. Let $R = F[t]$. We can assume that α is integral over R. Let $f = x^n + c_{n-1}x^{n-1} + \ldots + c_0$ be the minimal polynomial of α over $F(t)$. The integrality of α over R (with R integrally closed) assures that $f \in R[x]$. Let

$$\mathcal{P}_f \;=\; \{\pi \in F[t] \mid \pi \text{ is irreducible and monic in } F[t] \text{ and } \pi | f(r) \text{ for some } r \in R\}.$$

We will show that $|\mathcal{P}_f| = \infty$. Assume first that $c_0 = 1$, and write $f = xh(x)+1$ with $h \in R[x]$ and $\deg(h) = n-1$. Suppose $|\mathcal{P}_f| = \{\pi_1, \ldots, \pi_k\}$. Let $s = t\pi_1 \ldots \pi_k$. Since h has only finitely many roots in R, there is a natural number ℓ with $h(s^\ell) \neq 0$. Then $f(s^\ell) = s^\ell h(s^\ell) + 1$ has positive degree in t, so is not a unit of R. If p is an irreducible monic factor of $f(s^\ell)$, then $p \in \mathcal{P}_f$, but $p \nmid s$, so $p \notin \{\pi_1, \ldots, \pi_k\}$, a contradiction. Hence $\mathcal{P}_f$ cannot be finite if $c_0 = 1$.

Now assume $c_0 \neq 1$. Let $f(c_0x) = c_0 g(x)$. So $g \in R[x]$ with $\deg(g) = \deg(f) \geq 1$, and g has constant term 1. By the previous case, $|\mathcal{P}_g| = \infty$. But since$f(c_0r) = c_0g(r)$, we have $\mathcal{P}_g \subseteq \mathcal{P}_f$. So, $|\mathcal{P}_f| = \infty$, as claimed.

Now, take any $\pi \in P_f$, and let V be the DVR $R_{(\pi)}$ which is the localization of R at its prime ideal (π). Let $M = \pi V$, which is the maximal ideal of V; so $\overline{V} = V/M$. Assume first that the ring $V[\alpha]$ is integrally closed.

Since $\pi | f(r)$ for some $r \in R$, the image $\overline{f}$ of f in $\overline{V}[x]$ has a root $\overline{r}$ in $\overline{V}$. Note that $fF(t)[x] \cap V[x] = fV[x]$, by the division algorithm as f is monic in $V[x]$. Hence, $V[\alpha] \cong V[x]/fV[x]$ and

$$V[\alpha]/MV[\alpha] \;\cong\; V[x]/(f, M) \;\cong\; \overline{V}[x]/(\overline{f}). \tag{2}$$

Because $\overline{f}(\overline{r}) = 0, x - \overline{r}$ is an irreducible factor of $\overline{f}$ in $\overline{V}[x]$. Let N be the maximal ideal of $V[\alpha]$ containing $MV[\alpha]$ corresponding to $(x - \overline{r})/(f)$ in $\overline{V}[x]/(\overline{f})$ in the isomorphism given by (2). Let W be the localization $V[\alpha]_N$. Then W is a DVR, as $V[\alpha]$ is the integral closure of V in L. Furthermore, $W \cap F(t) = V$ and $\overline{W} \cong V[\alpha]/N \cong \overline{V}[x]/(x - \overline{r}) \cong \overline{V}$. Thus, the desired W exists for $V = R_{(\pi)}$ whenever $\pi \in \mathcal{P}_f$ and $R_{(\pi)}[\alpha]$ is integrally closed.

To complete the proof we show that the needed integral closure property of $R_{(\pi)}[\alpha]$ occurs for all but finitely many $\pi \in \mathcal{P}_f$. Let T be the integral closure of R in L; so T is a finitely generated R-module ([2], V, §3.2, Th. 2). We have $R[\alpha] \subseteq T$, and T and $R[\alpha]$ each have quotient field L. So, $T/R[\alpha]$ is a finitely generated torsion R-module; hence it has nonzero annihilator in R. Therefore, there is $b \in R$ with $b \neq 0$ and $bT \subseteq R[\alpha]$. Hence, $R[\alpha][1/b] = T[1/b]$, which is integrally closed. For any monic irreducible $\pi \in R$, if $\pi \nmid b$ then the DVR $R_{(\pi)}$ is a localization of $R[1/b]$. Hence, $R_{(\pi)}[\alpha]$ is a localization of $R[1/b][\alpha]$, so $R_{(\pi)}[\alpha]$ is integrally closed. There are only finitely many monic irreducibles of R dividing b. For all other π in the infinite set $\mathcal{P}_f$, we have $R_{(\pi)}(\alpha)$ is integrally closed. □

Remark. For the result of Prop. 6, it is not sufficient to assume that $[L\!:\!F(t)] < \infty$. For example, suppose $\mathrm{char}(F) = p \neq 0$ and $[F^{1/p}\!:\!F] \geq p^2$. Take any field K with $F \subseteq K \subseteq F^{1/p}$ and $p^2 \leq [K\!:\!F] < \infty$, and let $L = K(t)$. Take any discrete valuation ring V of $F(t)$ and any extension of V to a DVR W of L. Identify $\overline{V}$ and K with their canonical images in $\overline{W}$. Since $\overline{V} = F(\beta)$ for some β, the Theorem of the Primitive Element shows that $\overline{V} \cap K = F(\gamma)$, for some $\gamma \in K$. Since $\gamma^p \in F$, we have $[F(\gamma)\!:\!F] \leq p < [K\!:\!F]$, so $\overline{V}$ doesn't contain all of K. Because $K \subseteq \overline{W}$, this shows that $\overline{W} \neq \overline{V}$.

Theorem 7. *Let F be a field which is finitely generated but not algebraic over some subfield F_0. Then, F is NKNT.*

Proof. We need to show that for each finite degree extension field K of F and each noncommutative finite dimensional division algebra D with center K, we have $\mathrm{NK}_1(D)$ is nontrivial. As in the proof of Theorem 5, we can assume that $K = F$ and that F is a finite degree extension of $F_0(t)$, with t transcendental over F_0. Since NK_1 respects the primary decomposition of D, by Corollary 3 it suffices to consider the case where $\mathrm{ind}(D) = p^k$, where p is a prime number with $p \neq \mathrm{char}(F_0)$.

Let L be any maximal subfield of D and let S be the separable closure of $F_0(t)$ in L. Then, $S = F_0(t)(\alpha)$ for some α. By Proposition 6, applied to the field extension $F_0(t) \subseteq S$, there is a DVR V of $F_0(t)$ (with $F_0 \subseteq V$) which has an extension to a DVR W of S with $\overline{V} = \overline{W}$. Because L is purely inseparable over S, W has a unique extension to a DVR Y of L, and $\overline{Y}$ is purely inseparable over $\overline{W}$. Let $Z = Y \cap F$, which is a DVR of F. Since $\overline{W} = \overline{V} \subseteq \overline{Z} \subseteq \overline{Y}$, we have $\overline{Y}$ is purely inseparable over $\overline{Z}$. If $\mathrm{char}(F_0) = 0$, it follows that $\overline{Y} = \overline{Z}$; hence $[\overline{Y} : \overline{Z}] = 1 \neq p^k = [L : F]$. If $\mathrm{char}(F_0) = q \neq 0$, then $[\overline{Y} : \overline{Z}] = q^\ell$ for some $\ell \geq 0$. Since $q \neq p$ by hypothesis, we again have $[\overline{Y} : \overline{Z}] \neq [L : F]$.

Let V_F (resp. V_L) be the integral closure of V in F (resp. L), and let Z_L be the integral closure of Z in L. Because the integral closure of $F_0[t]$ in F (resp. in L) is a finitely generated $F_0[t]$-module, by [2], V, §3.2, Th. 2, and V is a localization of $F_0[t]$ (or $F_0[t^{-1}]$), V_F and V_L are finitely generated V-modules, so V_L is a finitely generated V_F-module. Then, as Z is a localization of V_F, Z_L is a finitely generated Z-module. Since the conclusion of Proposition 4 fails for $Z \subseteq Y$ in the field extension $F \subseteq L$, we have $F^{*p^k} \subsetneqq N_{L/K}(L^*) \subseteq \mathrm{Nrd}(D^*)$, showing that $\mathrm{NK}_1(D)$ is nontrivial. □

Corollary 8. *If D is a noncommutative division algebra whose center is finitely generated over its prime field or over an algebraically closed field, then* $\mathrm{NK}_1(D) \neq 1$. *Hence,* $\mathrm{CK}_1(D) \neq 1$ *and D^* contains a maximal proper normal subgroup.*

Proof. This is immediate from the Theorem and the comments in the introduction. □

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