The Witt groups of the spheres away from two

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Abstract

We calculate the Witt groups of the spheres up to 2-primary torsion.

1 Introduction

Let A be a Gorenstein ring of finite Krull dimension with $\frac{1}{2} \in A$. Denote the coordinate ring of the n-sphere over A by

$$S^n_A := A[X_1, \ldots, X_{n+1}]/(X_1^2 + \cdots + X_{n+1}^2 - 1),$$

and let

$$\mathbb{P}S^n_A := \operatorname{Proj} A[X_0, \ldots, X_{n+1}]/(X_0^2 + \cdots + X_n^2 - X_{n+1}^2)$$

be the corresponding projective scheme.

We are interested in determining the (total) coherent Witt groups of these varieties, in terms of the Witt groups of the base A. As a first step we obtain the following two theorems, but we hope to determine soon also the 2-primary torsion. We abbreviate:

$$\overline{\mathrm{W}}^i(X) := \tilde{\mathrm{W}}^i(X) \otimes \mathbb{Z}[1/2], \qquad \overline{\mathrm{W}}^{\mathrm{tot}}(X) := \bigoplus_{0 \leq i \leq 3} \overline{\mathrm{W}}^i(X).$$

Theorem 1.1. *We have isomorphisms*

$$\overline{\mathrm{W}}^i(S^n_A) \simeq \overline{\mathrm{W}}^i(\mathbb{P}S^n_A) \simeq \overline{\mathrm{W}}^i(A) \oplus \overline{\mathrm{W}}^{i-n}(A)$$

for all i.

If the base A is regular, then S^n_A is also regular and its coherent and derived (locally free) Witt groups coincide. The tensor product induces a natural structure of $\mathbb{Z}/4$-graded rings on $\mathrm{W}^{\mathrm{tot}}(A)$ and $\mathrm{W}^{\mathrm{tot}}(S^n)$ (denote it by $\star$). If $q : \operatorname{Spec} S^n \to \operatorname{Spec} A$ denotes the canonical projection, then q^* makes $\mathrm{W}^{\mathrm{tot}}(S^n)$ into a $\mathbb{Z}/4$-graded $\mathrm{W}^{\mathrm{tot}}(A)$-algebra. All this remains true of course for $\overline{\mathrm{W}}^{\mathrm{tot}}$. Our geometric proof of Theorem 1.1 lets us easily find this multiplicative structure:

Theorem 1.2. *Let moreover A be regular. Then*

$$\overline{\mathrm{W}}^{\mathrm{tot}}(S^n_A) \simeq \overline{\mathrm{W}}^{\mathrm{tot}}(A)[\alpha]/(\alpha^2),$$

with α sitting in degree n.

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Remarks. (a) Theorem 1.1 follows also from the general results of G.W. Brumfiel in [Br1] and [Br2], even in greater generality (*i.e.*, for any commutative ring A with $\frac{1}{2} \in A$). In *loc. cit.* Brumfiel sketches the developement of a theory $\mathrm{KO}^{-n}(\operatorname{Sper} A)$ inspired by the usual (real) topological K-theory of spaces, and which depends only on the real spectrum $\operatorname{Sper} A$ (equipped with its sheaf of abstract semi-algebraic functions). One of the main results of Brumfiel is that there are natural isomorphisms

$$\mathrm{W}_n^K(A) \otimes \mathbb{Z}[1/2] \simeq \mathrm{KO}^{-n}(\operatorname{Sper} A) \otimes \mathbb{Z}[1/2] \quad (n \geq 0),$$

where W_n^K are Karoubi's Witt groups. By identifying $\overline{\mathrm{W}}^n(A) \simeq \mathrm{W}_n^K(A) \otimes \mathbb{Z}[\frac{1}{2}]$ on the left hand side ([HS, Lemma A.3]), and by reducing Brumfiel's theory to the usual K-theory of some space on the right hand side, one can use topological results to calculate Balmer-Witt groups tensored with $\mathbb{Z}[\frac{1}{2}]$. We can avoid all this, our approach here being much simpler and completely geometric in nature. Our main tools will be the (12-periodic) localization long exact sequence, homotopy invariance and dévissage (recalled in Theorem A.13).

(b) By Proposition 2.1 below, we see that our calculation produces non-trivial groups only when the base ring A has characteristic 0 (*i.e.,* when $\mathbb{Z} \subset A$).

(c) In an Appendix at the end of the paper it is shown how to construct a canonical injective resolution of $\mathcal{O}_X$, for any Gorenstein scheme X. As a consequence, one can define coherent Witt groups which depend only on the scheme X, independently of any choice (Definition A.4). Similarly, one can define canonical transfers (push-forwards) for finite morphisms (Theorem A.10) . Moreover if X is regular, we will have *canonical* isomorphisms $\mathrm{W}^i(X) \simeq \tilde{\mathrm{W}}^i(X)$ between derived and coherent Witt groups (Theorem A.8).

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2 Proof of Theorem 1.1

The case $n = 0$ is trivial. We have isomorphisms

$$\operatorname{Spec} S^0 \simeq \operatorname{Proj} A[X_0, X_1]/(X_0^2 - X_1^2) \simeq \operatorname{Spec}(A \times A)$$

and thus $\tilde{\mathrm{W}}^i(S^0) \simeq \tilde{\mathrm{W}}^i(\mathbb{P}S^0) \simeq \tilde{\mathrm{W}}^i(A) \oplus \tilde{\mathrm{W}}^i(A)$.

For the rest of the proof we can assume $n \geq 1$. The following Proposition is the main component of our calculation.

Proposition 2.1. *Let X be a finite dimensional Gorenstein scheme with $\frac{1}{2} \in \mathcal{O}_X(X)$, such that none of its residue fields admits a total ordering (i.e., it has no 'real points'). Let $\mathcal{L}$ be a line bundle over X. Then all the coherent Witt groups $\tilde{\mathrm{W}}^i(X, \mathcal{L})$ of X with values in $\mathcal{L}$ are 2-primary torsion groups.*

Proof. For any field K, the classical Pfister local-global principle implies that $\mathrm{W}(K)$ is 2-primary torsion iff K is nonreal (i.e., iff K doesn't admit a total ordering, iff -1 is a sum of squares in K). Therefore, with the above hypothesis the first page of the Gersten-Witt spectral sequence for $(X, \mathcal{L})$ (see Balmer-Walter [BW, Thm. 7.2], Gille [Gi1, Thm. 3.14]) contains only 2-primary torsion groups, because the latter are sums of Witt groups $\mathrm{W}(k(x))$ of the residue fields of X. But the spectral sequence converges to the groups $\tilde{\mathrm{W}}^i(X, \mathcal{L})$ because X is finite dimensional. $\square$

Corollary 2.2. *Let X be a finite dimensional Gorenstein scheme, and let $X_{\mathbb{Q}} := X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$. Then the canonical projection $X_{\mathbb{Q}} \to X$ induces isomorphisms $\overline{\mathrm{W}}^i(X) \simeq \overline{\mathrm{W}}^i(X_{\mathbb{Q}})$ for all i.*

Proof. Let first $X = \operatorname{Spec} R$ be affine. Notice that a ring R satisfies the non-reality hypothesis of Proposition 2.1 if -1 is a sum of squares in R. Thus we can assume that the characteristic of R is 0, the other case being trivial. Consider for $m \geq 2$ the localization long exact sequence

$$\cdots \longrightarrow \tilde{\mathrm{W}}^i_{(m)}(A) \longrightarrow \tilde{\mathrm{W}}^i(A) \longrightarrow \tilde{\mathrm{W}}(A[1/m]) \longrightarrow \cdots .$$

By dévissage and Proposition 2.1 we have $\overline{\mathrm{W}}^i_{(m)}(A) \simeq \overline{\mathrm{W}}^i(A/m) \simeq 0$, so we obtain $\overline{\mathrm{W}}^i(A) \simeq \overline{\mathrm{W}}^i(A[1/m])$. Now $A \otimes \mathbb{Q} \simeq \operatorname{colim}_m A[1/m]$ and Witt groups commute with filtering colimits ([Gi2, Thm. 1.6]), therefore $\overline{\mathrm{W}}^i(A) \simeq \overline{\mathrm{W}}^i(A \otimes \mathbb{Q})$. The global case is obtained with Mayer-Vietoris, or can be done directly with the same reasoning as above. $\square$

Notation. In the following, we will write

$$C^n = C^n_A := \frac{A[X_1, \ldots, X_n]}{(1 + X_1^2 + \cdots + X_n^2)}.$$

An immediate corollary of Proposition 2.1 is that $\overline{\mathrm{W}}^i(C^n) \simeq 0$ for all i.

Now, the intersection of the projective sphere $\mathbb{P}S^n = V(\Sigma_k X_k^2 - X_{n+1}^2) \subset \mathbb{P}^{n+1}$ with the affine open $D(X_{n+1}) = \{X_{n+1} \neq 0\} \subset \mathbb{P}^{n+1}$ is isomorphic to the affine sphere $\operatorname{Spec} S^n$. We have

Lemma 2.3. *The open immersion* $\operatorname{Spec} S^n \simeq D(X_{n+1}) \hookrightarrow \mathbb{P}S^n$ *induces isomorphisms* $\overline{\mathrm{W}}^i(\mathbb{P}S^n) \simeq \overline{\mathrm{W}}^i(S^n)$.

Proof. We have a localization sequence

$$\cdots \longrightarrow \tilde{\mathrm{W}}^i_{(X_{n+1})}(\mathbb{P}S^n) \longrightarrow \tilde{\mathrm{W}}^i(\mathbb{P}S^n) \longrightarrow \tilde{\mathrm{W}}^i(S^n) \longrightarrow \cdots . \tag{1}$$

Denote by $\mathbb{P}C^n$ the closed projective subscheme $V(X_{n+1}) \subset \mathbb{P}S^n$, i.e. $\mathbb{P}C^n = V(X_0^2 + \cdots + X_n^2) \subset \mathbb{P}^n$. By dévissage we have isomorphisms

$$\tilde{\mathrm{W}}^i_{(X_{n+1})}(\mathbb{P}S^n) \simeq \tilde{\mathrm{W}}^{i-1}(\mathbb{P}C^n, \mathcal{L}_n), \tag{2}$$

where $\mathcal{L}_n$ is some line bundle over $\mathbb{P}C^n$ $(n \geq 1)$. On the affine opens $D(X_i) = \{X_i \neq 0\}$ $(i = 0, \ldots, n)$, the scheme $\mathbb{P}C^n$ is isomorphic to C^n; thus $\mathbb{P}C^n$ clearly satisfies the hypothesis of Proposition 2.1 and therefore $\overline{\mathrm{W}}^i(\mathbb{P}C^n, \mathcal{L}_n) = 0$. Then the exactness of (1) implies the claim. $\square$

Lemma 2.4. *Write $f := 1 + X_1^2 + \cdots + X_n^2 \in A[X_1, \ldots, X_n]$, and denote by $\mathbb{A}^n_f$ the corresponding open in the affine plane. Then we have isomorphisms $S^n_{1-X_{n+1}} \simeq \mathbb{A}^n_f$.*

Proof. The isomorphism is given by the stereographic projection. Indeed, writing $U := S^n_{1-X_{n+1}}$ and $V := A[Y_1, \ldots, Y_n]_{1+Y_1^2+\cdots+Y_n^2}$ for the localized rings, we define a morphism $V \to U$ by

$$Y_\ell \mapsto \frac{X_\ell}{1 - X_{n+1}} \quad (\ell = 1, \ldots, n).$$

Its inverse is given by

$$X_{n+1} \mapsto \frac{\Sigma_k Y_k^2 - 1}{\Sigma_k Y_k^2 + 1} \quad , \quad X_\ell \mapsto \frac{2Y_\ell}{\Sigma_k Y_k^2 + 1} \quad (\ell = 1, \ldots, n).$$

(Recall that 2 is invertible in A.) ☐

Lemma 2.5. *The projection $p : \mathbb{A}^n_f \to \operatorname{Spec} A$ induces an isomorphism $p^* : \overline{\mathrm{W}}^i(A) \simeq \overline{\mathrm{W}}^i(\mathbb{A}^n_f)$.*

Proof. We use the localization long exact sequence:

$$\cdots \longrightarrow \overline{\mathrm{W}}^i_{(f)}(\mathbb{A}^n) \longrightarrow \overline{\mathrm{W}}^i(\mathbb{A}^n) \longrightarrow \overline{\mathrm{W}}^i(\mathbb{A}^n_f) \longrightarrow \cdots.$$

Since $f = 1 + X_1^2 + \cdots + X_n^2$ is a regular element, by dévissage we have

$$\overline{\mathrm{W}}^i_{(f)}(\mathbb{A}^n) \simeq \overline{\mathrm{W}}^i(C^n).$$

Because of Proposition 2.1, the latter is zero. Now we use homotopy invariance of coherent Witt groups to conclude. ☐

Lemma 2.6. *There are split short exact sequences*

$$0 \longrightarrow \overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n) \longrightarrow \overline{\mathrm{W}}^i(S^n) \longrightarrow \overline{\mathrm{W}}^i(A) \longrightarrow 0.$$

Proof. Consider the long exact sequence associated to the localization $S^n \to S^n_{1-X_{n+1}}$:

$$\cdots \to \overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n) \to \overline{\mathrm{W}}^i(S^n) \to \overline{\mathrm{W}}^i(S^n_{1-X_{n+1}}) \to \overline{\mathrm{W}}^{i+1}_{(1-X_{n+1})}(S^n) \to \cdots,$$

and consider the projection $q : \operatorname{Spec} S^n \to \operatorname{Spec} A$. We then have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} S^n \arrow[dr, "q"'] & \operatorname{Spec} S^n_{1-X_{n+1}} \arrow[l, "i"'] \arrow[d, "p"] \\
& \operatorname{Spec} A
\end{tikzcd}$$

where i is the inclusion. Using Lemma 2.5 and the fact that $\operatorname{Spec} S^n_{1-X_{n+1}} \simeq \mathbb{A}^n_f$ (Lemma 2.4), we obtain a commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{W}}^i(S^n) \arrow[r, "i^*"] & \overline{\mathrm{W}}^i(\mathbb{A}^n_f) \\
& \overline{\mathrm{W}}^i(A) \arrow[u, "p^*", "\simeq"'] \arrow[ul, "q^*"]
\end{tikzcd}$$

The above long exact sequence yields finally the split sequences

$$0 \longrightarrow \overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n) \longrightarrow \overline{\mathrm{W}}^i(S^n) \longrightarrow \overline{\mathrm{W}}^i(A) \longrightarrow 0.$$

$\square$

Corollary 2.7. *For any i, we have $\overline{\mathrm{W}}^i(S^n) \simeq \overline{\mathrm{W}}^i(A) \oplus \overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n)$.*

Next we compute $\overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n)$. In order to do this, we introduce some more notation:

Notation.

$$B^n = B^n_A := \frac{A[X_1, \dots, X_n]}{(X_1^2 + \cdots + X_n^2)}.$$

Thus we have isomorphisms $S^n/(1 - X_{n+1}) \simeq B^n$. For $n \geq 1$, the element $1 - X_{n+1} \in S_n$ is regular and we can use dévissage to obtain:

$$\tilde{\mathrm{W}}^i_{(1-X_{n+1})}(S^n) \simeq \tilde{\mathrm{W}}^{i-1}(B^n). \tag{3}$$

Notice *en passant* that the rings B^n are singular. This makes the use of coherent (rather that derived) Witt groups necessary for our calculation, even when A is regular.

Lemma 2.8. *For $n \geq 1$ and any i, we have $\overline{\mathrm{W}}^i(B^n) \simeq \overline{\mathrm{W}}^{i-n+1}(A)$.*

Proof. We will exploit the recursive property $B^n/(X_n) \simeq B^{n-1}$. Consider the long exact sequence

$$\cdots \longrightarrow \tilde{\mathrm{W}}^i_{(X_n)}(B^n) \longrightarrow \tilde{\mathrm{W}}^i(B^n) \longrightarrow \tilde{\mathrm{W}}^n(B^n_{X_n}) \longrightarrow \cdots$$

associated to the localization $B^n \to B^n_{X_n}$. Notice that $B^n_{X_n} \simeq C^{n-1}[X_n, X_n^{-1}]$. It is a result of Gille that $\tilde{\mathrm{W}}^i(R[T, T^{-1}]) \simeq \tilde{\mathrm{W}}^i(R) \oplus \tilde{\mathrm{W}}^i(R)$ for R a finite dimensional Gorenstein ring (see [Gi1, Thm. 5.6]). Thus we have

$$\overline{\mathrm{W}}^i(B^n_{X_n}) \simeq \overline{\mathrm{W}}^i(C^{n-1}) \oplus \overline{\mathrm{W}}^i(C^{n-1}) = 0,$$

where the vanishing due to Proposition 2.1. For $n \geq 2$, the element $X_n \in B^n$ is regular, so dévissage yields

$$\tilde{\mathrm{W}}^i_{(X_n)}(B^n) \simeq \tilde{\mathrm{W}}^{i-1}(B^n/X_n) = \tilde{\mathrm{W}}^{i-1}(B^{n-1}).$$

Altogether, we obtain the formula $\overline{\mathrm{W}}^i(B^n) \simeq \overline{\mathrm{W}}^{i-1}(B^{n-1})$ ($n \geq 2$). We finish the proof by remarking that $\tilde{\mathrm{W}}^j(B^1) \simeq \tilde{\mathrm{W}}^j(A)$ for all j. (For example, one can use the generalization of affine dévissage to zero dimensional ideals, see [Gi2, Thm. 3.5]: $\tilde{\mathrm{W}}^j(A[X]/X^2) = \tilde{\mathrm{W}}^j_{(X)}(A[X]/X^2) \simeq \tilde{\mathrm{W}}^j((A[X]/X^2)/X) = \tilde{\mathrm{W}}^j(A)$.) $\square$

Finally, we have $\overline{\mathrm{W}}^i(S^n) \simeq \overline{\mathrm{W}}^i(A) \oplus \overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n)$ by Corollary 2.7 and $\overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n) \simeq \overline{\mathrm{W}}^{i-n}(A)$ by (3) and the last Lemma. Together with Lemma 2.3, this ends to proof of Theorem 1.1.

3 Proof of Theorem 1.2

From now on, the base A and thus also S^n_A will be assumed to be regular rings. We will still denote by $q : \operatorname{Spec} S^n \to \operatorname{Spec} A$ the structure morphism.

For any ring R, we will denote by $e_R \in \mathrm{W}^0(R)$ the multiplicative unit of $\mathrm{W}^{\mathrm{tot}}(R)$; this is just the diagonal form $< 1 >= [\mathrm{id} : R \to R]$ (we make the usual identification $R = R^\vee$). In this proof we will also abbreviate

$$P := S^n/(X_1, \ldots, X_n)$$

$$\mathfrak{n} := (X_{n+1} - 1) \qquad \mathfrak{s} := (X_{n+1} + 1),$$

so that $P/\mathfrak{n} \simeq A$ and $P/\mathfrak{s} \simeq A$ are the North Pole and the South Pole of the sphere. Write i_P, i_N and i_S for the corresponding closed immersions. We will further write

$$\alpha_N := i_{N*}(e_{A/\mathfrak{n}}) \in \mathrm{W}^n(S^n), \quad \alpha_S := i_{S*}(e_{A/\mathfrak{s}}) \in \mathrm{W}^n(S^n),$$

and we will keep the same notation for the images in $\overline{\mathrm{W}}^n(S^n)$ of these forms.

The next Lemma is just a corollary of the proof of Theorem 1.1.

Lemma 3.1.
$$\overline{\mathrm{W}}^{\mathrm{tot}}(S^n) = \overline{\mathrm{W}}^{\mathrm{tot}}(A) \cdot e_{S^n} \oplus \overline{\mathrm{W}}^{\mathrm{tot}}(A) \cdot \alpha_N$$

Proof. From the proof of Lemma 2.6 we see that $q^* : \overline{\mathrm{W}}^{\mathrm{tot}}(A) \to \overline{\mathrm{W}}^{\mathrm{tot}}(S^n)$ is injective. Since $q^*(e_A) = e_{S^n}$, we recognize the first direct summand. By the proof of Lemma 2.8 and functoriality of the transfer (Remark A.12, see also Remark A.14), we see that the other summand is the image of i_{N*} : $\overline{\mathrm{W}}^{\mathrm{tot}}(P/\mathfrak{n}) \to \overline{\mathrm{W}}^{\mathrm{tot}}(S^n)$. This image is $\overline{\mathrm{W}}^{\mathrm{tot}}(A) \cdot \alpha_N$ by the following Lemma. □

Lemma 3.2. *Let $i : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} S^n$ be the closed immersion of an A-point, let $i_* : \mathrm{W}^0(A) \to \mathrm{W}^n(S^n)$ be the induced transfer morphism. Then for every form $\beta \in \mathrm{W}^0(A)$:*
$$i_*(\beta) = q^*(\beta) \star i_*(e_A) \in \mathrm{W}^n(S^n).$$

Proof. This follows from the projection formula for coherent Witt groups (Gille [Gi3, Thm. 5.2]), applied to the morphism i:

$$i_*(\beta) = i_*(\underbrace{i^* q^*}_{\mathrm{id}}(\beta) \star e_A) = q^*(\beta) \star i_*(e_A).$$

□

Proposition 3.3. *The relation $\alpha_N = -\alpha_S$ holds in $\mathrm{W}^n(S^n)$.*

As an immediate consequence of this, the form $\alpha_N^2 = -\alpha_N \star \alpha_S$ is supported on the intersection of the North and the South Poles, which is empty: $\operatorname{Spec} P/\mathfrak{n} \cap \operatorname{Spec} P/\mathfrak{s} = \emptyset$; so it is trivially equal to zero.

To prove Proposition 3.3, we first recall straight from [Gi3, § 9] some facts about forms on Koszul complexes. For any ring R and any regular sequence $(x_1, \ldots, x_n)$ in R, we will denote by $K_\bullet(x_1, \ldots, x_n)$ the Koszul complex for this sequence, and we set it in (homological) degrees from n to 0. We are interested in the situation when R is an S-algebra $S \to R$ over some other ring S, and the sequence is such that $R/(x_1, \ldots, x_n) \simeq S$. (Below we will specialize to the P-algebra S^n and the regular sequence $(X_1, \ldots, X_n)$ in S^n.) For $1 \le i \le n$ and any unit $r \in R^\times$, the complex $K_\bullet(x_i)$ can be equipped by the following symmetric 1-form:

$$\begin{tikzcd}
K_\bullet(x_i): \arrow[d, "\ell_r :="] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r, "\cdot x_i"] \arrow[d, "\cdot r"'] & R \arrow[r] \arrow[d, "\cdot(-r)"] & 0 \arrow[r] \arrow[d] & \cdots \\
K_\bullet(x_i): & \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r, "\cdot(-x_i)"] & R \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

For any choice of n units in R, the product $(K_\bullet(x_1), \ell_r) \star \cdots \star (K_\bullet(x_n), \ell_r)$ is a symmetric n-space on the complex $K_\bullet(x_1, \ldots, x_n) \simeq K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$. We denote its form by $\ell_{r_1 \cdot \ldots \cdot r_n}$.

By displaying an explicit Lagrangian, it is easy to see that $[K_\bullet(x_i), \ell_{r_i}] = 0 \in \mathrm{W}^n(R)$, and therefore $[K_\bullet(x_1, \ldots, x_n), \ell_{r_1 \cdot \ldots \cdot r_n}] = 0 \in \mathrm{W}^n(R)$ for all symmetric spaces as above.

Lemma 3.4. *If $\phi : K_\bullet(x_1, \ldots, x_n) \to \operatorname{Hom}_R(K_\bullet(x_1, \ldots, x_n), R)[n]$ is a quasi-isomorphism of complexes, then there exists a unit $r \in R^\times$ such that $\ell_{r \cdot 1 \cdot \ldots \cdot 1}$ is chain homotopic to ϕ.*

Proof. This is a slight generalization of [Gi3, Lemma 9.1]. The same proof goes through. ☐

We have an isomorphism $\mathrm{W}^0(P) \simeq \mathrm{W}^0(P/\mathfrak{n}) \oplus \mathrm{W}^0(P/\mathfrak{s})$, induced by $P \simeq P/\mathfrak{n} \times P/\mathfrak{s}$, which identifies e_P with (e_N, e_S). Under this isomorphism, the transfer $i_{P*} : \mathrm{W}^0(P) \to \mathrm{W}^n(S^n)$ identifies with (i_{N*}, i_{S*}), and in particular

$$\alpha_N + \alpha_S = i_{N*}(e_N) + i_{S*}(e_S) = i_{P*}(e_P).$$

But the last term is zero in $\mathrm{W}^n(S^n)$. In fact, $i_{P*}(e_P)$ can be represented by $(F_\bullet, \psi)$, where $F_\bullet$ is a projective resolution of the S^n-module $P = S^n/(X_1, \ldots, X_n)$, and where ψ is a symmetric quasi-isomorphism between $F_\bullet$ and its n-shifted dual, lying above the morphism $\mathrm{id} : A \to A$. Since $(X_1, \ldots, X_n)$ is a regular sequence in S^n, we can take $F_\bullet$ to be the Koszul complex for this sequence. By the above Lemma (or by direct inspection) we have $i_{A*}(e_A) = [K_\bullet(X_1, \ldots, X_n), \ell_1] = 0 \in \mathrm{W}^n(S^n)$. ☐

A Appendix: The canonical injective resolution

A.1 The local case

We first recall some basic facts about injective modules. Our source here is [Bou].

Let A be a ring and consider the module $F = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. Then F is an injective A-module ([Bou, Corollary 1]). If M is any A-module, then we have a canonical homomorphism

$$e_M : M \to F^{\mathrm{Hom}_A(M,F)}$$

given by $e_M(m) = (\phi(m))_{\phi \in \mathrm{Hom}_A(M,F)}$. Observe that $F^{\mathrm{Hom}_A(M,F)}$ is injective and e_M is injective ([Bou, Corollary 2]). Let $I_0(M) := F^{\mathrm{Hom}_A(M,F)}$, $L_0(M) := \mathrm{Coker}(e_M)$ and $q_M : I_0(M) \to L_0(M)$ be the canonical projection. By definition, we obtain an exact sequence

$$0 \longrightarrow M \xrightarrow{e_M} I_0(M) \xrightarrow{q_M} L_0(M) \longrightarrow 0.$$

We define inductively $I_{-n}(M)$ by $I_0(L_{-n+1}(M))$, $L_{-n}(M)$ by $L_0(L_{-n+1}(M))$ and homomorphisms $\delta^M_{-n} : I_{-n}(M) \to I_{-n-1}(M)$ by

$$\begin{cases} \delta^M_{-n} = 0 & \text{if } n < 0 \\ \delta^M_0 = e_{L_0(M)} \circ q_M & \text{if } n = 0 \\ \delta^M_{-n} = e_{L_{-n}(M)} \circ q_{L_{-n+1}(M)} & \text{if } n > 0. \end{cases}$$

We then obtain an injective resolution of M:

$$0 \longrightarrow M \xrightarrow{e_M} I_0(M) \xrightarrow{\delta^M_0} I_{-1}(M) \xrightarrow{\delta^M_{-1}} I_{-2}(M) \longrightarrow \cdots$$

This resolution is called *canonical injective resolution* of M. Suppose now that M is of (finite) injective dimension n. Then using [Bou, Proposition 11], we see that $L_{-n+1}(M)$ is an injective module. Therefore we have the following definition:

Definition A.1. Let M be an A-module of injective dimension n. Then the injective resolution

$$0 \longrightarrow M \xrightarrow{e_M} I_0(M) \xrightarrow{\delta^M_0} \cdots \xrightarrow{\delta^M_{-n+2}} I_{-n+1}(M) \xrightarrow{q_{L_{-n+2}}} L_{-n+1}(M) \longrightarrow 0$$

is called *canonical finite injective resolution* of M.

Example 1. Suppose that A is a Gorenstein ring. Then by definition, A is of finite injective dimension (equal to the Krull dimension of the ring) and therefore has a canonical finite injective resolution. Any invertible A-module L is also of finite injective dimension.

A.2 The global case

Let X be a scheme. Consider the constant sheaf (of abelian groups) $\mathbb{Q}/\mathbb{Z}$. Then $\mathcal{F} := \mathrm{Hom}(\mathcal{O}_X, \mathbb{Q}/\mathbb{Z})$ is a sheaf of $\mathcal{O}_X$-modules over X.

Lemma A.2. *The sheaf $\mathcal{F}$ is injective in the category $\mathcal{M}_{qc}$ of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. We consider a covering of X by open connected affine subscheme $\mathrm{Spec}\, A_i$. Since the restriction of $\mathrm{Hom}(\mathcal{O}_X, \mathbb{Q}/\mathbb{Z})$ to $\mathrm{Spec}\, A_i$ is $\mathrm{Hom}(A_i, \mathbb{Q}/\mathbb{Z})$ and the latter is an injective module by the above section, we see that $\mathrm{Hom}(\mathcal{O}_X, \mathbb{Q}/\mathbb{Z})$ is injective. $\square$

Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F})$ and $\mathcal{F}$ as sheaves of sets. Then the sheaf $\mathrm{Hom}_{ShSets}(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F}), \mathcal{F})$ of morphisms of sheaves of sets can be seen as an $\mathcal{O}_X$-module (since $\mathcal{F}$ is an $\mathcal{O}_X$-module). We put $\mathcal{F}^{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{F})} := \mathrm{Hom}_{ShSets}(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F}), \mathcal{F})$. Then we have a canonical homomorphism

$$e_{\mathcal{M}} : \mathcal{M} \to \mathcal{F}^{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{F})}$$

defined over an open U by $e_{\mathcal{M}}(U) = e_{\mathcal{M}(U)}$. As in the local case, observe that $\mathcal{F}^{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{F})}$ is an injective $\mathcal{O}_X$-module and that $e_{\mathcal{M}}$ is injective. Mimicking the construction of the local case, we obtain an injective resolution of $\mathcal{M}$

$$0 \longrightarrow \mathcal{M} \xrightarrow{e_{\mathcal{M}}} I_0(\mathcal{M}) \xrightarrow{\delta_0^{\mathcal{M}}} I_{-1}(\mathcal{M}) \xrightarrow{\delta_{-1}^{\mathcal{M}}} I_{-2}(\mathcal{M}) \longrightarrow \cdots$$

called *canonical injective resolution* of $\mathcal{M}$. If $\mathcal{M}$ is of (finite) injective dimension n, we see that $L_{-n+1}(\mathcal{M}) := \mathrm{Coker}(\delta_{-n+2}^{\mathcal{M}})$ is also injective. To avoid overloading the notations, we put $I_{-j} := I_{-j}(\mathcal{M})$ and $\delta_{-j}^{\mathcal{M}} := \delta_{-j}$ for any $j \leq n-1$; we also put $I_{-n} := L_{-n+1}(\mathcal{M})$ and $q_{L_{-n+2}^{\mathcal{M}}} := \delta_{-n+1}$. We have:

Definition A.3. Let X be a scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module of injective dimension n. Then the injective resolution

$$0 \longrightarrow \mathcal{M} \xrightarrow{e_{\mathcal{M}}} I_0 \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{-n+2}} I_{-n+1} \xrightarrow{\delta_{-n+1}} I_{-n} \longrightarrow 0$$

is called *canonical finite injective resolution* of $\mathcal{M}$.

Example 2. If X is a Gorenstein scheme, then $\mathcal{O}_X$ has a canonical finite injective resolution. Any invertible $\mathcal{O}_X$-module has the same property.

A.3 Coherent Witt groups

Our main references for this section are [Bal] and [Gi1].

Let X be an n-dimensional Gorenstein scheme such that 2 is a global unit over X. Let $D^b_c(\mathcal{M}_{qc})$ be the triangulated category of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules whose homology sheaves are coherent. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and

$$0 \longrightarrow \mathcal{L} \xrightarrow{e_{\mathcal{L}}} I_0 \longrightarrow \cdots \longrightarrow I_{-n} \longrightarrow 0$$

be the canonical finite injective resolution of $\mathcal{L}$. For any $\mathcal{M}_\bullet \in D^b_c(\mathcal{M}_{qc})$, we have a dual complex $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_\bullet, I_\bullet)$ whose groups are

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_\bullet, I_\bullet)_i := \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}_{-i-j}, I_{-j})$$

and differentials are given by $f \mapsto f d^{\mathcal{M}}_{-i-j+1} + (-1)^{i+1} d^I_{-j} f$. There is a canonical homomorphism $\varpi_{\mathcal{M}_\bullet}$ of $\mathcal{M}_\bullet$ to its bidual for any $\mathcal{M}_\bullet$, and it turns out that it is an isomorphism ([Gi1, Theorem 2.6.4]). Therefore $(D^b_c(\mathcal{M}_{qc}), \mathrm{Hom}_{\mathcal{O}_X}(\_, I_\bullet), 1, \varpi)$ is a triangulated category with duality in the sense of Balmer ([Bal, Definition 4.1]) and one can define the Witt groups of this category ([Bal, Definition 4.3]).

Definition A.4. Let $\mathcal{O}_X$ be a Gorenstein scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module and $I_\bullet$ be the canonical finite injective resolution of $\mathcal{L}$. Then the coherent Witt groups $\tilde{\mathrm{W}}^i(X, \mathcal{L})$ of X are defined to be the groups

$$\tilde{\mathrm{W}}^i(X, \mathcal{L}) := \mathrm{W}^i(D^b_c(\mathcal{M}_{qc}), \mathrm{Hom}_{\mathcal{O}_X}(\_, I_\bullet), 1, \varpi).$$

In the case where $\mathcal{L} = \mathcal{O}_X$, we put $\tilde{\mathrm{W}}^i(X) := \tilde{\mathrm{W}}^i(X, \mathcal{O}_X)$.

Remark A.5. The original definition of coherent Witt groups is due to Gille ([Gi1, Definition 2.6.10]), who chooses a finite injective resolution $J_\bullet$ of $\mathcal{L}$ and puts

$$\tilde{\mathrm{W}}^i(X, \mathcal{L}) := \mathrm{W}^i(D^b_c(\mathcal{M}_{qc}), \mathrm{Hom}_{\mathcal{O}_X}(\_, J_\bullet), 1, \varpi^{J_\bullet}).$$

It is clear that this definition depends on the choice of a finite injective resolution of $\mathcal{L}$. Since we have defined a canonical finite injective resolution, Definition A.4 does not depend on any choice.

Remark A.6. If

$$0 \longrightarrow \mathcal{L} \xrightarrow{j} J_0 \longrightarrow J_{-1} \longrightarrow \cdots \longrightarrow J_{-n} \longrightarrow 0$$

is any finite injective resolution of $\mathcal{L}$, then there is a unique quasi-isomorphism $\varphi : I_\bullet \to J_\bullet$ making the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "e_{\mathcal{L}}"] \arrow[d, equal] & I_\bullet \arrow[d, "\varphi"] \\ 0 \arrow[r] & \mathcal{L} \arrow[r, "j"'] & J_\bullet . \end{tikzcd}$$

This quasi-isomorphism induces an isomorphism of Witt groups

$$\varphi : \tilde{\mathrm{W}}^i(X, \mathcal{L}) \to \mathrm{W}^i(D^b_c(\mathcal{M}_{qc}), \mathrm{Hom}_{\mathcal{O}_X}(\_, J_\bullet), 1, \varpi^{J_\bullet}).$$

Suppose now that X is a regular scheme (and 2 is a global unit over X). Consider the triangulated category $D^b(\mathcal{P}(X))$ of bounded complexes of coherent locally free $\mathcal{O}_X$-modules. The usual duality $\mathrm{Hom}_{\mathcal{O}_X}(\_, \mathcal{L})$ induces a 1-duality on $D^b(\mathcal{P}(X))$ and there is a canonical natural isomorphism $\varpi_{P_\bullet} : P_\bullet \to \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Hom}_{\mathcal{O}_X}(P_\bullet, \mathcal{L}), \mathcal{L})$ for any $P_\bullet \in D^b(\mathcal{P}(X))$. One can check that $(D^b(\mathcal{P}(X)), \mathrm{Hom}_{\mathcal{O}_X}(\_, \mathcal{L}), 1, \varpi)$ is a triangulated category with duality in the sense of [Bal, Definition 4.1]. Thus we have the following definition (see [Bal]):

Definition A.7. Let X be a regular scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. The derived Witt groups of X are defined to be the groups

$$\mathrm{W}^i(X,\mathcal{L}) := \mathrm{W}^i(D^b(\mathcal{P}(X)), \mathrm{Hom}_{\mathcal{O}_X}(\_,\mathcal{L}), 1, \varpi).$$

When $\mathcal{L} = \mathcal{O}_X$, we simply put $\mathrm{W}^i(X) := \mathrm{W}^i(X, \mathcal{O}_X)$.

In particular, a regular scheme is Gorenstein. Hence there is a canonical quasi-isomorphism $e_{\mathcal{L}} : \mathcal{L} \to I_\bullet$. This quasi-isomorphism and the inclusion $D^b(\mathcal{P}(X)) \subset D^b_c(\mathcal{M}_{qc})$ yield isomorphisms $\mathrm{W}^i(X,\mathcal{L}) \to \tilde{\mathrm{W}}^i(X,\mathcal{L})$ ([Gi1, Corollary 2.6.11]). Thus we get the following result:

Theorem A.8. *Let X be a regular scheme and let $\mathcal{L}$ be a line bundle over X. Then we have canonical isomorphisms* $\mathrm{W}^i(X,\mathcal{L}) \simeq \tilde{\mathrm{W}}^i(X,\mathcal{L})$.

A.4 Transfers

The reference here is [Gi3]. Let X and Y be Gorenstein schemes. For simplicity, we assume that they are connected. Suppose that $f : X \to Y$ is a finite morphism and let $d = \dim(Y) - \dim(X)$. Let $\overline{f} : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ be the morphism of locally ringed space induced by f. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M})$ is a quasi-coherent $f_*\mathcal{O}_X$-module and $\overline{f}^*\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M})$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, if

$$0 \longrightarrow \mathcal{L} \xrightarrow{e_{\mathcal{O}_Y}} I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-n} \longrightarrow 0$$

is the canonical finite injective resolution of an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ then $\overline{f}^*\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, I_\bullet)$ is a complex of quasi-coherent $\mathcal{O}_X$-modules. Moreover, for any i the module $J_{-i} := \overline{f}^*\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, I_{-i})$ is injective ([Gi3, Chapter 4]). Now consider the complex (where $d = \dim(Y) - \dim(X)$):

$$0 \longrightarrow \mathrm{Ker}(\delta_{-d}) \xrightarrow{\iota} J_{-d} \longrightarrow J_{-d-1} \longrightarrow \cdots \longrightarrow J_{-n} \longrightarrow 0.$$

It turns out that $\mathrm{Ker}(\delta_{-d}) = \overline{f}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{L})$ and that $J_\bullet$ is a finite injective resolution of this $\mathcal{O}_X$-module, which is invertible ([Gi3, Corollary 6.3]). The following theorem is due to Gille:

Theorem A.9 (Transfer)**.** *Let X and Y be connected Gorenstein schemes and let $f : X \to Y$ be a finite morphism of relative dimension d. Then the functor $f_* : D^b_c(\mathcal{M}_{qc}(X)) \to D^b_c(\mathcal{M}_{qc}(Y))$ induces homomorphisms*

$$Tr_{X/Y} : \mathrm{W}^i(D^b_c(\mathcal{M}_{qc}(X)), \mathrm{Hom}_{\mathcal{O}_X}(\_, J_\bullet), 1, \varpi^{J_\bullet}) \to \tilde{\mathrm{W}}^{i+d}(Y, \mathcal{L}).$$

Proof. See [Gi3, Theorem 6.4]. □

Now $\overline{f}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{L})$ is an invertible $\mathcal{O}_X$-module and therefore it admits a canonical finite injective resolution

$$0 \longrightarrow \overline{f}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{e} I_0 \longrightarrow \cdots \longrightarrow I_{-n+d} \longrightarrow 0.$$

By Remark A.6, there are canonical isomorphisms

$$\varphi : \tilde{\mathrm{W}}^i(X, \overline{f}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{L})) \to \mathrm{W}^i(D^b_c(\mathcal{M}_{qc}(X)), \mathrm{Hom}_{\mathcal{O}_X}(\_, J_\bullet), 1, \varpi^{J_\bullet}).$$

We finally have the following result:

Theorem A.10 (Canonical transfer)**.** *Let X and Y be connected Gorenstein schemes and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a finite morphism and $d = \dim Y - \dim X$. Then the functor $f_* : D^b_c(\mathcal{M}_{qc}(X)) \to D^b_c(\mathcal{M}_{qc}(Y))$ induces* canonical *homomorphisms*

$$f_* : \tilde{\mathrm{W}}^i(X, \overline{f}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{L})) \to \tilde{\mathrm{W}}^{i+d}(Y, \mathcal{L}).$$

Proof. The canonical transfer is defined to be the composition

$$\begin{tikzcd}
\tilde{\mathrm{W}}^i(X, \overline{f}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)) \arrow[r, "\varphi"] \arrow[dr, "f_*"'] & \mathrm{W}^i(D^b_c(\mathcal{M}_{qc}(X)), \mathrm{Hom}_{\mathcal{O}_X}(_, J_\bullet), 1, \varpi^{J_\bullet}) \arrow[d, "Tr_{X/Y}"] \\
 & \tilde{\mathrm{W}}^{i+d}(Y)
\end{tikzcd}$$

□

Remark A.11. In a recent preprint, Nenashev announced a canonical transfer associated to any inclusion $Y \hookrightarrow X$ of smooth schemes ([Ne, Definition 4.1]). It would be interesting to compare his canonical transfer with the one defined above.

Remark A.12. Let X, Y, Z be connected Gorenstein schemes and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be finite morphisms. Then using the fact that $Tr_{Y/Z} \circ Tr_{X/Y} = Tr_{X/Z}$ for some good choices of injective resolutions (see [Gi1]) it is not hard to see that $(gf)_* = g_*f_*$. Thus the canonical transfer is functorial.

The following theorem is also useful:

Theorem A.13 (Dévissage)**.** *Let Y be a connected Gorenstein scheme and $j : X \hookrightarrow Y$ be a connected closed subscheme of Y of codimension d. Then the canonical transfer gives canonical isomorphisms*

$$j_* : \tilde{\mathrm{W}}^i(X, \overline{j}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(j_*\mathcal{O}_X, \mathcal{O}_Y)) \to \tilde{\mathrm{W}}^{i+d}_X(Y).$$

Proof. See [Gi4, Theorem 3.2]. □

Remark A.14. It happens sometimes that $\overline{j}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(j_*\mathcal{O}_X, \mathcal{O}_Y)$ is isomorphic to $\mathcal{O}_Y$, *e.g.* when $Y = \mathrm{Spec}\, A$ is affine and $j : \mathrm{Spec}\, A/I \to \mathrm{Spec}\, A$ is a closed immersion whose defining ideal I can be given by a regular sequence $(a_1, ..., a_d)$ in A. Every choice of a sequence gives an isomorphism $\mathcal{O}_Y \simeq \overline{j}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(j_*\mathcal{O}_X, \mathcal{O}_Y)$ and therefore an isomorphism of Witt groups

$$\phi : \tilde{\mathrm{W}}^i(X) \to \tilde{\mathrm{W}}^i(X, \overline{j}^*\mathrm{Ext}^d_{\mathcal{O}_Y}(j_*\mathcal{O}_X, \mathcal{O}_Y)).$$

One would usually make this choice tacitly and will still write $f_* : \tilde{\mathrm{W}}^i(X) \simeq \tilde{\mathrm{W}}^i_X(Y)$ for the composite isomorphism $f_* \circ \phi$.

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