KNESER-TITS FOR A RANK 1 FORM OF E_6 (AFTER VELDKAMP)

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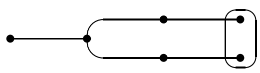
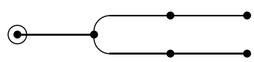
Abstract. We prove the Kneser-Tits Conjecture for groups of index ${}^2E^{29}_{6,1}$ using an argument inspired by a 1968 paper by Veldkamp.

The notion of *simple* for an algebraic group is different from the notion of simple for abstract groups. Recall that an abstract group Γ is *projectively simple* if $\Gamma/Z(\Gamma)$ is simple as an abstract group. For a given field k, the Kneser-Tits Conjecture asserts: *For every simply connected quasi-simple k-isotropic algebraic group G, the abstract group $G(k)$ is projectively simple.*

A good survey of the conjecture is given in [PR, §7.2]. Here are a few highlights. Many cases of the conjecture for classical groups are part of "geometric algebra" as in the books by E. Artin and J. Dieudonné. The conjecture holds for k algebraically closed, for the real numbers (E. Cartan), and for non-archimedean locally compact fields (V.P. Platonov). It fails wildly if the simply connected hypothesis is dropped. Some groups of inner type A_n provide counterexamples to the conjecture; these amount to central division algebras with nontrivial SK_1. In order to prove the conjecture for a particular field k, G. Prasad and M.S. Raghunathan showed that it suffices to consider the groups of k-rank 1.

For k a number field, no counterexamples are known. In order to prove the conjecture in that case, it remains only to prove it for groups with the following Tits indexes:

${}^2E^{29}_{6,1}$   ${}^2E^{35}_{6,1}$

(The conjecture has long been known for the classical groups, cf. [PR, p. 410]. The trialitarian groups are treated in [Pra].) We remark that when k is a totally imaginary number field or a global function field, the two indexes displayed above do not occur [Gi, p. 315, Th. 9b], hence the conjecture is proved in that case. The conjecture is still open for number fields with real embeddings, like the rational numbers.

In fact, one of the two "open" cases was—essentially—settled in 1968. The purpose of this paper is to give a proof of that case, i.e., to prove the following theorem.

Date: 20 February 2006.

2000 *Mathematics Subject Classification.* 20G15.

Theorem (Veldkamp)**.** *For every field k of characteristic $\neq 2, 3$ and every simply connected k-group G of index ${}^2E_{6,1}^{29}$, the abstract group $G(k)$ is projectively simple.*

This theorem is 9.5(i) in F.D. Veldkamp's paper [V 68]—his paper is only missing an argument that his groups have index ${}^2E_{6,1}^{29}$ and that every simply connected group of index ${}^2E_{6,1}^{29}$ is one of his groups. We present a proof from a different viewpoint that is inspired by his and uses somewhat modernized language. We feel that this is worthwhile, partially because his result does not seem to have been incorporated into the literature. For example, it is not mentioned in Tits's excellent survey [T 78]. Also, some delicate aspects of his proof can be avoided with modern techniques.

In [V 69], Veldkamp modified his proof slightly in order to include also the cases where G is quasi-split or of index ${}^2E_{6,2}^{16'}$. But these cases are already covered by Tits's survey, so we only consider groups as in the theorem.

We exclude characteristics 2 and 3—as Veldkamp did—in order to use convenient facts about quadratic forms and Jordan algebras. For the reader interested in global function fields, this hypothesis is harmless, as there are no groups of index ${}^2E_{6,1}^{29}$ over such a field.

Notation and conventions. Throughout this paper, C denotes an octonion k-algebra and K is a quadratic field extension of k. We occasionally write C also for the quadratic norm form on C. We write C^K for the "K-associate" of the norm on C as defined in [KMRT, p. 499]: if the norm form on C is $\langle 1 \rangle \oplus q$ and $K = k(\sqrt{\alpha})$, then C^K is the quadratic form $\langle 1 \rangle \oplus \langle \alpha \rangle q$.

1. Outline of proof

For a semisimple group G, write $G(k)^+$ for the subgroup generated by the k-points of the unipotent radicals of the parabolic k-subgroups of G. For G quasi-simple, $G(k)^+$ is projectively simple [T 64].

Fix G as in the theorem and a rank 1 k-split torus S in G. Write H for the centralizer of S in G; it is reductive of type ${}^2D_4$. As in [T 64, 3.2(18)], we have

$$G(k) = H(k) \cdot G(k)^+. \tag{1.1}$$

In §3 below, we prove that

$$D(k) \subseteq G(k)^+ , \tag{1.2}$$

where D is the stabilizer of a particular vector in the irreducible 54-dimensional representation of G. In sections 4 and 5, we observe that

$$H(k) \subseteq D(k) \cdot G(k)^+. \tag{1.3}$$

Combining (1.1) through (1.3), we find that $G(k)^+ = G(k)$, which proves the theorem.

2. Explicit description of G

Given a group G_0 of index ${}^2E_{6,1}^{29}$ as in the theorem, write K for the quadratic extension of k over which G_0 is of inner type. Write C for the octonion k-algebra determined by the Rost invariant of G_0 as in [GP, §2]. Note that C is not K-split and the semisimple anisotropic kernel of G_0 is $\mathrm{Spin}(C^K)$ [GP, 2.6]. We now give an explicit description of G_0 by Galois descent.

Write A for the Albert k-algebra of hermitian 3-by-3 matrices with entries in C. (See [SV] or [KMRT] for background about Albert algebras.) It has a nondegenerate symmetric bilinear form "tr" defined by

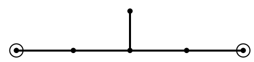
$$\mathrm{tr}\left(\begin{pmatrix} \varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3 \end{pmatrix}, \begin{pmatrix} \nu_1 & d_3 & \cdot \\ \cdot & \nu_2 & d_1 \\ d_2 & \cdot & \nu_3 \end{pmatrix}\right) = \sum_{i=1}^{3} \left[\varepsilon_i \nu_i + C(c_i, d_i)\right],$$

where C denotes the symmetric bilinear form deduced from the norm on C. (When writing elements of A, we replace some entries with a "$\cdot$". No information is lost, because these entries are determined by the condition that the matrix is hermitian.)

We write ${}^1G$ for the group of isometries of the cubic form

$$\begin{pmatrix} \varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3 \end{pmatrix} \mapsto \varepsilon_1\varepsilon_2\varepsilon_3 + \mathrm{trace}_C(c_1c_2c_3) - \sum_{i=1}^{3} \varepsilon_i C(c_i)$$

on A. This group is simply connected quasi-simple of index

${}^1E_{6,2}^{28}$ 

It acts on the 54-dimensional vector space $A \oplus A$ via the homomorphism ρ defined by

$$\rho(g)(a_1, a_2) = (ga_1, g^\dagger a_2),$$

where $g^\dagger$ is defined by the equation $\mathrm{tr}(gx, g^\dagger y) = \mathrm{tr}(x, y)$ for all $x, y \in A$. The map $g \mapsto g^\dagger$ is an automorphism of ${}^1G$ of order 2, see [J, p. 76] or [SV, 7.3.1]; it is outer because it is not the identity on the center of ${}^1G$.

Abbreviate $A \otimes K$ as A_K and write ι for the non-identity k-automorphism of K. Consider the k-space V of elements of $A_K \oplus A_K$ fixed by

$$(a_1, a_2) \mapsto (\tau\iota a_2, \tau\iota a_1),$$

where $\tau \in {}^1G(k)$ is defined by

$$\tau\begin{pmatrix} \varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \pi(c_2) & \cdot \\ \cdot & \varepsilon_3 & \pi(c_1) \\ \pi(c_3) & \cdot & \varepsilon_2 \end{pmatrix},$$

where π denotes the canonical involution on C.

Define G to be the group ${}^1G$ with a twisted $\mathrm{Gal}(K/k)$-action given by

$$\iota * g = \tau\iota(g)^\dagger\tau^{-1} \quad \text{for } g \in G(K),$$

where the action on the right is the usual one on ${}^1G(K)$. Note that ρ is a k-homomorphism $G \to GL(V)$.

2.1. **Proposition.** *The group G constructed above is simply connected quasi-simple of index ${}^2E_{6,1}^{29}$. It is k-isomorphic to the given group G_0.*

Proof. The group G is simply connected quasi-simple of type E_6 because it is so over K. It is of type ${}^2E_6$ because it is obtained by twisting the group ${}^1G$ of inner type by the outer automorphism $\dagger$.

Since C is not K-split, G has index ${}^1E_{6,2}^{28}$ over K. Every circled vertex in the k-index of G must also be circled in the K-index, so G is anisotropic or is of index ${}^2E_{6,1}^{29}$. Although it is easy to show that G is isotropic, we must do some more work in order to identify the semisimple anisotropic kernel of G.

Let Rel be the group of related triples of proper similarities of C as defined in [Ga, §7]; it is a reductive group of type ${}^1D_4$ with a 2-dimensional center. There is an injection $\psi\colon \mathrm{Rel} \to {}^1G$ defined by

$$\psi_{(t_1,t_2,t_3)} \begin{pmatrix} \varepsilon_1 & c_3 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_3 \end{pmatrix} = \begin{pmatrix} \mu(t_1)^{-1}\varepsilon_1 & t_3c_3 & \cdot \\ \cdot & \mu(t_2)^{-1}\varepsilon_2 & t_1c_1 \\ t_2c_2 & \cdot & \mu(t_3)^{-1}\varepsilon_3 \end{pmatrix},$$

where $\mu(t_i) \in k^\times$ satisfies $C(t_ic_i) = \mu(t_i)C(c_i)$ for all $c_i \in C$. We identify Rel with its image in ${}^1G$. We remark that

$$\psi^\dagger_{(t_1,t_2,t_3)} = \psi_{(\mu(t_1)^{-1}t_1,\mu(t_2)^{-1}t_2,\mu(t_3)^{-1}t_3)}.$$

The image of $s\colon \mathbb{G}_m \to {}^1G$ defined by

$$s(\lambda) = \psi_{(1,\lambda,\lambda^{-1})}$$

is a rank 1 torus S in the center of Rel. We claim that Rel is the centralizer in G of S. To see this, consider an element $g \in {}^1G$ that centralizes S. Write e_i for the element of A whose only nonzero entry is a 1 in the (i,i) place. The weight spaces of S in A—e.g., ke_2 and ke_3—are invariant under g. Since $s(\lambda)^\dagger = s(\lambda^{-1})$, the element $g^\dagger$ also commutes with S, hence

$$g(e_3 \times A) = (g^\dagger e_3) \times A = e_3 \times A.$$

The space $e_3 \times A$ is the direct sum of the S-weight spaces

$$ke_1, \quad ke_2, \quad \text{and} \quad \begin{pmatrix} 0 & C & \cdot \\ \cdot & 0 & 0 \\ 0 & \cdot & 0 \end{pmatrix},$$

with weights 0, -2, and -1. Therefore, g leaves the subspace ke_i invariant for all i, hence g is in Rel by [A, p. 254, Cor.].

The map ψ defines a map from a twisted form H of Rel into G, where the twisted ι-action on H sends $(t_1,t_2,t_3) \in H(K)$ to a triple

$$(\pi\iota(\mu(t_1)^{-1}t_1)\pi, \pi\iota(\mu(t_3)^{-1}t_3)\pi, \pi\iota(\mu(t_2)^{-1}t_2)\pi).$$

Since

$$\iota * s(\lambda) = \tau s(\iota(\lambda)^{-1})\tau = s(\iota(\lambda)),$$

S is a rank 1 k-split torus in G. By the preceding paragraph, H is the centralizer in G of S.

Finally, we claim that the semisimple anisotropic kernel of G is isomorphic to the group $\mathrm{Spin}(C^K)$. To see this, note that ψ restricts to an inclusion $\mathrm{Spin}(C) \hookrightarrow {}^1G$, where $\mathrm{Spin}(C)$ consists of the triples (t_1, t_2, t_3) such that $\mu(t_i) = 1$ for all i. This gives rise to an inclusion of a twisted form T of $\mathrm{Spin}(C)$ in G, where ι acts on a related triple (t_1, t_2, t_3) in $T(K)$ via

$$\iota * (t_1, t_2, t_3) = (\pi\iota(t_1)\pi, \pi\iota(t_3)\pi, \pi\iota(t_2)\pi).$$

That is, T is the spin group of the quadratic form obtained by restricting the norm on $C \otimes K$ to the elements fixed by the map $v \mapsto \pi\iota v$, which is C^K by [KMRT, 36.21(1)].

Since $\mathrm{Spin}(C^K)$ is also the semisimple anisotropic kernel of G_0, the last sentence of the proposition follows from Tits's Witt-type theorem [Sp 98, 16.4.2]. □

2.2. *Remark.* Our explicit construction of a group of index ${}^2E_{6,1}^{29}$ is different from the one in Veldkamp's paper. The groups arising as in his 3.3(3) are indeed of index ${}^2E_{6,1}^{29}$ by [GP, 9.6], and all groups of index ${}^2E_{6,1}^{29}$ are obtained as in his 3.3(3) by [GP, 2.8].

3. The subgroup D

Write e for $(e_1, e_1) \in V$, and let D be the subgroup of G that fixes the vector $e \in V$. Over K, it is isomorphic to $\mathrm{Spin}(\langle 1, -1\rangle \oplus C)$ by [Sp 62, Prop. 4].

3.1. **Lemma.** *D is k-isomorphic to the spin group of a 10-dimensional quadratic form of Witt index 1.*

Proof. From the description of D over K, we can conclude that D is—as a k-group—quasi-simple simply connected of type D_5 and of k-rank ≤ 1. The k-rank of D is exactly 1 because D contains the rank 1 k-split torus S from the proof of Prop. 2.1.

To complete the proof, it suffices to show that the vector representation of D is k-defined. Suppose not. Then D is k-isomorphic to the spin group of a 5-dimensional, isotropic skew-hermitian form over a quaternion division k-algebra that is split by K. It follows that the K-rank of D is at least 2, which is a contradiction. □

A classical result from geometric algebra—[D, §II.9(C)]—implies that $D(k)^+$ is all of $D(k)$, see [PR, pp. 409, 410]. The proof of (1.2) is completed by the following lemma, pointed out to me by Gopal Prasad.

3.2. **Lemma.** *Let G' and G be isotropic quasi-simple simply connected k-groups such that G' is a subgroup of G. Then $G'(k)^+$ is contained in $G(k)^+$.*

Proof. Fix a nontrivial k-split torus S in G'. Since S is contained in the split simply connected subgroup of G' (respectively, G) from [BT, 7.2], it follows that $S(k)$ is contained in $G'(k)^+$ (resp., $G(k)^+$). But the conjugates of $S(k)$

under $G'(k)$ (respectively, $G(k)$) generate $G'(k)^+$ (resp., $G(k)^+$) because $G'(k)^+$ (resp., $G(k)^+$) is projectively simple. $\square$

4. Multipliers

Consider the k-subspace $\{(\mu e_1, \iota(\mu)e_1) \mid \mu \in K\}$ of V. It is an H-invariant subspace of V, and for $h \in H$, we define $\gamma(h) \in K^\times$ by

$$\rho(h)e = (\gamma(h)e_1, \iota(\gamma(h))e_1).$$

We have

$$1 = \mathrm{tr}(e_1, e_1) = \mathrm{tr}(\rho(h)e) = \gamma(h)\iota(\gamma(h)),$$

so γ defines a k-homomorphism $H \to R^1_{K/k}$, where $R^1_{K/k}$ denotes the rank 1 torus whose k-points are the norm 1 elements of $K^\times$. The purpose of this section is to prove:

4.1. **Lemma.** *The image of $\gamma : H(k) \to R^1_{K/k}(k)$ consists of elements $\lambda\iota(\lambda)^{-1}$ for $\lambda \in K^\times$ such that $\lambda\iota(\lambda) \in k^\times$ is a norm from C.*

Proof. From the explicit description of H as a twist of Rel, we find that the kernel of γ is generated by $\mathrm{Spin}(C^K)$ and S. Projection on the first entry defines a surjection $\ker\gamma \to SO(C^K)$ whose restriction to $\mathrm{Spin}(C^K)$ is the vector representation. We compute the image of $\lambda\iota(\lambda)^{-1} \in R^1_{K/k}(k)$ under the composition

$$R^1_{K/k}(k) \longrightarrow H^1(k, \ker\gamma) \longrightarrow H^1(k, SO(C^K)), \tag{4.2}$$

where the first map is induced by the short exact sequence

$$1 \longrightarrow \ker\gamma \longrightarrow H \xrightarrow{\gamma} R^1_{K/k} \longrightarrow 1.$$

Viewed as a point of $R^1_{K/k}$ over the separable closure k_{sep} of k, the element $\lambda\iota(\lambda)^{-1}$ is the image of

$$t := \left(\sqrt{N_\lambda}\lambda^{-1}, \sqrt{\lambda}, \sqrt{\lambda N_\lambda^{-1}}\right) \quad \in H(k_{\mathrm{sep}})$$

for some fixed square root of λ in k_{sep} and $N_\lambda := \lambda\iota(\lambda)$. For $\sigma \in \mathrm{Gal}(k_{\mathrm{sep}}/k)$, the first entry of $t^{-1}\sigma(t)$ is $\sqrt{N_\lambda}^{-1}\sigma(\sqrt{N_\lambda})$. That is, the image of $\lambda\iota(\lambda)^{-1}$ under the composition (4.2) is the image of N_λ under the map

$$k^\times/k^{\times 2} \cong H^1(k, Z(SO(C^K))) \to H^1(k, SO(C^K)).$$

If $\lambda\iota(\lambda)^{-1}$ is in the image of $H(k)$, it has trivial image in $H^1(k, \ker\gamma)$ and consequently in $H^1(k, SO(C^K))$. It follows that N_λ is a similarity of C^K. The following proposition completes the proof. $\square$

4.3. **Proposition.** *If a quadratic form ϕ is Witt-equivalent to a form similar to $\sigma + \pi$ where σ is an n-Pfister form and* $\dim\pi < 2^n$, *then every similarity of ϕ is represented by σ.*

Proof. Since ϕ is Witt-equivalent to a form similar to $\sigma + \pi$, the two forms have the same group of similarities. So we may assume that ϕ equals $\sigma + \pi$. Let V_ν be the vector space underlying ν for $\nu = \sigma, \pi$ and put $V := V_\sigma \oplus V_\pi$. We are given an $f \in GL(V)$ such that $\sigma(fv) + \pi(fv) = \mu(\sigma(v) + \pi(v))$ for some $\mu \in k^\times$ and all $v \in V$, and we want to show that μ is represented by σ.

Now $\dim V_\sigma = \dim(f(V_\sigma)) = 2^n$ and $\dim(V_\sigma + f(V_\sigma))$ is at most $\dim V < 2^{n+1}$. Therefore, there are nonzero $x, y \in V_\sigma$ such that $f(x) = y$. Then

$$\mu\sigma(x) = \sigma(f(x)) = \sigma(y).$$

We may assume that σ is anisotropic—otherwise, σ automatically represents μ—hence $\sigma(x)$ and $\sigma(y)$ are nonzero. The set of nonzero elements represented by σ is a group [L, X.1.8], so μ is represented by σ. □

5. Conclusion of the proof

This section contains a proof of (1.3), i.e., we prove:

5.1. Lemma. *$H(k) \subseteq D(k) \cdot G(k)^+$.*

Proof. Fix $h \in H(k)$. We will produce an element g of $G(k)^+$ such that $\rho(g)e = (\gamma(h)e_1, \iota(\gamma(h))e_1)$. Then $g^{-1}h$ will belong to $D(k)$ and the lemma will follow.

By Lemma 4.1, $\gamma(h)$ is of the form $\lambda\iota(\lambda)^{-1}$ for some $\lambda \in K^\times$ such that $\lambda\iota(\lambda)$ is a norm from C. Fix a quadratic subfield ℓ of C such that $\lambda\iota(\lambda)$ is a norm from ℓ. Since C is not split by K, $K \otimes \ell$ is a biquadratic extension field of k which we denote simply by $K\ell$. As described in [J, §5], there is an injective k-homomorphism

$$\phi\colon R_{\ell/k}(SL_3) \to {}^1G \quad \text{via } \phi_g(a) = ga\pi(g)^t,$$

for $g \in SL_3(\ell)$ and $a \in A$, where juxtaposition denotes usual matrix multiplication, t denotes the transpose, and π means to apply the nontrivial ℓ/k-automorphism to the entries of g. Moreover,

$$\phi_g^\dagger = \phi_{\pi(g)^{-t}} \quad \text{and} \quad \tau\phi_g\tau^{-1} = \phi_{(2\,3)g(2\,3)},$$

where $(2\,3)$ denotes the matrix $\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)$. (The first equation is from [J, p. 77]. The second equation is verified in the same manner, i.e., by checking it for elementary matrices g.)

We now twist the $\mathrm{Gal}(K/k)$-action on $R_{\ell/k}(SL_3)$ and ${}^1G$ simultaneously to produce a k-injection $\phi\colon SU \to G$, for some k-group SU. Specifically,

$$R_{\ell/k}(SL_3)(K) = SL_3(K\ell),$$

and for $g \in SL_3(K\ell)$, we put

$$\iota * g = (2\,3)\iota\pi(g)^{-t}(2\,3).$$

Writing M for the subfield of $K\ell$ fixed by $\iota\pi$ and consulting [KMRT, pp. 23ff, 42ff], we recognize that S is the transfer from M to k of the special unitary group of a 3-dimensional, isotropic $K\ell/M$-hermitian form. That is, SU is the transfer from M to k of the quasi-split simply connected group of type ${}^2A_2$.

Because $\lambda\iota(\lambda) \in k^\times$ is a norm from ℓ and K, the Biquadratic Lemma (see e.g. [W, 2.14]) gives an element $\gamma \in K\ell$ such that $\gamma\pi(\gamma) = \alpha\lambda$ for some $\alpha \in k^\times$. Consider the element

$$g := \begin{pmatrix} \gamma\,\iota\pi(\gamma)^{-1} & & \\ & \gamma^{-1} & \\ & & \iota\pi(\gamma) \end{pmatrix} \quad \in SL_3(K\ell).$$

Note that g is in $SU(k)$. On the other hand,

$$\phi_g(e_1) = \gamma\,\iota\pi(\gamma)^{-1}\,\pi(\gamma)\,\iota(\gamma)^{-1} e_1 = (\alpha\lambda)\,\iota(\alpha\lambda)^{-1} e = \lambda\,\iota(\lambda)^{-1} e_1.$$

Since SU is k-quasi-split, $SU(k)^+$ is all of $SU(k)$ [St, §8]. By Lemma 3.2, ϕ_g is in $G(k)^+$. This proves Lemma 5.1, which in turn completes the proof of the main theorem. □

Acknowledgements. I am grateful to Gopal Prasad for raising my interest in this question and for many helpful discussions.

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