GRADED HERMITIAN FORMS AND SPRINGER'S THEOREM

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Abstract. An analogue of Springer's theorem on the Witt group of quadratic forms over a complete discretely valued field is proved for Hermitian forms over division algebras over a Henselian field, including some cases where the residue characteristic is 2. Residue forms are defined by means of vector space valuations as Hermitian forms on the graded modules associated with the induced filtrations.

Introduction

In the algebraic theory of quadratic forms, a fundamental result due to Springer [Sp_1] (see also [S, Ch. 6, §2]) yields an isomorphism from the Witt group of any complete discretely valued field F onto the direct sum of two copies of the Witt group of the residue field $\overline{F}$, provided the characteristic of $\overline{F}$ is different from 2:

$$W(F) \simeq W(\overline{F}) \oplus W(\overline{F}). \tag{0.1}$$

(Springer also considered the case where $\overline{F}$ is a perfect field of characteristic 2 and the characteristic of F is 0, but his result in this case has a different form.) Springer's theorem has been generalized in various ways, most recently by Larmour [L], who proved an analogue for the Witt group of Hermitian or skew-Hermitian forms over division algebras with involution over a field with Henselian valuation with residue characteristic not 2. In this paper we give another approach to Larmour's generalization. We work in terms of valuations on vector spaces and the graded structures arising from the filtrations determined by the valuations. Valuations on vector spaces were used in Springer's original papers [Sp_1], [Sp_2], and also appear in the work of Goldman and Iwahori [GI] and of Bruhat and Tits [BT_1], [BT_2]. But, the use of associated graded structures is new here, and it seems to considerably illuminate the earlier approaches. Besides reproving Larmour's theorem, we are able to prove the analogous result in many cases where the residue characteristic is 2. See Def. 4.1 for a precise description of these cases—they appear to be all the cases where our approach yields a result like Springer's theorem. However, our results do *not* cover the very complicated case of quadratic forms over valued fields of residue characteristic 2, as treated for instance by Jacob in [J] and Aravire and Jacob in [AJ].

We think our approach sheds an interesting light even on the classical case. Indeed, a discrete valuation on a field F defines a $\mathbb{Z}$-filtration whose associated graded ring $gr(F)$ is $\overline{F}[t, t^{-1}]$, the ring of Laurent polynomials in one indeterminate over the residue field; for a suitably defined Witt ring $W_g\big(gr(F)\big)$ of graded forms over $gr(F)$, the isomorphism (0.1) can be viewed as an isomorphism

$$W(F) \simeq W_g\big(gr(F)\big).$$

The graded rings associated with the filtration induced by a valuation on a division algebra have the property that every homogeneous element is invertible; they are therefore called *graded division rings* (although they are not division rings). The first section develops the theory of graded Hermitian forms over graded division rings with involution. It is well-known (cf. [S, Ch. 7], [K, Ch. 1, §6]) that the fundamental properties of Hermitian forms over a division ring of characteristic not 2, such as Witt cancellation, hold also for even (also called trace-valued) forms over a division ring of characteristic 2.

The second author is partially supported by the National Fund for Scientific Research (Belgium) and by the European Community under contract HPRN-CT-2002-00287, KTAGS. The third author would like to thank the second author and UCL for their hospitality while the work for this paper was carried out.

We show in Prop. 1.4 that analogues of these fundamental properties hold also for graded Hermitian forms over a graded division ring; again, we must restrict to even forms when the characteristic is 2. It is convenient to state our results in terms of even forms. But, "even" is only a restriction in characteristic 2; for, when the characteristic is different from 2, all forms are even. Given a graded division ring E with torsion-free abelian grade group Γ_E, an involution σ on E preserving the gradation, and a central element ϵ such that $\epsilon\sigma(\epsilon) = 1$, we define the Witt group $W_g^+(E, \sigma, \epsilon)$ of even nondegenerate graded ϵ-Hermitian forms for σ over E. Graded ϵ-Hermitian forms have a canonical orthogonal decomposition determined by the grade group, which yields a (non-canonical) decomposition of $W_g^+(E, \sigma, \epsilon)$ into a direct sum indexed by $\frac{1}{2}\Gamma_E/\Gamma_E$ of Witt groups of the homogeneous component of E of degree 0, with respect to various involutions, see Prop. 1.5. The main difference in the graded setting is that graded hyperbolic planes are not all isometric: They are isometric if and only if they have the same grade set.

In Section 2 we discuss value functions, which are analogues of valuations for vector spaces, and their associated graded vector spaces. For a vector space M over a division ring D, the most useful value functions $\alpha\colon M \to \Gamma \cup \{\infty\}$ are those for which there exists a base $\{ m_1, \ldots, m_k \}$ such that

$$\alpha\big(\sum_{i=1}^{k} m_i d_i\big) \;=\; \min_{1\le i\le k}\big(\alpha(m_i d_i)\big)\,.$$

Such a base is called a *splitting base* of α, and value functions for which there exists a splitting base are called *norms*. Given any two norms α and β on a vector space M, we show the existence of a common splitting base (Th. 2.8) and use it to define a norm which we call the average of α and β. Our principal results in this section are known in the complete discrete case; they were observed by Goldman and Iwahori [GI] and by Bruhat and Tits [BT$_1$].

The main results of this paper are given in Sections 3 and 4. In Section 3, we consider norms α on vector spaces over a valued division algebra D with involution τ which are compatible with a given λ-Hermitian form h, in the sense that there is an induced nondegenerate graded λ'-Hermitian form h'_α on the associated graded vector space (for the induced involution τ' on $gr(D)$). However, there is a fundamental obstruction when the residue characteristic is 2, in that the form h'_α induced by an even form h may not be even. In Prop. 3.15, we spell out conditions on the valuation and on the pair (τ, λ) which guarantee that the form h'_α is even for every even form h and every compatible norm α. Under these conditions, we show in Th. 3.11 that the Witt equivalence class of h'_α does not depend on the choice of compatible norm α, and that the correspondence $h \mapsto h'_\alpha$ yields a well-defined and canonical group epimorphism

$$\Theta\colon\ W^+(D, \tau, \lambda) \to W_g^+(gr(D), \tau', \lambda'). \tag{0.2}$$

The graded form h'_α may be viewed as a generalized residue form of h; it actually encapsulates *all* the residue forms of h, which appear as the components in the canonical orthogonal decomposition of h'_α.

The notion of compatible norm is due to Springer [Sp$_1$], [Sp$_2$], though it was not expressed in terms of associated graded forms. This notion also appears in [GI] and [BT$_2$]. Our definition in Def. 3.1 follows [BT$_2$] rather than [Sp$_2$] and [GI] in that we require that $\alpha(m) + \alpha(n) \le v\big(h(m, n)\big)$ for all m, n in the vector space, instead of $2\alpha(m) \le v\big(h(m, m)\big)$.

The results in Section 3 do not require a Henselian hypothesis. In Section 4, we obtain the analogue of Springer's theorem, Th. 4.6, which asserts that the map Θ of (0.2) is an isomorphism when the subfield of the center of D fixed under τ is Henselian and the residue characteristic is different from 2. Furthermore, a form is anisotropic iff its associated graded form is anisotropic. These results also hold in the good cases when the residue characteristic is 2: For these, we need a tameness assumption on D and that the isometry group of the forms be of unitary or symplectic type, see Def. 4.1. When D is tame, the good cases for residue characteristic 2 are exactly those cases where induced graded forms of even forms are always even. Finally, under the same hypotheses as in our generalization of Springer's theorem, we show in Prop. 4.9 that the residues h'_α, ℓ'_β of two Hermitian forms h, ℓ with respect to compatible norms α, β are isometric if and only if there is an isometry between h and ℓ which preserves the norms.

1. Graded division rings, vector spaces, and Hermitian forms

Let Γ be a divisible torsion-free abelian group. Let $E = \bigoplus_{\gamma\in\Gamma} E_\gamma$ be a Γ-graded ring, i.e., E is an associative ring with each E_γ an additive subgroup of E and $E_\gamma \cdot E_\delta \subseteq E_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. The set of homogeneous elements of E is $E^h = \bigcup_{\gamma\in\Gamma} E_\gamma$. The grade set of E is $\Gamma_E = \{\, \gamma \in \Gamma \mid E_\gamma \neq (0)\,\}$.

Assume now that the graded ring E is a *graded division ring*, i.e., every nonzero homogeneous element of E is a unit. Then, E_0 is a division ring, and for each $\gamma \in \Gamma_E$, E_γ is a 1-dimensional left and right E_0-vector space. Also, Γ_E is a subgroup of Γ. Note that the center of E, denoted $Z(E)$, inherits a grading from E, and $Z(E)$ is a graded field, i.e, a commutative graded division ring.

Let $S = \bigoplus_{\gamma\in\Gamma} S_\gamma$ be a graded right E-module; that is, S is a right E-module with each S_γ an additive subgroup of S and $S_\gamma \cdot E_\delta \subseteq S_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. The homogeneous elements of S are those in $\bigcup_{\gamma\in\Gamma} S_\gamma$. Since E is a graded division ring, slight variations of the usual ungraded arguments show: S is a free E-module with a base consisting of homogeneous elements; every two such bases have the same cardinality; every homogeneous generating set of S as an E-module contains a base; every set of E-independent homogeneous elements of S can be enlarged to a homogeneous base. All this is easy to prove, and is well-known (see, e.g. [HW$_2$, §1]). Because of these analogues with the ungraded case, S is called a *graded right E-vector space*, and $dim_E(S)$ is defined to be the number of elements in any homogeneous base of S.

The grade set of S is $\Gamma_S = \{\, \gamma \in \Gamma \mid S_\gamma \neq (0)\}$. Note that Γ_S need not be a subgroup of Γ, but it is a union of cosets of Γ_E. Indeed, there is a canonical decomposition of S according to the cosets of Γ_E in Γ_S: For $\gamma \in \Gamma$, let $[\gamma] = \gamma + \Gamma_E \subseteq \Gamma$. Let

$$S_{[\gamma]} \ = \bigoplus_{\delta\in\Gamma_E} S_{\gamma+\delta}\,.$$

Then, $S_{[\gamma]}$ is clearly a graded E-subspace of S, and if $S_{[\gamma]} \neq (0)$, then $\Gamma_{S_{[\gamma]}} = [\gamma]$. We call $S_{[\gamma]}$ the *$[\gamma]$-component* of S. Observe that $dim_E(S_{[\gamma]}) = dim_{E_0}(S_{\gamma+\delta})$ for each $\delta \in \Gamma_E$. We have

$$S \ = \bigoplus_{[\gamma]\in\Gamma_S/\Gamma_E} S_{[\gamma]}\,. \tag{1.1}$$

It is easy to see that if T is another graded right E-vector space, then $T \cong S$ (graded, i.e., grade-preserving, E-vector space isomorphism) iff $\Gamma_T = \Gamma_S$ and $dim_E(T_{[\gamma]}) = dim_E(S_{[\gamma]})$ for each $\gamma \in \Gamma_T$.

Let $\sigma\colon E \to E$ be a *graded involution* on E, i.e., σ is an antiautomorphism of E with $\sigma\circ\sigma = id$ and $\sigma(E_\gamma) = E_\gamma$ for each $\gamma \in \Gamma_E$. As usual, σ is said to be of the *first kind* if $\sigma|_{Z(E)} = id$, and of the *second kind* otherwise. (If σ is of the first kind, then it is of either orthogonal type or symplectic type. This is discussed after Remark. 3.12 below.) Take any $\epsilon \in Z(E)_0$ with $\epsilon\sigma(\epsilon) = 1$. (Of course, if σ is of the first kind, then necessarily $\sigma(\epsilon) = \epsilon$, so $\epsilon = \pm1$.) A *graded ϵ-Hermitian form* for σ on a finite-dimensional graded right E-vector space S is a bi-additive function $k\colon S \times S \to E$ such that for all $s, t \in S$, $c, d \in E$, $\gamma, \delta \in \Gamma_S$,

$$k(sc, td) \ = \ \sigma(c)k(s,t)d\,; \tag{1.2a}$$

$$k(t,s) \ = \ \epsilon\sigma(k(s,t))\,; \tag{1.2b}$$

$$k(S_\gamma, S_\delta) \ \subseteq \ E_{\gamma+\delta}\,. \tag{1.2c}$$

Let T be a graded subspace of S, i.e., T is an E-submodule of S with $T = \bigoplus_{\gamma\in\Gamma} T_\gamma$, with each T_γ a subgroup of S_γ. Set $T^\perp = \{\, s \in S \mid k(s,t) = 0 \text{ for all } t \in T\,\}$. Clearly $T^\perp$ is a graded subspace of S. As in the ungraded case, we have $dim_E(T) + dim_E(T^\perp) = dim_E(S) + dim_E(S^\perp)$; hence $T^{\perp\perp} = T$ for any graded subspace T of S. We say that k is *nondegenerate* if $S^\perp = (0)$. We say that k is *isotropic* if it has an isotropic vector, i.e., a nonzero $s \in S$ with $k(s,s) = 0$. A significant fact that follows from the assumption that Γ is torsion-free is that whenever k is isotropic, it has a *homogeneous* isotropic vector. For, since Γ is torsion-free, it can be given a total ordering making it into an ordered abelian group. Then, with respect to this ordering, any nonzero $s = \sum s_\gamma$ (with each $s_\gamma \in S_\gamma$) has a leading term, which is the nonzero s_δ with the smallest δ. Clearly, if s is isotropic, then its leading term is a homogeneous isotropic vector.

If k is nondegenerate, we say that k is *metabolic* if S has a totally isotropic graded subspace T (i.e., $k(T,T) = 0$) with $dim_E(T) = \frac{1}{2}dim(S)$. We say that k is *hyperbolic* if it is nondegenerate and S has two complementary totally isotropic graded subspaces. Clearly, every hyperbolic space is an orthogonal sum of two-dimensional hyperbolic graded subspaces. If $\ell\colon U \times U \to E$ is another graded ϵ-Hermitian form for σ, we write $k \cong \ell$ if k and ℓ are *graded isometric*, i.e., there is a graded (i.e., grade-preserving) E-vector space isomorphism $f\colon S \to U$ with f an isometry between k and ℓ. We write $k \perp \ell$ for the orthogonal sum of k an ℓ on $S \oplus U$: $(k \perp \ell)\big((s,u),(s',u')\big) = k(s,s') + \ell(u,u')$.

For any $s \in S$, condition (1.2b) shows that $k(s,s) = \epsilon\sigma(k(s,s))$. We say that the form k is *even* if

$$\text{for every } s \in S \text{ there is } c \in E \text{ with} \quad k(s,s) = c + \epsilon\sigma(c)\,. \tag{1.3}$$

If $char(E) \neq 2$, then every form is even (take $c = \frac{1}{2}k(s,s)$). This is also true whenever σ is of the second kind. (For, then there is $z \in Z(E)$ with $z + \sigma(z) = 1$. Then take $c = zk(s,s)$.) Just as in the ungraded case, we will see that many results holding when $char(E) \neq 2$ continue to be true for even forms when $char(E) = 2$.

The compatibility of the graded Hermitian form k with the gradings on S and E assures that k is well-behaved with respect to the canonical decomposition (1.1) of S. Note that because $E_\rho = (0)$ for $\rho \notin \Gamma_E$, condition (1.2c) shows

$$k(S_{[\gamma]}, S_{[\delta]}) = 0 \qquad \text{whenever } \gamma + \delta \notin \Gamma_E\,. \tag{1.4}$$

For $\gamma \in \Gamma$, we write $k_{[\gamma]}$ for $k|_{S_{[\gamma]}}$.

PROPOSITION 1.1. *Assume k is nondegenerate. Then,*

(i) *If $\gamma \in \frac{1}{2}\Gamma_E$, then $k_{[\gamma]}$ is nondegenerate and ${S_{[\gamma]}}^\perp = \bigoplus_{[\delta]\neq[\gamma]} S_{[\delta]}$.*

(ii) *If $\gamma \notin \frac{1}{2}\Gamma_E$, then $k_{[\gamma]}$ is totally isotropic, ${S_{[\gamma]}}^\perp = \bigoplus_{[\delta]\neq[-\gamma]} S_{[\delta]}$, $dim_E(S_{[-\gamma]}) = dim_E(S_{[\gamma]})$, and $k|_{S_{[\gamma]}+S_{[-\gamma]}}$ is nondegenerate and hyperbolic.*

(iii) *$S = \perp_{[\gamma]\in\frac{1}{2}\Gamma_E/\Gamma_E} S_{[\gamma]} \;\perp\; \perp_{[\delta]}(S_{[\delta]} + S_{[-\delta]})$, where the second orthogonal sum is taken with one summand for each pair $[\delta], [-\delta]$ with $[\delta] \notin \frac{1}{2}\Gamma_E/\Gamma_E$.*

(iv) *k is anisotropic iff $\Gamma_S \subseteq \frac{1}{2}\Gamma_E$ and $k_{[\gamma]}$ is anisotropic for each $\gamma \in \frac{1}{2}\Gamma_E$.*

(v) *k is metabolic (resp. hyperbolic, resp. even) iff $k_{[\gamma]}$ is metabolic (resp. hyperbolic, resp. even) for each $\gamma \in \frac{1}{2}\Gamma_E$.*

Proof. (i) Suppose $\gamma \in \frac{1}{2}\Gamma_E$. Formula (1.4) shows that $\bigoplus_{[\delta]\neq[\gamma]} S_{[\delta]} \subseteq S_{[\gamma]}^\perp$; this inclusion is an equality by dimension count. Then, $k_{[\gamma]}$ is nondegenerate, as $S_{[\gamma]} \cap S_{[\gamma]}^\perp = (0)$.

(ii) Suppose $\gamma \notin \frac{1}{2}\Gamma_E$. In this case, formula (1.4) shows that $\bigoplus_{[\delta]\neq[-\gamma]} S_{[\delta]} \subseteq S_{[\gamma]}^\perp$. Hence,

$$dim_E(S_{[\gamma]}) \;=\; dim_E(S) - dim_E(S_{[\gamma]}^{\perp}) \;\leq\; dim_E(S) - \sum_{[\delta]\neq[-\gamma]} dim_E(S_{[\delta]}) \;=\; dim_E S_{[-\gamma]}\,. \tag{1.5}$$

The same argument, using $-\gamma$ in place of γ, shows the reverse inequality to (1.5). Hence, $dim_E(S_{[\gamma]}) = dim_E(S_{[-\gamma]})$, and equality holds in (1.5). Therefore, the inclusion for $S_{[\gamma]}^{\perp}$ is an equality. Then, $\big(S_{[\gamma]} + S_{[-\gamma]}\big)^{\perp} = S_{[\gamma]}^{\perp} \cap S_{[-\gamma]}^{\perp} = \bigoplus_{[\delta]\neq[\pm\gamma]} S_{[\delta]}$. Since this shows $\big(S_{[\gamma]} + S_{[-\gamma]}\big) \cap \big(S_{[\gamma]} + S_{[-\gamma]}\big)^{\perp} = (0)$, $k|_{S_{[\gamma]}+S_{[-\gamma]}}$ is nondegenerate. $S_{[\gamma]} + S_{[-\gamma]}$ is hyperbolic since it contains the complementary totally isotropic graded subspaces $S_{[\gamma]}$ and $S_{[-\gamma]}$.

(iii) is clear from (1.4).

(iv) is clear from (ii) and the fact that if k is isotropic, then it contains a homogeneous isotropic vector.

(v) Suppose k is metabolic, say with totally isotropic graded subspace W with $dim_E(W) = \frac{1}{2}dim_E(S)$. For each $\gamma \in \frac{1}{2}\Gamma_E$, $W_{[\gamma]}$ is a totally isotropic subspace of $S_{[\gamma]}$ with respect to the nondegenerate form $k_{[\gamma]}$, so $dim_E(W_{[\gamma]}) \leq \frac{1}{2}dim_E(S_{[\gamma]})$. Likewise, $dim_E(W_{[\delta]}+W_{[-\delta]}) \leq \frac{1}{2}dim_E(S_{[\delta]}+S_{[-\delta]})$ for each $\delta \in \Gamma - \frac{1}{2}\Gamma_E$. Since $\sum_{\rho\in\Gamma} dim_E(W_{[\rho]}) = dim_E(W) = \frac{1}{2}\sum_{\rho\in\Gamma} dim_E(S_{[\rho]})$, all these inequalities must be equalities. Hence, $k_{[\gamma]}$ is metabolic for each $\gamma \in \frac{1}{2}\Gamma_E$. Likewise, if k is hyperbolic, with complementary totally isotropic graded subspaces W and U, then for each $\gamma \in \frac{1}{2}\Gamma_E$, $W_{[\gamma]}$ and $U_{[\gamma]}$ are complementary totally isotropic subspaces of $S_{[\gamma]}$, so $k_{[\gamma]}$ is hyperbolic. Also, any subform of an even form is even. This proves one direction of (v). The converse is clear using (ii), since any orthogonal sum of metabolic (resp. hyperbolic, resp. even) forms is metabolic (resp. hyperbolic, resp. even) and any hyperbolic form is even. □

We now show how graded Hermitian forms for E for σ are related to Hermitian forms over the division ring E_0 with respect to various involutions on E_0.

Let

$\mathcal{GH}(E,\sigma,\epsilon)$ be the category of pairs (S,k) where S is a finite-dimensional graded right E-vector space and $k\colon S\times S \to E$ is a nondegenerate graded ϵ-Hermitian form on S for σ; the morphisms are graded isometries.

For any $\gamma \in \frac{1}{2}\Gamma_E$, let

$\mathcal{GH}(E,\sigma,\epsilon;[\gamma])$ be the category of pairs $(S,k) \in \mathcal{GH}(E,\sigma,\epsilon)$ with $\Gamma_S = \gamma+\Gamma_E$ or $S=(0)$; the morphisms are graded isometries.

For any involution $\widehat{\sigma}$ on E_0 and any $\widehat{\epsilon}$ in E_0 with $\widehat{\epsilon}\widehat{\sigma}(\widehat{\epsilon}) = 1$, let

$\mathcal{H}(E_0,\widehat{\sigma},\widehat{\epsilon})$ be the category of pairs (U,h) where U is a finite-dimensional right E_0-vector space and $h\colon U\times U \to E_0$ is a nondegenerate $\widehat{\epsilon}$-Hermitian form on U for $\widehat{\sigma}$; the morphisms are isometries.

Let

$\mathcal{GH}^+(E,\sigma,\epsilon)$ be the full subcategory of $\mathcal{GH}(E,\sigma,\epsilon)$ consisting of pairs (S,k) with k even.

Likewise, define $\mathcal{GH}^+(E,\sigma,\epsilon;[\gamma])$ (resp. $\mathcal{H}^+(E_0,\widehat{\sigma},\widehat{\epsilon})$) to be the subcategory of even forms in $\mathcal{GH}(E,\sigma,\epsilon;[\gamma])$ (resp. $\mathcal{H}(E_0,\widehat{\sigma},\widehat{\epsilon})$). Recall that if $char(E) \neq 2$, then $\mathcal{GH}^+(E,\sigma,\epsilon) = \mathcal{GH}(E,\sigma,\epsilon)$, and likewise for the other two plus categories. We will write $(S,k) \in \mathcal{GH}(E,\sigma,\epsilon)$ if (S,k) is an object in this category. We often abbreviate (S,k) to k.

If S is any graded right E-vector space and $\delta \in \Gamma$, let $S(\delta)$ denote the δ-shift of S, i.e., $S(\delta) = S$ as a right E-vector space, but with the grading shifted according to the rule

$$S(\delta)_\gamma \;=\; S_{\gamma+\delta}\,.$$

Clearly, $\mathit{dim}_E(S(\delta)) = \mathit{dim}_E(S)$ and $\Gamma_{S(\delta)} = -\delta + \Gamma_S$.

Note that for any $\gamma \in \Gamma_E$ there is a nonzero $r \in E_\gamma$ with $\sigma(r) = \pm r$. For, take any nonzero $s \in E_\gamma$. We can choose $r = s + \sigma(s)$ if this is nonzero. Otherwise, choose $r = s$. In either case, there is a new graded involution $\widetilde{\sigma}$ on E, given by $\widetilde{\sigma} = \mathit{int}(r) \circ \sigma$, i.e., $\widetilde{\sigma}(c) = r\sigma(c)r^{-1}$. If $\gamma = -2\rho$ with $\rho \in \Gamma_E$, and $(S,k) \in \mathcal{GH}(E,\sigma,\epsilon)$, then we can define a form $\widetilde{k}$ on the shifted space $S(\rho)$ by $\widetilde{k}(s,t) = rk(s,t)$. It is easy to check that $\widetilde{k}$ is a graded $\widetilde{\epsilon}$-Hermitian form for $\widetilde{\sigma}$, where $\widetilde{\epsilon} = \epsilon$ if $\sigma(r) = r$ and $\widetilde{\epsilon} = -\epsilon$ if $\sigma(r) = -r$; that is, $\widetilde{\epsilon} = \epsilon\sigma(r)r^{-1}$.

Proposition 1.2.

(i) *Let σ_0 be the restriction of σ to E_0, so σ_0 is an involution on the division ring E_0. There is a canonical equivalence of categories $\Psi\colon \mathcal{GH}(E,\sigma,\epsilon;[0]) \to \mathcal{H}(E_0,\sigma_0,\epsilon)$ given by $(S,k) \mapsto (S_0, k|_{S_0})$.*

(ii) *For any $\gamma,\rho \in \frac{1}{2}\Gamma_E$, choose any nonzero $r \in E_{-2\rho}$ with $\sigma(r) = \pm r$. Let $\widetilde{\sigma} = \mathit{int}(r)\circ\sigma$ and $\widetilde{\epsilon} = \epsilon\sigma(r)r^{-1}$. There is an equivalence of categories $\mathcal{GH}(E,\sigma,\epsilon;[\gamma]) \to \mathcal{GH}(E,\widetilde{\sigma},\widetilde{\epsilon};[\gamma-\rho])$ given by $(S,k) \mapsto (S(\rho),\widetilde{k})$, where $\widetilde{k}(s,t) = rk(s,t)$.*

These equivalences respect dimension and orthogonal sums and send anisotropic (resp. metabolic, resp. hyperbolic, resp. even) forms to anisotropic (resp. metabolic, resp. hyperbolic, resp. even) forms.

Proof. (i) If $(S,k) \in \mathcal{GH}(E,\sigma,\epsilon;[0])$ then $(S_0,k|_{S_0}) \in \mathcal{H}(E_0,\sigma_0,\epsilon)$. (To see that $k|_{S_0}$ is nondegenerate, note that S_0 generates S as a graded E-vector space; so for $s \in S_0$, $k(s,S_0) = 0$ implies $k(s,S) = 0$, hence $s = 0$.) The functor in the reverse direction $\Theta\colon \mathcal{H}(E_0,\sigma_0,\epsilon) \to \mathcal{GH}(E,\sigma,\epsilon;[0])$ is given by scalar extension: Map $(U,\ell) \in \mathcal{H}(E_0,\sigma_0,\epsilon)$ to $(U\otimes_{E_0}E, k) \in \mathcal{GH}(E,\sigma,\epsilon;[0])$, where $U\otimes_{E_0}E = \bigoplus_{\gamma\in\Gamma} U_\gamma$ with $U_\gamma \cong U\otimes_{E_0}E_\gamma$; k is the scalar extension of ℓ, defined by, for all $s_i,t_j \in U_0, d_i,c_j \in E$,

$$k\big(\sum_i s_i\otimes d_i,\ \sum_j t_j\otimes c_j\big) \;=\; \sum_{i,j}\sigma(d_i)\ell(s_i,t_j)c_j\,.$$

It is routine to verify that k is well-defined and satisfies axioms (1.2a)–(1.2c), and that $k|_{(U\otimes_{E_0}E)_0} \cong \ell$ under the canonical isomorphism $(U\otimes_{E_0}E)_0 \cong U$. If k were degenerate, then the 0-component of the graded vector space $(U\otimes_{E_0}E)^\perp$ would correspond to a nonzero E_0-vector space in $U^\perp$, contrary to the nondegeneracy of ℓ. Clearly, the compositions $\Theta\circ\Psi$ and $\Psi\circ\Theta$ are isomorphic to the identity functors on $\mathcal{GH}(E,\sigma,\epsilon;[0])$ and $\mathcal{H}(E_0,\sigma_0,\epsilon)$, so we have the desired equivalence of categories.

(ii) This is clear. The inverse morphism $\mathcal{GH}(E,\widetilde{\sigma},\widetilde{\epsilon};[\gamma-\rho]) \to \mathcal{GH}(E,\sigma,\epsilon;[\gamma])$ is given by $(S,\widetilde{k}) \mapsto (S(-\rho),\widehat{k})$, where $\widehat{k}(s,t) = r^{-1}\widetilde{k}(s,t)$. □

If U is any right E_0-vector space, let $U^* = \mathit{Hom}_{E_0}(U,E_0)$, made into a right E_0-vector space via σ_0, i.e., for $u^* \in U^*$, $y \in U$, and $c \in E_0$, $(u^*c)(y) = \sigma_0(c)u^*(y)$.

Remark 1.3. Take any $\gamma \in \Gamma$ with $\gamma \notin \frac{1}{2}\Gamma_E$ and any finite-dimensional right graded E-vector space S with $\Gamma_S = [\gamma]\cup[-\gamma]$ and $\mathit{dim}_{E_0}(S_\gamma) = \mathit{dim}_{E_0}(S_{-\gamma})$. If $k\colon S\times S \to E$ is any nondegenerate graded ϵ-Hermitian form for σ then there is an E_0-vector space isomorphism $\varphi\colon S_\gamma \to (S_{-\gamma})^*$ given by

$$\varphi(s)(t) = k(s,t). \tag{1.6}$$

Then, k is completely determined by φ, and every E_0-isomorphism $\varphi\colon S_\gamma \to (S_{-\gamma})^*$ determines a unique ϵ-Hermitian form k on S for σ such that (1.6) holds, given by $k(s_1+t_1,s_2+t_2) = \varphi(s_1,t_2)+\epsilon\sigma(\varphi(s_2,t_1))$ for all $s_i \in S_{[\gamma]}$, $t_i \in S_{[-\gamma]}$. Details are left to the reader.

Much of the theory of (even) Hermitian forms over division rings carries over to graded Hermitian forms over graded division rings. We collect here some of the basic properties that we will need below. The principal difference in the graded setting is that hyperbolic forms of the same dimension need not be isometric. (One needs also that the underlying graded vector spaces be graded isomorphic.)

Proposition 1.4.

(i) (*isometry extension*) *Let* $(S,k) \in \mathcal{GH}^+(E,\sigma,\epsilon)$. *Suppose* T *and* U *are graded subspaces of* S *such that* $k|_T$ *is nondegenerate, and that there is a graded isomorphism* $f\colon T \to U$ *which is a graded isometry between* $k|_T$ *and* $k|_U$. *Then, there is a graded isomorphism* $g\colon S \to S$ *which is an isometry for* k, *such that* $g|_T = f$.

(ii) (*Witt Cancellation*) *For* $k_1, k_2, k_3 \in \mathcal{GH}^+(E,\sigma,\epsilon)$, *if* $k_1 \perp k_3 \cong k_2 \perp k_3$, *then* $k_1 \cong k_2$.

(iii) *If* $(S,k) \in \mathcal{GH}^+(E,\sigma,\epsilon)$ *and* k *is metabolic, with a totally isotropic graded subspace* W *with* $dim_E(W) = \frac{1}{2}dim_E(S)$, *then* W *has a complementary totally isotropic subspace in* S; *hence,* S *is hyperbolic.*

(iv) (*diagonalizability*) *Every anisotropic form in* $\mathcal{GH}^+(E,\sigma,\epsilon)$ *is isometric to an orthogonal sum of 1-dimensional forms.*

(v) *For any* $k \in \mathcal{GH}^+(E,\sigma,\epsilon)$, $k \perp -k$ *is hyperbolic.*

(vi) *For any* $k \in \mathcal{GH}^+(E,\sigma,\epsilon)$, *we have* $k \cong k_{an} \perp k_{hyp}$, *where* k_{an} *is anisotropic and* k_{hyp} *is hyperbolic. The forms* k_{an} *and* k_{hyp} *are unique up to isometry.*

(vii) *If* $k_1, k_2, \ell_1, \ell_2 \in \mathcal{GH}^+(E,\sigma,\epsilon)$ *with* ℓ_1 *and* ℓ_2 *hyperbolic and* $k_1 \perp \ell_1 \cong k_2 \perp \ell_2$, *then* $k_{1\,an} \cong k_{2\,an}$.

(viii) *For any* $k_1, k_2 \in \mathcal{GH}^+(E,\sigma,\epsilon)$, *if* $k_1 \perp -k_2$ *is hyperbolic, then* $k_{1\,an} \cong k_{2\,an}$.

Proof. These can presumably be proved by mimicking the ungraded proofs as in [K] or [S]. But, that is not necessary since we will instead use the ungraded results and apply Prop. 1.2 to get the corresponding graded ones.

(i) In view of Prop. 1.1, it suffices to prove that (a) for each $\gamma \in \frac{1}{2}\Gamma_E$ the isometry $f|_{T_{[\gamma]}} : T_{[\gamma]} \to U_{[\gamma]}$ extends to an isometry $S_{[\gamma]} \to S_{[\gamma]}$; and (b) for each $\delta \in \Gamma$, $\delta \notin \frac{1}{2}\Gamma_E$, the isometry $T_{[\delta]} + T_{[-\delta]} \to U_{[\delta]} + U_{[-\delta]}$ extends to an isometry $S_{[\delta]} + S_{[-\delta]} \to S_{[\delta]} + S_{[-\delta]}$. For case (a), we have from [K, Cor. (6.4.5)] that the isometry extension result holds for forms in $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$ for all involutions $\widehat{\sigma}$ on E_0. Hence by Prop. 1.2(i) we have isometry extension for forms in $\mathcal{GH}^+(E, \widetilde{\sigma}, \widetilde{\epsilon}; [0])$ for all graded involutions $\widetilde{\sigma}$ on E. Hence, by Prop. 1.2(ii) isometry extension holds for forms in $\mathcal{GH}^+(E,\sigma,\epsilon;[\gamma])$ for any γ in $\frac{1}{2}\Gamma_E$. This settles case (a). For case (b), take any $\delta \in \Gamma$ with $\delta \notin \frac{1}{2}\Gamma_E$, and without loss of generality assume $S = S_{[\delta]} + S_{[-\delta]}$; then, $dim_E(S_{[\delta]}) = dim_E(S_{[-\delta]})$, as k is nondegenerate. Because $f|_{T_{[\delta]}+T_{[-\delta]}}$ is nondegenerate, we have $T_{[\delta]} \cap (T_{[-\delta]})^\perp = (0)$, so by dimension count $S_{[\delta]} = T_{[\delta]} \oplus (T_{[-\delta]})^\perp$; likewise, $S_{[\delta]} = U_{[\delta]} \oplus (U_{[-\delta]})^\perp$. Let $f_1\colon (T_{[-\delta]})^\perp \to (U_{[-\delta]})^\perp$ be any graded E-vector space isomorphism, and let $g_0 = f|_{T_{[\delta]}} \oplus f_1\colon T_{[\delta]} \oplus (T_{[-\delta]})^\perp \to U_{[\delta]} \oplus (U_{[-\delta]})^\perp$. Let $\varphi\colon S_{[\delta]} \to S^*_{[-\delta]} = Hom_E(S_{[-\delta]}, E)$ be the map given by $\varphi(s)(t) = k(s,t)$; so φ is an E-isomorphism when $S^*_{[-\delta]}$ is made a right E-module via τ. Let $g_1\colon S_{[-\delta]} \to S_{[-\delta]}$ be the unique graded E- isomorphism satisfying the condition that for the dual map $g_1^*\colon S^*_{[-\delta]} \to S^*_{[-\delta]}$ we have $g_1^* = \varphi \circ g_0^{-1} \circ \varphi^{-1}$. Set $g = g_0 \oplus g_1\colon S_{[\delta]} + S_{[-\delta]} \to S_{[\delta]} + S_{[-\delta]}$. The condition $g_1^* \circ \varphi \circ g_0 = \varphi$ says that g is an isometry for k. The definition of g gives $g|_{T_{[\delta]}} = f|_{T_{[\delta]}}$. We need to verify the same equality on $T_{[-\delta]}$. Since g is a k-isometry and $g((T_{[-\delta]})^\perp) = (U_{[-\delta]})^\perp$, we have $g(T_{[-\delta]}) = U_{[-\delta]}$. Take any $t \in T_{[-\delta]}$ and any $u \in U_{[\delta]}$, and let $t' = f^{-1}(u) \in T_{[\delta]}$. Then,

$$k(u, g(t)) = k(f(t'), g(t)) = k(g(t'), g(t)) = k(t', t) = k(f(t'), f(t)) = k(u, f(t))\,.$$

Thus, $g(t) - f(t) \in U_{[-\delta]} \cap (U_{[\delta]})^\perp = (0)$. Hence, $g|_{T_{[-\delta]}} = f|_{T_{[-\delta]}}$, completing the verification that $g|_T = f$. This completes case (b).

(ii) This is immediate from (i).

(iii) Let W be a graded subspace of S which is totally isotropic with respect to the form k, with $dim_E(W) = \frac{1}{2}dim_E(S)$. Then, for each $\gamma \in \frac{1}{2}\Gamma_E$, the proof of Prop. 1.1(v) shows that $W_{[\gamma]}$ is a totally isotropic subspace of $S_{[\gamma]}$ with $dim_E(W_{[\gamma]}) = \frac{1}{2}dim_E(S_{[\gamma]})$. Because maximal totally isotropic subspaces of metabolic forms in $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$ have complementary totally isotropic subspaces [K, proof of Prop. (3.7.1)], it follows by Prop. 1.2 that this is also true in $\mathcal{GH}^+(E, \sigma, \epsilon; [\gamma])$. Hence, there is a totally isotropic graded subspace $U_{[\gamma]}$ of $S_{[\gamma]}$ which is complementary to $W_{[\gamma]}$. Also, for $\delta \notin \frac{1}{2}\Gamma_E$, the proof of Prop. 1.1(v) shows that $dim_E(W_{[\delta]} + W_{[-\delta]}) = \frac{1}{2}dim_E(S_{[\delta]} + S_{[-\delta]})$. Since $W_{[-\delta]} \subseteq W_{[\delta]}^{\perp} \cap S_{[-\delta]}$, dimension count shows that this inclusion is an equality. Choose any complementary graded subspace $U_{[\delta]}$ of $W_{[\delta]}$ in $S_{[\delta]}$, and set $U_{[-\delta]} = U_{[\delta]}^{\perp} \cap S_{[-\delta]}$. Then, $U_{[\delta]} + U_{[-\delta]}$ is a maximal totally isotropic graded subspace of $S_{[\delta]} + S_{[-\delta]}$. Moreover,

$$U_{[-\delta]} \cap W_{[-\delta]} \;=\; \big(U_{[\delta]}^{\perp} \cap S_{[-\delta]}\big) \cap \big(W_{[\delta]}^{\perp} \cap S_{[-\delta]}\big) \;=\; \big(U_{[\delta]} + W_{[\delta]}\big)^{\perp} \cap S_{[-\delta]} \;=\; S_{[\delta]}^{\perp} \cap S_{[-\delta]} \;= (0)\,.$$

Therefore, $U_{[\delta]} + U_{[-\delta]}$ is a totally isotropic graded subspace of $S_{[\delta]} + S_{[-\delta]}$ which is complementary to $W_{[\delta]} + W_{[-\delta]}$. Let U be the sum of the $U_{[\gamma]}$ for $\gamma \in \frac{1}{2}\Gamma_E$ and the $U_{[\delta]} + U_{[-\delta]}$, one for each pair $[\delta], [-\delta]$ with $\delta \notin \frac{1}{2}\Gamma_E$. Then U is totally isotropic since the summands are totally isotropic and pairwise orthogonal, and U is complementary to W in S.

(iv) It is known [K, Prop. (6.2.4)] that every form in $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$ is diagonalizable, except when $\widehat{\sigma}$ is trivial on E_0 and $\widehat{\epsilon} = -1$. But, in that case, all the forms in $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$ are hyperbolic. Hence by Prop. 1.2(i) all the anisotropic forms in $\mathcal{GH}^+(E, \widetilde{\sigma}, \widetilde{\epsilon}; [0])$ are diagonalizable, for every graded involution $\widetilde{\sigma}$. Hence, by Prop. 1.2(ii) the same is true for anisotropic forms in $\mathcal{GH}^+(E, \sigma, \epsilon; [\gamma])$ for each $\gamma \in \frac{1}{2}\Gamma_E$. The desired result then holds by Prop. 1.1(iv) and (iii).

(v) For any $(S, k) \in \mathcal{GH}^+(E, \sigma, \epsilon)$ and any $\gamma \in \frac{1}{2}\Gamma_E$, we have $(k \perp -k)_{[\gamma]} = k_{[\gamma]} \perp -k_{[\gamma]}$, which is hyperbolic by virtue of Prop. 1.2 and the corresponding result for $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$ for each involution $\widehat{\sigma}$ on E_0 [K, Prop. (3.5.3)]. For each $\delta \notin \frac{1}{2}\Gamma_E$, the $[\delta] \cup [-\delta]$-piece of $k \perp -k$ is automatically hyperbolic, see Prop. 1.1(ii). Hence, $k \perp -k$ is an orthogonal sum of hyperbolic forms, so it is hyperbolic.

(vi) By Prop. 1.2 and the corresponding result for every $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$, see [K, Prop. (6.3.2)], for each $\gamma \in \frac{1}{2}\Gamma_E$ we have $k_{[\gamma]} \cong k_{[\gamma]\,an} \perp k_{[\gamma]\,hyp}$ with $k_{[\gamma]\,an}$ anisotropic and $k_{[\gamma]\,hyp}$ hyperbolic, each of them in $\mathcal{GH}^+(E, \sigma, \epsilon; [\gamma])$, and $k_{[\gamma]\,an}$ and $k_{[\gamma]\,hyp}$ are unique up to isometry. Set $k_{an} = \perp_{[\gamma] \in \frac{1}{2}\Gamma_E/\Gamma_E} k_{[\gamma]\,an}$ and $k_{hyp} = \perp_{[\gamma] \in \frac{1}{2}\Gamma_E/\Gamma_E} k_{[\gamma]\,hyp} \;\perp\; k_{irr}$, where k_{irr} is the sum of the $k_{[\delta]}$ for $[\delta] \in \Gamma/\Gamma_E - \frac{1}{2}\Gamma_E/\Gamma_E$. Then, $k \cong k_{an} \perp k_{hyp}$, and by Prop. 1.1 (iv) and (v), k_{an} is anisotropic and k_{hyp} is hyperbolic. For the uniqueness, suppose $k \cong k_1 \perp k_2$ with k_1 anisotropic and k_2 hyperbolic. Then, for $\gamma \in \frac{1}{2}\Gamma_E$, $k_{[\gamma]\,an} \cong (k_1 \perp k_2)_{[\gamma]\,an} \cong k_{1\,[\gamma]}$, by the uniqueness in $\mathcal{GH}^+(E, \sigma, \epsilon; [\gamma])$, since $k_{1\,[\gamma]}$ is anisotropic and $k_{2\,[\gamma]}$ is hyperbolic, by Prop. 1.1 (iv) and (v). Hence, $k_1 \cong k_{an}$ by Prop. 1.1(iv); then $k_2 \cong k_{hyp}$ by part (ii) above.

(vii) Since $k_{1\,hyp} \perp \ell_1$ is hyperbolic, the uniqueness part of (vi) shows that

$$k_{1\,an} \cong (k_1 \perp \ell_1)_{an} \cong (k_2 \perp \ell_2)_{an} \cong k_{2\,an}\,.$$

(viii) Apply (vii) to $k_1 \perp (k_2 \perp -k_2) \cong k_2 \perp (k_1 \perp -k_2)$ using (v). □

For each of the categories defined preceding Prop. 1.2 there is an associated Witt group: Let $\mathcal{C}$ be any of $\mathcal{GH}^+(E, \sigma, \epsilon)$, $\mathcal{GH}^+(E, \sigma, \epsilon; [\gamma])$, or $\mathcal{H}^+(E_0, \widehat{\sigma}, \widehat{\epsilon})$. For $(S, k) \in \mathcal{C}$, we write $c\ell(k)$ for the isometry class of (S, k) (meaning graded isometry class in the graded case). The set $\mathcal{I}so(\mathcal{C})$ of isometry classes of forms in $\mathcal{C}$ is a cancellative monoid with respect to the operation induced by orthogonal sum. The Witt group $W(\mathcal{C})$ is the group $\mathcal{I}so(\mathcal{C})/\!\sim$ of equivalence classes with respect to the equivalence relation $c\ell(k_1) \sim c\ell(k_2)$ iff there are hyperbolic forms ℓ_1 and ℓ_2 in $\mathcal{C}$ with $k_1 \perp \ell_1 \cong k_2 \perp \ell_2$. Let $[k]$ denote the equivalence class of $c\ell(k)$. Prop. 1.4(vii) in the graded case or [K, Prop. (6.3.2)] in the ungraded case show that $[k_1] = [k_2]$ in $W(\mathcal{C})$ iff $k_{1\,an} \cong k_{2\,an}$. The (well-defined, associative) operation in $W(\mathcal{C})$ is: $[k_1] + [k_2] = [k_1 \perp k_2]$, and Prop. 1.4(v) shows that $W(\mathcal{C})$ is actually a group. Set

$$\begin{aligned} W_g^+(E,\sigma,\epsilon) &= W(\mathcal{GH}^+(E,\sigma,\epsilon))\,; \\ W_g^+(E,\sigma,\epsilon;[\gamma]) &= W(\mathcal{GH}^+(E,\sigma,\epsilon;[\gamma]))\,,\ \text{for each } [\gamma] \in \tfrac{1}{2}\Gamma_E/\Gamma_E; \\ W^+(E_0,\widehat{\sigma},\widehat{\epsilon}) &= W(\mathcal{H}^+(E_0,\widehat{\sigma},\widehat{\epsilon}))\,,\ \text{the usual even Witt group of } \mathcal{H}^+(E_0,\widehat{\sigma},\widehat{\epsilon})\,. \end{aligned}$$

Proposition 1.5. *For any graded involution σ on E and any $\epsilon \in Z(E_0)$ with $\epsilon\sigma(\epsilon) = 1$,*

(i) $W_g^+(E,\sigma,\epsilon;[0]) \cong W^+(E_0,\sigma|_{E_0},\epsilon)$, *canonically.*

(ii) $W_g^+(E,\sigma,\epsilon) \cong \bigoplus_{[\gamma]\in\frac{1}{2}\Gamma_E/\Gamma_E} W_g^+(E,\sigma,\epsilon;[\gamma])$, *canonically.*

(iii) *For any $\gamma,\rho \in \frac{1}{2}\Gamma_E$, choose any nonzero $r \in E_{-2\rho}$ with $\sigma(r) = \pm r$, and let $\widetilde{\sigma} = int(r)\circ\sigma$ and $\widetilde{\epsilon} = \epsilon\sigma(r)r^{-1}$. Then, $W_g^+(E,\sigma,\epsilon;[\gamma]) \cong W_g^+(E,\widetilde{\sigma},\widetilde{\epsilon};[\gamma-\rho])$.*

(iv) *For each $[\gamma] \in \frac{1}{2}\Gamma_E/\Gamma_E$, choose any $\gamma \in [\gamma]$ and any nonzero $r_\gamma \in E_{-2\gamma}$ with $\sigma(r) = \pm r$, let $\sigma_{[\gamma]}$ be the graded involution $int(r_\gamma)\circ\sigma$, and let $\sigma_{[\gamma]\,0} = \sigma_{[\gamma]}|_{E_0}$ and $\epsilon_{[\gamma]} = \epsilon\sigma(r_{[\gamma]})r_{[\gamma]}^{-1}$. Then,*

$$W_g^+(E,\sigma,\epsilon) \cong \bigoplus_{[\gamma]\in\frac{1}{2}\Gamma_E/\Gamma_E} W^+(E_0,\sigma_{[\gamma]\,0},\epsilon_{[\gamma]})\,.$$

Proof. (i) is immediate from Prop. 1.2(i).

(ii) For $[k] \in W_g^+(E,\sigma,\epsilon)$ the map $[k] \mapsto \big([k_{[\gamma]}]\big)_{[\gamma]\in\frac{1}{2}\Gamma_E/\Gamma_E}$ is well-defined and gives the asserted isomorphism, by Prop. 1.1.

(iii) is immediate from Prop. 1.2(ii).

(iv) We have by (iii) and (i), $W_g^+(E,\sigma,\epsilon;[\gamma]) \cong W_g^+(E,\sigma_{[\gamma]},\epsilon_{[\gamma]};[0]) \cong W^+(E_0,\sigma_{[\gamma]\,0},\epsilon_{[\gamma]})$. With this, (iv) follows from (ii). □

2. Norms on vector spaces over valued fields

Let D be a division ring finite dimensional over its center F, and suppose D has a valuation v. That is, we have a divisible totally ordered abelian group Γ and an element ∞ (with $\infty > \gamma$ for all $\gamma \in \Gamma$ and $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$) and $v\colon D \to \Gamma \cup \{\infty\}$ is a function satisfying, for all $c, d \in D$,

$$v(d) = \infty \text{ iff } d = 0; \tag{2.1a}$$

$$v(cd) = v(c) + v(d); \tag{2.1b}$$

$$v(c+d) \geq min\big(v(c), v(d)\big). \tag{2.1c}$$

It is immediate that $v(1) = v(-1) = 0$, and if $v(c) \neq v(d)$ then $v(c+d) = v(c-d) = min\big(v(c), v(d)\big)$. Let $D^\times = D - \{0\}$. Let $\Gamma_D = v(D^\times)$, the value group of v, which is a subgroup of Γ.

For each $\gamma \in \Gamma_D$ define the abelian groups

$$D^{\geq\gamma} = \{\, d \in D \mid v(d) \geq \gamma \,\}, \quad D^{>\gamma} = \{\, d \in D \mid v(d) > \gamma \,\}, \quad \text{and } D_\gamma = D^{\geq\gamma}/D^{>\gamma}\,.$$

The *associated graded ring* of (v on) D is

$$gr(D) = \bigoplus_{\gamma\in\Gamma} D_\gamma\,,$$

with the multiplication induced by the multiplication in D. For any $d \in D^\times$, we write d' for the image $d + D^{>v(d)}$ of d in $D_{v(d)}$, and $(0_D)' = 0$ in $gr(D)$. Property 2.1b implies that

$$(cd)' = c'd' \quad \text{for all } c, d \in D\,.$$

So, in particular, we have $d'(d^{-1})' = 1'$ for any $d \in D^\times$. This shows that $gr(D)$ is a graded division ring, as described in §1. Clearly the grade group $\Gamma_{gr(D)} = \Gamma_D$. Note also that the valuation ring of v is $V_D = D^{\geq 0}$, and the unique maximal left (and right) ideal M_D of V_D is $M_D = D^{>0}$. Hence, $D_0 = V_D/M_D = \overline{D}$, the residue division ring of D.

Now, let M be any finite-dimensional right D-vector space. We call a function $\alpha\colon M \to \Gamma \cup \{\infty\}$ a *value function* (with respect to v on D) if for all $m, n \in M$ and $d \in D$,

$$\alpha(m) = \infty \ \text{ iff } \ m = 0; \tag{2.2a}$$

$$\alpha(md) = \alpha(m) + v(d); \tag{2.2b}$$

$$\alpha(m+n) \geq \min\big(\alpha(m), \alpha(n)\big)\,. \tag{2.2c}$$

It is immediate that $\alpha(-m) = \alpha(m)$ and that if $\alpha(m) \neq \alpha(n)$ then $\alpha(m+n) = \alpha(m-n) = \min\big(\alpha(m), \alpha(n)\big)$. The value set of α is $\Gamma_M = \{\alpha(m) \mid m \in M,\ m \neq 0\} \subseteq \Gamma$. This Γ_M need not be a group, but it is a union of cosets of Γ_D. For each $\gamma \in \Gamma$, define the abelian groups $M^{\geq\gamma}$, $M^{>\gamma}$, and M_γ just as for D above. The *associated graded vector space* of (α on) M is

$$gr(M) \ = \ \bigoplus_{\gamma\in\Gamma} M_\gamma\,.$$

When we need to specify the value function, we write $gr_\alpha(M)$. The module action of D on M induces a well-defined module action of $gr(D)$ on $gr(M)$, making $gr(M)$ into a graded right vector space over $gr(D)$. For nonzero $m \in M$, we write m' for the image $m + M^{>\alpha(m)}$ of m in $M_{\alpha(m)}$. We write $(0_M)' = 0$ in $gr(M)$. Clearly, for all $m \in M$, $d \in D$, we have

$$(md)' = m'd'\,.$$

Also, for nonzero $m, n \in M$, we frequently use the obvious fact that

$$(m+n)' \ = \begin{cases} m' & \text{if } \alpha(m) < \alpha(n); \\ n' & \text{if } \alpha(m) > \alpha(n); \\ m' + n' & \text{if } \alpha(m) = \alpha(n) \text{ and } m' + n' \neq 0\,. \end{cases} \tag{2.3}$$

Here is a fundamental way of constructing a value function on M: Take any base $\{m_1, \ldots, m_k\}$ of M as D-vector space, and take any $\gamma_1, \ldots, \gamma_k \in \Gamma$. Then, define $\alpha\colon M \to \Gamma \cup \{\infty\}$ by $\alpha(\sum m_i d_i) = \min\limits_{1\leq i\leq k}\big(\gamma_i + v(d_i)\big)$. That is, $\alpha(m_i) = \gamma_i$ and for all $d_1, \ldots, d_k \in D$,

$$\alpha\Big(\sum_{i=1}^{k} m_i d_i\Big) \ = \ \min_{1\leq i\leq k}\big(\alpha(m_i) + v(d_i)\big)\,. \tag{2.4}$$

It is easy to check that α satisfies the axioms for a value function on M. In fact, we will be exclusively interested in the value functions arising this way.

Definition 2.1. Given a value function α on M, a base $\{m_1, \ldots, m_k\}$ of M for which formula (2.4) holds is called a *splitting base* of α. We say that the value function α is a *norm* on M (with respect to the valuation v on D) if there is a splitting base for α.

The associated graded vector space elucidates the notion of splitting bases:

Proposition 2.2. *Let α be a value function on M, and let $m_1, \ldots, m_\ell \in M - \{0\}$. Then,*

(i) *$m'_1, \ldots, m'_\ell$ are $gr(D)$-linearly independent in $gr(M)$ iff $\alpha\big(\sum_{i=1}^{\ell} m_i d_i\big) \ = \ \min\limits_{1\leq i\leq\ell}\big(\alpha(m_i) + v(d_i)\big)$ for all $d_1, \ldots, d_\ell \in D$.*

(ii) *If $m'_1, \ldots, m'_\ell$ are $gr(D)$-linearly independent in $gr(M)$, then $m_1, \ldots, m_\ell$ are D-linearly independent in M.*

Proof. Suppose $m_1',\dots,m_\ell'$ are $gr(D)$-linearly dependent in $gr(M)$, say $\sum_{i=1}^{\ell} m_i'c_i = 0$ in $gr(M)$, with some $c_i \neq 0$. Then, each homogeneous component of the sum is 0. Suppose some $m_i'c_i$ has nonzero γ-component. Then, after throwing out the summands with trivial γ-component and renumbering the summands, we have $0 = \sum_{i=1}^{j} m_i'd_i' = \sum_{i=1}^{j}(m_id_i)'$ in M_γ, for some nonzero $d_i \in D$ with each $\alpha(m_id_i) = \gamma$. This means that $\alpha\big(\sum_{i=1}^{j} m_id_i\big) > \gamma = \min_{1\le i\le j}\big(\alpha(m_i)+v(d_i)\big)$, so the condition on the m_i in (i) fails to hold.

On the other hand, suppose $m_1',\dots,m_\ell'$ are $gr(D)$-linearly independent in $gr(M)$. Take any linear combination $m = \sum_{i=1}^{\ell} m_id_i$ with the $d_i \in D$ and some $d_i \neq 0$. Let $\gamma = \min_{1\le i\le l}\big(\alpha(m_id_i)\big)$. After reordering the m_i, there is a $j \ge 1$ with $\alpha(m_id_i) = \gamma$ for $1 \le i \le j$ and $\alpha(m_id_i) > \gamma$ for $i > j$. Let $n = \sum_{i=1}^{j} m_id_i$. Since all the $\alpha(m_id_i) = \gamma$ for $1 \le i \le j$ and $\sum_{i=1}^{j}(m_id_i)' = \sum_{i=1}^{j} m_i'd_i' \neq 0$ in M_γ by the independence of the m_i', it follows that $\alpha(n) = \gamma$. We have $m - n = \sum_{i=j+1}^{\ell} m_id_i \in M^{>\gamma}$. Since $\alpha(m-n) > \gamma = \alpha(n) = \alpha(-n)$, we have $m - n \neq -n$, so $m \neq 0$, and $\alpha(m) = \alpha(n + (m-n)) = \min\big(\alpha(m), \alpha(m-n)\big) = \gamma$. This shows that the m_i satisfy the condition in (i), completing the proof of (i). It also shows (as we saw $m \neq 0$) that $m_1,\dots,m_\ell$ are D-linearly independent in M, proving (ii). □

COROLLARY 2.3. *Let α be a value function on M. Then,*

(i) *$\dim_{gr(D)} gr(M) \le \dim_D(M)$, and equality holds iff α is a norm.*
(ii) *Suppose α is a norm. Then $\{m_1,\dots,m_k\}$ is a splitting base for α iff $\{m_1',\dots,m_k'\}$ is a homogeneous base of $gr(M)$ as a graded $gr(D)$-vector space.*
(iii) *Suppose α is a norm with splitting base $\{m_1,\dots,m_k\}$. Take any nonzero $n = \sum_{i=1}^{k} m_id_i \in M$. For any j with $\alpha(m_jd_j) = \alpha(n)$ the set $\{n\} \cup \{m_i \mid i \neq j\}$ is a splitting base for α.*

Proof. (i) and (ii) are immediate from Prop. 2.2.

(iii) Let $\gamma = \alpha(n)$. Then, the image n' of n in $gr(M)$ lies in M_γ, and we have $n' = \sum_{i\in I} m_i'd_i'$, where $I = \{i \mid \alpha(m_id_i) = \gamma\}$, and each of these summands is nonzero. By hypothesis, one of the summands is $m_j'd_j'$. We can use this equation to express m_j' as a linear combination of n' and the m_i' with $i \neq j$. Thus, the usual exchange argument applies to show that $\{n'\}\cup\{m_i' \mid i \neq j\}$ is a homogeneous $gr(D)$-base of $gr(M)$. Therefore, by part(ii), $\{n\} \cup \{m_i \mid i \neq j\}$ is a splitting base for α. □

REMARK 2.4. Let $F \subseteq L$ be fields with $[L:F] < \infty$, let v be any valuation on F, and let α be any valuation on L extending v. Let Γ_F, Γ_L be the value groups of v and α, and let $\overline{F}$ and $\overline{L}$ be the residue fields of the associated valuation rings. Of course, α is a value function on L, viewed as an F-vector space, with respect to v. It is easy to prove, and well-known (cf. [HW₂] or [Bl]) that $[gr(L):gr(F)] = [\overline{L}:\overline{F}]\,|\Gamma_L:\Gamma_F|$. The quantities on the right are the residue degree and the ramification index of α/v. The Fundamental Inequality in valuation theory (see, e.g., [B, VI.8.1, Lemma 2]) says that

$$[\overline{L}:\overline{F}]\,|\Gamma_L:\Gamma_F| \;\le\; [L:F]\,. \tag{2.5}$$

Thus,

$$[gr(L):gr(F)] \;\le\; [L:F]\,, \tag{2.6}$$

which agrees with Cor. 2.3(i). Now, it is standard in valuation theory that whenever v has more than one extension to L, then the inequality (2.5) is strict, so we have a strict inequality in (2.6); then the value function α on L is not a norm, by Cor. 2.3. This provides an abundant source of examples of value functions which are not norms. On the other hand, suppose α is the only extension of v to L. One says that α/v is *defectless* if equality holds in (2.5). Ostrowski's Theorem, deducible from [EP, Cor. 5.3.8], yields that α/v is defectless whenever $char(\overline{F}) \nmid [L:F]$. If v is maximally complete (i.e., v has no immediate extensions to any larger field), then it is known by [EP, Th. 5.2.5] and [Sg, Th. 11, p. 55] that for any field $L \supseteq F$ with $[L:F] < \infty$, v has a unique extension to L and equality holds in (2.5). In fact, a variation on the argument in [Sg, proof of Th. 11, p. 55] shows that when v is maximally complete, then every value function for v on any finite-dimensional vector space over F is a norm. This is indicated in [BT_1, p. 299]. Examples of maximally complete valuations are complete discrete valuations and the usual valuations on iterated Laurent series fields $K((x_1))\dots((x_n))$.

Proposition 2.5. *Suppose α is a norm on M. Let N be any nonzero D-subspace of M. Then, $\alpha|_N$ is a norm on N. Moreover, any splitting base of $\alpha|_N$ can be enlarged to a splitting base of α.*

Proof. Let $k = dim_D(M) = dim_{gr(D)}(gr(M))$. For the first assertion we argue by induction on $dim_D(N)$. If $dim_D(N) = 1$, then $1 \le dim_{gr(D)}(gr(N)) \le dim_D(N) = 1$ by Cor. 2.3(ii); hence equality holds, so $\alpha|_N$ is a norm. Now assume $dim_D(N) > 1$. Take any nonzero $n \in N$. Then, $n' \neq 0$ in $gr(M)$, so there exist $m_2, \dots, m_k \in M$ such that $\{ n', m_2', \dots, m_k' \}$ is a homogeneous base for the graded $gr(D)$-vector space $gr(M)$. Let $P = D$-span of $\{ m_2, \dots, m_k \}$. By Prop. 2.2(ii) $dim_D(P) = k-1$. Since $dim_{gr(D)}(gr(P)) \le dim_D(P) = k-1$, by Cor. 2.3(i), the $gr(D)$-linearly independent elements $\{ m_2', \dots, m_k' \}$ span $gr(P)$. Since $n' \notin gr(P)$, we have $N \not\subseteq P$. Let $Q = N \cap P$; so $dim_D(Q) = dim_D(N) - 1$. By induction we have $\alpha|_Q$ is a norm, so $dim_{gr(D)}(gr(Q)) = dim_D(N) - 1$. But, $n' \notin gr(Q)$, as $gr(Q) \subseteq gr(P)$. Hence, $dim_{gr(D)}(gr(N)) \ge 1 + dim_{gr(D)}(gr(Q)) = dim_D(N)$. By Cor. 2.3(i), $\alpha|_N$ is a norm.

The second assertion of the proposition follows easily from Cor. 2.3(ii). □

Remark 2.6. If α is a norm on M and N is a D-subspace of M, then a complementary subspace P of N in M (i.e., $P \cap N = (0)$ and $P + N = M$) is called a *splitting complement* of N if $\alpha(n+p) = min\big(\alpha(n), \alpha(p)\big)$ for all $n \in N$, $p \in P$. It is easy to see that splitting complements always exist. Indeed, a complement P of N is a splitting complement iff $gr(P)$ is a complement of $gr(N)$ as graded subspaces of $gr(M)$.

Let M and N be finite-dimensional right D-vector spaces with respective norms α and β, and let $f\colon M \to N$ be any nonzero D-linear map. Define

$$j_{\alpha,\beta}(f) \;=\; min\{\, \beta(f(m)) - \alpha(m) \mid m \in M,\ m \neq 0 \,\}. \tag{2.7}$$

We show that this minimum exists, so $j_{\alpha,\beta}$ is well-defined: Let $\{ m_1, \dots, m_k \}$ be any splitting base of M for α. For $m \in M$, $m \neq 0$, write $m = \sum_{i=1}^{k} m_i d_i$. Then,

$$\begin{aligned} \beta(f(m)) \;&\ge\; \min_{1\le i\le k} \big(\beta(f(m_i)) + v(d_i)\big) \;=\; \min_{1\le i\le k} \big(\beta(f(m_i)) - \alpha(m_i) + \alpha(m_i) + v(d_i)\big) \\ &\ge\; \min_{1\le i\le k} \big(\beta(f(m_i)) - \alpha(m_i)\big) + \min_{1\le i\le k} \big(\alpha(m_i) + v(d_i)\big) \\ &=\; \min_{1\le i\le k} \big(\beta(f(m_i)) - \alpha(m_i)\big) + \alpha(m)\,. \end{aligned} \tag{2.8}$$

Thus, $\beta(f(m)) - \alpha(m) \ge \min_{1\le i\le k} \big(\beta(f(m_i)) - \alpha(m_i)\big)$ for all nonzero $m \in M$. This shows that $j_{\alpha,\beta}$ exists, and that

$$j_{\alpha,\beta}(f) \;=\; \min_{1\le i\le k} \big(\beta(f(m_i)) - \alpha(m_i)\big)\,. \tag{2.9}$$

For short, let $j = j_{\alpha,\beta}$. By definition of j, for each $\gamma \in \Gamma$, we have $f(M^{\ge\gamma}) \subseteq N^{\ge\gamma+j}$, so also $f(M^{>\gamma}) \subseteq N^{>\gamma+j}$. Therefore, f induces a well-defined map $M_\gamma \to N_{\gamma+j}$ for each $\gamma \in \Gamma$; these combine to give the *associated graded map* $f'\colon gr_\alpha(M) \to gr_\beta(N)$. This f' is given on homogeneous elements by, for any $m \in M$,

$$f'(m + M_{\alpha(m)}) \;=\; f(m) + N_{\alpha(m)+j}\,. \tag{2.10}$$

It is easy to check that f' is a $gr(D)$-module homomorphism which shifts all grades by j. Hence, $ker(f')$ (resp. $im(f')$) is a graded subspace of $gr(M)$ (resp. $gr(N)$).

PROPOSITION 2.7. *Let $f\colon M \to N$ be a nonzero D-linear map, and let $f'\colon gr_\alpha(M) \to gr_\beta(N)$ be the associated graded map just described. Then,*

(i) *$gr(ker(f)) \subseteq ker(f')$ and $im(f') \subseteq gr(im(f))$.*
(ii) *$im(f') = gr(im(f))$ iff $dim_{gr(D)}(im(f')) = dim_{gr(D)}(gr(im(f)))$ iff $gr(ker(f)) = ker(f')$ iff for every $n \in im(f)$ there is $m \in M$ with $f(m) = n$ and $\alpha(m) = \beta(n) - j_{\alpha,\beta}(f)$.*

Proof. (i) is clear from the definitions. Because the subspaces involved are graded, one has only to check the inclusions for homogeneous elements.

(ii) The fact that each condition implies the next is obvious from dimension considerations, except that the next to last implies the last. We now prove that. Suppose $gr(ker(f)) = ker(f')$. Let P be a splitting complement of $ker(f)$ for α. Then $f|_P\colon P \to im(f)$ is an isomorphism. Since $gr(P) \cap ker(f') = gr(P) \cap gr(ker(f)) = (0)$, we have $f'|_{gr(P)}$ is injective. Take any splitting base $\{\, p_1, \ldots, p_k \,\}$ for $\alpha|_P$. Since $f'(p_i') \neq 0$, we have $\beta(f(p_i)) = \alpha(p_i) + j$, where $j = j_{\alpha,\beta}(f)$, and $f(p_i)' = f'(p_i')$. The injectivity of $f'|_{gr(P)}$ shows that the set $\{\, f(p_1)', \ldots, f(p_k)' \,\}$ is $gr(D)$-independent in $gr(N)$. For any $n \in im(f)$, take the $m \in P$ with $f(m) = n$. Write $m = \sum_{i=1}^{k} p_i d_i$. Then $n = \sum_{i=1}^{k} f(p_i) d_i$. From the $gr(D)$-independence of the p_i' and of the $f(p_i)'$, Prop. 2.2(i) yields

$$\beta(n) \;=\; \min_{1\le i\le k}\big(\beta(f(p_i)) + v(d_i)\big) \;=\; \min_{1\le i\le k}\big(\alpha(p_i) + j + v(d_i)\big) \;=\; \alpha(m) + j\,.$$

This proves the last condition in (ii). Since the last condition in (ii) clearly implies the first, the cycle of implications is now complete. □

THEOREM 2.8. *Let M be a finite-dimensional D-vector space with two norms α and β. Then, there is a subset of M which is a splitting base for α and also a splitting base for β.*

Proof. We argue by induction on $dim_D(M)$. Since the 1-dimensional case is clear, assume $dim_D(M) > 1$. Let $\{\, m_1, \ldots, m_k \,\}$ be a splitting base of M for α and $\{\, n_1, \ldots, n_k \,\}$ a splitting base for β. Write each $n_j = \sum_{i=1}^{k} m_i d_{ij}$. Choose s, $1 \le s \le k$, so that $\min_{1\le i\le k}\big(\alpha(n_i) - \beta(n_i)\big) = \alpha(n_s) - \beta(n_s)$. Choose r, $1 \le r \le k$, so that $\alpha(n_s) = \alpha(m_r d_{rs})$; so $d_{rs} \neq 0$. By Cor. 2.3(iii), $\{\, n_s \,\} \cup \{\, m_i \mid i \neq r \,\}$ is a splitting base for α. Let $P = \sum_{i\neq r} m_i D$. So, P is a splitting complement to $n_s D$ for α. We show that this is also true for β. For $j \neq s$, let

$$p_j \;=\; n_j \;-\; n_s d_{rs}^{-1} d_{rj} \;\in P\,.$$

Now, for any j, we have

$$\alpha(m_r) + v(d_{rj}) - \beta(n_j) \;\ge\; \alpha(n_j) - \beta(n_j) \;\ge\; \alpha(n_s) - \beta(n_s) \;=\; \alpha(m_r) + v(d_{rs}) - \beta(n_s)\,.$$

Hence,

$$\beta(n_j) \;\le\; \beta(n_s) - v(d_{rs}) + v(d_{rj}) \;=\; \beta(n_s d_{rs}^{-1} d_{rj})\,.$$

This shows that for $j \neq s$ we have in $gr_\beta(M)$, $p_j' = n_j'$ or $p_j' = n_j' - n_s'(d_{rs}^{-1} d_{rj})'$. We know from Cor. 2.3(ii) that $\{\, n_1', \ldots, n_k' \,\}$ is a homogeneous $gr(D)$-base for $gr_\beta(M)$. Whichever values the p_j' take it is clear that the set $\{\, n_s' \,\} \cup \{\, p_j' \mid j \neq s \,\}$ spans $gr_\beta(M)$, so it is a homogeneous base. Therefore, by Cor. 2.3(ii) $\{\, n_s \,\} \cup \{\, p_j \mid j \neq s \,\}$ is a splitting base of M for β. The D-linearly independent set $\{\, p_j \mid j \neq s \,\}$ must span P, since $dim_D(P) = k - 1$. Therefore, P is a splitting complement to $n_s D$

for β. By induction, P has a simultaneous splitting base for α and β. This set combined with n_s gives a simultaneous splitting base for α and β on M. □

The existence of common splitting bases for norms in the case of complete discrete valuations with finite residue fields was proved by Goldman and Iwahori in [GI, Prop. 1.3] by an argument attributed to Weil. It was noted more generally for arbitrary rank 1 valuations by Bruhat and Tits in [BT_1, Prop. 1.26; App.].

If α and β are two norms on the same D-vector space M, we define

$$\alpha \leq \beta \quad \text{if } \alpha(m) \leq \beta(m) \text{ for all } m \in M. \tag{2.11}$$

The existence of common splitting bases allows one to define convex combinations of norms. In the next section we will use the average:

Definition 2.9. Let α and β be two norms on a D-vector space M, and choose some subset $\{ m_1, \ldots, m_k \}$ of M which is a splitting base for both α and β. Define the *average* of α and β, $\mathrm{av}_{\alpha,\beta}\colon M \to \Gamma \cup \{\infty\}$ by

$$\mathrm{av}_{\alpha,\beta}\big(\sum_{i=1}^{k} m_i d_i\big) \;=\; \min_{1\leq i\leq k}\big(\tfrac{1}{2}\alpha(m_i) + \tfrac{1}{2}\beta(m_i) + v(d_i)\big)\,.$$

Thus, $\mathrm{av}_{\alpha,\beta}$ is the norm on M with splitting base $\{ m_1, \ldots, m_k \}$ such that $\mathrm{av}_{\alpha,\beta}(m_i) = \frac{1}{2}\alpha(m_i) + \frac{1}{2}\beta(m_i)$ for all i.

Proposition 2.10. *Let α and β be norms on a D-vector space M.*

(i) *For all $m \in M$, $\mathrm{av}_{\alpha,\beta}(m) \geq \frac{1}{2}\alpha(m) + \frac{1}{2}\beta(m)$.*

(ii) *The definition of $\mathrm{av}_{\alpha,\beta}$ is independent of the choice of common splitting base of M for α and β. Any common splitting base for α and β is also a splitting base for $\mathrm{av}_{\alpha,\beta}$.*

Proof. Assume $\mathrm{av}_{\alpha,\beta}$ has been defined using the common splitting base $\{ m_1, \ldots, m_k \}$.

(i) For any $m = \sum_{i=1}^{k} m_i d_i \in M$, we have

$$\begin{aligned}\mathrm{av}_{\alpha,\beta}(m) \;&=\; \min_{1\leq i\leq k}\big(\tfrac{1}{2}\alpha(m_i) + \tfrac{1}{2}\beta(m_i) + v(d_i)\big)\\ &\geq\; \tfrac{1}{2}\min_{1\leq i\leq k}\big(\alpha(m_i) + v(d_i)\big) + \tfrac{1}{2}\min_{1\leq j\leq k}\big(\beta(m_j) + v(d_j)\big) \;=\; \tfrac{1}{2}\alpha(m) + \tfrac{1}{2}\beta(m)\,.\end{aligned}$$

(ii) Let $\{ n_1, \ldots, n_k \}$ be any common splitting base for α and β, and define $\mu\colon M \to \Gamma \cup \{\infty\}$ by $\sum_{j=1}^{k} n_j c_j \mapsto \min_{1\leq j\leq k}\big(\frac{1}{2}\alpha(n_j) + \frac{1}{2}\beta(n_j) + v(c_j)\big)$. For each n_j, we have $\mu(n_j) = \frac{1}{2}\alpha(n_j) + \frac{1}{2}\beta(n_j) \leq \mathrm{av}_{\alpha,\beta}(n_j)$ by (i). Hence, for any $m = \sum_{j=1}^{k} n_j c_j$ in M,

$$\begin{aligned}\mu(m) \;&=\; \min_{1\leq j\leq k}\big(\mu(n_j) + v(c_j)\big) \;\leq\; \min_{1\leq j\leq k}\big(\mathrm{av}_{\alpha,\beta}(n_j) + v(c_j)\big) \;=\; \min_{1\leq j\leq k}\big(\mathrm{av}_{\alpha,\beta}(n_j c_j)\big)\\ &\leq \mathrm{av}_{\alpha,\beta}\big(\sum_{j=1}^{k} n_j c_j\big) = \mathrm{av}_{\alpha,\beta}(m)\,.\end{aligned}$$

Thus, $\mu \leq \mathrm{av}_{\alpha,\beta}$. Symmetrically, we have $\mathrm{av}_{\alpha,\beta} \leq \mu$, so equality holds. This shows that the definition of $\mathrm{av}_{\alpha,\beta}$ is independent of the choice of common splitting base for α and β. If we take any common splitting base for α and β, we could use that splitting base for defining $\mathrm{av}_{\alpha,\beta}$ and it is then clear that that base is also a splitting base for $\mathrm{av}_{\alpha,\beta}$. □

3. Graded Hermitian forms induced by norms

Let D be a division ring finite-dimensional over its center F, and suppose D has a valuation $v\colon D \to \Gamma \cup \{\infty\}$. Let τ be an involution on D such that $v(\tau(d)) = v(d)$ for all $d \in D$. Let $K = Z(D)$, and let $F = K^\tau = \{\, c \in K \mid \tau(c) = c\,\}$, which is a subfield of K with $[K\!:\!F] < 2$ and K Galois over F. We say τ is of the first kind if $F = K$, and of the second kind otherwise. Because $v \circ \tau$ is a valuation on D and every valuation on F has at most one extension to D by [E$_1$] or [W$_1$], the hypothesis that $v \circ \tau = v$ reduces to $v \circ \tau|_F = v|_F$. This holds automatically if τ is of the first kind; when τ is of the second kind, this condition is equivalent to: $v|_F$ has a unique extension to a valuation on K. Because τ is compatible with v, τ induces a well-defined graded involution τ' on $gr(D)$. Fix some $\lambda \in D$ with $\lambda\tau(\lambda) = 1$ (so $v(\lambda) = 0$). Let λ' be the image of λ in $gr(D)_0 = \overline{D}$.

Let M be a finite dimensional right D-vector space, and let $h\colon M \times M \to D$ be a nondegenerate λ-Hermitian form for τ. Let $\alpha\colon M \to \Gamma \cup \{\infty\}$ be a norm on M.

Definition 3.1. (a) We say α is *bounded by* h, denoted $\alpha \prec h$, if for all $m, n \in M$,

$$\alpha(m) + \alpha(n) \ \le\ v(h(m,n))\,. \tag{3.1}$$

(b) If $\alpha \prec h$, we say that α is *compatible with* h, denoted $\alpha \mathrel{\overline{\prec}} h$, if for each $m \in M$ there is $n \in M$ with $\alpha(m) + \alpha(n) = v(h(m,n))$.

The condition that $\alpha \prec h$ can be restated: For all $\gamma, \delta \in \Gamma_D$,

$$h(M^{\ge\gamma}, M^{\ge\delta}) \ \subseteq\ D^{\ge\gamma+\delta}\,. \tag{3.2}$$

(3.2) shows that h also maps $M^{>\gamma} \times M^{\ge\delta}$ and $M^{\ge\gamma} \times M^{>\delta}$ into $D^{>\gamma+\delta}$. Therefore, h induces a well-defined bi-additive map $h'_\alpha\colon M_\gamma \times M_\delta \to D_{\gamma+\delta}$. This h'_α is given by: For any $m, n \in M$ with $\alpha(m) = \gamma$ and $\alpha(n) = \delta$,

$$h'_\alpha(m', n') \ = \begin{cases} h(m,n)' & \text{if } \ v(h(m,n)) = \alpha(m) + \alpha(n)\,, \\ 0 & \text{if } \ v(h(m,n)) > \alpha(m) + \alpha(n)\,. \end{cases} \tag{3.3}$$

Now extend h'_α biadditively to a map also denoted $h'_\alpha\colon gr(M) \times gr(M) \to gr(D)$. Easy calculations show that h'_α is a graded λ'-Hermitian form on $gr(M)$ for the graded involution τ' on $gr(D)$.

Remark 3.2. The condition that $\alpha \prec h$ is exactly what is needed to assure that the associated graded form h'_α defined by (3.3) is well-defined. The stronger condition that $\alpha \mathrel{\overline{\prec}} h$ can be restated: For every $\gamma \in \Gamma_{gr(D)}$ and every nonzero $m' \in M_\gamma$ (where $m \in M$ with $\alpha(m) = \gamma$) there is a nonzero $n \in N$ with $h'_\alpha(m', n') \ne 0$. This is equivalent to: h'_α is a nondegenerate form. It is clear that if $\alpha \mathrel{\overline{\prec}} h$ and N is any subspace of M, then $\alpha|_N \prec h|_N$. We have $\alpha|_N \mathrel{\overline{\prec}} h|_N$ iff $h'_\alpha|_{gr(N)}$ is nondegenerate.

Lemma 3.3. *Let α be a norm on M. If $\{m_1, \ldots, m_k\}$ is any splitting base of M for α, then $\alpha \prec h$ iff for all i, j,*

$$\alpha(m_i) + \alpha(m_j) \ \le\ v(h(m_i, m_j))\,. \tag{3.4}$$

Proof. Condition (3.4) holds by definition if $\alpha \prec h$. Conversely, suppose we have (3.4). For any $m, n \in M$, write $m = \sum m_i d_i$ and $n = \sum m_i c_i$ with all $d_i, c_i \in D$. Then,

$$\begin{aligned} v(h(m,n)) \ &= \ v\big(\sum_{i,j} \tau(d_i) h(m_i, n_i) c_j\big) \ \ge \ \min_{i,j}\big(v(\tau(d_i)) + v(h(m_i, m_j)) + v(c_j)\big) \\ &\ge \ \min_{i,j}\big(v(d_i) + \alpha(m_i) + \alpha(m_j) + v(c_j)\big) \\ &\ge \ \min_i\big(v(d_i) + \alpha(m_i)\big) \ + \ \min_j\big(\alpha(m_j) + v(c_j)\big) \ = \ \alpha(m) + \alpha(n)\,. \end{aligned}$$

So, $\alpha \prec h$. $\square$

For any λ-Hermitian form on M and any norm α on M there is the h-dual norm $\alpha^\sharp$ defined by

$$\alpha^\sharp(n) \;=\; min\{v(h(n,m)) - \alpha(m) \mid m \in M\}\,. \tag{3.5}$$

To see that $\alpha^\sharp$ is well-defined, note that h induces an isomorphism $\varphi\colon M \to M^* = Hom_D(M,D)$ given by $\varphi(m)(n) = h(m,n)$. This φ is actually a right D-vector space isomophism when we turn the left D-vector space M^* into a right D-vector space via τ, i.e., for $d \in D$, $f^* \in M^*$ define $f^* \cdot d$ by $(f^* \cdot d)(m) = \tau(d)f^*(m)$. Observe that when we view v as a D-norm on D with respect to v, then $\alpha^\sharp(n) = j_{\alpha,v}(\varphi(n))$ for the function $j_{\alpha,v}$ defined in (2.7) (with $N = D$ and $\beta = v$). The well-definition of $j_{\alpha,v}$, proved in the calculation preceding (2.9), yields that $\alpha^\sharp$ is well-defined. Formula (2.9) shows that for any splitting base $\{\,m_1,\dots,m_k\,\}$ of M and any $n \in M$, we have

$$\alpha^\sharp(n) \;=\; \min_{1\le i\le k}\big(v(h(n,m_i)) - \alpha(m_i)\big)\,. \tag{3.6}$$

Lemma 3.4(i) below shows that $\alpha^\sharp$ is a norm on M.

Lemma 3.4. *Let α,β be norms on M. Then,*

(i) *If $\{\,m_1,\dots,m_k\,\}$ is any splitting base for α, let $\{\,m_1^\sharp,\dots,m_k^\sharp\,\}$ be the h-dual base of M, defined by $h(m_i^\sharp,m_j) = \delta_{ij}$ (Kronecker delta). Then, $\{\,m_1^\sharp,\dots,m_k^\sharp\,\}$ is a splitting base of M for $\alpha^\sharp$, and $\alpha^\sharp(m_i^\sharp) = -\alpha(m_i)$, for all i. Hence, $\alpha^\sharp$ is a norm on M.*

(ii) $\alpha^{\sharp\sharp} = \alpha$.

(iii) *If $\alpha \le \beta$ then $\beta^\sharp \le \alpha^\sharp$.*

(iv) $(av_{\alpha,\beta})^\sharp \;=\; av_{\alpha^\sharp,\beta^\sharp}$.

Proof. (i) This follows by an easy direct calculation, using (3.6).

(ii) follows from (i) since the h-dual of the h-dual base $\{\,m_1^\sharp,\dots,m_k^\sharp\,\}$ of (i) is $\{\,m_1^{\sharp\sharp},\dots,m_k^{\sharp\sharp}\,\}$, where each $m_i^{\sharp\sharp} = \lambda m_i$.

(iii) is clear from the definition.

(iv) Let $\{\,m_1,\dots,m_k\,\}$ be a common splitting base for α and β, which exists by Th. 2.8. By Prop. 2.10(ii) this set is also a splitting base for $av_{\alpha,\beta}$. Then by (i) $\{\,m_1^\sharp,\dots,m_k^\sharp\,\}$ is a splitting base for $\alpha^\sharp$ and $\beta^\sharp$ (so also for $av_{\alpha^\sharp,\beta^\sharp}$) and for $(av_{\alpha,\beta})^\sharp$. Part (i) shows that $av_{\alpha^\sharp,\beta^\sharp}$ and $(av_{\alpha,\beta})^\sharp$ agree on the $m_i^\sharp$, so they must coincide. □

Proposition 3.5. *Let α be a norm on M.*

(i) *$\alpha \prec h$ iff $\alpha \le \alpha^\sharp$.*

(ii) *$\alpha \mathrel{\overline{\prec}} h$ iff $\alpha = \alpha^\sharp$.*

(iii) *If $\alpha \mathrel{\overline{\prec}} h$, then α is maximal in $\{\,\beta \mid \beta$ is a norm of M and $\beta \prec h\,\}$.*

(iv) *$av_{\alpha,\alpha^\sharp} \mathrel{\overline{\prec}} h$.*

(v) *If $\alpha \prec h$, then $\alpha \le av_{\alpha,\alpha^\sharp}$.*

Proof. (i) and (ii) are immediate from the definitions.

(iii) Suppose $\alpha \mathrel{\overline{\prec}} h$ and β is a norm with $\beta \prec h$ and $\alpha \le \beta$. Then, $\alpha \le \beta \le \beta^\sharp \le \alpha^\sharp = \alpha$, by (i), Lemma 3.4(iii), and (ii) of this proposition. Hence, $\beta = \alpha$.

(iv) By Lemma 3.4(iv) and (ii), $(av_{\alpha,\alpha^\sharp})^\sharp = av_{\alpha^\sharp,\alpha^{\sharp\sharp}} = av_{\alpha^\sharp,\alpha} = av_{\alpha,\alpha^\sharp}$. So, (iv) follows from (ii) of this proposition.

(v) Let $\{\,m_1,\dots,m_k\,\}$ be a common splitting base for α and $\alpha^\sharp$, so it is also a splitting base for $av_{\alpha,\alpha^\sharp}$ by Prop. 2.10(ii). Because $\alpha \prec h$ we have from (i) above that $\alpha(m_i) \le \alpha^\sharp(m_i)$. Hence, $\alpha(m_i) \le \frac{1}{2}\alpha(m_i) + \frac{1}{2}\alpha^\sharp(m_i) = av_{\alpha,\alpha^\sharp}(m_i)$. Since this is true for all the m_i in a common splitting base for α and $av_{\alpha,\alpha^\sharp}$, we must have $\alpha \le av_{\alpha,\alpha^\sharp}$. □

Prop. 3.5 shows that the norms compatible with h are precisely the ones that are maximal among the norms bounded by h. Moreover, parts (iv) and (v) show that every norm bounded by h is less than or equal to a norm compatible with h. This was shown previously for discrete valuations by Bruhat and Tits in [BT_2, pp. 160–162], where the norms we have defined as compatible with h are called "maximinorantes" for h, i.e., maximal among norms bounded by h. Earlier still, norms maximal among those bounded by h were considered by Springer in [Sp_2] and by Goldman and Iwahori in [GI] for certain complete discrete valuations; but their definition was somewhat different, since for $\alpha \prec h$ they require only the weaker condition that $2\alpha(m) \leq v(h(m,m))$ for all $m \in M$.

Corollary 3.6. *If h is any nondegenerate λ-Hermitian form for τ on a D-vector space M, then there is a norm α on M with $\alpha \precsim h$.*

Proof. This is immediate from Prop. 3.5(iv). □

Examples 3.7. (i) Suppose $\{m_1, \ldots, m_k\}$ is an orthogonal base for h on M. For any $\gamma_1, \ldots, \gamma_k \in \Gamma$, let α be the norm on M with splitting base $\{m_1, \ldots, m_k\}$ such that each $\alpha(m_i) = \gamma_i$. Then, by Lemma 3.3, $\alpha \prec h$ iff each $\alpha(m_i) + \alpha(m_i) \leq v(h(m_i, m_i))$, iff each $\gamma_i \leq \frac{1}{2} v(h(m_i, m_i))$. When this holds, we have $h'_\alpha(m'_i, m'_j) = 0$ whenever $i \neq j$ and

$$h'_\alpha(m'_i, m'_i) = \begin{cases} h(m_i, m_i)' & \text{if } \ v(h(m_i, m_i)) = 2\gamma_i \,, \\ 0 & \text{if } \ v(h(m_i, m_i)) > 2\gamma_i \,. \end{cases}$$

Since the diagonal form h'_α is nondegenerate iff each $h'_\alpha(m'_i, m'_i) \neq 0$, we have $\alpha \precsim h$ iff each $\gamma_i = \frac{1}{2} v(h(m_i, m_i))$.

(ii) Suppose h is a hyperbolic λ-Hermitian form on M. Then, M has complementary totally isotropic subspaces N and P of the same dimension. Let $\{n_1, \ldots, n_\ell\}$ be any D-vector space base of N, and let $\{p_1, \ldots, p_\ell\}$ be the corresponding base of P such that $h(n_i, p_j) = \delta_{ij}$ (Kronecker delta) for all i, j. Take any $\gamma \in \Gamma$. Let α be the norm on M with splitting base $\{n_1, \ldots, n_\ell, p_1, \ldots, p_\ell\}$ such that $\alpha(n_i) = \gamma$ and $\alpha(p_i) = -\gamma$ for $1 \leq i \leq \ell$. Then, $\alpha \precsim h$, and $gr(M) = gr(N) \oplus gr(P)$, with $\Gamma_{gr(N)} = [\gamma]$ and $\Gamma_{gr(P)} = [-\gamma]$. Also, $gr(N)$ and $gr(P)$ are complementary totally h'_α-isotropic subspaces of $gr(M)$, so h'_α is hyperbolic. To see that $\alpha \prec h$, one can check (3.4) for the given splitting base of α. Since $h'_\alpha(n'_i, p'_j) = \delta_{ij}$ and $h'_\alpha(n'_i, n'_j) = 0 = h'_\alpha(p'_i, p'_j)$ for all i, j it is clear that h'_α is nondegenerate, which verifies $\alpha \precsim h$.

(iii) Suppose M and N are finite-dimensional right D-vector spaces with respective nondegenerate λ-Hermitian forms (for the involution τ on D) h and k, and respective norms α and β. Then on $M \oplus N$ we have the nondegenerate λ-Hermitian form $h \perp k$ for τ given by

$$(h \perp k)\big((m_1, n_1), (m_2, n_2)\big) \ = \ h(m_1, m_2) + k(n_1, n_2)\,.$$

There is also the value function $\alpha \oplus \beta$ on $M \oplus N$ given by $(\alpha \oplus \beta)(m, n) = min(\alpha(m), \beta(n))$. Then, $\alpha \oplus \beta$ is a norm on $M \oplus N$, with $gr(M \oplus N) \cong gr(M) \oplus gr(N)$. Furthermore, if $\alpha \prec h$ and $\beta \prec k$, then $\alpha \oplus \beta \prec h \perp k$. When this occurs, we have $(h \perp k)'_{\alpha \oplus \beta} \cong h'_\alpha \perp k'_\beta$. Since an orthogonal sum of nondegenerate graded Hermitian forms is nondegenerate, it follows that if $\alpha \precsim h$ and $\beta \precsim k$, then $\alpha \oplus \beta \precsim h \perp k$. All this is easy to verify.

These examples give another way of seeing the existence of norms compatible with any given λ-Hermitian form h. For, by [S, p. 259, Th. 6.3; p. 264, Th. 8.1], h is diagonalizable or hyperbolic. The first case is covered by Ex. 3.7(i) and the second by Ex. 3.7(ii).

Proposition 3.8. *Let h be a nondegenerate λ-Hermitian form for τ on M and let α be a norm on M with $\alpha \precsim h$. Let N be any subspace of M. Let $gr(N)^\perp$ be the orthogonal of $gr(N)$ in $gr(M)$ with respect to h'_α. Then,*

(i) *$gr(N)^\perp = gr(N^\perp)$ in $gr(M)$.*

(ii) *Suppose $h|_N$ is nondegenerate. Then, $\alpha|_N \precsim h|_N$ iff $N^\perp$ is a splitting complement of N with respect to α, iff $\alpha|_{N^\perp} \precsim h|_{N^\perp}$.*

Proof. (i) It is clear from (3.3) that $gr(N^\perp) \perp gr(N)$ with respect to h'_α, i.e., $gr(N^\perp) \subseteq gr(N)^\perp$. Since $\alpha \precsim h$, we have h'_α is a nondegenerate form on $gr(M)$. Hence, as $\alpha|_N$ and $\alpha|_{N^\perp}$ are norms by Prop. 2.5, we have

$$\begin{aligned} dim_{gr(D)}(gr(N^\perp)) &= dim_{gr(D)}(gr(M)) - dim_{gr(D)}(gr(N)) = dim_D(M) - dim_D(N) \\ &= dim_D(N^\perp) = dim_{gr(D)}(gr(N^\perp)). \end{aligned}$$

Hence, $gr(N^\perp) = gr(N)^\perp$.

(ii) The nondegeneracy of $h|_N$ implies that $N \cap N^\perp = (0)$, so $h|_{N^\perp}$ is also nondegenerate. Since h'_α is nondegenerate, we have $\alpha|_N \precsim h|_N$ iff $gr(N) \cap gr(N)^\perp = (0)$, iff (by (i)) $gr(N) \cap gr(N^\perp) = (0)$, iff (by Remark 2.6) $N^\perp$ is a splitting complement of N. This condition is symmetric in N and $N^\perp$. So it holds iff $\alpha|_{N^\perp} \precsim h|_{N^\perp}$. □

PROPOSITION 3.9. *Suppose the λ-Hermitian form h on M is hyperbolic. For any norm α on M with $\alpha \precsim h$, the associated graded form h'_α is metabolic. If h'_α is even, then it is hyperbolic.*

Proof. Since h is hyperbolic, M has a totally isotropic subspace N with $dim_D(N) = \frac{1}{2} dim_D(M)$. Then, $gr(N)$ is a graded subspace of $gr(M)$ with $dim_{gr(D)}(gr(N)) = \frac{1}{2} dim_{gr(D)}(gr(M))$. Furthermore, $gr(N)$ is totally isotropic for h'_α, as $h'_\alpha(m', n') = 0$ for all homogeneous elements m', n' of $gr(N)$. Hence, h'_α is metabolic. If h'_α is even, it is also hyperbolic, by Prop. 1.4(iii). □

Prop. 3.9 indicates the importance of knowing that associated graded forms are even.

DEFINITION 3.10. For a division algebra D with valuation v and involution τ compatible with v and any $\lambda \in D$ with $\lambda\tau(\lambda) = 1$, we say that (v, τ, λ) *preserves even forms* if for any $(S, h) \in \mathcal{H}^+(D, \tau, \lambda)$, and any norm α on S with $\alpha \precsim h$, the associated graded form h'_α is even.

It is clear that whenever $char(\overline{D}) \neq 2$, (v, τ, λ) preserves even forms since all forms are even in characteristic different from 2. We will show in Prop. 3.15 below other significant cases where (v, τ, λ) preserves even forms.

THEOREM 3.11.

(i) *Let h be a nondegenerate λ-Hermitian form for τ on M. If α and β are any two norms on M with $\alpha \precsim h$ and $\beta \precsim h$, and if h'_α and h'_β are even, then the anisotropic parts of h'_α and h'_β are isometric.*

(ii) *Suppose (v, τ, λ) preserves even forms. Then, the map $h \mapsto h'_{\alpha\,an}$ (for any norm α with $\alpha \precsim h$) gives a well-defined group epimorphism $\Theta\colon W^+(D, \tau, \lambda) \to W_g^+(gr(D), \tau', \lambda')$. Furthermore, there is a canonical "first residue map" $W^+(D, \tau, \lambda) \to W^+(\overline{D}, \overline{\tau}, \lambda')$, where $\overline{\tau}$ is the involution on $\overline{D}$ induced by τ.*

Proof. (i) On the D-vector space $M \oplus M$ there is the hyperbolic λ-Hermitian form $h \perp -h$ and the norm $\alpha \oplus \beta$. Since $\alpha \precsim h$ and $\beta \precsim -h$, Ex. 3.7(iii) shows that $\alpha \oplus \beta \precsim h \perp -h$ and $(h \perp -h)'_{\alpha\oplus\beta} \cong h'_\alpha \perp (-h)'_\beta \cong h'_\alpha \perp -h'_\beta$. Therefore, by Prop. 3.9 $h'_\alpha \perp -h'_\beta$ is hyperbolic. Hence, h'_α and h'_β have isometric anisotropic parts, by Prop. 1.4(viii).

(ii) Take any $(M, h), (N, \ell) \in \mathcal{H}^+(D, \tau, \lambda))$ with ℓ hyperbolic, and norms α on M, β on N, and δ on $M \oplus N$ with $\alpha \precsim h$, $\beta \precsim \ell$, and $\delta \precsim h \perp \ell$. We have, by (i),

$$(h \perp \ell)'_{\delta\,an} \cong (h \perp \ell)'_{\alpha\oplus\beta\,an} \cong (h'_\alpha \perp \ell'_\beta)_{an} \cong h'_{\alpha\,an},$$

as ℓ'_β is hyperbolic by Prop. 3.9. Therefore, the map Θ is well-defined. It is clearly a group homomorphism. It is surjective since any anisotropic form in $\mathcal{GH}^+(gr(D), \tau', \lambda')$ is an orthogonal sum of 1-dimensional forms (see Prop. 1.4(iv)) whose Witt group classes clearly lie in $im(\Theta)$ (see Ex. 3.7(i)).

We have canonical maps,

$$W_g^+(gr(D),\tau',\lambda') \xrightarrow{\cong} \bigoplus_{[\gamma]\in\frac{1}{2}\Gamma_{gr(D)}/\Gamma_{gr(D)}} W_g^+(gr(D),\tau',\lambda';[\gamma]) \longrightarrow W_g^+(gr(D),\tau',\lambda';[0])$$
$$\xrightarrow{\cong} W^+(gr(D)_0,\tau'|_{gr(D)_0},\lambda') \xrightarrow{=} W^+(\overline{D},\overline{\tau},\lambda'),$$

where the first map is the isomorphism of Prop. 1.5(ii), the second is projection onto the [0]-component, the third is the isomorphism of Prop. 1.5(i), and the fourth expresses the equality $gr(D)_0=\overline{D}$, in which $\tau'|_{gr(D)_0}=\overline{\tau}$. The first residue map is the composition of these maps with Θ. □

Remark 3.12. For a diagonal λ-Hermitian form $h=\langle d_1,\ldots,d_k\rangle$ for τ, with each $d_i\in Symd(D,\tau,\lambda)$, the image of h under the first residue map of Th. 3.11(ii) is computable as follows: We can reorder the d_i so that $v(d_1),\ldots,v(d_j)\in 2\Gamma_D$ and $v(d_{j+1}),\ldots,v(d_k)\notin 2\Gamma_D$. For each $i\le j$ choose any $s_i\in D$ with $v(s_i)=-\frac{1}{2}v(d_i)\in\Gamma_D$. Let $c_i=\tau(s_i)d_is_i$; so $v(c_i)=0$. Then the first residue of h is the class of $\langle\overline{c_1},\ldots,\overline{c_j}\rangle$ in $W^+(\overline{D},\overline{\tau},\overline{\lambda})$. The theorem shows that the Witt class of this form is well-defined, independent of the choice of diagonalization of h and independent of the choices of the s_i. There are also second residue maps obtainable by projection onto the other components in the direct sum of Prop. 1.5(ii). But these are not canonical because of the choices of the r_γ. Notice also that these second residues live in $W^+(\overline{D},\overline{\tau_{[\gamma]}},\overline{\lambda_{[\gamma]}})$ where the involutions on $\overline{D}$ and the $\overline{\lambda}$'s in $\overline{D}$ can vary for the different $[\gamma]\in\frac{1}{2}\Gamma_D/\Gamma_D$.

When the involution τ on D is of the first kind, it is of either symplectic type or orthogonal type (see the definitions in [KMRT, Def. (2.5)]), and $\lambda=\pm1$. We say that (τ,λ) is a *symplectic pair* if τ is of symplectic type and $\lambda=1$ or τ is of orthogonal type and $\lambda\neq1$. (So, when $char(D)=2$, (τ,λ) is a symplectic pair iff τ is of symplectic type.) This terminology is used because (τ,λ) is a symplectic pair iff the isometry groups of all λ-Hermitian forms for τ are symplectic groups.

For any kind or type of involution, in investigating preservation of even forms, we need to work with the set of λ-symmetrized elements of D: Set

$$Symd(D,\tau,\lambda) \;=\; \{\,d+\lambda\tau(d)\mid d\in D\,\}. \tag{3.7}$$

Then, $Symd(D,\tau,\lambda)$ is a vector space over $F=K^\tau$, where $K=Z(D)$. It is known (see [KMRT, Prop. (2.6), Prop. (2.17), Prop. (2.7)]) that if $dim_K(D)=n^2$,

$$dim_F(Symd(D,\tau,\lambda)) \;=\; \begin{cases} n^2 & \text{if } \tau \text{ is of the second kind;}\\ n(n-1)/2 & \text{if } \tau \text{ is of the first kind with } (\tau,\lambda) \text{ a symplectic pair;}\\ n(n-1)/2 & \text{if } \tau \text{ is of the first kind and } char(D)=2;\\ n(n+1)/2 & \text{otherwise.}\end{cases} \tag{3.8}$$

The analogous result holds in the graded situation: As in §1, let E be a graded division ring finite dimensional over its center $Z(E)$, let σ be a graded involution on E, and let $\epsilon\in Z(E)_0$ with $\epsilon\sigma(\epsilon)=1$. Let $R=Z(E)$, and let $S=Z(E)^\sigma$, which is a graded subfield of R with $[R{:}S]=1$ or 2, depending on whether σ is of the first or the second kind. The graded division ring E has no zero divisors, as Γ_E is totally ordered. (If we had $ab=0$ for nonzero elements a,b of E, then the product of their least degree homogeneous components would be 0; but E has no homogeneous zero divisors.) Thus, the integral domain S has a quotient field, call it Q. We have $E\otimes_S Q$ has no zero divisors and is finite dimensional over its center $R\otimes_S Q$, so it is a division ring, with $dim_{R\otimes_S Q}(E\otimes_S Q)=dim_R(E)$. The involution σ on E extends to an involution $\widetilde{\sigma}=\sigma\otimes id$ on $D\otimes_S Q$, and clearly $\widetilde{\sigma}$ is of the same kind (first or second) as σ. When σ is of the first kind, we define the type of σ (orthogonal or symplectic) to be that of $\widetilde{\sigma}$.

We say that (σ, ϵ) is a symplectic pair if $(\widetilde{\sigma}, \epsilon)$ is a symplectic pair for $E \otimes_S Q$. Analogously to (3.7), define

$$Symd(E, \sigma, \epsilon) \; = \; \{\, c + \epsilon\sigma(c) \mid c \in E \,\}\,. \tag{3.9}$$

Clearly $Symd(E, \sigma, \epsilon)$ is a graded S-vector subspace of E and $Symd(E, \sigma, \epsilon) \otimes_S Q \cong Symd(E \otimes_S Q, \widetilde{\sigma}, \epsilon)$. By applying (3.8) to $Symd(E \otimes_S Q, \widetilde{\sigma}, \epsilon)$, it follows that if $n^2 = dim_R(E) = dim_{R\otimes_S Q}(E \otimes_S Q)$ then $dim_S(Symd(E, \sigma, \epsilon))$ satisfies the formulas analogous to those in (3.8) in all four cases. Notice also that if $char(E) \neq 2$ and σ is of the first kind, then $dim_S(Symd(E, \sigma, \epsilon))$ distinguishes the type of σ directly within E without reference to Q. When $char(E) = 2$ and σ is of the first kind, then one has (see [KMRT, Prop. (2.6)(2)]) that $\widetilde{\sigma}$ is symplectic iff $Trd(c) = 0$ for all $c \in E \otimes_S Q$ such that $\widetilde{\sigma}(c) = c$, where Trd is the reduced trace. Furthermore, this holds iff $1 \in Symd(E \otimes_S Q, \widetilde{\sigma}, \epsilon)$. The corresponding criteria apply within E for σ to be symplectic, since for the reduced trace Trd on $E \otimes_S Q$, we have $Trd(E_\gamma) \subseteq R_\gamma$ for each $\gamma \in \Gamma_E$. (This follows because the minimal polynomial over $R \otimes_S Q$ of a homogeneous element of R has homogeneous coefficients in R, as shown by the proof of [HW$_1$, Prop. 2.2].)

The following proposition will be needed in determining preservation of even forms when $char(\overline{D}) = 2$. Since $Symd(D, \tau, \lambda)$ is an F-vector subspace of D, the valuation v restricts to a value function on $Symd(D, \tau, \lambda)$ with respect to the valuation $v|_F$ on F. So, it has an associated graded $gr(F)$-vector space $gr(Symd(D, \tau, \lambda)))$, which is a graded subspace of $gr(D)$. We say that D is *defectless* over F if $dim_{gr(F)}(gr(D)) = dim_F(D)$. Equivalently, D is defectless over F iff v is a norm on D with respect to $v|_F$ when D is viewed as a vector space over F. When this occurs, v restricts to a norm on $Symd(D, \tau, \lambda)$, by Prop. 2.5.

PROPOSITION 3.13.

(i) $Symd(gr(D), \tau', \lambda') \;\subseteq\; gr(Symd(D, \tau, \lambda))\,.$

(ii) *Suppose D is defectless over F. Then, the following conditions are equivalent:*

- (a) (v, τ, λ) *preserves even forms.*
- (b) $Symd(gr(D), \tau', \lambda') \;=\; gr(Symd(D, \tau, \lambda))\,.$
- (c) $dim_{gr(D)}\big(Symd(gr(D), \tau', \lambda')\big) \;=\; dim_D(Symd((D, \tau, \lambda)))\,.$
- (d) *For every $d \in Symd(D, \tau, \lambda)$ there is an $a \in D$ with $a + \lambda\tau(a) = d$ and $v(a) = v(d)$.*

Proof. (i) For $d \in D^\times$ with image d' in gr(D), we have $d' + \lambda'\tau'(d') = \big(d + \lambda\tau(d)\big)'$ if $v(d + \lambda\tau(d)) = v(d)$, and $d' + \lambda'\tau'(d') = 0$ otherwise. This proves the desired inclusion for homogeneous elements; the inclusion then holds throughout these graded vector spaces.

(ii) (The defectless assumption is not needed for (a) $\Leftrightarrow$ (b).) (b) $\Rightarrow$ (a) Suppose condition (b) holds. Take any $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$ and any norm α on M with $\alpha \precsim h$, and form $gr(M)$ with respect to α. For any nonzero homogeneous element $\widetilde{m}$ of $gr(M)$ there is a nonzero $m \in M$ with $m' = \widetilde{m}$. We have $h'_\alpha(\widetilde{m}, \widetilde{m}) = h(m, m)'$ or $= 0$, by (3.3). In either case, $h'_\alpha(\widetilde{m}, \widetilde{m}) \in gr(Symd(D, \tau, \lambda))$. Condition (b) yields $h'_\alpha(\widetilde{m}, \widetilde{m}) \in Symd(gr(D), \tau', \lambda')$. Because $gr(M)$ is generated as an abelian group by its homogeneous elements, it follows that $h'_\alpha(s, s) \in Symd(gr(D), \tau', \lambda')$ for all $s \in gr(M)$; so h'_α is an even form, proving (a).

(a) $\Rightarrow$ (b) Suppose (b) does not hold. Then, there is a homogeneous $\widetilde{a} \in gr(Symd(D, \tau, \lambda))$ with $\widetilde{a} \notin Symd(gr(D), \tau', \lambda')$. We have $\widetilde{a} = a'$ for some $a \in Symd(D, \tau, \lambda)$. On the 1-dimensional D-vector space D, define an even λ-Hermitian form h for τ by $h(d, e) = \tau(d)ae$. Any norm α on D is defined by choosing $\gamma \in \Gamma$ and setting $\alpha(d) = v(d) + \gamma$ for all $d \in D$. Then, $\{\,1\,\}$ is a splitting base of α and an orthogonal base for h, so Ex. 3.7(i) shows that $\alpha \precsim h$ iff $\gamma = \frac{1}{2}v(h(1, 1)) = \frac{1}{2}v(a)$. When this holds, we have $h'_\alpha(1', 1') = h(1, 1)' = \widetilde{a} \notin Symd(gr(D), \tau', \lambda')$. So, h'_α is not even for the unique v-norm on D which is compatible with h; thus, (a) does not hold.

(b) $\Leftrightarrow$ (c) As noted above, since D is defectless over F, v is a $v|_F$-norm for the F-vector space D and for its subspace $Symd(D, \tau, \lambda)$. Hence, $dim_{gr(F)}\big(gr(Symd(D, \tau, \lambda))\big) = dim_F(Symd(D, \tau, \lambda))$. Therefore,

(c) is equivalent to: $dim_{gr(F)}\big(gr(Symd(D,\tau,\lambda))\big) = dim_{gr(F)}\big(Symd(gr(D),\tau',\lambda')\big)$. In view of (i), this is clearly equivalent to (b).

(b) $\Leftrightarrow$ (d) We again use the fact that v is a $v|_F$-norm for D. Let $f\colon D \to D$ be the F-linear map given by $c \mapsto c + \lambda\tau(c)$. In the notation of (2.7), let

$$j \;=\; j_{v,v}(f) = min\{\, v(c+\lambda\tau(c)) - v(c) \mid c \in D^\times \,\}\,.$$

Clearly, $j \geq 0$. Suppose first that $j > 0$. So, for any $d \in D^\times$, we have $v(d+\lambda\tau(d)) \geq v(d) + j > v(d)$. Therefore, condition (c) holds only for $d = 0$. Also, taking $d = 1$, we have $v(1+\lambda) > v(1) = 0$, i.e. $\lambda' = -1$. Furthermore, for any $d \in D^\times$, we have $v(d - \tau(d)) = v(d + \lambda\tau(d) - (1+\lambda)\tau(d)) > v(d)$. This means that $\tau'(d') = d'$ in $gr(D)$, and hence $Symd(gr(D),\tau',\lambda') = 0$. Thus, condition (b) holds iff $Symd(D,\tau,\lambda) = 0$ iff condition (c) holds.

Now, suppose $j = 0$. Then, the graded map f' on $gr(D)$ induced by f (see (2.10)) maps D_γ into D_γ for each $\gamma \in \Gamma_D$, and for $c \in D^\times$,

$$f'(c') \;=\; \begin{cases} (c+\lambda\tau(c))' \;=\; c' + \lambda'\tau'(c'), & \text{if } v(c+\lambda\tau(c)) = v(c);\\ \qquad 0 \qquad\quad\;\, =\; c' + \lambda'\tau'(c'), & \text{if } v(c+\lambda\tau(c)) > v(c).\end{cases}$$

So, $im(f') = Symd(gr(D),\tau',\lambda')$, while clearly $gr(im(f)) = gr(Symd(D,\tau,\lambda))$. Thus, condition (b) holds iff $im(f') = gr(im(f))$, iff, by Prop. 2.7(ii), for every $d \in Symd(D,\tau,\lambda) = im(f)$ there is an $a \in D$ with $d = f(a) = a + \lambda\tau(a)$ and $v(a) = v(d) + j = v(d)$, which is condition (d). $\square$

In some cases, preservation of even forms requires an assumption of tameness of the valuation. Let K^h be the Henselization of K with respect to $v|_K$. (If $v|_K$ is a discrete valuation, we could replace K^h by the completion of K with respect to v.) Let $D^h = D \otimes_K K^h$. By Morandi's theorem [M, Th. 2], because $v|_K$ extends to a valuation on D, D^h is a division ring; furthermore, the Henselian valuation v^h on K^h extends uniquely to a valuation on D^h with $\overline{D^h} \cong \overline{D}$ and $\Gamma_{D^h} = \Gamma_D$. Tameness of division algebras over Henselian fields is described in [JW, §6] and in [W$_2$, §3]. We say that D is *tame* with respect to v if D^h is tame, i.e., if D^h is split by the maximal tamely ramified extension field of K^h. By [Bl, Cor. 4.4] or [HW$_2$, Prop. 4.3] (applied to D^h), we have

$$D \text{ is tame iff } dim_{gr(K)}(gr(D)) \;=\; dim_K(D) \text{ and } Z(gr(D)) = gr(K)\,. \tag{3.10}$$

In many cases arising here, we have $dim_K(D)$ a power of 2 and $char(\overline{D}) = 2$; the condition of tameness is then equivalent to: D is split by the maximal unramified extension of K^h; equivalently, D^h has a maximal subfield which is unramified over K^h. When the involution τ is of the second kind, we sometimes require that D be tame over $F = K^\tau$. This means that D is tame and K is tame over F i.e., as $[K\!:\!F] = 2$, either $[\overline{K}\!:\!\overline{F}] = 2$ and $\overline{K}$ is separable over $\overline{F}$, or $|\Gamma_K\!:\!\Gamma_F| = 2$, with the latter case not allowed if $char(\overline{F}) = 2$.

Remark 3.14. Whenever D is tame over F the involutions τ and τ' are of the same kind. For, if τ is of the first kind, then $\tau'|_{Z(gr(D))} = id$, as $Z(gr(D)) = gr(Z(D))$ by (3.10). On the other hand, if τ is of the second kind and $char(\overline{D}) = 2$, then the tameness implies that K is unramified over F. Since τ induces the nontrivial automorphism of K/F, the residue involution $\overline{\tau}$ induces the nontrivial automorphism of $\overline{K}/\overline{F}$, so τ' is not the identity on $gr(Z(D))$. If τ is of the second kind and $char(\overline{D}) \neq 2$, then there is $c \in K$ with $\tau(c) = -c$. So, in $gr(K) = Z(gr(D))$, we have $\tau'(c') = -c' \neq c'$, showing that τ' is of the second kind.

Proposition 3.15. (v,τ,λ) *preserves even forms in each of the following cases:*

(i) $char(\overline{D}) \neq 2$.
(ii) τ' *is of the second kind.*
(iii) $char(D) = 2$ *and* D *is tame over* F.
(iv) $char(D) = 0$, $char(\overline{D}) = 2$, D *is tame, and* (τ,λ) *is a symplectic pair.*

Proof. (i) and (ii) When $char(\overline{D}) \neq 2$ or τ' is of the second kind, then all forms in $\mathcal{H}(gr(D), \tau', \lambda')$ are even.

(iii) and (iv) In case (iii), we may assume that τ (hence also τ') is of the first kind, since the second kind case is covered by (ii). Let $n^2 = dim_K(D)$ and $n'^2 = dim_{Z(gr(D))}(gr(D))$. Because D is tame over F, we have $Z(gr(D)) = gr(K)$, $n = n'$, and $dim_F(D) = dim_{gr(F)}(gr(D))$ (see (3.10)). The last equality shows that D is defectless over F. Since $n = n'$, the dimension formula (3.8) and the analogous graded formula yield in each case that $dim_{gr(F)}(Symd(gr(D), \tau', \lambda')) = dim_F(Symd(D, \tau, \lambda))$. Hence, (v, τ, λ) preserves even forms, by Prop. 3.13 (ii)(c)⇒(a). □

Note that the proof of Prop. 3.15 shows that when D is tame over F, (v, τ, λ) does *not* preserve even forms except in the cases listed in the proposition.

4. Henselian valuations

Classically, Springer's Theorem [Sp$_1$] for quadratic forms over a field with complete discrete valuation (with residue characteristic not 2) says that a form is anisotropic iff its two residue forms are anisotropic, and that its class in the Witt group is determined by the Witt classes of the residue forms. This corresponds to having not just a map Θ as in Th. 3.11 but having Θ an isomorphism. It is well known (see, e.g., [AK, p. 174], [Kne, Th. 12.1.5, sentence after (12.2.1)]) that Springer's theorem is valid for Henselian valuations (with any value group) as well as for complete discrete valuations. We will in this section prove that when v on F is Henselian, then Θ is actually an isomorphism whenever $char(\overline{F}) \neq 2$, and sometimes even when $char(\overline{F}) = 2$.

Recall that a valuation v on a field F is *Henselian* if Hensel's Lemma holds for v. Equivalently, (cf. [EP, Th. 4.1.3]), v is Henselian iff v has a unique extension to a valuation on each field L algebraic over F. It follows immediately that the extension of v to any such L is also Henselian. Furthermore, the uniqueness of the extension allows one to see that v extends uniquely to each division algebra finite-dimensional over F (cf. [W$_1$]).

Throughout this section, $D, v, \tau, \lambda, K, F, \tau', \lambda'$ will have the same meaning as in §3. When we say that v is Henselian, we mean that $v|_F$ is Henselian. This assures that the valuation on K is also Henselian, and that v extends, uniquely, to a valuation on D. The uniqueness of these valuations guarantees that for any involution τ on D with $K^\tau = F$, we have $v \circ \tau = v$.

Our principal results in this section which hold whenever $char(\overline{D}) \neq 2$ also hold in the following cases when $char(\overline{D}) = 2$:

Definition 4.1. *The good cases when* $char(\overline{D}) = 2$ are when D is tame over F, and, if τ is of the first kind and $char(D) = 0$, (τ, λ) is a symplectic pair.

Note that when D is tame over F and $char(\overline{D}) = 2$, these good cases are exactly those where (v, τ, λ) preserves even forms—see Prop. 3.15 and the remark after its proof. Also recall Remark 3.14 that the tameness assumption guarantees that τ and τ' are of the same kind.

The case of anisotropic associated graded Hermitian forms is quite special, and requires no Henselian assumption:

Proposition 4.2. *Let* $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$. *Let* $\beta(m) = \frac{1}{2}v(h(m, m)) \in \frac{1}{2}\Gamma_D \cup \{\infty\}$, *for all* $m \in M$. *Then, the following conditions are equivalent:*

(i) *There is a norm* α *on* M *with* $\alpha \precsim h$ *and* h'_α *anisotropic.*
(ii) β *is a norm on* M *with* $\beta \precsim h$.
(iii) *For all* $m, n \in M$, $2v(h(m, n)) \geq v(h(m, m)) + v(h(n, n))$.

When these conditions hold, β *is the only norm on* M *which is compatible with* h, *and* h'_β *is anisotropic.*

Proof. Note first that for any norm α on M with $\alpha \precsim h$ and any nonzero $m \in M$, (3.3) shows

$$h'_\alpha(m', m') \neq 0 \quad \text{iff} \quad \alpha(m) = \beta(m)\,. \tag{4.1}$$

(i) $\Rightarrow$ (ii) If $\alpha \precsim h$ and h'_α is anisotropic, then (4.1) shows $\beta = \alpha$. Hence, β is a norm on M with $\beta \precsim h$.

(ii) $\Rightarrow$ (i) and (iii) Suppose (ii) holds. Then, h'_β has no isotropic homogeneous elements, by (4.1), so h'_β is anisotropic. Then (i) holds with $\alpha = \beta$. Since $\beta \prec h$, we have, by the definition, $\beta(m) + \beta(n) \le v(h(m,n))$ for all $m, n \in M$. This is the inequality in (iii).

(iii) $\Rightarrow$ (ii) Suppose (iii) holds. By Cor. 3.6 there is a norm α on M with $\alpha \precsim h$. Since $\alpha \prec h$, we have $\alpha(m) + \alpha(m) \le v(h(m,m))$ for all $m \in M$, i.e., $\alpha(m) \le \beta(m)$. Suppose there were a nonzero $n \in M$ with $\alpha(n) < \beta(n)$. Then, for every $m \in M$, the inequality in (iii) says that $v(h(n,m)) \ge \beta(n) + \beta(m) > \alpha(n) + \alpha(m)$. This contradicts the definition of $\alpha \precsim h$. Thus, we must have $\beta = \alpha$, proving (ii).

When the conditions (i)–(iii) hold, the proof of (iii) $\Rightarrow$ (ii) shows that β is the only norm on M compatible with h, and the proof of (ii) $\Rightarrow$ (i) shows that h'_β is anisotropic. □

PROPOSITION 4.3. *Suppose (v, τ, λ) preserves even forms. Then, the following conditions are equivalent:*

(i) *For every form $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$ and every norm α on M with $\alpha \precsim h$, if h is anisotropic, then h'_α is anisotropic.*
(ii) *For every form $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$ and every norm α on M with $\alpha \precsim h$, if h'_α is hyperbolic, then h is hyperbolic.*
(iii) *The canonical map $\Theta\colon W^+(D, \tau, \lambda) \to W^+_g(gr(D), \tau', \lambda')$ of Th. 3.11(ii) is an isomorphism.*
(iv) *For each nonzero $a, c \in Symd(D, \tau, \lambda)$ with $v(c) > v(a)$, the diagonal form $\langle a, -(a+c)\rangle \in \mathcal{H}^+(D, \tau, \lambda)$ is isotropic.*
(v) *For each nonzero $a \in Symd(D, \tau, \lambda)$ let $\tau_a = int(a^{-1}) \circ \tau$; then for each $m \in Symd(D, \tau_a, 1)$ with $v(m) > 0$ there is $d \in D$ with $\tau_a(d)d = 1 + m$.*

Proof. (i)$\Rightarrow$(ii) Suppose $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$ and α is a norm on M with $\alpha \precsim h$ and h'_α hyperbolic. We argue by induction on $dim_D(M)$. There is a two-dimensional graded subspace N' of $gr(M)$ with $h'_\alpha|_{N'}$ hyperbolic. Let N be a subspace of M with $gr(N) = N'$. Then $\alpha|_N \precsim h|_N$ by Remark 3.2 since $h'_\alpha|_{gr(N)}$ is nondegenerate, and $\big(h|_N\big)'_{\alpha|_N}$ is hyperbolic, so isotropic. By (i), $h|_N$ is isotropic, hence metabolic, hence hyperbolic by Prop. 1.4(iii), since it is even. By Prop. 3.8, $\alpha|_{N^\perp} \precsim h|_{N^\perp}$ and $gr(N^\perp) = N'^\perp$ in $gr(M)$. Therefore, $\big(h|_{gr(N^\perp)}\big)'_{\alpha|_{gr(N^\perp)}}$ is hyperbolic, since h'_α and $h'_\alpha|_{N'}$ are hyperbolic. By induction, $h|_{N^\perp}$ is hyperbolic. So, $h = h|_N \perp h|_{N^\perp}$ is hyperbolic.

(ii)$\Leftrightarrow$(iii) This is clear since the kernel of $W^+(D, \tau, \lambda) \to W^+_g(gr(D), \tau', \lambda')$ consists of Witt group equivalence classes of forms h with compatible norm α with h'_α hyperbolic.

(ii)$\Rightarrow$(iv) Given a and c as in (iv), let M be a two-dimensional right D-vector space with base $\{ m_1, m_2 \}$ and let h be the even λ-Hermitian form for τ with $h(m_1, m_1) = a$, $h(m_1, m_2) = 0$, and $h(m_2, m_2) = -(a+c)$. So, $h = \langle a, -(a+c)\rangle$. Let α be the norm on M with splitting base $\{ m_1, m_2 \}$ such that $\alpha(m_1) = \alpha(m_2) = \frac{1}{2}v(a) = \frac{1}{2}v(-(a+c))$. Then (see Ex. 3.7(i)) $\alpha \precsim h$ and we have $h'_\alpha(m'_1, m'_1) = a' = -h'_\alpha(m_2, m_2)$ and $h'_\alpha(m'_1, m'_2) = 0$. The 2-dimensional even form h'_α is clearly hyperbolic, so by (ii) h is hyperbolic, so isotropic.

(iv)$\Rightarrow$(i) Assume (iv) holds but not (i). Then, there is $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$ and a norm α on M with $h \precsim \alpha$ with h anisotropic but h'_α isotropic. There is a 2-dimensional subspace N' of $gr(M)$ with $h'_\alpha|_{N'}$ hyperbolic. Let N be any 2-dimensional D-subspace of M with $gr(N) = N'$. We have $\alpha|_N \precsim h|_N$ as $h'_\alpha|_{gr(N)}$ is nondegenerate. Take any orthogonal base $\{ n_1, n_2 \}$ of N for $h|_N$, which exists as $h|_N$ is anisotropic (see Prop. 1.4(iv)). Say $h(n_1, n_1) = a$ and $h(n_2, n_2) = b$, with $a, b \in Symd(D, \tau, \lambda)$. Let β be the norm on N with splitting base $\{ n_1, n_2 \}$ such that $\beta(n_1) = \frac{1}{2}v(a)$ and

$\beta(n_2) = \frac{1}{2}v(b)$. Then $\beta \precsim h|_N$ by Ex. 3.7(i), and for the form h'_β on $gr_\beta(N)$ we have $h'_\beta(n'_1, n'_1) = a'$, $h'_\beta(n'_2, n'_2) = b'$, and $h'_\beta(n'_1, n'_2) = 0$. Because $\big(h|_N\big)'_{\alpha|_N}$ is isotropic, Prop. 4.2 shows that h'_β must also be isotropic; so, it has a homogeneous isotropic vector of the form $n'_1 d' + n'_2$ for some $d \in D^\times$. Then, $0 = h'_\beta(n'_1 d' + n'_2, n'_1 d' + n'_2) = \tau'(d')a'd' + b' = (\tau(d)ad)' + b'$. Lifting back to D, this yields $b = -\tau(d)ad - e$ with $v(e) > v(\tau(d)ad) = v(a) + 2v(d)$. Then, by replacing the base vector n_2 by $n_2 d^{-1}$, we have

$$h|_N \;=\; \langle a, b\rangle \;\cong\; \langle a, \tau(d^{-1})bd^{-1}\rangle \;=\; \langle a, -a - \tau(d^{-1})ed^{-1}\rangle \;=\; \langle a, -(a + c)\rangle\,,$$

where $c = \tau(d^{-1})ed^{-1}$. We have $v(c) = v(e) - 2v(d) > v(a)$, and $c \in \mathit{Symd}(D, \tau, \lambda)$ since a and b and hence e lie in $\mathit{Symd}(D, \tau, \lambda)$. By (iv), $h|_N$ is isotropic, contradicting the choice of h.

(iv)⇔(v) Let $h = \langle a, -(a + c)\rangle \in \mathcal{H}^+(D, \tau, \lambda)$ with $a, c \in \mathit{Symd}(D, \tau, \lambda)$ and $v(c) > v(a)$. Then, the scaled form $a^{-1}h = \langle a^{-1}a, -a^{-1}(a+c)\rangle = \langle 1, -(1+a^{-1}c)\rangle$ lies in $\mathcal{H}^+(D, \tau_a, 1)$, where $\tau_a = \mathit{int}(a^{-1}) \circ \tau$. We have $1, (1 + ac^{-1}) \in \mathit{Symd}(D, \tau_a, 1)$, so $a^{-1}c \in \mathit{Symd}(D, \tau_a, 1)$, with $v(a^{-1}c) > 0$. If (v) holds, then $a^{-1}h$ is isotropic, so h is isotropic, proving (iv). Conversely, for $a \in \mathit{Symd}(D, \tau, \lambda)$ and $m \in \mathit{Symd}(D, \tau_a, 1)$ with $v(m) > 0$, we have $am \in \mathit{Symd}(D, \tau, \lambda)$ with $v(am) > v(a)$. If (iv) holds, then the form $\langle a, -(a + am)\rangle \in \mathcal{H}^+(D, \tau, \lambda)$ is isotropic, so the scaled form $a^{-1}\langle a, -(a + am)\rangle = \langle 1, -(1 + m)\rangle \in \mathcal{H}^+(D, \tau_a, 1)$ is also isotropic. Then, $1 + m = \tau(d)d$ for some $d \in D^\times$, proving (v). □

Lemma 4.4. *Let F be a field with a Henselian valuation v, and let L be a finite degree separable extension field of F with L unramified over F with respect to v. Let $\mathfrak{m}_F$ (resp. $\mathfrak{m}_L$) be the maximal ideal of the valuation ring of v on F (resp. L). Then, $1 + \mathfrak{m}_F \subseteq N_{L/F}(1 + \mathfrak{m}_L)$, where $N_{L/F}$ is the norm from L to F.*

Proof. This is known. See [E$_2$, Prop. 2], where it is pointed out that there is a proof of this contained in [Y, Lemma 4.1] which is valid for all Henselian valuations not just for discrete Henselian valuations. For the convenience of the reader we give the short proof. That L is unramified over F means that the residue field $\overline{L}$ is separable over $\overline{F}$ and $[\overline{L}:\overline{F}] = [L:F]$. Take any nonzero $a \in \overline{L}$ with $\overline{L} = \overline{F}(a)$. Let $\widetilde{f} = x^k + \widetilde{c}_{k-1}x^{k-1} + \cdots + \widetilde{c}_0 \in \overline{F}[x]$ be the minimal polynomial of a over $\overline{F}$. Choose any $c_0, \ldots, c_{k-1} \in F$ with $v(c_i) \geq 0$ and $\overline{c_i} = \widetilde{c}_i$ in $\overline{F}$, and let $f = x^k + c_{k-1}x^{k-1} + \cdots + c_0 \in F[x]$. Since the image $\widetilde{f}$ of f in $\overline{L}[x]$ has the simple root a and v on L is Henselian, by Hensel's Lemma f has a root b in L with $v(b) = 0$ and $\overline{b} = a$ in $\overline{L}$. Then, $\overline{F(b)} = \overline{L}$, as $a \in \overline{F(b)}$, so $[F(b):F] \geq [\overline{F(b)}:\overline{F}] = [\overline{L}:\overline{F}] = [L:F] = k$. Therefore, $F(b) = L$ and f must be the minimal polynomial of b over F; hence, $N_{L/F}(b) = (-1)^{k-1}c_0$. For any $m \in \mathfrak{m}_F$, let $g = x^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0(1 + m) \in F[x]$. The same reasoning for g as just given for f shows that g has a root d in L with $v(d) = 0$ and $\overline{d} = a$ in $\overline{L}$, and $N_{L/F}(d) = (-1)^{k-1}c_0(1 + m)$. Then, $db^{-1} \in 1 + \mathfrak{m}_L$ and $N_{L/F}(db^{-1}) = 1 + m$. □

In fact, the inclusion in Lemma 4.4 is an equality. The reverse inclusion is not hard to prove, but not included here because we do not need it.

Proposition 4.5. *Let A be a central simple algebra over a field F and let σ be a symplectic involution on A. Take any $a \in \mathit{Symd}(A, \sigma, 1)$ such that $F(a)$ is a field. Then, σ restricts to a symplectic involution on the centralizer $C_A(F(a))$.*

Proof. This is known by [KMRT, Prop. (4.12)] if $\mathit{char}(F) \neq 2$ (or if a is separable over F; it suffices in these cases that $\sigma(a) = a$). Thus, we may assume that $\mathit{char}(F) = 2$. Let $L = F(a)$. Assume the result is false, i.e., that $\sigma|_{C_A(L)}$ is of orthogonal type. There is a splitting field S of A with S linearly disjoint to L over F. For example, we could take S to be the function field over F of the Severi-Brauer variety $SB(A)$, which is a regular extension of F by [Ja, Th. 3.2.11 and Th. 3.7.12], so linearly disjoint to every algebraic extension of F. By replacing A by $S \otimes_F A$, L by $S \otimes_F L$, and σ by $id \otimes \sigma$ (which does not change the type of the involution), we may assume that A is split, say $A = \mathit{End}_F(V)$ for some

F-vector space V. Because $L \subset End_F(V)$, we may view V as a vector space over L. Let $s\colon L \to F$ be any nonzero F-linear map. By [KMRT, Ex. (4.11)], there is a nondegenerate symmetric L-bilinear form $b\colon V \times V \to L$ such that σ is the adjoint involution to the nondegenerate F-bilinear transfer form $s_*b = s \circ b\colon V \times V \to F$. Then, $\sigma|_{C_A(L)}$ is the adjoint involution to b, by [KMRT, Prop. (4.7)]. Because $\sigma|_{C_A(L)}$ is of orthogonal type, the form b is not alternating; so V has an orthogonal L-vector space base $\{ y_1, \ldots, y_k \}$. On the other hand, s_*b is alternating, as σ is symplectic. We claim: for any $y \in V$,

$$s_*b(y, ay) \;=\; 0\,. \tag{4.2}$$

For, as $a \in Symd(A, \sigma, 1)$, we may write $a = c + \sigma(c)$ for some $c \in A$. Then,

$$\begin{aligned} s_*b(y, ay) &= s_*b(y, cy) + s_*b(y, \sigma(c)y) \;=\; s_*b(y, cy) + s_*b(cy, y) \\ &= s_*b(y, cy) + s_*b(y, cy) \;=\; 0\,, \end{aligned}$$

as claimed. Then, for any integer $i \geq 0$, as $\sigma(a) = a$,

$$s_*b(y, a^i y) \;=\; \begin{cases} s_*b(a^{i/2}y, a^{i/2}y) \;=\; 0, & \text{for } i \text{ even, as } s_*b \text{ is alternating;} \\ s_*b(a^{(i-1)/2}y, a^{(i-1)/2}ay) \;=\; 0, & \text{for } i \text{ odd, by (4.2).} \end{cases} \tag{4.3}$$

Hence, $s_*b(y, Ly) = 0$ for every $y \in V$. Thus, for any of the y_i, we have

$$s_*b(y_i, V) \;=\; s_*b(y_i, Ly_i) \;+\; \sum_{j \neq i} s_*b(y_i, Ly_j) \;=\; 0 + 0,$$

by (4.3) and the orthogonality of the y_j. This contradicts the nondegeneracy of s_*b. So $\sigma|_{C_A(L)}$ must be symplectic. □

Theorem 4.6. *Let D be a division algebra with valuation v, and involution τ, with λ, K, F, τ', and λ' as defined at the beginning of §3. Assume v is Henselian. Then, whenever char$(\overline{D}) \neq 2$ and also in the good cases when char$(\overline{D}) = 2$ (see Def. 4.1) the canonical map $\Theta\colon W^+(D, \tau, \lambda) \to W_g^+(gr(D), \tau', \lambda')$ of Th. 3.11(ii) is an isomorphism; the other conditions in Prop. 4.3 also hold.*

Proof. When *char*$(\overline{D}) \neq 2$, and also in the good cases when *char*$(\overline{D}) = 2$, Prop. 3.15 shows that (v, τ, λ) preserves even forms. We prove that condition (v) of Prop. 4.3 holds. Then, the theorem follows by Prop. 4.3. The proof is divided into three cases.

Case I. Suppose *char*$(\overline{D}) \neq 2$. For any $a \in Symd(D, \tau, \lambda)$ and $m \in Symd(D, \tau_a, 1)$ with $v(m) > 0$, we have $\tau_a(m) = m$, so τ is the identity on $F(m)$. By Hensel's Lemma (applied to $x^2 - (1+m) \in F(m)[x]$), $1 + m$ has a square root, say b, in $F(m)$. Then, $\tau_a(b)b = b^2 = 1 + m$, as desired.

Case II. Suppose τ and τ' are of the second kind and K is unramified over F. For $a \in Symd(D, \tau, \lambda)$, τ_a is also of the second kind with $K^{\tau_a} = F$, as $\tau_a|_K = \tau|_K$. Take any $m \in Symd(D, \tau_a, 1)$ with $v(m) > 0$; so $\tau_a(m) = m$. Let $L = F(m)$. Because τ_a acts nontrivially on K but trivially on L, the field $K \cdot L$ is not L. Hence, $[K \cdot L : L] = 2$ and $\tau_a|_{K \cdot L}$ is the nonidentity L-automorphism of $K \cdot L$. Also, $K \cdot L$ is unramified over L as K is unramified over F by [EP, Th. 5.2.7, Rem. 5.2.8]. By Lemma 4.4 applied to $K \cdot L/L$, there is $d \in K \cdot L$ with $1 + m = N_{K \cdot L/L}(d) = \tau_a(d)d$, as desired.

Case III. Now suppose *char*$(\overline{D}) = 2$, τ and τ' are of the first kind, D is tame. If *char*$(D) = 0$, assume also that (τ, λ) is a symplectic pair. For any nonzero $a \in Symd(D, \tau, \lambda)$, the pair $(\tau_a, 1)$ is a symplectic pair by [KMRT, Prop. (2.7)], i.e., τ_a is a symplectic involution. (When *char*$(D) = 2$, this holds even if (τ, λ) is not a symplectic pair.) For the case *char*$(D) = 2$, so $\lambda = 1$, to invoke [KMRT, Prop. (2.7)] we need that $a^{-1} \in Symd(D, \tau, \lambda)$. But, if $a = e + \tau(e)$, then $\tau(a) = a$, so $a^{-1} = a^{-1}ea^{-1} + \tau(a^{-1}ea^{-1}) \in Symd(D, \tau, \lambda)$.) Take any $m \in Symd(D, \tau_a, 1)$ with $v(m) > 0$. By Prop. 4.5, $\tau_a|_{C_D(K(m))}$ is of symplectic type. Let $L = K(m)$ and let $C = C_D(L)$. Then, $L = Z(C)$ and $C \neq L$, since $\tau_a|_C$ is a symplectic involution. Moreover, we claim that C is tame. We verify this using the criteria in (3.10): Since $[gr(D) : gr(K)] = [D : K]$, v is a $v|_K$ norm for D viewed as a K-vector space; so $[gr(L) : gr(F)] = [L : F]$ by Prop. 2.5 and Cor. 2.3(i). Hence, $[gr(D) : gr(L)] =$

$[gr(D):gr(K)]/[gr(L):gr(K)] = [D:K]/[L:K] = [D:L]$. Likewise, $[gr(D):gr(C)] = [D:C]$; hence, $[gr(C):gr(L)] = [C:L]$. Clearly, $gr(C) \subseteq C_{gr(D)}(gr(L))$; in fact, equality holds here by dimension count, by the Double Centralizer Theorem and its graded analogue [HW$_2$, Prop. 1.5]. Another application of the graded Double Centralizer Theorem then shows that $gr(L) = Z(gr(C))$. Thus, C is tame by (3.10), as claimed. (This follows also from [JW, p. 166, last line].) We now have that $(C, \tau_a|_C, 1)$ is a good case for $char(\overline{C}) = 2$ as in Def. 4.1. Now, $1 \in Symd(C, \tau_a|_C, 1)$ as τ_a is a symplectic involution, by [KMRT, Prop. 2.6] if $char(C) = 2$, and trivially if $char(C) \neq 2$ (then $1 = \frac{1}{2} + \tau_a(\frac{1}{2})$). So, by applying Prop. 3.15(iii) and (iv) and Prop. 3.13(ii) to C, we obtain a $c \in C$ with $v(c) = v(1) = 0$ and $c + \tau_a(c) = 1$. Let $M = L(c)$; so $\tau_a(M) = M$. Let $N = M^{\tau_a}$. Then, $[M:N] = 2$, as $c \notin N$, and τ_a is the nonidentity automorphism of M. The automorphism induced by τ_a on the residue field $\overline{M}$ is nontrivial, as it sends $\overline{c}$ to $\overline{c} + 1$. Therefore, M is unramified over N. Hence, by Lemma 4.4, there is $d \in M$ with $1 + m = N_{M/N}(d) = \tau_a(d)d$, as desired.

Cases I, II, and III cover all the cases stated in the theorem, since when D is tame over F, τ and τ' are of the same kind, by Remark 3.14. □

EXAMPLE 4.7. Let $D = \left(\frac{-1,-1}{\mathbb{Q}_2}\right)$, the Hamilton quaternion division algebra over the dyadic local field $\mathbb{Q}_2$. Let $\{1, i, j, k\}$ be the standard base of D. Let $u = (-1+i+j+k)/2$ and $s = i - j$. Then, $u^2 + u + 1 = 0$, $sus^{-1} = -u - 1$, and $s^2 = -2$. From this it is clear that D is the cyclic algebra $(\mathbb{Q}_2(u)/\mathbb{Q}_2, \rho, -2)$, where $\rho = int(s)|_{F(u)}$. The complete discrete (so Henselian) 2-adic valuation on $\mathbb{Q}_2$ has value group $\Gamma_{\mathbb{Q}_2} = \mathbb{Z}$ and residue field $\overline{\mathbb{Q}_2} = \mathbb{F}_2$. In the extension of v to D, we must have $v(u) = 0$ and $v(s) = \frac{1}{2}$; so $\overline{D} = \mathbb{F}_4$ and $\Gamma_D = \frac{1}{2}\mathbb{Z}$. Even though D is ramified over $\mathbb{Q}_2$, with ramification index equal to the residue characteristic, D is tame over $\mathbb{Q}_2$ since it is inertially split, i.e., it has a maximal subfield $\mathbb{Q}_2(u)$ which is unramified over $\mathbb{Q}_2$. The graded field $gr(\mathbb{Q}_2)$ of $\mathbb{Q}_2$ with respect to v is the Laurent polynomial ring $\mathbb{F}_2[t, t^{-1}]$, where $t = 2'$, which has grade 1. We have $gr(D)_0 = \overline{D} = \mathbb{F}_4$ and $gr(D) = \mathbb{F}_4\{s', s'^{-1}\}$, a twisted Laurent polynomial ring, where conjugation by s' induces the Frobenius automorphism φ on $\mathbb{F}_4$, $s'^2 = t$, and s' has grade equal to $v(s) = \frac{1}{2}$. Clearly, $Z(gr(D)) = \mathbb{F}_2[t, t^{-1}] = gr(\mathbb{Q}_2)$. Let τ be the unique symplectic involution on D and let $\lambda = 1$, so (τ, λ) is a symplectic pair for D. We have $\tau(u) = -u - 1$ and $\tau(s) = -s$. The induced graded involution τ' on $gr(D)$ is of the first kind, and is given by $\tau'(u') = u' + 1$ (which shows that τ' is symplectic) and $\tau'(s') = s'$. We have $\tau'|_{gr(D)_0} = \varphi$, which is an involution of the second kind on $gr(D)_0$, even though τ' itself is of the first kind. Since $s' \in gr(D)_{\frac{1}{2}}$ and $\tau'(s') = s'$, we can use $\widetilde{\tau} = int(s') \circ \tau'$ in computing the $[\frac{1}{4}]$-component of the Witt group. Note that $\widetilde{\tau}|_{gr(D)_0} = id$. Thus, for the Witt group of even (i.e., all) τ-Hermitian forms on D, we have by Th. 4.6 and Prop. 1.5,

$$\begin{aligned} W^+(D, \tau, 1) \;\cong\; W_g^+(gr(D), \tau', 1) \;&\cong\; W_g^+(gr(D), \tau', 1; [0]) \;\oplus\; W_g^+(gr(D), \tau', 1; [\tfrac{1}{4}]) \\ &\cong\; W^+(gr(D)_0, \tau', 1) \;\oplus\; W^+(gr(D)_0, \widetilde{\tau}, 1) \\ &=\; W^+(\mathbb{F}_4, \varphi, 1) \;\oplus\; W^+(\mathbb{F}_4, id, 1) \;\cong\; \mathbb{Z}/2\mathbb{Z} \;\oplus\; (0)\,. \end{aligned}$$

For the last isomorphism, we use that $Symd(\mathbb{F}_4, \varphi, 1) = \mathbb{F}_2$ and $Symd(\mathbb{F}_4, id, 1) = (0)$. Of course, this Witt group could also have been calculated using Jacobson's theorem [S, Th. 1.7, p. 352], which gives an injection of $W^+(D, \tau, 1)$ into the Witt group of quadratic forms over $\mathbb{Q}_2$ via the transfer map.

There are other involutions on this D as well, all of orthogonal type. For example, let $d = i + j + k$ and let $\widehat{\tau} = int(d) \circ \tau$. So, $\widehat{\tau}(u) = -u - 1$ and $\widehat{\tau}(s) = s$. Note that even though $\widehat{\tau}$ is orthogonal, its associated graded involution $\widehat{\tau}'$ coincides with τ', which is symplectic. But, $(\widehat{\tau}, 1)$ is not a symplectic pair, and we cannot hope to use $\widehat{\tau}'$ to compute $W^+(D, \widehat{\tau}, 1)$ since $(D, \widehat{\tau}, 1)$ does not preserve even forms. Indeed, $Symd(D, \widehat{\tau}, 1) = \{\, a + bs \mid a \in \mathbb{Q}_2,\ b \in \mathbb{Q}_2(u) \,\}$, which has dimension 3 over $\mathbb{Q}_2$, while $Symd(gr(D), \widehat{\tau}', 1) = gr(\mathbb{Q}_2)$. With respect to any compatible norm α for the even Hermitian form

$h = \langle s \rangle$ for $\widehat{\tau}$, the associated graded form h'_α is not even. On the other hand, $(\widehat{\tau}, -1)$ is a symplectic pair, and Th. 4.6 shows that $W^+(D, \widehat{\tau}, -1) \cong W^+_g(gr(D), \widehat{\tau}', 1)$, which we just computed, as $\widehat{\tau}' = \tau'$.

The isometry group of an even form h acts on the family of norms compatible with h. We will show in Cor. 4.10 below that under the hypotheses of Th. 4.6 the action is as transitive as possible, in that two norms are in the same orbit iff they have isometric associated graded forms. The next lemma, giving a canonical form for norms compatible with hyperbolic planes, is a building block for the group action result.

Lemma 4.8. *Suppose we have $char(\overline{D}) \neq 2$ or one of the good cases when $char(\overline{D}) = 2$. Suppose $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$ with $dim_D(M) = 2$ and h hyperbolic, and let α be a norm on M with $\alpha \precsim h$. Then, there is a splitting base $\{ m, n \}$ for α with m and n isotropic, $h(m, n) = 1$, and $\alpha(n) = -\alpha(m)$. Furthermore, for any $\delta \in \Gamma_{gr(M)}$, m can be chosen so that $\alpha(m) = \delta$.*

Proof. Take any isotropic vector $m \in M$. Then, m' is isotropic in $gr(M)$. By Prop. 1.4(iii) the maximal totally isotropic graded subspace $m' gr(M)$ of $gr(M)$ has complementary totally isotropic graded subspace, call it P. Take any nonzero homogeneous element $\widehat{p} \in P$. Then, $h'_\alpha(m', \widehat{p}) \neq 0$ since $(m' gr(D))^\perp = m' gr(D)$; also, $h'_\alpha(m', \widehat{p})$ is homogeneous (hence a unit) in $gr(D)$, since m' and $\widehat{p}$ are each homogeneous in $gr(M)$. Let $\widetilde{p} = \widehat{p}\, h'_\alpha(m', \widehat{p})^{-1}$. Then $\widetilde{p}$ is homogeneous and isotropic, and $h'_\alpha(m', \widetilde{p}) = 1$. Because $\widetilde{p}$ is homogeneous and nonzero, there is a nonzero $p \in M$ with $p' = \widetilde{p}$. Since $h'_\alpha(m', p') = 1 \in gr(D)_0$, by (3.3) $\alpha(m) + \alpha(p) = v(h(m, p)) = 0$. If p is isotropic, then set $n = p\, h(m, p)^{-1}$; then, n is isotropic, $h(m, n) = 1$, and $\alpha(n) = \alpha(p) - v(h(m, p)) = -\alpha(m) - 0$, as desired.

Now suppose p is anisotropic. By replacing p by $p\, h(m, p)^{-1}$, we may assume that $h(m, p) = 1$, while not changing p'. Because h is even, there is $d \in D$ with $d + \lambda\tau(d) = -h(p, p)$. Moreover, d can be chosen with $v(d) = v(h(p, p))$. This is clear if $char(\overline{D}) \neq 2$ (take $d = -\frac{1}{2}h(p, p)$), and holds by Prop. 3.15(iii) and (iv) and Prop. 3.13(ii) in the good cases when $char(\overline{D}) = 2$. Let $n = md + p$. Then, $h(n, n) = d + \lambda\tau(d) + h(p, p) = 0$. Because $h'_\alpha(p', p') = 0$, we have $2\alpha(p) < v(h(p, p)) = v(d)$. Hence,

$$\alpha(m) \; + \; v(d) \; = \; -\alpha(p) + v(d) \; > \; \alpha(p) = -\alpha(m)\,. \tag{4.4}$$

Because $\{ m', p' \}$ is a homogeneous base of $gr(M)$, by Cor. 2.3 $\{ m, p \}$ is a splitting base for α on M. Therefore, by (4.4),

$$\alpha(n) \; = \; min\big(\alpha(md), \alpha(p)\big) \; = \; min\big(\alpha(m) + v(d), -\alpha(m)\big) \; = \; -\alpha(m)\,.$$

Since $h(m, n) = h(m, p) = 1$, we have $v(h(m, n)) = 0 = \alpha(m) + \alpha(n)$, so $h'_\alpha(m', n') = h(m, n)' \neq 0$. Therefore, as m' is isotropic, m' and n' must be $gr(D)$-linearly independent in $gr(M)$, so they form a homogeneous base of $gr(M)$. Therefore, by Cor. 2.3(ii) $\{ m, n \}$ is a splitting base for α on M. Let $\gamma = \alpha(m)$. Then, $\Gamma_{gr(M)} = [\gamma] \cup [-\gamma]$, where $[\gamma] = \gamma + \Gamma_{gr(D)}$. Note that for any nonzero $c \in D$, we have $\{ mc, n\tau(c)^{-1} \}$ is a splitting base for α with mc and $n\tau(c)^{-1}$ isotropic and $h(mc, n\tau(c)^{-1}) = 1$, and $\alpha(mc) = \gamma + v(c) = -\alpha(n\tau(c)^{-1})$. Therefore, by interchanging m and n if necessary and multiplying by a constant in D, we can arrange to find an m with $\alpha(m)$ having any desired value in $\Gamma_{gr(M)}$. □

Proposition 4.9. *Suppose v is Henselian and we have $char(\overline{D}) \neq 2$ or one of the good cases when $char(\overline{D}) = 2$. Let (M, h), $(N, \ell) \in \mathcal{H}^+(D, \tau, \lambda)$, and let α be a norm on M with $\alpha \precsim h$ and β a norm on N with $\beta \precsim \ell$. Then, there is an isometry $f \colon M \to N$ between h and ℓ with $\alpha = \beta \circ f$ iff $h'_\alpha \cong \ell'_\beta$ (graded isometry).*

Proof. $\Rightarrow$ If there is an isometry f as described, then f induces a map $f' \colon gr(M) \to gr(N)$ which is clearly a graded isometry between h'_α and ℓ'_β.

$\Leftarrow$ Suppose there is a graded isometry $g\colon gr(M) \to gr(N)$. Consider first the special case where h'_α is anisotropic. Then, h is also anisotropic. We have $h'_\alpha \perp -\ell'_\beta$ is hyperbolic, so $h \perp -\ell$ is hyperbolic, by Th. 4.6(i) and Prop. 4.3. Therefore, there is an isometry $f\colon M \to N$ between h and ℓ. Because α and $\beta \circ f$ are each norms on M compatible with h, and h'_α is anisotropic, we have $\beta \circ f = \alpha$, by Prop. 4.2.

Now consider another special case: Assume $dim_D(M) = 2$ and h'_α is hyperbolic. Then, h is hyperbolic by Th. 4.6(i) and Prop. 4.3. For any $\gamma \in \Gamma_{gr(M)}$, Lemma 4.8 says there is a splitting base $\{ m_1, m_2 \}$ for α on M with $h(m_1, m_1) = h(m_2, m_2) = 0$, $h(m_1, m_2) = 1$, and $\alpha(m_1) = \gamma = -\alpha(m_2)$. Since $gr(N) = gr(M)$ and ℓ'_β is also hyperbolic, there is a splitting base $\{ n_1, n_2 \}$ for β satisfying the same conditions relative to ℓ and β. Then, the D-linear map $f\colon M \to N$ given by $f(m_i) = n_i$, $i = 1, 2$ is an isometry between f and ℓ with $\alpha = \beta \circ f$.

We can now prove the general case. We have by Prop. 1.4(vi) and (iv), $gr(M) = \mathop{\perp}\limits_{i=1}^{k} M'_i$, where $h'_\alpha|_{M'_0}$ is anisotropic and for each $i \geq 1$, $h'_\alpha|_{M'_i}$ is hyperbolic with $dim_{gr(D)}(gr(M'_i)) = 2$. We construct a "good lift" of the M'_i to D-subspaces M_i of M. Let M_0 be any D-subspace of M with $gr(M_0) = M'_0$ in $gr(M)$. Then, since $h'_\alpha|_{gr(M_0)}$ is nondegenerate, Prop. 3.8 shows that $gr(M_0^\perp) = M_0'^\perp = \bigoplus_{i=1}^{k} M'_i$. Let M_1 be any subspace of $M_0^\perp$ with $gr(M_1) = M'_1$. Iterating this, we see we can choose a subspace M_j of $\big(\bigoplus_{i=0}^{j-1} M_i\big)^\perp$ with $gr(M_j) = M_j$. Then, $M = \mathop{\perp}\limits_{i=0}^{k} M_i$ with respect to h and because $gr(M)$ is a direct sum of the $gr(M_i)$, $\alpha = \mathop{\oplus}\limits_{i=0}^{k} \alpha|_{M_i}$. Likewise, let $N'_i = g(M'_i) \subseteq gr(N)$. Construct a good lift of the N'_i to subspaces N_i of N with $N = \mathop{\perp}\limits_{i=0}^{k} N_i$ with respect to ℓ and $gr(N_i) = N'_i$, so $\beta = \mathop{\oplus}\limits_{i=0}^{k} \beta|_{N_i}$. By the special cases considered above, for each i there is $f_i\colon M_i \to N_i$ which is an isometry between $h|_{M_i}$ and $\ell|_{N_i}$ with $\alpha|_{M_i} = \beta|_{N_i} \circ f_i$. Then, the map $f = \oplus f_i$ has the desired properties. □

Corollary 4.10. *Suppose v is Henselian and we have $char(\overline{D}) \neq 2$ or one of the good cases when $char(\overline{D}) = 2$. Let $(M, h) \in \mathcal{H}^+(D, \tau, \lambda)$, and let α, β be norms on M with $\alpha \mathrel{\overline{\prec}} h$ and $\beta \mathrel{\overline{\prec}} h$. Then, there is a D-linear map $f\colon M \to M$ which is an isometry for h with $\alpha = \beta \circ f$ iff $h'_\alpha \cong h'_\beta$ (graded isometry).*

Proof. This is immediate from Prop. 4.9. □

Cor. 4.10 shows that when h is isotropic, the action of the isometry group of h on the set of norms compatible with h is not transitive. This is because hyperbolic forms of the same dimension in $\mathcal{GH}^+(gr(D), \tau', \lambda')$ are not isometric unless they satisfy the added condition of being isomorphic as graded vector spaces. Ex. 3.7(ii) shows that there are many nonisomorphic possibilities for h'_α when h is hyperbolic. Note by contrast that Springer in [Sp$_2$] and Goldman and Iwahori in [GI, Th. 4.16] did prove transitivity of the isometry group action for their different notion of norms compatible with a quadratic form.

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