COHOMOLOGICAL INVARIANTS OF QUATERNIONIC SKEW-HERMITIAN FORMS

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Abstract. We define a complete system of invariants $e_{n,Q}, n \geq 0$ for quaternionic skew-hermitian forms, which are twisted versions of the invariants e_n for quadratic forms. We also show that quaternionic skew-hermitian forms defined over a field of 2-cohomological dimension at most 3 are classified by rank, discriminant, Clifford invariant and Rost invariant.

Introduction

A recent breakthrough in the theory of quadratic forms is the proof by Vovoedsky and al. of Milnor's conjectures (see Section 1.1). In particular, their results yield to a complete system of cohomological invariants $(e_n)_{n\geq 0}$ for quadratic forms. However, no complete system of invariants for (skew)-hermitian forms over a division algebra with involution is known. In fact, very few invariants have been constructed. The major advance in this direction is the construction by Rost of a cohomological invariant $H^1(\_, G) \to H^3(\_, \mathbb{Q}/\mathbb{Z}(2))$, where G is a semi-simple simply connected linear algebraic group, which then has been used by Bayer-Fluckiger and Parimala in [3] to construct a Rost invariant for skew-hermitian forms over a division algebra with a symplectic involution.

In this paper, we show how to 'twist' reasonable cohomological invariants for quadratic forms into cohomological invariants for skew-hermitian forms defined over a quaternion algebra Q endowed with its canonical involution γ, that we will call 'quaternionic skew-hermitian forms' for short. The construction is based on the fact that the unramified cohomology of the conic associated to Q comes from the base field. In particular, we obtain some invariants $e_{n,Q}$, which are twisted versions of the invariants e_n. We then prove that these invariants $e_{n,Q}$ form a complete system of invariants of quaternionic skew-hermitian forms. As an application, we show hat quaternionic skew-hermitian forms defined over a field of 2-cohomological dimension at most 3 are classified by rank, discriminant, Clifford invariant and Rost invariant, defined by Bayer-Fluckiger and Parimala in [2] and [3].

The paper is organized as follows. The first part collects all the results needed in the sequel: in Section 1.1, we recall Vovoedsky's results on Milnor conjectures and useful corollaries which are need in the sequel. In Section 1.2, we recall some basic notions on residue maps and unramified cohomology, and compute the unramified cohomology group of function fields of conics. In Section 1.3, we state briefly the

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properties of the residue maps for quadratic forms. The second part of the paper describes how to construct cohomological invariants for quaternionic skew-hermitian forms from cohomological invariants of quadratic forms and give applications: in Section 2.1, we briefly recall the properties of rank, discriminant, Clifford invariant and Rost invariant. In Section 2.2, we explain how to twist invariants of quadratic forms to get invariants of quaternionic skew-hermitian forms. Sections 2.3 and 2.4 give examples and applications to the classification problem of such forms. Finally, in Section 2.5 we list some open problems relative to these new invariants.

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1. Preliminaries

1.1. **Milnor and Bloch-Kato conjectures.** Let k be a field. Let $T^M(k)$ be the graded algebra

$$T^M(k) := \mathbb{Z} \oplus k^\times \oplus (k^\times)^{\otimes 2} \oplus \cdots,$$

and let $\mathcal{I}^M(k)$ be the two-sided ideal generated by the elements $a \otimes (1-a), a \in k, a \neq 0, 1$. The quotient ring $T^M(k)/\mathcal{I}^M(k)$ is a graded ring, denoted by $K_*^M(k)$.

Assume that $\mathrm{char}(k) \neq 2$, and let $k_n^M(k) = K_n^M(k)/2$. If $a_1, \cdots, a_n \in k^\times$, we will denote by $\{a_1, \cdots, a_n\}$ the class of $a_1 \otimes \cdots \otimes a_n$, and by $\langle\langle a_1, \cdots, a_n\rangle\rangle$ the quadratic form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$. We will also denote by $I^n(k)$ the n^{th}-power of the fundamental ideal $I(k)$ of the Witt ring $W(k)$ of k.

If $a \in k^\times$, and m is an integer which is prime to $\mathrm{char}(k)$, we will denote by $(a)_m$ the corresponding cohomology class in $H^1(k, \mu_m)$. For $n \geq 1$, we define two group homomorphisms h_n and s_n by

$$h_n : k_n^M(k) \to H^n(k, \mu_2), \{a_1, \cdots, a_n\} \mapsto (a_1)_2 \cup \cdots \cup (a_n)_2$$

$$s_n : k_n^M(k) \to I^n(k)/I^{n+1}(k), \{a_1, \cdots, a_n\} \mapsto \langle\langle a_1, \cdots, a_n\rangle\rangle + I^{n+1}(k)$$

Finally, if m is an integer which is prime to $\mathrm{char}(k)$, we define a norm-residue homomorphism $u_{n,m}$ by

$$u_{n,m} : K_n^M(k)/m \to H^n(k, \mu_m^{\otimes n}), \{a_1, \cdots, a_n\} \mapsto (a_1)_m \cup \cdots \cup (a_n)_m$$

One can check that these homomorphisms are well-defined. The Bloch-Kato conjecture states that $u_{n,m}$ is an isomorphism for all n, m. It has been proved for $n = 2$ and m arbitrary by Merkurjev and Suslin [11]. The following result is due to Voevodsky ([17]):

Theorem 1 (Milnor's conjecture). *For all $n \geq 1$, h_n is an isomorphism.*

A classical dévissage argument due to Tate (see, [7], Proposition 1.1.(c)) then yields to:

Theorem 2 (Bloch-Kato's conjecture for $p = 2$). *Let k be a field of characteristic different from 2. For all $n \geq 1$ and all $m = 2^r, r \geq 1$, $u_{n,m}$ is an isomorphism.*

For a detailled account and historical comments on Milnor and Bloch-Kato conjectures, see [7] for example.

For any sequence $\underline{a} = (a_1, \cdots, a_n)$ of elements of $k^\times$, let $Q_{\underline{a}}$ denote the projective quadric associated to $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_{n-1}\rangle - \langle a_n\rangle$. We denote by $k(Q_{\underline{a}})$ its function field. Let $(Q_{\underline{a}})_{(0)}$ denote the set of closed points of $Q_{\underline{a}}$. If $x \in (Q_{\underline{a}})_{(0)}$, we denote by $\kappa(x)$ its residue field.

In [13], Orlov, Vishik and Voevodsky proved:

Theorem 3. *Let k be a field of characteristic different from 2. For $n \geq 1$, we have an exact sequence*

$$\coprod_{x\in(Q_{\underline{a}})_{(0)}} k_*^M(\kappa(x)) \xrightarrow{\mathrm{Tr}_{\kappa(x)/k}} k_*^M(k) \xrightarrow{\cdot\underline{a}} k_{*+n}^M(k) \longrightarrow k_{*+n}^M(k(Q_{\underline{a}})),$$

where $\mathrm{Tr}_{k(x)/k}$ denotes the transfert map and $\cdot\underline{a}$ denotes the multiplication by the symbol $\{a_1, \cdots, a_n\}$.

This sequence has been established in characteristic zero in [13], but one can check that all the results used are proved in [17] in any characteristic different from 2.

As a corollary, they get:

Theorem 4 (Milnor's conjecture for quadratic forms)**.** *Let k be a field of characteristic different from 2. For $n \geq 1$, s_n is an isomorphism.*

See also [12] for a different proof of this result.

For $n \geq 1$, we define a map $e_n : I^n(k) \to H^n(k, \mu_2)$ on generators by

$$e_n(\langle\langle a_1, \cdots, a_n\rangle\rangle) = (a_1)_2 \cup \cdots \cup (a_n)_2$$

From all the previous results, we obtain the following:

Theorem 5. *Let k be a field of characteristic different from 2. For $n \geq 1$, the map e_n is well-defined and induces a group isomorphism*

$$\overline{e}_n : I^n(k)/I^{n+1}(k) \simeq H^n(k, \mu_2)$$

We also define $e_0 : W(k) \to \mathbb{Z}/2\,\mathbb{Z} \simeq H^0(k, \mu_2)$ by

$$e_0(q) = \dim(q) + 2\,\mathbb{Z}$$

By definition of $I(k)$, $\overline{e}_0 : W(k)/I(k) \to \mathbb{Z}/2\,\mathbb{Z}$ is a group isomorphism.

The following result is a direct corollary of the previous theorem and Arason-Pfister Haupsatz (see [15] p.156):

Theorem 6. *Let k be a field of characteristic different from 2, and let q be a non-degenerate quadratic form over k. Then q is hyperbolic if and only if $e_n(q) = 0$ for all $n \geq 0$.*

1.2. **Residues and unramified cohomology.**

1.2.1. *Residues.* Let K be a field equipped with a discrete valuation υ, and let π be a local parameter. Let C be a finite $\Gamma_{\kappa(\upsilon)}$-module whose order is prime to $n = \mathrm{char}(\kappa(\upsilon))$ and satisfying $nC = 0$. We set $C(-1) = \mathrm{Hom}(\mu_n, C)$. For $d \geq 1$, there exists a group homomorphism

$$\partial_\upsilon : H^d(K, C) \to H^{d-1}(\kappa(\upsilon), C(-1)),$$

called *the residue map with respect to* υ. The residue of a cohomology class $\alpha \in H^d(K, C)$ may be computed as follows. Let K_υ be the completion fo K at υ. Then π is also a local parameter for the unique valuation on K_υ extending υ, we have an injection $H^d(\kappa(x), C) \hookrightarrow H^d(K_\upsilon, C)$ and a unique decomposition

$$\mathrm{Res}_{K_\upsilon/K}(\alpha) = \alpha_0 + (\pi)_n \cup \alpha_1,$$

with $\alpha_0 \in H^d(\kappa(\upsilon), C)$ and $\alpha_1 \in H^{d-1}(\kappa(\upsilon), C(-1))$. Then $\partial_\upsilon(\alpha) = \alpha_1$ (the result does not depend on the choice of a local parameter).

If $C = \mu_2$, $C(-1)$ identifies with μ_2 for all $d \geq 1$, and the residue map can be described as the unique group homomorphism satisfying the following properties: for every $a_1, \cdots, a_{d-1}, b_1, \cdots, b_d \in \mathcal{O}_\upsilon^\times$, we have

$$\partial_\upsilon((\pi)_2 \cup (a_1)_2 \cdots \cup (a_{d-1})_2) = (\overline{a}_1)_2 \cdots \cup (\overline{a}_{d-1})_2$$
$$\partial_x((b_1)_2 \cdots \cup (b_d)_2) = 0,$$

where $\overline{u}$ denotes the image of $u \in \mathcal{O}_\upsilon$ in $\kappa(\upsilon)$ under the canonical projection.

We say that $\alpha \in H^d(K, C)$ is *unramified with respect to* υ is $\partial_\upsilon(\alpha) = 0$.

1.2.2. *Unramified cohomology.* Let X be a smooth proper irreducible variety defined over k. We denote by $X^{(1)}$ the set of points of codimension 1 in X. The ring $\mathcal{O}_{X,x}$ is then a discrete valuation ring. We will denote by υ_x the corresponding discrete valuation and by π_x a local parameter. For $n \geq 2$ prime to the characteristic of k, let

$$r_n : \Gamma_k \to \mathrm{Aut}\mu_n(k_s)$$

be the action of Γ_k of $\mu_n(k_s)$. Let $s_n : \Gamma_k \to \mathrm{Aut}\mu_n(k_s)$ be the homomorphism such that $s_n(\sigma) = r_n(\sigma)^{-1}$ for all $\sigma \in \Gamma_k$, and let $\mu_n^{\otimes -1}(k_s)$ be the corresponding Galois module. We also set $\mu_n^{\otimes 0}(k_s) = \mathbb{Z}/n\,\mathbb{Z}$.

For $d \geq 1$, applying the results of the previous paragraph, we get a residue homomorphism

$$\partial_x : H^d(k(X), \mu_n^{\otimes d-1}) \to H^{d-1}(\kappa(x), \mu_n^{\otimes d-2})$$

We say that $\alpha \in H^d(k(X), \mu_n^{\otimes d-1})$ is *unramified at* x if $\partial_x(\alpha) = 0$. This notion does not depend on the choice of a local parameter.

We say that $\alpha \in H^n(k(X), \mu_n^{\otimes d})$ is *unramified* if it is unramified at every $x \in X^{(1)}$. We denote by $H^d_{nr}(k(X)/k, \mu_n^{\otimes d-1})$ is the subgroup of unramified cohomology classes. It is a birational invariant of X. In particular, if X is a rational variety, then the restriction map induces an isomorphism $H^d(k, \mu_n^{\otimes d-1}) \simeq H^d_{nr}(k(X), \mu_n^{\otimes d-1})$.

Therefore if F/k is a finitely generated extension, we can define the group of unramified elements $H^d_{nr}(F/k, \mu_n^{\otimes d-1})$ by

$$H^d_{nr}(F/k, \mu_n^{\otimes d-1}) = H^d_{nr}(k(X), \mu_n^{\otimes d-1}),$$

where X is any irreducible smooth proper model of F/k. We refer to [5] for more details.

For every integer $i \geq 0$, we put

$$\mathbb{Q}/\mathbb{Z}(i-1) = \varinjlim \mu_n^{\otimes i-1}(k_s),$$

where the limit is taken over the integers prime to the characteristic of k. Thus $H^d(k(X), \mathbb{Q}/\mathbb{Z}(d-1))$ is the direct limit of the groups $H^d(k(X), \mu_n^{\otimes d-1})$ with respect to the maps

$$H^d_{nr}(k(X), \mu_n^{\otimes d-1}) \to H^d(k(X), \mu_m^{\otimes d-1})$$

The residue maps for the various n then fit together to give rise to a residue map

$$\partial_x : H^d(k(X), \mathbb{Q}/\mathbb{Z}(d-1)) \to H^{d-1}(k(X), \mathbb{Q}/\mathbb{Z}(d-2))$$

We then define the groups $H^d_{nr}(k(X), \mathbb{Q}/\mathbb{Z}(d-1))$ and $H^d_{nr}(F/k, \mathbb{Q}/\mathbb{Z}(d-1))$ as previously.

Notice that the isomorphism $k_n^M(k) \simeq H^n(k, \mu_{2^m}^{\otimes n})$ shows immediately that the group $H^d(k, \mu_{2^m}^{\otimes d-1})$ identifies to the 2^m-torsion subgroup of $H^d(k, \mathbb{Q}/\mathbb{Z}(d-1))$, and that the map $H^d(k, \mu_{2^m}^{\otimes d-1}) \to H^d(k, \mu_{2^n}^{\otimes d-1})$ is injective for all $n \geq m$.

Similarly, for all integers $n \geq m$ prime to $\mathrm{char}(k)$, $H^3(k, \mu_m^{\otimes 2})$ identifies to the m-torsion subgroup of $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$, and the map $H^3(k, \mu_n^{\otimes 2}) \to H^3(k, \mu_m^{\otimes 2})$ is injective for all $n \geq m$.

The next result will be useful in the sequel:

Proposition 7. *Let Q be a (not necessarily division) quaternion algebra over k, let $SB(Q)$ be the associated Severi-Brauer variety, and let $k(Q)$ be its function field. For all $d \geq 1$, the restriction map induces a group isomorphism*

$$\mathrm{Res}_{k(Q)/k} : H^d(k, \mathbb{Q}/\mathbb{Z}(d-1))/[Q] \cup H^{d-2}(k, \mu_2) \simeq H^d_{nr}(k(Q), \mathbb{Q}/\mathbb{Z}(d-1)),$$

where $[Q] \cup H^{d-2}(k, \mu_2)$ is viewed as a subgroup of $H^d(k, \mathbb{Q}/\mathbb{Z}(d-1))$.

Remark 8. If $d = 1$, $[Q] \cup H^{d-2}(k, \mu_2)$ has to be understood as the trivial group.

Proof. If $Q = (a, b)$, then $SB(Q)$ is the projective quadric associated to the quadratic form $q = \langle 1, -a, -b \rangle$. By [8] Proposition A.1, the restriction map

$$H^d(k, \mathbb{Q}/\mathbb{Z}(d-1)) \to H^d_{nr}(k(Q), \mathbb{Q}/\mathbb{Z}(d-1))$$

is surjective. As already pointed out in [8] the kernel of this map is exactly the kernel of the map

$$H^d(k, \mu_2) \to H^d_{nr}(k(Q), \mu_2).$$

If $d = 1$, this last kernel is trivial since k is algebraically closed in $k(Q)$. If $d \geq 2$, the result follows from Milnor's conjecture and the exactness of the sequence

$$k^M_{d-2}(k) \xrightarrow{\cdot\{a,b\}} k^M_d(k) \longrightarrow k^M_d(k(q))$$

$\square$

1.3. **Residues for quadratic forms.** We recall briefly here how to define residue homomorphisms for quadratic froms and their basic properties. Let K be a field equipped with a discrete valuation υ, let $\mathcal{O}_\upsilon$ the corresponding valuation ring and let π be a local parameter. We assume that K and $\kappa(\upsilon)$ both have characteristic different from 2. Then every (non-degenerate) quadratic form q over K can be written as

$$q \simeq \langle a_1, \cdots, a_m, \pi a_{m+1}, \cdots, \pi a_n \rangle, a_i \in \mathcal{O}_\upsilon^\times$$

We then set

$$\partial_{1,\pi}(q) := \langle \overline{a}_1, \cdots, \overline{a}_m \rangle \in W(\kappa(\upsilon))$$

$$\partial_{2,\pi}(q) := \langle \overline{a}_{m+1}, \cdots, \overline{a}_n \rangle \in W(\kappa(\upsilon))$$

One can show that $\partial_{i,\pi}(q)$ only depends on the Witt class of q (and in particular of its isomorphism class) and that $\partial_i : W(K) \to W(\kappa(\upsilon))$ is a group homomorphism for $i = 1, 2$.

Let π' be another local parameter for υ. Then $\pi' = u\pi$ for some $u \in \mathcal{O}_\upsilon^\times$, and we easily have

$$\partial_{1,\pi'} = \partial_{1,\pi}, \partial_{2,\pi'} = \overline{u}\partial_{2,\pi}$$

See [15], Chapter 6 for more details.

We say that q is *unramified with respect to* υ if $\partial_{2,\pi}(q) = 0$. This notion does not depend on the choice of π by the equality above. Finally, if $q \in I^n(K)$ then $\partial_{2,\pi}(q) \in I^{n-1}(\kappa(\upsilon))$ and we have

$$\partial_\upsilon(e_n(q)) = e_{n-1}(\partial_{2,\pi}(q))$$

See [1], Satz 4.11 for a proof. In particular, if $q \in I^n(K)$ is unramified with respect to υ then so is $e_n(q)$.

Finally, if X is a proper smooth variety defined over a field k and $x \in X^{(1)}$, we say that q is *unramified at* x if q is unramified with respect to υ_x.

2. Cohomological invariants of quaternionic skew-hermitian forms

If (V, h) is a hermitian space over a division algebra with involution (D, σ), only few invariants are defined: the rank, the signatures, the Clifford invariant and the Rost invariant, the two last invariants being defined when σ is orthogonal. The discriminant, Clifford invariant and Rost invariant may be considered as 'twisted versions' of the e_1, e_2 and e_3 for quadratic forms in the orthogonal case, and have been defined by Bayer-Fluckiger and Parimala in [2] and [3].

In the first section, we recall how extend the definition of these invariants to other cases using scaling of hermitian spaces. We also define higher invariants for skew-hermitian forms over (Q, γ), where Q is a quaternion division algebra and γ is the canonical involution on Q. We then prove a classification result for these forms over an arbitrary base field. We also setup a conjecture on classification of hermitian forms when the base field k has cohomological dimension at most 3, and prove it in a particular case.

2.1. **Invariants of hermitian forms.** Let (V, h) be a hermitian form over (A, σ), where A is a central simple algebra with an orthogonal involution over k.

In [2] and [3], the rank, the discriminant, the Clifford invariant and the Rost invariant of h are defined. The rank $rk(h)$ is an integer; it is the dimension of the A-vector space V. The discriminant $d(h)$ is an element of $k^\times/k^{\times 2}$. The Clifford invariant $\mathcal{C}\ell(h)$ is an element of $\mathrm{Br}_2(k)/\langle[A]\rangle$ and is defined only for hermitian forms h of even rank with trivial discriminant. The Rost invariant $R(h)$ is an element of $H^3(k, \mu_2^{\otimes 4})/H^1(k, \mu_2) \cup [A]$ and is defined for hermitian forms h with even rank rank, trivial discriminant and trivial Clifford invariant.

If A is split, then h corresponds to a quadratic form q_h via Morita equivalence, whose isomorphism class only depends on the isomorphism class of h.

In this case, we have

$$rk(h) = \dim(q_h), d(h) = \mathrm{disc}(q_h), \mathcal{C}\ell(h) = c(q_h), R(h) = e_3(q_h)$$

See [2],[3] for more details.

We now recall how to extend the definition of these invariants to the case of skew-hermitian forms over algebras with symplectic involutions. We first recall the definition of 'scaling'.

Let (A, σ) be a central simple algebra with involution of first kind of any type, and let $a \in A^\times$ such that $\sigma(a) = \varepsilon' a, \varepsilon' = \pm 1$. Let $\tau = \mathrm{Int}(a^{-1}) \circ \sigma$. If (V, h) is a ε-hermitian form over (A, σ), we define $\phi_a(V, h)$ to be (V, ah). One can check that $\phi_a(V, h)$ is a $\varepsilon\varepsilon'$-hermitian space over (A, τ). It is shown in [2] and [3] that all the previous invariants are unchanged under scaling, provided that $\varepsilon = \varepsilon' = 1$.

Now we may extend the definitions of $\iota(h) = rk(h), d(h), \mathcal{C}\ell(h), R(h)$ to the case where h is a skew-hermitian form over (A, σ), with σ symplectic as follows:

Let $a \in A^\times$ such that $\sigma(a) = -a$. Then $\phi_a(h)$ is a hermitian form over (A, τ), where $\tau = \mathrm{Int}(a^{-1}) \circ \sigma$ is orthogonal. We then set

$$\iota(h) = \iota(\phi_a(h))$$

We claim that it does not depend on the choice of a. Indeed, if $a' \in A^\times$ such that $\sigma(a') = -a'$, and if $\tau' = \mathrm{Int}(a'^{-1}) \circ \sigma$, then set $u = aa'^{-1}$. Then $\tau(u) = u$ and $\tau' = \mathrm{Int}(u^{-1}) \circ \tau$. We then have $\iota(\phi_a(h)) = \iota(\phi_u(\phi_a(h))) = \iota(\phi_{a'}(h))$.

2.2. **Descent of cohomological invariants.** Let Q be a quaternion k-algebra, and let γ be the standard (symplectic) involution on Q. Let us denote by $\mathfrak{C}_k$ the category of field extensions of k, by **Sets** the category of sets, and by **AbGrps** the category of abelian groups. We will denote by $\mathbf{Quad}, \mathbf{Herm}_Q^-$, the functors from $\mathfrak{C}_k$ with values in **Sets** defined by

$$\mathbf{Quad}(L/k) = \{ \text{ isomorphism classes of quadratic forms over } L\},$$

$$\mathbf{Herm}_Q^-(L/k) = \{ \text{ isomorphism classes of skew-hermitian forms over } (Q_L, \gamma_L) \},$$

and by $\mathbf{W}, \mathbf{W}_Q^-$, the functors from $\mathfrak{C}_k$ with values in **AbGrps** defined by

$$\mathbf{W}(L/k) = W(L), \mathbf{W}_Q^-(L/k) = W^{-1}(Q_L, \gamma_L),$$

where $W(L)$ denotes the Witt group of quadratic forms over L and $W^{-1}(Q_L, \gamma_L)$ denotes the Witt group of skew-hermitian forms over (Q_L, γ_L).

Let (V, h) be a skew-hermitian form over (Q, γ) of rank n. If Q is split, then $(Q, \gamma) \simeq (M_2(k), \sigma_0)$, where σ_0 is the symplectic involution σ_0 on $M_2(k)$ defined by

$$\sigma_0(M) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

In this case, h corresponds to a quadratic form q_h of dimension $2n$ over k via Morita equivalence. The properties of Morita equivalence show that h is uniquely determined by q_h, and that for two skew-hermitian forms h, h', we have $h \simeq h'$ if and only if $q_h \simeq q_{h'}$ (resp. $q_h \sim q_{h'}$). Moreover q_h is hyperbolic if and only if h is hyperbolic, so we also have $h \sim h'$ if and only if $q_h \sim q_{h'}$.

Now let us go back to the general case. Since all symplectic involutions on a split algebra are isomorphic and $Q \otimes k(Q)$ is split, we have an isomorphism

$$(Q \otimes k(Q), \gamma \otimes id_{k(Q)}) \simeq (M_2(k(Q)), \sigma_0)$$

Hence $h_{k(Q)}$ corresponds via Morita equivalence to a quadratic form $q_{h_{k(Q)}}$ of dimension $2n$ over $k(Q)$. One can easily show that the isomorphism class of $q_{h_{k(Q)}}$ does not depend on the choice of the isomorphism above (this follows from the general properties of Morita equivalence. See also [15], p.361-362 for an explicit description of Morita equivalence in this specific case).

We now use this quadratic form to 'twist' subfunctors of **Quad** or **W**.

Notation: For any subfunctor **F** of **Quad** (resp. **W**), we denote by $\mathbf{F}_Q^-$ the subfunctor of $\mathbf{Herm}_Q^-$ (resp. $\mathbf{W}_Q^-$) defined by

$$\mathbf{F}_Q^-(L/k) = \{h \in \mathbf{Herm}_Q^-(L) | q_{h_{k(Q_L)}} \in \mathbf{F}(L/k)\},$$

respectively

$$\mathbf{F}_Q^-(L/k) = \{h \in \mathbf{W}_Q^-(L) | q_{h_{k(Q_L)}} \in \mathbf{F}(L/k)\}$$

Definition 1. Let **F** be a subfunctor of **Quad** (resp. **W**), and let $\iota : \mathbf{F} \to H^d(\_, \mu_2)$ be a cohomological invariant (i.e. a natural transformation of functors) of degree $d \geq 1$. We say that ι is *reasonable* if for every field extension L/k, every point x of $SB(Q_L)$ of codimension 1 and every $q \in \mathbf{F}(k(Q_L)/k)$ unramified at x, $\iota_L(q)$ is unramified at x.

Examples:

- Since $\mathbf{Quad}_{2n} \simeq H^1(\_, \mathbf{O}_{2n})$, every invariant of $\mathbf{Quad}_{2n}$ is reasonable by the Compatibility Theorem of [6], p.29.

- Let **F** be the subfunctor of **W** defined by

$$\mathbf{F}(L/k) = I^n(L), n \geq 1$$

where $I^n(L)$ is the n^{th} power of the fundamental ideal $I(L)$ of the ring $W(L)$.
The results of the first section show that e_n is a reasonable invariant.

Proposition 9. *Let* **F** *be a subfunctor of* **Quad** *or* **W**, *and let* $\iota : \mathbf{F} \to H^d(\_, \mu_2)$ *be a reasonable cohomological invariant of degree* $d \geq 1$. *Let* $h \in \mathbf{F}_Q^-(k)$. *Then the following holds:*

1) $\iota_{k(Q)}(q_{h_{k(Q)}}) \in H^d_{nr}(k(Q), \mu_2)$.

2) *If $d = 1$, there exists a unique element $\iota_Q(h) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$, satisfying*

$$\mathrm{Res}_{k(Q)/k}(\iota_Q(h)) = \iota(q_{h_{k(Q)}}).$$

3) *If $d \geq 2$, there exists a unique element $\iota_Q(h) \in H^d(k, \mu_4^{\otimes d})/[Q] \cup H^{d-2}(k, \mu_2)$ satisfying*

$$\mathrm{Res}_{k(Q)/k}(\iota_Q(h)) = \iota(q_{h_{k(Q)}}).$$

Here $[Q] \cup H^{d-2}(k, \mu_2)$ is identified with a subgroup of $H^d(k, \mu_4^{\otimes d-1})$.

Proof. We will prove the first part using arguments quite similar to those used in [4, Section 3.3]. Let $x \in SB(Q)$ be a point of codimension 1. Then the residue field $\kappa(x)$ splits Q, hence $h_{\kappa(x)}$ corresponds to a quadratic form $q_{h_{\kappa(x)}}$ on $\kappa(x)$. Similarly, $h_{k(Q)_x}$ corresponds to a quadratic form $q_{h_{k(Q)_x}}$ on $k(Q)_x$ (where $k(Q)_x$ is the completion of $k(Q)$ at x). Moreover it is clear that $q_{h_{k(Q)_x}} \simeq (q_{h_{k(Q)}})_{k(Q)_x}$, since $h_{k(Q)_x} \simeq (h_{k(Q)})_{k(Q)_x}$. Therefore we have $q_{h_{k(Q)_x}} \in \mathbf{F}(k(Q)_x/k)$ and

$$\mathrm{Res}_{k(Q)_x/k(Q)}(\iota_{k(Q)}(q_{h_{k(Q)}})) = \iota_{k(Q)_x}((q_{h_{k(Q)}})_{k(Q)_x}) = \iota_{k(Q)_x}(q_{h_{k(Q)_x}})$$

Hence it is enough to prove that $\iota_{k(Q)_x}(q_{h_{k(Q)_x}})$ is unramified at x. Since $\kappa(x)$ and $k(Q)_x$ have same characteristic (since they both contain k) and $k(Q)_x$ is complete, we have $k(Q)_x \simeq \kappa(x)((t))$; in particular we have an injection $\kappa(x) \hookrightarrow k(Q)_x$. Hence $h_{k(Q)_x}$ is the image of $h_{\kappa(x)}$ under the scalar extension $\kappa(x) \hookrightarrow k(Q)_x$. Consequently, $q_{h_{k(Q)_x}}$ is the image of $q_{h_{\kappa(x)}}$ under the scalar extension $\kappa(x) \hookrightarrow k(Q)_x$. Since $k(Q)_x \simeq \kappa(x)((t))$, non-zero elements of $\kappa(x)$ are units, hence it follows from the definition of the residue maps that $q_{h_{k(Q)_x}}$ is then unramified. Since ι is reasonable, we are done.

By definition of the residue maps on $H^d(k(Q), \mathbb{Q}/\mathbb{Z}(d-1))$, it follows that $\iota(q_h) \in H^d_{nr}(k(Q), \mathbb{Q}/\mathbb{Z}(d-1))$. By Proposition 7, there exists $\alpha \in H^d(k, \mathbb{Q}/\mathbb{Z}(d-1))$ such that $\mathrm{Res}_{k(Q)/k}(\alpha) = \iota(q_h)$. Hence we get $\mathrm{Res}_{k(Q)/k}(2\alpha) = 0$, since $\iota(q_h)$ is killed by 2. If $d = 1$, we know that the restriction map is injective, so we get $2\alpha = 0$. Hence α lies in the 2-torsion subgroup of $H^1_{nr}(k(Q), \mathbb{Q}/\mathbb{Z}(d-2))$, that is $\alpha \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. We set $\iota_Q(h) = \alpha$ is this case. If $d \geq 2$, $2\alpha \in [Q] \cup H^{d-2}(k, \mu_2)$, and so $4\alpha = 0$. Thus α lies in the 4-torsion subgroup of $H^d_{nr}(k(Q), \mathbb{Q}/\mathbb{Z}(d-2))$, that is $\alpha \in H^d(k, \mu_4^{\otimes d-1})$. In this case, we define $\iota_Q(h)$ as the class of α modulo $[Q] \cup H^{d-2}(k,_m u_2)$. The uniqueness of $\iota_Q(h)$ follows from Proposition 7. □

Remark 10. If Q is the split quaternion algebra, then the class $\iota_Q(h)$ that we obtain is just $\iota(q_h)$.

Remark 11. If $d = 1$, the class $\iota_Q(h)$ may also be considered as a class in $H^1(k, \mu_4^{\otimes 0}) = H^1(k, \mathbb{Z}/4\mathbb{Z})$ by composing with the injective map $H^1(k, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(k, \mathbb{Z}/4\mathbb{Z})$. We will often do so in order to harmonize notation.

It is immediate that we obtain a natural transformation of functors

$$\iota_Q : \mathbf{F}_Q^- \to H^d(\_, \mu_4^{\otimes d})/[Q] \cup H^{d-2}(\_, \mu_2),$$

called the *Q-twist of ι*.

By definition, ι_Q satisfies the following property: for all field extension L/k and all $h \in \mathbf{F}^-(L/k)$, we have

$$\iota_Q(h) = \mathrm{Res}_{L(Q)/L}(q_{h_{L(Q)}})$$

We will denote by $\mathbf{H}_Q^d[d-1] : \mathfrak{C}_k \to \mathbf{AbGrps}$ the functor defined by

$$\mathbf{H}_Q^d[d-1](L/k) = H^d(L, \mu_4^{\otimes d-1})/[Q_L] \cup H^{d-2}(L, \mu_2).$$

2.3. **Examples.** Let $\mathbf{F} = \mathbf{Quad}_{2n}$, the subfunctor of **Quad** consisting of isomorphism classes of quadratic forms of dimension $2n$, so $\mathbf{F}_Q^-$ is just the subfunctor $\mathbf{Herm}_{n,Q}^-$ of $\mathbf{Herm}_Q^-$ consisting of isomorphism classes of skew-hermitian forms over (Q, γ) of rank n. Since every cohomological invariant of $\mathbf{Quad}_{2n}$ is reasonable, we can twist the Stiefel-Withney classes $w_1, \cdots, w_{2n}$ of quadratic forms of dimension $2n$ to get twisted invariants $w_{1,Q}, \cdots, w_{2n,Q}$, that we still call *Stiefel-Whitney classes.*

Now assume that $\mathbf{F}$ is the subfunctor of $\mathbf{W}$ defined by

$$\mathbf{F}(L/k) = I^n(L), n \geq 1$$

We will denote by $I_{n,Q}^-(L)$ the group $\mathbf{F}_Q^-(L/k)$.

As pointed out before, e_n is a reasonable invariant. Twisting e_n, we get an invariant $e_{n,Q} : I_{n,Q}^- \to \mathbf{H}_Q^n[n-1]$ for all $n \geq 1$.

We also set $e_{0,Q}(h) = \dim(q_{h_{k(Q)}}) \in \mathbb{Z}/4\mathbb{Z}$, so we get an invariant

$$e_{0,Q} : W_Q^- \to \mathbb{Z}/4\mathbb{Z}$$

Notice that we have $e_{0,Q}(h) = 2rk(h) \in \mathbb{Z}/4\mathbb{Z}$.

Remark 12. The reader may wonder why we defined $e_0(h)$ as the class modulo of $\dim(q_{h_{k(Q)}}) = 2rk(h)$ modulo 4, rather than by the class of $rk(h)$ modulo 2. The reason is that we want an invariant of dimension 0 which measures a first obstruction to hyperbolicity, as the dimension modulo 2 does in the case of quadratic forms. In the case of quaternionic skew-hermitian forms, $q_{h_{k(Q)}}$ is always even-dimensional, whatever the parity of $rk(h)$ is. In particular, if h has odd rank, we still have $rk(h) = 0 \in \mathbb{Z}/2\mathbb{Z}$, whereas h is not hyperbolic. However in this case, $\dim(q_{h_{k(Q)}}) \not\equiv 0[4]$.

2.4. **Applications.** We now prove that $(e_{n,Q})_{n\geq 0}$ is a complete system of invariants for quaternionic skew-hermitian forms. Recall that by definition, we have $e_0(h) = \dim(q_{h_{k(Q)}}) \in \mathbb{Z}/4\mathbb{Z}$, and the invariants $e_{n,Q}, n \geq 1$ satisfy the following property: for every non-degenerate skew-hermitian form h over (Q, γ), $e_{n,Q}(h)$ is the unique element of $H^d(k, \mu_4^{\otimes d})/[Q] \cup H^{d-2}(k, \mu_2)$ which is mapped to $e_n(q_{h_{k(Q)}})$ via the restriction homomorphism

$$\mathrm{Res}_{k(Q)/k} : H^d(k, \mu_4^{\otimes d})/[Q] \cup H^{d-2}(k, \mu_2) \to H^d(k(Q), \mu_2)$$

where $q_{h_{k(Q)}}$ is the quadratic form over $k(Q)$ corresponding to $h_{k(Q)}$ via Morita equivalence (see Section 2.2).

Theorem 13. *Let k be a field of characteristic different from 2, and let Q be a quaternion algebra over k. Let h be a non-degenerate skew-hermitian over (Q, γ). Then h is hyperbolic if and only if $e_{n,Q}(h) = 0$ for all $n \geq 0$.*

Proof. Assume that h is hyperbolic. Then h has even rank, so $\dim(q_{h_{k(Q)}})$ is a multiple of 4 and then $e_{0,Q}(h) = 0$. Moreover, since h is hyperbolic, $h_{k(Q)}$ is hyperbolic as well, and so is $q_{h_{k(Q)}}$ by the properties of Morita equivalence. Therefore

$e_n(q_{h_{k(Q)}}) = 0$ for all $n \geq 1$. By definition of $e_{n,Q}$ it means that $e_{n,Q}(h) = 0$ for all $n \geq 1$. Conversely, assume that $e_{n,Q}(h) = 0$ for all $n \geq 0$. Then the quadratic form $q_{h_{k(Q)}}$ is even-dimensional and satisfies

$$e_n(q_{h_{k(Q)}})) = \mathrm{Res}_{k(Q)/k}(e_{n,Q}(h)) = \mathrm{Res}_{k(Q)/k}(0) = 0 \text{ for all } n \geq 1$$

By Milnor's conjecture for quadratic forms (see Section 1.1, Theorem 6), $q_{h_{k(Q)}}$ is hyperbolic. The properties of Morita equivalence imply in turn that $h_{k(Q)}$ is hyperbolic. By [14], Proposition 3.3, we have an injection

$$W^-(Q,\gamma) \hookrightarrow W^-(Q_{k(Q)}, \gamma_{k(Q)})$$

Therefore, h is hyperbolic. □

We now compare the invariants $e_{n,Q}(h), n = 1, 2, 3$ with the invariants $d(h), \mathcal{C}\ell(h)$ and $R(h)$ defined in [2] and [3], or more precisely with their extended versions to skew-hermitians over division algebras with symplectic involutions (see Section 2.1).

Lemma 14. *For every quaternionic skew-hermitian form h, we have*

$$e_1(h) = d(h), e_2(h) =, \mathcal{C}\ell(h) \text{ and } e_3(h) = R(h)$$

Proof. We know that we have

$$\mathrm{disc}(q_{h_{k(Q)}}) = e_1(q_{h_{k(Q)}}) = d(h_{k(Q)}) = \mathrm{Res}_{k(Q)/k}(d(h)) \in k(Q)^\times/k(Q)^{\times 2}$$

Therefore, we get $e_{1,Q}(h) = d(h)$, since $e_{1,Q}(h)$ is the unique class satisfying this property. The properties of the Clifford invariant and the Rost invariant show that we have

$$\mathrm{Res}_{k(Q)/k}(\mathcal{C}\ell(h)) = e_2(q_{h_{k(Q)}}), \mathrm{Res}_{k(Q)/k}(R(h)) = e_3(q_{h_{k(Q)}}),$$

so we get for the same reason as above

$$e_{2,Q}(h) = \mathcal{C}\ell(h), e_{3,Q}(h) = R(h)$$

□

Proposition 15. *Assume that $cd_2(k) \leq 3$, and let h be a hermitian form over a division k-algebra with an orthogonal involution (D, σ) such that* $\mathrm{ind}(D) \leq 2$. *Then h is hyperbolic if and only if it has even rank, trivial discriminant, trivial Clifford invariant and trivial Rost invariant.*

Proof. The direct implication is well-known, and is proved in [2] and [3]. Let us prove the other one. If $D = k$, then h is a quadratic form, and the assumptions on h imply that $e_n(h) = 0$ for $n = 0, 1, 2, 3$. We also have $e_n(h) = 0$ for $n \geq 4$ since $cd_2(k) \leq 3$, hence h is hyperbolic. If $D = Q$ is a division quaternion algebra, the fact that h has even rank implies that $e_{0,Q}(h) = 0$. Morever, the previous lemma show that $e_n(h) = 0$ for $n = 1, 2, 3$ under our assumptions. Finally, since $cd_2(k) \leq 3$, $e_{n,Q}(h) = 0$ for $n \geq 4$, and we conclude by using the previous theorem. □

Conjecture: The previous result remains true without any restriction on $\mathrm{ind}(D)$.

2.5. **Open problems.** We end this paper by listing some questions relative to the invariants and the functors defined in the previous sections.

1) In [6], p.43, it is proved that every comological invariant with values in $H^*(/k, \mu_2)$ can be written in a unique way as a linear combination of the Stiefel-Whitney classes w_i with coefficients in $H^*(k, \mu_2)$.

Is it possible to describe all the cohomological invariants of $\mathbf{Herm}^-_{n,Q}$ with values in $H^*_Q[*-1]$ in terms of the $w_{i,Q}$'s ?

2) For every $n \geq 1$, let $\mathbf{Pf}^-_{n,Q}$ be the subfunctor of $\mathbf{Herm}^-_{2^{n-1},Q}$ defined by

$$\mathbf{Pf}^-_{n,Q}(L/k) = \{h \in \mathbf{Herm}^-_{2^{n-1},Q}(L/k) | q_{h_{k(Q)}} \text{ is a } n\text{-fold Pfister form } \}$$

If Q is a division quaternion algebra and L/k is a field extension, is $I^-_{n,Q}(L/k)$ generated by the Witt classes of the elements of $\mathbf{Pf}^-_{n,Q}(L/k)$?

3) Is $e_{n,Q}$ surjective for every $n \geq 0$? If not, can we describe its image ?

Final remark: The method used here to twist the invariants e_n cannot be generalized to construct a complete system of invariants for skew-hermitian forms over diivsion algebras (D, σ), where σ is symplectic, since the restriction map

$$H^d(k, \mathbb{Q}/\mathbb{Z}(d-1)) \to H^d_{nr}(k(SB(D)), \mathbb{Q}/\mathbb{Z}(d-1))$$

is no longer surjective in general. However, it does not mean necessarily that such twists do not exist. For example, the Rost invariant defined by Bayer-Fluckiger and Parimala proves the existence of a twist for e_3.

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